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Nonequilibrium Phase Transition of Elementary Cellular Automata with a Single Conserved Quantity

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1. Introduction

Conservation laws are one of the most important concepts in physics. In a Hamiltonian dynamical system, the state of which is represented by a point in the phase space, energy is conserved and the motion of the system is restricted on a surface where energy is a constant or, if more conserved quantities exist, a submanifold of lower dimensions. If the system has as many conserved quantities as the degrees of freedom, it is integrable and the motion of the system is completely understood. In many cases, the existence of a conserved quantity is connected with some symmetry by Noether's theorem. There are many cases where finding a hidden symmetry and the corresponding conservation law deepen the understanding of a physical system. Contrastingly, nonexistence of redundant conserved quantities is important in statistical mechanics. It is because statistical mechanics is constructed on the three hypotheses; preservation of phase space volume (Liouville's theorem), energy conservation, and ergodic property of the motion, and the last one implies that the system does not have a conserved quantity except energy and mass. In dissipative systems, energy is no more conserved but the mass or the number of particles is still conserved in many interesting systems including granular matter.

Considering the wide applications of cellular automata to physics, it is natural to study conserved quantities in cellular automata (CA). Hattori and the author gave a necessary and sufficient condition for a CA to have an additive conserved quantity which is a sum of local quantities and whose value does not change under the time evolution of CA (Hattori and Takesue 1991). Exact definition of the additive conserved quantity will be given in Section 2. In that paper, the condition was applied to Wolfram's elementary CA (ECA) and their reversible variants, elementary reversible CA (ERCA) and the table of additive conserved quantities for every rule was obtained. Generalization to staggered quantities was done in (Takesue 1995). These works clarified that the number of conserved quantities depends on rules. Moreover, it is often the case that not only the sum but also each summand itself is conserved (Takesue 1989). It corresponds to class 2 in Wolfram's classification, which is characterized by separated simple stable or periodic structures. In the light of physics, we are more interested in such a CA that has a small number of additive conserved quantities whose summand is not invariant.

Dynamical behavior of the conserved quantities is very different between reversible and irreversible CA. Reversible CA satisfy the preservation of the phase space volume owing to the discreteness of states. Thus, by regarding an additive conserved quantity as a Hamiltonian, we can construct Gibbs statistical mechanics on the CA. Then its connection with dynamics is an interesting subject. The formation of the canonical distribution in a subsystem (Takesue 1987), Fourier's law and Kubo formula for the thermal conduction (Takesue 1990a), relaxation to equilibrium (Takesue 1990b), and Boltzmann-type equations (Takesue 1997) were discussed for ERCA. In particular, two rules (rules 26R and 94R) were found to show diffusive behavior in the macroscopic scale. Furthermore, we can devise two-dimensional CA which conserve the Ising Hamiltonian with or without other degrees of freedom (Creutz 1985; Vichniac 1984). Using those CA, dynamics at the critical point was studied (Saito et al. 1999).

Concerning irreversible CA, additive conserved quantities should be considered as the particle number rather than energy. In this context, number-conserving CA, where the number of 1s is conserved, are well studied. One of the interesting behavior is density classification. Originally, density classification problem meant searching a CA that has the following property: If an initial density of 1s exceeds a given threshold, the CA evolves into the state of all 1s, and if the density is below the threshold, the state of all 0s is reached. A number of rules were proposed, but finally it was proved that perfect classification is impossible for one-dimensional two-state CA (Land and Belew 1995). However, M. S. Capcarrere et al found that ECA rule 184 in Wolfram's notation performs the classification if the output condition is loosened (Capcarrere 2001). Rule 184 is a number-conserving CA. Thus, there must be a block 11 if the density of 1s is above $1/2$ and a block 00 if the density is below $1/2$. Because rule 184 tends to place 0 and 1 as alternately as possible, blocks 11 disappear and one or more block 00 remain below $1/2$ by the time $N/2$, where N is the system size. Therefore the presence of a block 00 after $N/2$ indicates that the density is below $1/2$. Similarly, blocks 11 indicates the density is above $1/2$. Note that the behavior is observed under the cyclic boundary condition. In a recent paper (Takesue 2008), the author showed that ECAs with a single additive conserved quantity classify the density of the conserved quantity. Moreover, the paper showed preliminary results that the same rules can show a kind of nonequilibrium phase transition when some stochastic boundary conditions are employed. In this chapter, we will focus on the latter phenomenon in the ECAs.

In the next section, the additive conserved quantities are defined and the conservation condition is derived. The condition is applied to the ECAs to find the rules with a single additive conserved quantity. The nonequilibrium phase transition discussed here is originally found in a continuous Markov chain called the asymmetric simple exclusion process (ASEP). In Section 3, the ASEP and its phase transition are introduced and the mechanism of the phase transition is clarified by the so-called domain wall theory. We discuss what stochastic boundary condition is suited for the ECAs in Section 4. In Section 5, the probability distribution of patterns are calculated and the domain wall theory is applied to the ECAs. Section 6 describes diffusive behavior of the domain wall observed just on the phase transition line. The last section is devoted to discussion and conclusion.

2. Elementary CA and conserved quantities

Wolfram's elementary cellular automata (ECA) (Wolfram 1983) are a class of one-dimensional cellular automata with two possible states for each cell and local update rules which depend only on three neighbor cells. That is, if $x_i^t \in \{0, 1\}$ denotes the value of cell i at time t , the

evolution of an elementary CA is written as

$$x_i^{t+1} = f(x_{i-1}^t x_i^t x_{i+1}^t) \quad (1)$$

with some fixed function $f : \{0,1\}^3 \rightarrow \{0,1\}$. Thus there are $2^{2^3} = 256$ different rules in the ECAs, but the rules that are transformed into each other by the left-right inversion or the exchange of 0 and 1 or their composite are isomorphic and accordingly the ECA are classified into 88 equivalence classes.

Let us impose the cyclic boundary condition of period N on the ECA and denote the configuration at time t by $x^t = (x_0^t x_1^t \dots x_{N-1}^t)$. Now we consider a function of x^t of the form

$$\Phi(x^t) = \sum_{i=0}^{N-1} E(x_i^t x_{i+1}^t \dots x_{i+k}^t) \quad (2)$$

where $E(x_0 x_1 \dots x_k)$ is some function of $k+1$ variables. If $\Phi(x^1) = \Phi(x^0)$ holds for any $x^0 \in \{0,1\}^N$, this quantity Φ is called an additive conserved quantity of range k and the corresponding E the conserved density.

In (Hattori and Takesue 1991) a necessary and sufficient condition for E to be a conserved density was derived as follows. First, assume that Φ is an additive conserved quantity of range k . Then, for any $x = (x_i)$, the following equality must hold:

$$\sum_{i=0}^{N-1} [G(x_i x_{i+1} \dots x_{i+k+2}) - E(x_i \dots x_{i+k})] = 0, \quad (3)$$

where G is the function of $k+3$ variables defined as

$$G(x_0 x_1 \dots x_{k+2}) = E(f(x_0 x_1 x_2) f(x_1 x_2 x_3) \dots f(x_k x_{k+1} x_{k+2})) \quad (4)$$

and indices are understood mod N . Equality (3) holds if we assume $x_0 = 0$.

$$\sum_{i=0}^{N-1} [G(x_i x_{i+1} \dots x_{i+k+2}) - E(x_i \dots x_{i+k})] \Big|_{x_0=0} = 0. \quad (5)$$

Subtraction of Eq. (5) from Eq. (3) leaves only an N -independent number of terms.

$$\sum_{i=0}^{k+2} [G(x_{i-k-2} \dots x_i) - G(x_{i-k-2} \dots x_{-1} 0 x_1 \dots x_i)] + \sum_{i=0}^k [E(x_{i-k} \dots x_i) - E(x_{i-k} \dots x_{-1} 0 x_1 \dots x_i)] = 0 \quad (6)$$

