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Robust Control of Hybrid Systems

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1. Introduction

The term "hybrid systems" was first used in 1966 Witsenhausen introduced a hybrid model consisting of continuous dynamics with a few sets of transition. These systems provide both continuous and discrete dynamics have proven to be a useful mathematical model for various physical phenomena and engineering systems. A typical example is a chemical batch plant where a computer is used to monitor complex sequences of chemical reactions, each of which is modeled as a continuous process. In addition to the discontinuities introduced by the computer, most physical processes admit components (eg switches) and phenomena (eg collision), the most useful models are discrete. The hybrid system models arise in many applications, such as chemical process control, avionics, robotics, automobiles, manufacturing, and more recently molecular biology.

The control design for hybrid systems is generally complex and difficult. In literature, different design approaches are presented for different classes of hybrid systems, and different control objectives. For example, when the control objective is concerned with issues such as safety specification, verification and access, the ideas in discrete event control and automaton framework are used for the synthesis of control.

One of the most important control objectives is the problem of stabilization. Stability in the continuous systems or not-hybrid can be concluded starting from the characteristics from their fields from vectors. However, in the hybrid systems the properties of stability also depend on the rules of commutation. For example, in a hybrid system by commutation between two dynamic stable it is possible to obtain instabilities while the change between two unstable subsystems could have like consequence stability. The majority of the results of stability for the hybrid systems are extensions of the theories of Lyapunov developed for the continuous systems. They require the Lyapunov function at consecutive switching times to be a decreasing sequence. Such a requirement in general is difficult to check without calculating the solution of the hybrid dynamics, and thus losing the advantage of the approach of Lyapunov.

In this chapter, we develop tools for the systematic analysis and robust design of hybrid systems, with emphasis on systems that require control algorithms, that is, hybrid control systems. To this end, we identify mild conditions that hybrid equations need to satisfy so that their behavior captures the effect of arbitrarily small perturbations. This leads to new concepts of global solutions that provide a deep understanding not only on the robustness

properties of hybrid systems, but also on the structural properties of their solutions. Alternatively, these conditions allow us to produce various tools for hybrid systems that resemble those in the stability theory of classical dynamical systems. These include general versions of theorems of Lyapunov stability and the principles of invariance of LaSalle.

2. Hybrid systems: Definition and examples

Different models of hybrid systems have been proposed in the literature. They mainly differ in the way either the continuous part or the discrete part of the dynamics is emphasized, which depends on the type of systems and problems we consider. A general and commonly used model of hybrid systems is the hybrid automaton (see e.g. (Dang, 2000) and (Girard, 2006)). It is basically a finite state machine where each state is associated to a continuous system. In this model, the continuous evolutions and the discrete behaviors can be considered of equal complexity and importance. By combining the definition of the continuous system, and discrete event systems hybrid dynamical systems can be defined:

Definition 1 *A hybrid system H is a collection $H := (Q, X, \Sigma, U, F, R)$, where*

* *Q is a finite set, called the set of discrete states;*
* *$X \subseteq \Re^n$ is the set of continuous states;*
* *Σ is a set of discrete input events or symbols;*
* *$X \subseteq \Re^m$ is the set of continuous inputs;*
* *$F : Q \times X \times U \to \Re^n$ is a vector field describing the continuous dynamics;*
* *$R : Q \times X \times \Sigma \times U \to Q \times X$ describes the discrete dynamics.*
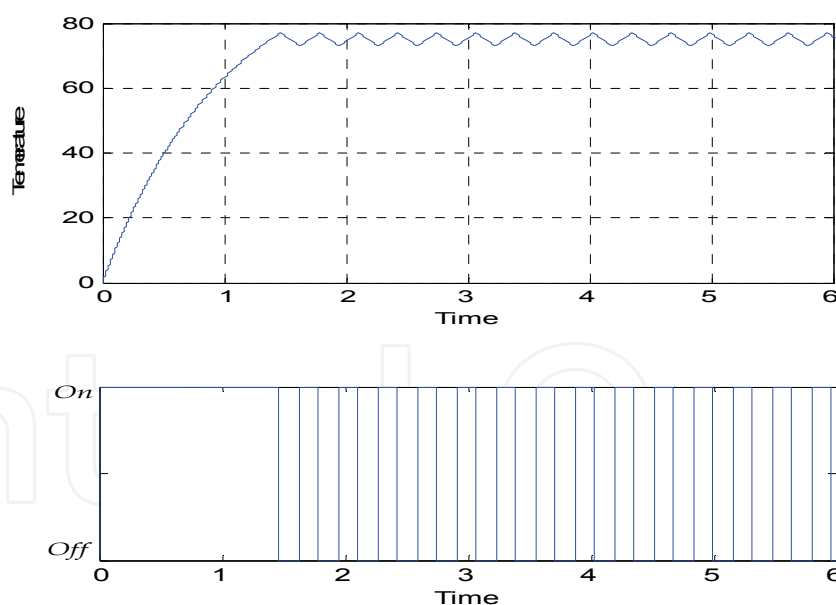