We can further put $x_{-k} = x_{-k+1} = \dots = x_{-1} = 0$ and utilize $G(00\dots 0) = E(0\dots 0)$ to obtain the following equality,

$$\begin{aligned} G(x_0 x_1 \dots x_{k+2}) - E(x_0 x_1 \dots x_k) &= \sum_{i=0}^{k+1} [-G(0\dots 0 x_0 x_1 \dots x_i) + G(0\dots 0 x_1 x_2 \dots x_{i+1})] \\ &\quad + \sum_{i=1}^k [E(0\dots 0 x_0 x_1 \dots x_{i-1}) - E(0\dots 0 x_1 x_2 \dots x_i)]. \end{aligned} \quad (7)$$

Clearly, this is a necessary condition for E to be a conserved density. In fact, it is also a sufficient condition, because Eq. (7) can be rewritten in the form of equation of continuity

$$E(x_i^{t+1} x_{i+1}^{t+1} \dots x_{i+k}^{t+1}) - E(x_i^t x_{i+1}^t \dots x_{i+k}^t) = J(x_{i-1}^t x_i^t \dots x_{i+k}^t) - J(x_i^t x_{i+1}^t \dots x_{i+k+1}^t), \quad (8)$$

where current function J is defined by

$$J(x_0x_1 \dots x_{k+1}) = \sum_{i=0}^{k+1} [E(0 \dots 0x_0x_1 \dots x_{i-1}) - G(0 \dots 0x_0x_1 \dots x_i)]. \tag{9}$$

Therefore, Eq. (7) is a necessary and sufficient condition, which is called the Hattori-Takesue condition.

It is evident that $E(x_0x_1 \dots x_k) = S(x_0 \dots x_{k-1}) - S(x_1 \dots x_k)$ leads to $\Phi(x) = 0$ for any function $S(x_0 \dots x_{k-1})$. To remove such trivial solutions from Eq.(7), we can assume that $E(0x_1 \dots x_k) = 0$ for any $(x_1 \dots x_k)$, and then the conservation condition is simplified as

$$G(x_0 \dots x_{k+2}) - E(x_0 \dots x_k) = \sum_{i=0}^{k+1} [G(0 \dots 0x_1x_2 \dots x_{k+2-i}) - G(0 \dots 0x_0x_1 \dots x_{k+1-i})]. \tag{10}$$

In some cases, however, E with the condition $E(0x_1 \dots p_k) = 0$ is not convenient for use and adding some surface term $S(x_0 \dots x_{k-1}) - S(x_1 \dots x_k)$ to it is preferable. Solutions of the equations (10) forms a vector space in the function space and we refer to its dimension as the number of additive conserved quantities of range k . This number increases with k , because the additive conserved quantities of range k include those of smaller ranges.

The conservation condition was generalized to staggered invariants, where factor $(-1)^i$ and/or $(-1)^t$ is introduced in the rhs of Eq.(2) (Takesue 1995). The numbers of additive and staggered conserved quantities of range $k = 6$ were listed for all 88 equivalence classes in that paper and those of $k = 9$ in (Takesue 2008). The numbers of conserved quantities depend on the rules. Some rules do not have a conserved quantity, some others have more than one, and the rules 11, 14, 35, 43, 56, 142, and 184 (and the rules isomorphic to them) have only one conserved quantity for each. Namely, each of them has a single additive conserved quantity, whose density E is not conserved, and no staggered invariants. Their respective conserved densities are listed in Table 1.

Rules	Conserved density
184	$E(x) = x$
14, 35, 43, 142	$E(xy) = (x - y)^2$
56	$E(xyz) = x + y + z - 3xyz$
11	$E(xyzw) = x(1 - y)[1 - z(1 - w)]$

Table 1. ECA rules with a single conserved quantity and their conserved densities

3. Phase transition in ASEP

The nonequilibrium phase transition which we will discuss was originally found in stochastic particle systems on a lattice. The famous example is the asymmetric simple exclusion process (ASEP) with open boundaries(Derrida et al. 1993; Sasamoto 1999?). It resembles ECA in one dimension and two possible states of a cell but time is continuous and the dynamics is stochastic. Consider a one-dimensional lattice composed of N cells. Each cell i is occupied by a particle ($\tau_i = 1$) or empty ($\tau_i = 0$). During an infinitesimal time interval dt each particle can hop to the right neighbor with probability dt and to the left neighbor with probability qdt , where q is a nonnegative number less than 1, provided that the destination is empty. Moreover, a particle is added to cell 1 with probability αdt if the cell is empty. Similarly, a particle is removed from cell N with probability βdt if the cell is occupied. From any initial

condition, the system goes to a steady state after some relaxation time. Since the steady state has a nonzero rightgoing current, it represents a nonequilibrium steady state with particle flow. The probability distribution of the system in the steady state was exactly obtained for the case $q = 0$, which is called the totally asymmetric exclusion process (TASEP), by Derrida et al (Derrida et al. 1993) using the method of matrix products. The result was extended to $q \neq 0$ by Sasamoto (Sasamoto 1999) and to the case where a particle can enter or exit from both ends by Uchiyama et al (Uchiyama 2004). It is remarkable that the ASEP shows phase transitions depending on the parameters for the boundary condition. Figure 1 shows the phase diagram of the TASEP. Region A ($\alpha < 1/2$ and $\alpha < \beta$) represents the low-density phase where the particle density is $\langle \tau_i \rangle \simeq \alpha$ and the current is $\alpha(1 - \alpha)$. Region B ($\beta < 1/2$, $\alpha > \beta$) is the high-density phase where the density is $\langle \tau_i \rangle \simeq 1 - \beta$ and the current is $\beta(1 - \beta)$. Region C ($\alpha > 1/2$, $\beta > 1/2$) is the maximal current phase where the density is $1/2$ and the current is $1/4$. The density changes discontinuously across the line $0 < \alpha = \beta < 1/2$. Thus, it is called the line of the first-order phase transition. The transitions between regions A and C and between B and C are the second-order, because the change of density is continuous. Similar phase diagrams are obtained for the general ASEP.

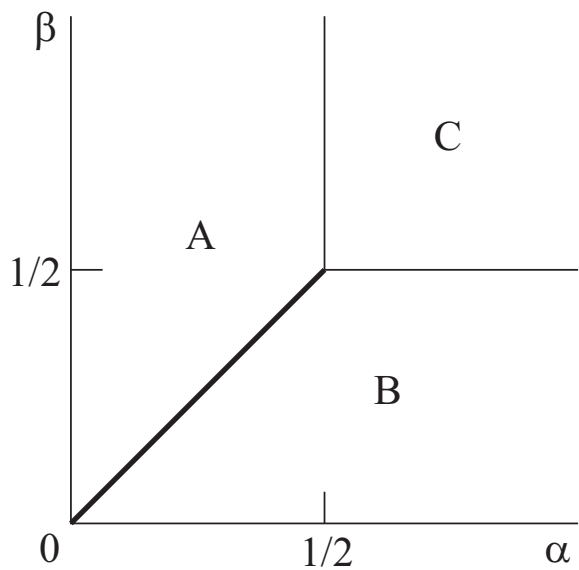


Fig. 1. The phase diagram of the TASEP. There are three phases: low-density phase (A), high-density phase (B) and the maximal current phase (C). The phase transition between the low and high density phases is the first-order and the particle density changes discontinuously. The phase transition between A and C and that between B and C are the second-order and the density changes continuously.

The mechanism of the phase transition is understood in terms of the behavior of a phase boundary. This is called the domain wall theory (Kolomeisky et al. 1998). Now we do not see the 0-1 sequence or discrete-time dynamics but consider coarse-grained density variation of the ASEP. That is, dynamical behavior of a cell is replaced by continuous change of locally averaged density profile. After some transient relaxation process, a part of the system can be regarded as belonging to one of the three phases. Let us assume that the system is composed of two different phases and that the left phase has density ρ_L and current J_L and the right phase has density ρ_R and current J_R . Then the velocity of the phase boundary (domain wall), V , is obtained in the same manner as the theory of shock wave in fluid mechanics and the

result is

$$V = \frac{J_R - J_L}{\rho_R - \rho_L}. \quad (11)$$

This is an outcome of the conservation law. If V is positive, the domain wall goes to the right end and the left phase prevails in the system. Conversely, if V is negative, the right phase dominates the system. Accordingly, the phase transition occurs when $V = 0$. This argument applies to the TASEP. Assume that the left phase is the low-density phase and the right one is the high-density phase. Then, $V = \beta - \alpha$. This means that the transition occurs at $\alpha = \beta$. The transition lines between the maximal current phase and the other phases are also successfully explained in the similar manner.

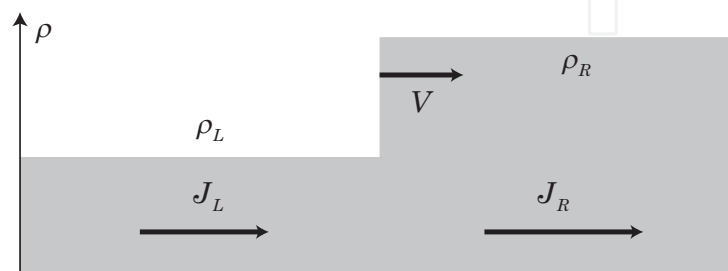


Fig. 2. Velocity of the domain wall is determined by the densities and currents in the two phase.

4. Stochastic boundary condition for the ECA

We will see that the ECAs with a single additive conserved quantity exhibit nonequilibrium phase transition of the same type as in the ASEP if an appropriate open boundary condition is employed. The first example is rule 184. This rule conserves the number of 1s as the ASEP does. Thus the following stochastic evolution is naturally devised. Consider the system of $N + 2$ cells, which are numbered from 0 through $N + 1$. In the evolution from time t to $t + 1$, the states of the cells 1 through N are updated according to the rule (1). For cells 0 and $N + 1$, the states are chosen with probability as

$$x_0^{t+1} = \begin{cases} 1 & \text{with probability } \alpha \\ 0 & \text{with probability } 1 - \alpha \end{cases}, \quad x_{N+1}^{t+1} = \begin{cases} 1 & \text{with probability } 1 - \beta \\ 0 & \text{with probability } \beta \end{cases}. \quad (12)$$

This is equivalent to the evolution of probability distribution as

$$p^{t+1}(x) = p_L(x_0)p_R(x_{N+1}) \sum_{x'_0, \dots, x'_{N+1}} \prod_{i=1}^N \delta(x_i, f(x'_{i-1}x'_i x'_{i+1})) p^t(x') \quad (13)$$

where $p^t(x)$ denotes the probability that $x^t = x = (x_i)_{0 \leq i \leq N+1}$, $\delta(x, y)$ is Kronecker's delta $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ if $x \neq y$, $p_L(u) = \alpha u + (1 - \alpha)(1 - u)$ and $p_R(v) = (1 - \beta)v + \beta(1 - v)$. For various α and β , we numerically computed time averages of the density of 1s, $\rho = N^{-1} \sum_{i=1}^N \langle x_i \rangle$, and current $J = (N - 1)^{-1} \sum_{i=1}^{N-1} \langle x_i(1 - x_{i+1}) \rangle$ in the steady states, where $\langle \cdot \rangle$ represents the time average. The result is shown in Fig. 3. Just as in the ASEP, region $\alpha < \beta$ is the low-density phase, where $\rho = \alpha$, $J = \alpha(1 - \alpha)$ and region $\alpha > \beta$ is the high-density phase, where $\rho = 1 - \beta$, $J = \beta(1 - \beta)$. The first order phase transition occurs at line $\alpha = \beta$. Note that the maximal current phase does not appear in the ECA.

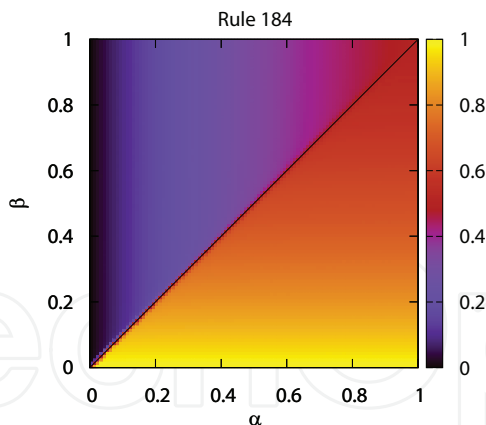


Fig. 3. Phase diagrams of rule 184 with the stochastic boundary condition (13). The average density of 1s is illustrated. The phase transition line $\alpha = \beta$ is also depicted.

We can adapt the boundary condition to the ECA with conserved density $E(xy) = (x - y)^2$ as

$$x_0^{t+1} = \begin{cases} 1 - x_1^{t+1} & \text{with probability } \alpha \\ x_1^{t+1} & \text{with probability } 1 - \alpha \end{cases}, \quad x_{N+1}^{t+1} = \begin{cases} 1 - x_N^{t+1} & \text{with probability } 1 - \beta \\ x_N^{t+1} & \text{with probability } \beta \end{cases} \quad (14)$$

This is equivalent to the following evolution of probability distributions.

$$p^{t+1}(x) = p_L(x_0|x_1)p_R(x_{N+1}|x_N) \sum_{x'_0, \dots, x'_{N+1}} \prod_{i=1}^N \delta(x_i, f(x'_{i-1}x'_ix'_{i+1})) p^t(x'), \quad (15)$$

where the conditional probabilities p_L and p_R are defined as

$$p_L(u|v) = \alpha \delta(E(uv), 1) + (1 - \alpha) \delta(E(uv), 0), \quad p_R(u|v) = (1 - \beta) \delta(E(vu), 1) + \beta \delta(E(vu), 0). \quad (16)$$

As shown in Table 1, the rules with the conserved density are rules 14, 35, 43 and 142. However, rule 43 is equivalent to rule 184 by block transformation $00, 11 \rightarrow 0$ and $01, 10 \rightarrow 1$. Namely, if we denote the transformation by b , the following equality holds:

$$b(f_{43}(x_0x_1x_2)f_{43}(x_1x_2x_3)) = f_{184}(b(x_0x_1)b(x_1x_2)b(x_2x_3)), \quad (17)$$

where f_{43} and f_{184} are respective rule functions. Thus, rule 43 of size $N + 2$ with the stochastic boundary condition (15) is transformed into rule 184 of size $N + 1$ with the stochastic boundary condition (13) and accordingly exhibits the same nonequilibrium phase transition. Similarly, rule 142 is transformed into rule 184 by another block transformation $01, 10 \rightarrow 0$ and $00, 11 \rightarrow 1$. In this case, the transformation is accompanied with change of parameter values $\alpha \rightarrow 1 - \alpha$, $\beta \rightarrow 1 - \beta$. The phase transition of the same type occurs in this rule also. Rules 14 and 35 are not equivalent to rule 184 and are considered in the next section. We only mention here that the current function for rule 14 is $J(xyz) = -xy - (1 - x)(1 - y)z$, and that for rule 35 is $J(xyz) = x(1 - y)(1 - z)$.