Fig. 1. A trajectory of the room temperature.

Example 1 **(Thermostat)**. The thermostat consists of a heater and a thermometer which maintain the temperature of the room in some desired temperature range (Rajeev, 1993). The lower and upper thresholds of the thermostat system are set at x_m and x_M such that $x_m \prec x_M$. The heater is maintained on as long as the room temperature is below x_M, and it is turned off whenever the thermometer detects that the temperature reaches x_M. Similarly, the heater remains off if the temperature is above x_m and is switched on whenever the

temperature falls to x_m (Fig. 1). In practical situations, exact threshold detection is impossible due to sensor imprecision. Also, the reaction time of the on/off switch is usually non-zero. The effect of these inaccuracies is that we cannot guarantee switching exactly at the nominal values x_m and x_M. As we will see, this causes non-determinism in the discrete evolution of the temperature.

Formally we can model the thermostat as a hybrid automaton shown in (Fig. 2). The two operation modes of the thermostat are represented by two locations '*on*' and '*off*'. The *on/off* switch is modeled by two discrete transitions between the locations. The continuous variable *x* models the temperature, which evolves according to the following equations.
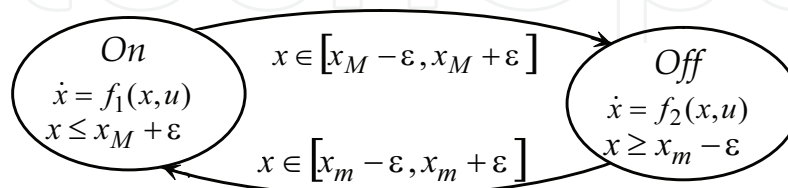

Fig. 2. Model of the thermostat.

- If the thermostat is on, the evolution of the temperature is described by:

$$\dot{x} = f_1(x,u) = -x + 4 + u \tag{1}$$

- When the thermostat is off, the temperature evolves according to the following differential equation:
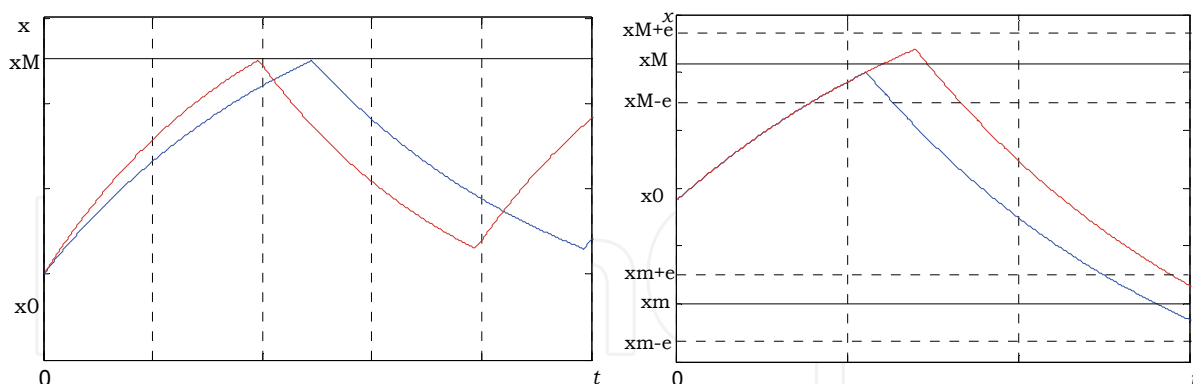
$$\dot{x} = f_2(x,u) = -x + u$$

Fig. 3. Two different behaviors of the temperature starting at x_0.