For rules 56, we must have further consideration. The additive conserved quantity for this rule has range 2. To make the conditional probability for boundary variables depend on E , there must be more than two stochastic variables for each boundary. One way is using more stochastic variables like $p_L(x_{-1}x_0|x_1)$ and the other way is increasing the variables of conditions like $p_L(x_0|x_1x_2)$. In either way, there is another problem. The conserved density

can take three values 0, 1 and 2. If we want to study general cases, two parameters per boundary are necessary. Thus, the boundary condition can be more complicated than Eqs. (13) or (15). However, we can avoid these problems. Carefully looking at the time evolution of rule 56, one can notice that block 111 does not appear in time $t \geq 1$. In fact, no preimages for block 111 exist for rule 56. This means that $E(xyz) = x + y + z - 3xyz$ is equivalent to $E(xyz) = x + y + z$ for $t \geq 1$. Thus, rule 56 conserves the number of 1s for $t \geq 1$. Such a quantity was called an eventually conserved quantity in (Hattori and Takesue 1991). In the case of stochastic boundary condition, block 111 can appear only at the ends of the system. No 111 appears in the interior of the system. Therefore, we can use the boundary condition (13) for rule 56. The rule function $f_{56} = x(1 - y) + (1 - x)yz$ satisfies the following equation,

$$f_{56}(xyz) - y = x(1 - y) - y(1 - z) - xyz. \quad (18)$$

This is interpreted as that the current function for the eventually conserved quantity is $J(xy) = x(1 - y)$.

Rule 11 has an eventually conserved quantity, too. In this case, block 101 has no preimages and under the absence of 101, $E(xy) = (x - y)^2$ becomes a conserved density. The corresponding current function is $J(xyz) = x(1 - y) + (1 - x)yz$. Therefore, the boundary condition (15) is appropriate to this rule.

5. Application of the domain wall theory

The domain wall theory is applied to the ECA with the stochastic boundary condition as follows. First we try to obtain the probability distribution of patterns of size three in the stationary states. Let p_i denote the probability distribution of block pattern starting from cell i . For example, $p_i(000)$ means that the probability that $x_i x_{i+1} x_{i+2} = 000$ and $p_i(0101)$ is the probability that $x_i x_{i+1} x_{i+2} x_{i+3} = 0101$. In the stationary state, those probabilities satisfy the following equations

$$p_i(x_1 x_2 x_3) = \sum_{x'_0, x'_1, x'_2, x'_3, x'_4} \delta(x_1 x_2 x_3, f(x'_0 x'_1 x'_2) f(x'_1 x'_2 x'_3) f(x'_2 x'_3 x'_4)) p_{i-1}(x'_0 x'_1 x'_2 x'_3 x'_4). \quad (19)$$

Assuming uniformity for p_i and introducing decoupling approximation for the probability distribution of larger size if necessary, we can obtain a set of stationary solutions which contain the average density of the additive conserved quantity as a parameter. In particular, it is useful to use the logic that if $p(x_0 x_1 x_2 x_3) = 0$, we must have $p(x_0 x_1 x_2) = 0$ or $p(x_1 x_2 x_3) = 0$ to close the equations consistently. Next, we try to adapt the solution to the left and right boundary conditions. If the boundary condition (15) is employed, the following equations must hold at the left boundary, namely

$$p_1(x_1 x_2 x_3) = \sum_{x'_0, x'_1, x'_2, x'_3, x'_4} \delta(x_1 x_2 x_3, f(x'_0 x'_1 x'_2) f(x'_1 x'_2 x'_3) f(x'_2 x'_3 x'_4)) p_0(x'_0 x'_1 x'_2 x'_3 x'_4) \quad (20)$$

and

$$p_0(x_0 x_1 x_2 x_3 x_4) = p_L(x_0 | x_1) p_1(x_1 x_2 x_3 x_4). \quad (21)$$

By solving the equations, the distribution in the left phase is determined and the average density ρ_L and current J_L of the additive conserved quantity are obtained as functions of parameter α . The right phase is determined in the similar manner and the average density ρ_R and the average current J_R is computed as functions of β . Once these quantities are obtained,

we can apply the domain wall theory and the line of phase transition is calculated from the condition $J_L = J_R$.

In (Takesue 2008) we showed how the above procedure works for rule 184. In that case, use of the probability distributions of size two is sufficient. In the following, we will discuss the four rules 35, 14, 56 and 11 in this order.

5.1 Rule 35

If we assume uniformity, Eq. (19) becomes

$$\begin{aligned} p(000) &= p(1100) + p(111), & p(001) &= p(100) + p(1101), \\ p(010) &= p(101) + p(1001), & p(011) &= p(1000), \\ p(100) &= p(011) + p(0100), & p(101) &= p(0101), \\ p(110) &= p(0001), & p(111) &= p(0000). \end{aligned} \quad (22)$$

The sixth equation means $p(1101) = 0$, so we must have $p(110) = 0$ or $p(101) = 0$. In the former case, the first and second equations mean that $p(000) = p(111)$ and $p(001) = p(100)$. Then, $p(0001) = p(1000) = 0$ is obtained from the eighth equation. Therefore, we have $p(000) = p(011) = p(111) = 0$ and the remainings are determined as

$$p(001) = p(100) = 1 - \rho, \quad p(010) = \frac{\rho}{2}, \quad p(101) = \frac{3}{2}\rho - 1, \quad (23)$$

where ρ is the expectation value of $E(xy)$ and this solution makes sense only when $\frac{2}{3} \leq \rho \leq 1$.

In the latter case, using approximation $p(0000) = \frac{p(000)^2}{p(00)}$, we arrive at

$$\begin{aligned} p(000) &= \frac{2-3\rho}{4}, & p(001) &= \frac{\rho}{2}, & p(010) &= \frac{\rho^2}{2-\rho}, & p(011) &= \frac{\rho(2-3\rho)}{2(2-\rho)}, \\ p(100) &= \frac{\rho}{2}, & p(101) &= 0, & p(110) &= \frac{\rho(2-3\rho)}{2(2-\rho)}, & p(111) &= \frac{(2-3\rho)^2}{4(2-\rho)}. \end{aligned} \quad (24)$$

This is the solution for $0 \leq \rho \leq \frac{2}{3}$.

The connection condition (20) is written as

$$\begin{aligned} p_1(000) &= p_1(1100) + p_1(111), & p_1(001) &= p_1(100) + p_1(1101), \\ p_1(010) &= p_1(101) + p_0(1001), & p_1(011) &= p_0(1000), \\ p_1(100) &= p_1(011) + p_1(0100), & p_1(101) &= p_1(0101), \\ p_1(110) &= p_0(0001), & p_1(111) &= p_0(0000). \end{aligned} \quad (25)$$

As in Eq. (21), p_0 is written as $p_0(xyzw) = (\alpha\delta_{x,1-y} + (1-\alpha)\delta_{xy})p_1(yzw)$. Then substitution of the solution for $0 \leq \rho \leq \frac{2}{3}$, (23), into p_1 satisfies the above equations if $\rho = \rho_L = \frac{2\alpha}{2+\alpha}$. Notice that as α varies from 0 to 1, ρ_L varies from 0 to $\frac{2}{3}$. The stationary solution for $\rho \geq \frac{2}{3}$, (24), cannot satisfy the above equations. At the right boundary $p_{N-2}(x_{N-2}x_{N-1}x_N)$ must satisfy

$$\begin{aligned} p_{N-2}(000) &= p_{N-2}(1100) + p_{N-2}(111), & p_{N-2}(001) &= p_{N-2}(100) + p_{N-2}(1101), \\ p_{N-2}(010) &= p_{N-2}(101) + p_{N-3}(1001), & p_{N-2}(011) &= p_{N-3}(1000), \\ p_{N-2}(100) &= p_{N-2}(011) + p_{N-2}(0100), & p_{N-2}(101) &= p_{N-2}(0101), \\ p_{N-2}(110) &= p_{N-3}(0001), & p_{N-2}(111) &= p_{N-3}(0000). \end{aligned} \quad (26)$$

Here the boundary condition is $p_{N-2}(xyzw) = [\beta\delta_{zw} + (1-\beta)\delta_{z,1-w}]p_{N-2}(xyz)$. In this case substitution of (24) into p_{N-2} satisfies the equations if $\rho = \rho_R = \frac{2}{2+\beta}$. Because $J(xyz) = x(1-y)(1-z)$, the flux in the left phase is $J_L = p_1(100) = \frac{\alpha}{2+\alpha}$ and that in the right phase is $J_R = p_{N-2}(100) = \frac{\beta}{2+\beta}$. Then, the velocity of the domain wall is obtained as

$$V = \frac{J_R - J_L}{\rho_R - \rho_L} = \frac{\beta - \alpha}{2 - \alpha - \alpha\beta} \quad (27)$$

Thus, if $\beta > \alpha$ the left phase prevails, and if $\beta < \alpha$ the right phase does.