The second source of non-determinism comes from the continuous dynamics. The input signal *u* of the thermostat models the fluctuations in the outside temperature which we cannot control. (Fig. 3 left) shows this continuous non-determinism. Starting from the initial temperature x_0, the system can generate a "*tube*" of infinite number of possible trajectories, each of which corresponds to a different input signal *u*. To capture uncertainty of sensors, we define the first guard condition of the transition from '*on*' to '*off*' as an interval $[x_M - \varepsilon, x_M + \varepsilon]$ with $\varepsilon \succ 0$. This means that when the temperature enters this interval, the thermostat can either turn the heater off immediately or keep it on for some time provided

that $x \leq x_M + \varepsilon$. (Fig. 3 right) illustrates this kind of non-determinism. Likewise, we define the second guard condition of the transition from '*off*' to '*on*' as the interval $\left[x_m - \varepsilon, x_m + \varepsilon\right]$. Notice that in the thermostat model, the temperature does not change at the switching points, and the reset maps are thus the identity functions.

Finally we define the two staying conditions of the '*on*' and '*off*' locations as $x \leq x_M + \varepsilon$ and $x \geq x_M - \varepsilon$ respectively, meaning that the system can stay at a location while the corresponding staying conditions are satisfied.

Example 2 (Bouncing Ball). Here, the ball (thought of as a point-mass) is dropped from an initial height and bounces off the ground, dissipating its energy with each bounce. The ball exhibits continuous dynamics between each bounce; however, as the ball impacts the ground, its velocity undergoes a discrete change modeled after an inelastic collision. A mathematical description of the bouncing ball follows. Let $x_1 := h$ be the height of the ball and $x_2 := \dot{h}$ (Fig. 4). A hybrid system describing the ball is as follows:

$$g(x) := \begin{bmatrix} 0 \\ -\gamma.x_2 \end{bmatrix}, D := \{x : x_1 = 0, \ x_2 \prec 0\} \ f(x) := \begin{bmatrix} x_2 \\ -g \end{bmatrix}, C := \{x : x_1 \geq 0\} \setminus D . \qquad (2)$$

This model generates the sequence of hybrid arcs shown in (Fig. 5). However, it does not generate the hybrid arc to which this sequence of solutions converges since the origin does not belong to the jump set D. This situation can be remedied by including the origin in the jump set D. This amounts to replacing the jump set D by its closure. One can also replace the flow set C by its closure, although this has no effect on the solutions.

It turns out that *whenever the flow set and jump set are closed, the solutions of the corresponding hybrid system enjoy a useful compactness property: every locally eventually bounded sequence of solutions has a subsequence converging to a solution.*
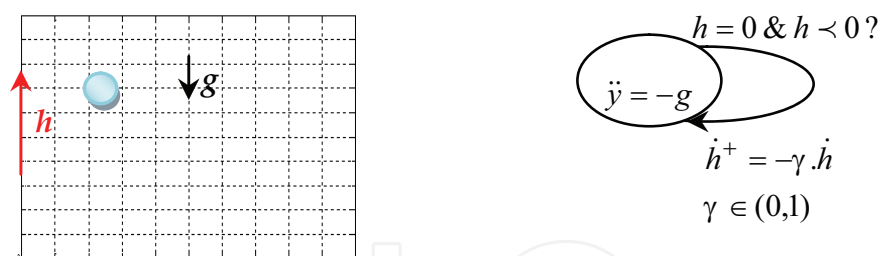
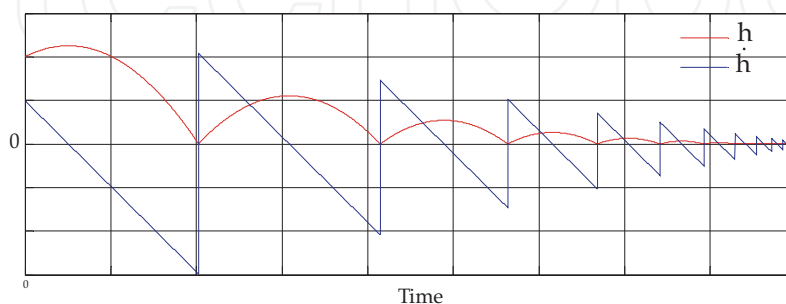

Fig. 4. Diagram for the bouncing ball system

Fig. 5. Solutions to the bouncing ball system

Consider the sequence of hybrid arcs depicted in (Fig. 5). They are solutions of a hybrid "bouncing ball" model showing the position of the ball when dropped for successively

lower heights, each time with zero velocity. The sequence of graphs created by these hybrid arcs converges to a graph of a hybrid arc with hybrid time domain given by $\{0\} \times \{$nonnegative integers$\}$ where the value of the arc is zero everywhere on its domain. If this hybrid arc is a solution then the hybrid system is said to have a "compactness" property. This attribute for the solutions of hybrid systems is critical for robustness properties. It is the hybrid generalization of a property that automatically holds for continuous differential equations and difference equations, where nominal robustness of asymptotic stability is guaranteed.

Solutions of hybrid systems are hybrid arcs that are generated in the following way: Let C and D be subsets of $\Re^n$ and let f, respectively g, be mappings from C, respectively D, to $\Re^n$. The hybrid system $H := (f, g, C, D)$ can be written in the form

$$
\begin{aligned}
\dot{x} &= f(x) & x &\in C \\
x^+ &= g(x) & x &\in D
\end{aligned}
\tag{3}
$$

The map f is called the "flow map", the map g is called the "jump map", the set C is called the "flow set", and the set D is called the "jump set". The state x may contain variables taking values in a discrete set (logic variables), timers, etc. Consistent with such a situation is the possibility that $C \cup D$ is a strict subset of $\Re^n$. For simplicity, assume that f and g are continuous functions. At times it is useful to allow these functions to be set-valued mappings, which will denote by F and G, in which case F and G should have a closed graph and be locally bounded, and F should have convex values.

In this case, we will write

$$
\begin{aligned}
\dot{x} &\in F & x &\in C \\
x^+ &\in G & x &\in D
\end{aligned}
\tag{4}
$$

A solution to the hybrid system (4) starting at a point $x_0 \in C \cup D$ is a hybrid arc x with the following properties:

1. $x(0,0) = x_0$;
2. given $(s,j) \in dom \ x$, if there exists $\tau \succ s$ such that $(\tau, j) \in dom \ x$, then, for all $t \in [s, \tau]$, $x(t,j) \in C$ and, for almost all $t \in [s, \tau]$, $\dot{x}(t,j) \in F(x(t,j))$;
3. given $(t,j) \in dom \ x$, if $(t, j+1) \in dom \ x$ then $x(t,j) \in D$ and $x(t, j+1) \in G(x(t,j))$.

Solutions from a given initial condition are not necessarily unique, even if the flow map is a smooth function.

3. Approaches to analysis and design of hybrid control systems

The analysis and design tools for hybrid systems in this section are in the form of Lyapunov stability theorems and LaSalle-like invariance principles. Systematic tools of this type are the base of the theory of systems for purely of the continuous-time and discrete-time systems. Some similar tools available for hybrid systems in (Michel, 1999) and (DeCarlo, 2000), the tools presented in this section generalize their conventional versions of continuous-time and discrete-time hybrid systems development by defining an equivalent concept of stability and provide extensions intuitive sufficient conditions of stability asymptotically.

3.1 LaSalle-like invariance principles

Certain principles of invariance for the hybrid systems have been published in (Lygeros et al., 2003) and (Chellaboina et al., 2002). Both results require, among other things, unique solutions which is not generic for hybrid control systems. In (Sanfelice et al., 2005), the general invariance principles were established that do not require uniqueness. The work in (Sanfelice et al., 2005) contains several invariance results, some involving integrals of functions, as for systems of continuous-time in (Byrnes & Martin, 1995) or (Ryan, 1998), and some involving nonincreasing energy functions, as in work of LaSalle (LaSalle, 1967) or (LaSalle, 1976). Such a result will be described here.