5.2 Rule 14

Equations for a uniform stationary state are

$$\begin{aligned} p(000) &= p(111) + p(0000) + p(11000), & p(001) &= p(0001) + p(1101) + p(11001), \\ p(010) &= p(101), & p(011) &= p(001), \\ p(100) &= p(011) + p(01000), & p(101) &= p(0101) + p(01001), \\ p(110) &= p(001), & p(111) &= 0. \end{aligned} \quad (28)$$

The first and the eighth equations imply that $p(01000) = 0$. Thus, $p(010) = 0$ or $p(100) = 0$ or $p(000) = 0$ must be satisfied to obtain a consistent solution. Case $p(010) = 0$ leads to

$$p(010) = p(101) = p(111) = 0, \quad p(001) = p(011) = p(100) = p(110) = \frac{\rho}{2}, \quad p(000) = 1 - 2\rho, \quad (29)$$

which is the solution for $0 \leq \rho \leq \frac{1}{2}$. Case $p(100) = 0$ leads to the solution $p(101) = p(010) = 2\rho$, $p(000) = 1 - 2\rho$ and the other entries are zeroes. In this case, block 000 and the other two cannot coexist in a system, because if 000 coexists with some 1s, 001 and 100 should have nonzero probability. Thus, this solution is not ergodic in the sense that time average for a system does not agree with expectation with respect to this probability distribution. Because we are interested in ergodic distribution only, we do not adopt this solution. In Case $p(000) = 0$ we have the following solution

$$p(000) = p(111) = 0, \quad p(001) = p(011) = p(100) = p(110) = \frac{1-\rho}{2}, \quad p(010) = p(101) = \rho - \frac{1}{2} \quad (30)$$

It corresponds to case $\frac{1}{2} \leq \rho \leq 1$. The connection condition at the right end is given as

$$\begin{aligned} p_{N-2}(000) &= p_{N-3}(111) + \beta[p_{N-2}(000) + p_{N-3}(1100)], \\ p_{N-2}(001) &= p_{N-3}(1101) + (1-\beta)[p_{N-2}(000) + p_{N-3}(1100)], \\ p_{N-2}(010) &= p_{N-3}(101), & p_{N-2}(011) &= p_{N-2}(001), \\ p_{N-2}(100) &= p_{N-3}(100) + \beta p_{N-3}(0100), \\ p_{N-2}(101) &= p_{N-3}(0101) + (1-\beta)p_{N-3}(0100), \\ p_{N-2}(110) &= p_{N-3}(001), & p_{N-2}(111) &= 0. \end{aligned} \quad (31)$$

Substitution of the stationary solution for $0 \leq \rho \leq \frac{1}{2}$, (29), into p_{N-2} satisfies these equations and the density of the conserved quantity is obtained as $\rho = \rho_R = \frac{2(1-\beta)}{4-3\beta}$. In addition, the average current is given as

$$J_R = -\rho_R = -\frac{2(1-\beta)}{4-3\beta}. \quad (32)$$

The connection condition at the left end is given as

$$\begin{aligned} p_1(000) &= (1 - \alpha)(p_1(111) + p_1(1000)) + p_1(0000), \\ p_1(001) &= p_1(0001) + (1 - \alpha)(p_1(101) + p_1(1001)), \\ p_1(010) &= \alpha p_1(01), & p_1(011) &= p_1(001), \\ p_1(100) &= \alpha(p_1(11) + p_1(1000)), & p_1(101) &= \alpha(p_1(101) + p_1(1001)), \\ p_1(110) &= (1 - \alpha)p_1(01), & p_1(111) &= 0. \end{aligned} \quad (33)$$

In this case, if either of the uniform stationary solutions is inserted into p_1 , the equations are not satisfied. Instead, if we assume that p_2 equals the stationary solution for $\frac{1}{2} \leq \rho \leq 1$, we have

$$\begin{aligned} p_1(000) &= \frac{(1 - \alpha)^2}{4 - 3\alpha + \alpha^2}, & p_1(001) &= \frac{1 - \alpha}{4 - 3\alpha + \alpha^2}, \\ p_1(010) &= \frac{\alpha}{4 - 3\alpha + \alpha^2}, & p_1(011) &= \frac{1 - \alpha}{4 - 3\alpha + \alpha^2}, \\ p_1(100) &= \frac{\alpha(1 - \alpha)}{4 - 3\alpha + \alpha^2}, & p_1(101) &= \frac{\alpha^2}{4 - 3\alpha + \alpha^2}, \\ p_1(110) &= \frac{1 - \alpha}{4 - 3\alpha + \alpha^2}, & p_1(111) &= 0. \end{aligned} \quad (34)$$

Note that only the property $p_2(000) = 0$ has been used to derive the above. The relation between α and ρ should be obtained by $p_1(000) + p_1(100) = p_2(000) + p_2(001) = \frac{1-\rho}{2}$, which leads to

$$\rho = \rho_L = \frac{2 - \alpha + \alpha^2}{4 - 3\alpha + \alpha^2}. \quad (35)$$

However, this is inconsistent with another condition $p_1(001) + p_1(101) = p_2(010) + p_2(011) = \frac{\rho}{2}$. Namely, the two conditions cannot be satisfied at the same time by the uniform solution. This inconsistency is resolved by considering a periodic solution for the stationary distribution. If we assume period two for the distribution and denote the distribution function starting from an even-numbered cell by p_e and that starting from an odd-numbered cell by p_o , Eqs. 28 are replaced with

$$\begin{aligned} p_e(000) &= p_o(111) + p_e(0000) + p_o(10000), & p_e(001) &= p_e(0001) + p_o(1101) + p_o(11001), \\ p_e(010) &= p_o(101), & p_e(011) &= p_e(001), \\ p_e(100) &= p_o(011) + p_o(01000), & p_e(101) &= p_o(0101) + p_o(01001), \\ p_e(110) &= p_o(001) & p_e(111) &= 0 \end{aligned} \quad (36)$$

and those with p_e and p_o interchanged. For $\rho \geq \frac{1}{2}$, we have the solution similar to (30) except

$$p_e(010) = p_o(101) = \rho - \frac{1}{2} + \epsilon, \quad p_e(101) = p_o(010) = \rho - \frac{1}{2} - \epsilon, \quad (37)$$

where ϵ is a parameter satisfying $-(\rho - 1/2) \leq \epsilon \leq \rho - 1/2$. Then, if we assume $p_2 = p_e$ with

$$\epsilon = -\frac{\alpha(1 - \alpha)}{2(4 - 3\alpha + \alpha^2)}, \quad (38)$$

the solution is consistently connected to Eq. 34. Thus, the average density (35) is still correct and the average current in the left phase is given by

$$J_L = -p_e(001) - p_e(110) - p_e(111) = -\frac{2(1-\alpha)}{4-3\alpha+\alpha^2} \quad (39)$$

The phase transition line is calculated via $J_L = J_R$ and the result is

$$\beta = \frac{\alpha(1+\alpha)}{1+\alpha^2}. \quad (40)$$

5.3 Rule 56

Equations for a uniform stationary state are

$$\begin{aligned} p(000) &= p(0000) + p(1111) + p(00010), & p(001) &= p(0010) + p(1110) + p(00011), \\ p(010) &= p(0011) + p(100) + p(1010), & p(011) &= p(1011), \\ p(100) &= p(1000) + p(0111) + p(10010), & p(101) &= p(0110) + p(1010) + p(10011), \\ p(110) &= p(1011), & p(111) &= 0. \end{aligned} \quad (41)$$

The first and the eighth equations leads to $p(00011) = 0$ and the second equation means $p(10011) = 0$. Thus $p(0011) = 0$ must hold. This implies that $p(001) = 0$ or $p(011) = 0$. In the former case, we have $p(001) = p(100) = p(111) = 0$. Then, for the state to be ergodic $p(000)$ must vanish and the remainings are determined as

$$p(010) = 2 - 3\rho, \quad p(011) = p(110) = 2\rho - 1, \quad p(101) = 1 - \rho, \quad (42)$$

where ρ is the density of 1s. This solution makes sense when $\frac{1}{2} \leq \rho \leq \frac{2}{3}$. Note that $p(111) = 0$ means the maximum value of ρ is $\frac{2}{3}$. In the latter case, $p(011) = p(110) = p(111) = 0$ and the others are

$$p(000) = 1 - 2\rho - a, \quad p(001) = p(100) = a, \quad p(010) = \rho, \quad p(101) = \rho - a, \quad (43)$$

where a is a real number in the region $0 \leq a \leq \min(\rho, 1 - 2\rho)$. This is the solution for $0 \leq \rho \leq \frac{1}{2}$. The connection at the left end is given by

$$\begin{aligned} p_1(000) &= (1-\alpha)[p_1(000) + p_1(0010)], \\ p_1(001) &= (1-\alpha)[p_1(010) + p_1(0011)] + \alpha p_1(110), \\ p_1(010) &= (1-\alpha)p_1(011) + p_1(100) + p_1(1010), & p_1(011) &= p_1(1011), \\ p_1(100) &= \alpha[p_1(000) + p_1(0010)], \\ p_1(101) &= \alpha[p_1(010) + p_1(0011)] + (1-\alpha)p_1(110), \\ p_1(110) &= \alpha p_1(011), & p_1(111) &= 0. \end{aligned} \quad (44)$$

This is satisfied by the bulk solution for $0 \leq \rho \leq \frac{1}{2}$ with $\rho = \rho_L = \frac{\alpha}{1+\alpha}$ and $a = \frac{\alpha(1-\alpha)}{1+\alpha}$. The average current in this phase is

$$J_L = p_1(100) + p_1(101) = \rho_L = \frac{\alpha}{1+\alpha}. \quad (45)$$

The connection at the right end is given as

$$\begin{aligned} p_{N-2}(000) &= p_{N-3}(0000) + \beta p_{N-3}(0001), \\ p_{N-2}(001) &= p_{N-3}(0010) + (1 - \beta)p_{N-3}(0001), \\ p_{N-2}(010) &= p_{N-2}(100) + p_{N-3}(0011) + \beta p_{N-2}(101), & p_{N-2}(011) &= p_{N-2}(1011), \\ p_{N-2}(100) &= p_{N-3}(1000) + \beta p_{N-3}(1001), \\ p_{N-2}(101) &= p_{N-3}(0110) + p_{N-3}(1010) + (1 - \beta)p_{N-3}(1001), \\ p_{N-2}(110) &= p_{N-3}(1011), & p_{N-2}(111) &= 0. \end{aligned} \quad (46)$$

This is satisfied by substitution of the stationary solution with $\rho = \rho_R = \frac{2-\beta}{3-\beta}$ into p_{N-2} . The average current is given as

$$J_R = p_{N-2}(100) + p_{N-2}(101) = 1 - \rho_R = \frac{1}{3 - \beta} \quad (47)$$

The phase transition occurs when $J_L = J_R$, which is given by

$$\beta = 2 - \frac{1}{\alpha}. \quad (48)$$

5.4 Rule 11

Let us assume period-two stationary solutions from the beginning. The equations are

$$\begin{aligned} p_e(000) &= p_o(111) + p_e(1010) + p_o(10100), & p_e(001) &= p_e(100) + p_e(1011), \\ p_e(010) &= p_o(1011) + p_o(10010), & p_e(011) &= p_o(1000) + p_o(10011), \\ p_e(100) &= p_o(011) + p_o(0010), & p_e(101) &= 0, \\ p_e(110) &= p_o(0011) + p_o(00010) & p_e(111) &= p_o(0000) + p_o(00010). \end{aligned} \quad (49)$$

and those with p_e and p_o interchanged. The sixth equation implies the second and third terms in the rhs of the first equation vanish. Thus we obtain $p_e(000) = p_o(111)$. In the same manner, $p_e(111) = p_o(000)$, $p_e(001) = p_e(100)$ and $p_o(001) = p_o(100)$ are obtained. The eighth equation is rewritten as $p_e(111) = p_o(000) - p_o(00010)$ and substitution of $p_e(111) = p_o(000)$ leads to $p_o(00010) = 0$. Thus at least one of $p_o(000)$, $p_e(001)$, or $p_o(010)$ must vanish. Similarly, one of $p_e(000)$, $p_o(001)$, or $p_e(010)$ must vanish. However, if $p_e(001) = 0$, $p_o(010)$ must vanish, because $p_o(01) = p_e(101) + p_e(001) = 0$. Thus, the case $p_e(001) = 0$ is included in the case $p_o(010) = 0$. Moreover, if $p_e(010) = 0$, $p_o(10010) = 0$ and accordingly $p_e(0010) = p_o(010) = 0$. Thus, it is sufficient to investigate the two cases $p_e(000) = p_o(000) = 0$ and $p_e(010) = p_o(010) = 0$. The former case leads to the following uniform (namely $p_e = p_o = p$) solution

$$\begin{aligned} p(000) &= p(101) = p(111) = 0, & p(001) &= p(100) = \frac{\rho}{2}, \\ p(010) &= 2\rho - 1, & p(011) &= p(110) = 1 - \frac{3}{2}\rho, \end{aligned} \quad (50)$$

where the average density ρ of eventually conserved quantity satisfies $\frac{1}{2} \leq \rho \leq \frac{2}{3}$. In the latter case, we have the period-two solution as

$$\begin{aligned} p_e(000) &= \frac{1}{2} - \rho + \epsilon, & p_o(000) &= \frac{1}{2} - \rho - \epsilon, \\ p_e(001) &= p_e(100) = \frac{\rho}{2} - \epsilon, & p_o(001) &= p_o(100) = \frac{\rho}{2} + \epsilon, \\ p_e(010) &= p_e(101) = 0, & p_o(010) &= p_o(101) = 0, \\ p_e(011) &= p_e(110) = \frac{\rho}{2} + \epsilon, & p_o(011) &= p_o(110) = \frac{\rho}{2} - \epsilon, \\ p_e(111) &= \frac{1}{2} - \rho - \epsilon, & p_o(111) &= \frac{1}{2} - \rho + \epsilon, \end{aligned} \quad (51)$$

where $0 \leq \rho \leq \frac{1}{2}$ and $0 \leq \epsilon \leq \min(\rho/2, 1/2 - \rho)$. Connection at the left end is given by

$$\begin{aligned} p_1(000) &= (1 - \alpha)p_1(11) + p_1(1010) + \alpha p_1(0100), & p_1(001) &= p_1(100), \\ p_1(010) &= \alpha[p_1(011) + p_1(0010)], & p_1(011) &= \alpha[p_1(000) + p_1(0011)], \\ p_1(100) &= \alpha p_1(11) + (1 - \alpha)p_1(010), & p_1(101) &= 0, \\ p_1(110) &= (1 - \alpha)[p_1(011) + p_1(0010)], & p_1(111) &= (1 - \alpha)[p_1(000) + p_1(0011)]. \end{aligned} \quad (52)$$

These equations can be solved with the assumption that p_2 is given by p_e in Eq. (51). As the result, p_1 is obtained as

$$\begin{aligned} p_1(000) &= K(1 - \alpha - \alpha^2 + 2\alpha^3), & p_1(001) &= K\alpha(1 + \alpha - 2\alpha^2), \\ p_1(010) &= K\alpha^2, & p_1(011) &= K\alpha, \\ p_1(100) &= K\alpha(1 + \alpha - 2\alpha^2), & p_1(101) &= 0, \\ p_1(110) &= K\alpha(1 - \alpha), & p_1(111) &= K(1 - \alpha), \end{aligned} \quad (53)$$

where $K = (2 + 2\alpha + \alpha^2 - 2\alpha^3)^{-1}$. The relations among ρ_L , ϵ and α are obtained via the consistency between p_1 and p_2 as

$$\rho_L = 1 - 2K = \frac{\alpha(2 + \alpha - 2\alpha^2)}{2 + 2\alpha + \alpha^2 - 2\alpha^3}, \quad \epsilon = \frac{\alpha^2(1 - \alpha)}{2 + 2\alpha + \alpha^2 - 2\alpha^3}. \quad (54)$$

The conserved current in the left domain is $J_L = p_e(011) + p_e(100) = \rho_L = K\alpha(2 + \alpha - 2\alpha^2)$. Connection at the right end is given by the condition

$$\begin{aligned} p_{N-2}(000) &= p_{N-3}(000) + \beta p_{N-3}(0001), & p_{N-2}(001) &= p_{N-2}(100) \\ p_{N-2}(010) &= (1 - \beta)p_{N-3}(1001), & p_{N-2}(011) &= p_{N-3}(1000) + \beta p_{N-3}(1001), \\ p_{N-2}(100) &= p_{N-3}(011) + p_{N-3}(0010), & p_{N-2}(101) &= 0 \\ p_{N-2}(110) &= p_{N-3}(0011) + (1 - \beta)p_{N-3}(0001), & p_{N-2}(111) &= p_{N-3}(0000) + \beta p_{N-3}(0001). \end{aligned} \quad (55)$$

These equations are satisfied by the substitution of (50) into p_{N-2} . Relation between ρ_R and β is given by

$$\rho_R = \frac{2}{3 + \beta}. \quad (56)$$

And the average current in the right phase is given as $J_R = 1 - \rho_R = \frac{1+\beta}{3+\beta}$. The phase transition occurs when $J_L = J_R$, that is

$$\beta = (2\alpha - 1)(1 - \alpha^2). \tag{57}$$

5.5 Numerical simulations

We carried out simulations of the four rules for various α and β . The result is illustrated in Fig. 4. As is seen from the figure, the agreement of the theoretically obtained phase transition line and numerical results is excellent. Values of the density and the current also show a nice agreement between the theory and numerical results.

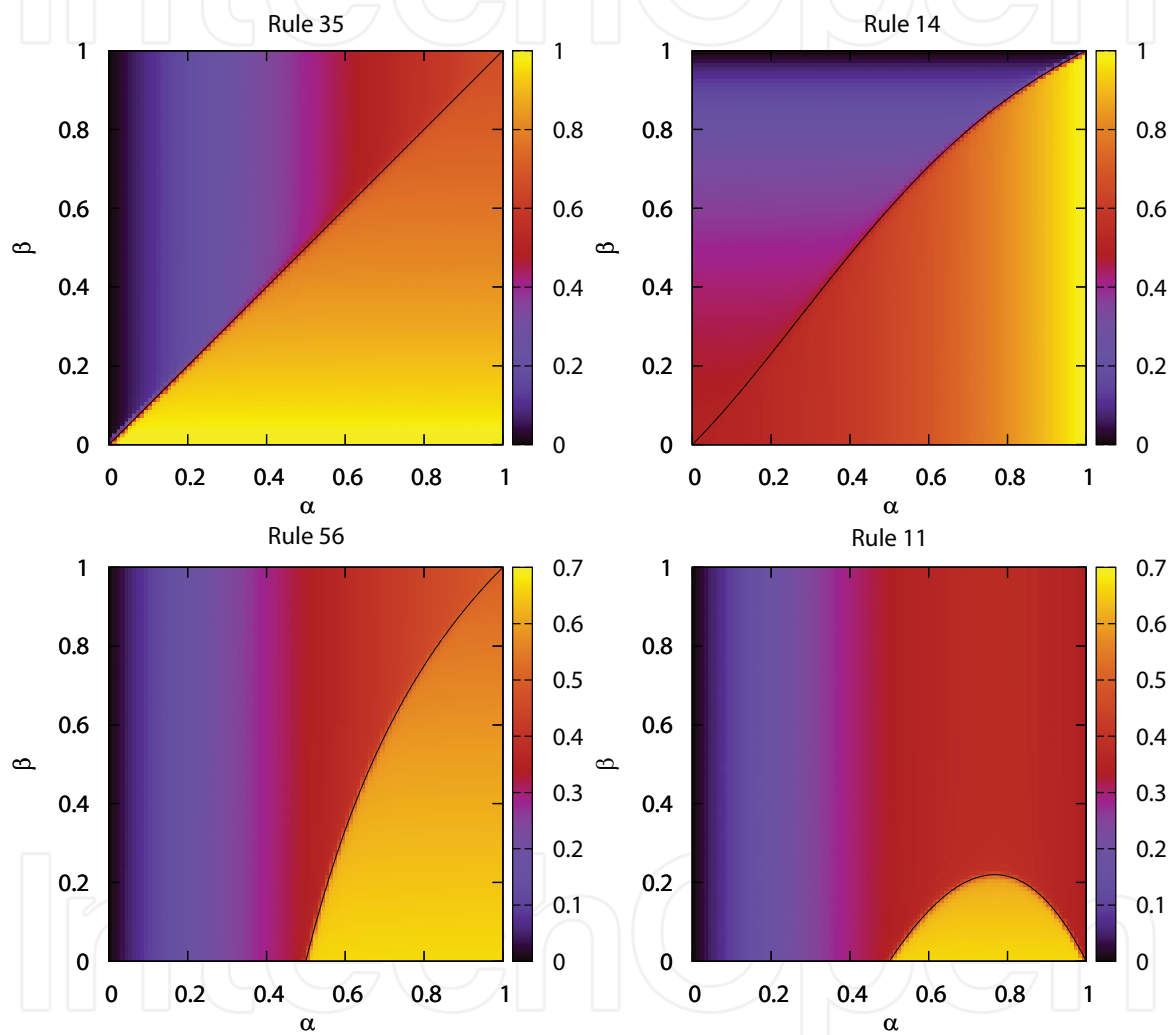


Fig. 4. Phase diagrams of the ECAs. The density of the conserved quantity for each rule and the theoretically obtained phase transition lines are illustrated.

6. Diffusion of the domain wall

The velocity of the domain wall V vanishes on the phase transition line. It does not mean that the domain wall stops somewhere but implies diffusive motion. Eq. (11) can be interpreted as that the domain wall undergoes a random walk with the rate of hopping to the right is

$J_R/(\rho_R - \rho_L)$ and that to the left is $J_L/(\rho_R - \rho_L)$. Then the diffusion constant is given by

$$D = \frac{J_R + J_L}{2(\rho_R - \rho_L)}. \quad (58)$$

On the phase transition line, the drift velocity vanishes and pure diffusive motion appears. This diffusive motion leads to interesting power-law in the power spectral density for the time sequence of the conserved density at a position (Takesue et al 2003). This is first observed for the TASEP and our ECA rules share this phenomenon. Now we consider the case where the conserved density is $E(xy) = (x - y)^2$. The case of $E(x) = x$ is simpler than that. The power spectral density is defined as follows. Record a time sequence of $E(x_i^t x_{i+1}^t)$ for a fixed i and $0 \leq t < T$. Fourier components of the time sequence are calculated as

$$\phi_n = \frac{1}{T} \sum_{t=0}^{T-1} (E(x_i^t x_{i+1}^t) - \rho_i) e^{-i\omega_n t}, \quad (59)$$

where $\omega_n = \frac{2\pi n}{T}$ ($n = 0, 1, 2, \dots, T$) and ρ_i is the expectation value of $E(x_i x_{i+1})$ in the stationary state. Then, the power spectral density is defined as

$$I(\omega_n) = T \langle |\phi_n|^2 \rangle, \quad (60)$$

where $\langle \rangle$ denotes the sample average. Now, we approximate the motion of the domain wall by Brownian motion. In this approximation, the density profile is represented as

$$\rho(x, t) = \rho_L + (\rho_R - \rho_L) \theta(x - X(t)), \quad (61)$$

where $\rho(x, t)$ is the coarse-grained density profile at time t and position x , $X(t)$ is a Brownian motion with diffusion constant D , $\theta(x)$ is the Heaviside function. Applying the Wiener-Khinchin theorem, we obtain the power spectral density as

$$I(\omega) = \frac{\sqrt{2D}}{2N} (\rho_R - \rho_L)^2 \omega^{-3/2}. \quad (62)$$

The power spectral density for rule 35 is shown in Fig. 5. The power law behavior is seen in a region of ω . The agreement with the theory is very good including the prefactor. Also in the other rules, the power spectral density exhibits the power law on the phase transition line. The prefactor agrees with the theoretical value in every case except in rule 11, where a small deviation is seen. The deviation is expected to decrease as the system size increases.

7. Discussion and conclusion

We have seen that the ECA with a single additive conserved quantity show a nonequilibrium phase transition of the same type as in the ASEP. The phase transition line is precisely determined by applying the domain wall theory. In these rules, there occurs only the first-order phase transition and no maximal current phase is observed. It is to be investigated why the maximal current phase does not exist in the ECA and in what kind of CA it appear. The connection between the stationary distribution and the boundary condition depends on the conditional probability. We discuss some possible generalizations. For example, we can consider the following conditional probability.

$$p_L(1|0) = \alpha, \quad p_L(0|0) = 1 - \alpha, \quad p_L(0|1) = \gamma, \quad p_L(1|1) = 1 - \gamma, \quad (63)$$

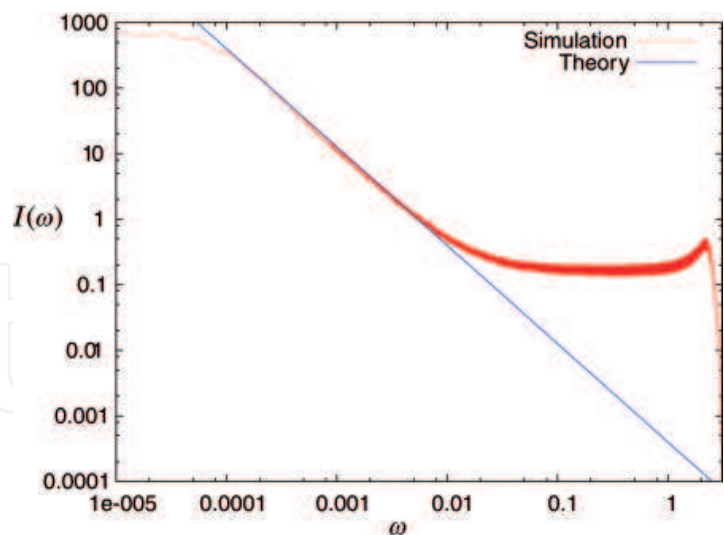


Fig. 5. Power spectral density for rule 35 with $\alpha = \beta = 0.5$. The system size $N = 200$ and the simulation time $T = 2^{20}$.

which is the general form of $p_L(x|y)$. In rule 11, this cause a small modification of the connected stationary state and the resulting phase transition line is represented as $\beta = (1 - \alpha^2)(\alpha + \gamma - 1)$. The similar generalization of the right boundary

$$p_R(0|0) = \delta, \quad p_R(1|0) = 1 - \delta, \quad p_R(0|1) = 1 - \beta, \quad p_R(1|1) = \beta \tag{64}$$

does not affect the connection.

In rules 11 and 14, the stationary distribution connecting with the left boundary is periodic with period two, which was numerically confirmed. This is caused from the fact that p_1 cannot satisfy a property of the stationary distribution, that is $p(010) = 0$ for rule 11 and $p(000) = 0$ for rule 14. However, this is resolved if we employ a suitable conditional probability of the form $p(x_0|x_1x_2)$. For rule 11, if we use the conditional probability given as

$$p'_L(1|01) = 0, \quad p'_L(0|01) = 1, \quad \text{and otherwise } p'_L(x|yz) = p_L(x|y), \tag{65}$$

where p_L is that defined in Eq. (16), $p_1(010) = 0$ is realized and p_1 agrees with the uniform solution of the stationary state ($\epsilon = 0$). The pase transition line is drastically changed into $\beta = 2\alpha - 1$. In the same manner, utilizing the conditional probability

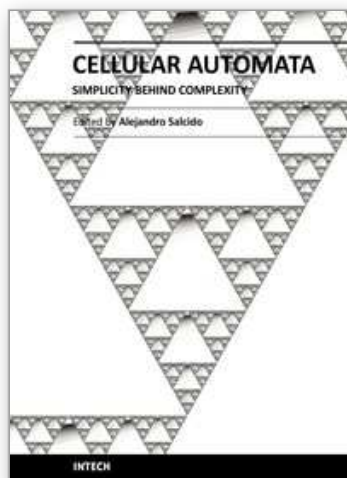
$$p'_L(1|11) = 0, \quad p'_L(0|11) = 1, \quad \text{and otherwise } p'_L(x|yz) = p_L(x|y) \tag{66}$$

for rule 14, p_1 connects with the uniform stationary solution and the phase transition occurs on the line $\beta = \frac{2\alpha}{1+\alpha}$.

8. References

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Cellular Automata - Simplicity Behind Complexity

Edited by Dr. Alejandro Salcido

ISBN 978-953-307-230-2

Hard cover, 566 pages

Publisher InTech

Published online 11, April, 2011

Published in print edition April, 2011

Cellular automata make up a class of completely discrete dynamical systems, which have become a core subject in the sciences of complexity due to their conceptual simplicity, easiness of implementation for computer simulation, and their ability to exhibit a wide variety of amazingly complex behavior. The feature of simplicity behind complexity of cellular automata has attracted the researchers' attention from a wide range of divergent fields of study of science, which extend from the exact disciplines of mathematical physics up to the social ones, and beyond. Numerous complex systems containing many discrete elements with local interactions have been and are being conveniently modelled as cellular automata. In this book, the versatility of cellular automata as models for a wide diversity of complex systems is underlined through the study of a number of outstanding problems using these innovative techniques for modelling and simulation.

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Shinji Takesue (2011). Nonequilibrium Phase Transition of Elementary Cellular Automata with a Single Conserved Quantity, Cellular Automata - Simplicity Behind Complexity, Dr. Alejandro Salcido (Ed.), ISBN: 978-953-307-230-2, InTech, Available from: <http://www.intechopen.com/books/cellular-automata-simplicity-behind-complexity/nonequilibrium-phase-transition-of-elementary-cellular-automata-with-a-single-conserved-quantity>

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