Suppose we can find a continuously differentiable function $V : \Re^n \to \Re$ such that

$$
\begin{aligned}
u_c(x) &:= \langle \nabla V(x), f(x) \rangle \leq 0 && \forall x \in \overline{C} \\
u_d(x) &:= V(g(x)) - V(x) \leq 0 && \forall x \in \overline{D}
\end{aligned}
\tag{5}
$$

Consider $x(\cdot, \cdot)$ a bounded solution with an unbounded hybrid time. Then there exists a value r in the range V so that x tends to the largest weakly invariant set inside the set

$$
M_r := V^{-1}(r) \bigcap \left(u_c^{-1}(0) \bigcup \left(u_d^{-1}(0) g(u_d^{-1}(0)) \right) \right)
\tag{6}
$$

where $u_d^{-1}(0)$: the set of points x satisfying $u_d(x) = 0$ and $g(u_d^{-1}(0))$ corresponds to the set of points $g(y)$ where $y \in u_d^{-1}(0)$.

The naive combination of continuous-time and discrete-time results would omit the intersection with $g(u_d^{-1}(0))$. This term, however, can be very useful for zeroing in set to which trajectories converge.

3.2 Lyapunov stability theorems

Some preliminary results on the existence of the non-smooth Lyapunov function for the hybrid systems published in (DeCarlo, 2000). The first results on the existence of smooth Lyapunov functions, which are closely related to the robustness, published in (Cai et al., 2005). These results required open basins of attraction, but this requirement has since been relaxed in (Cai et al. 2007). The simplified discussion here is borrowed from this posterior work.

Let $\mathcal{O}$ be an open subset of the state space containing a given compact set A and let $\omega : \mathcal{O} \to \Re_{\geq 0}$ be a continuous function which is zero for all $x \in A$, is positive otherwise, which grows without limit as its argument grows without limit or near the limit $\mathcal{O}$. Such a function is called a suitable indicator for the compact set A in the open set $\mathcal{O}$. An example of such a function is the standard function on $\Re^n$ which is an appropriate indicator of origin. More generally, the distance to a compact set A is an appropriate indicator for all A on $\Re^n$.

Given an open set $\mathcal{O}$, an appropriate indicator ω and hybrid data (f, g, C, D), a function $V : \mathcal{O} \to \Re_{\geq 0}$ is called a smooth Lyapunov function for $(f, g, C, D, \omega, \mathcal{O})$ if it is smooth and there exist functions α_1, α_2 belonging to the class-$\mathcal{K}_\infty$, such as

$$
\begin{aligned}
\alpha_1(\omega(x)) &\leq V(x) \leq \alpha_2(\omega(x)) && \forall x \in \mathcal{O} \\
\langle \nabla V(x), f(x) \rangle &\leq -V(x) && \forall x \in \overline{C} \bigcap \mathcal{O} \\
V(g(x)) &\leq e^{-1} V(x) && \forall x \in \overline{D} \bigcap \mathcal{O}
\end{aligned}
\tag{7}
$$

Suppose that such a function exists, it is easy to verify that all solutions for the hybrid system (f, g, C, D) from $\mathcal{O} \bigcap \left(\overline{C} \cup \overline{D} \right)$ satisfied

$$\omega(x(t,j)) \leq \alpha_1^{-1}\left(e^{-t-j}\alpha_2(\omega(x(0,0)))\right) \qquad \forall(t,j) \in dom\ x \qquad (8)$$

In particular,

- (pre-stability of A) for each $\varepsilon \succ 0$ there exists $\delta \succ 0$ such that $x(0,0) \in A + \delta B$ implies, for each generalized solution, that $x(t,j) \in A + \varepsilon B$ for all $(t,j) \in dom\ x$, and
- (before attractive A on $\mathcal{O}$) any generalized solution from $\mathcal{O} \cap \left(\overline{C} \cup \overline{D}\right)$ is bounded and if its time domain is unbounded, so it converges to A.

According to one of the principal results in (Cai et al., 2006) *there exists a smooth Lyapunov function for* $(f,g,C,D,\omega,\mathcal{O})$ *if and only if the set* A *is pre-stable and pre-attractive on* $\mathcal{O}$ *and* $\mathcal{O}$ *is forward invariant* (i.e., $x(0,0) \in \mathcal{O} \cap \left(\overline{C} \cup \overline{D}\right)$ implies $x(t,j) \in \mathcal{O}$ for all $(t,j) \in dom\ x$).

One of the primary interests in inverse Lyapunov theorems is that they can be employed to establish the robustness of the asymptotic stability of various types of perturbations.

4. Hybrid control application

In system theory in the 60s researchers were discussing mathematical frameworks so to study systems with continuous and discrete dynamics. Current approaches to hybrid systems differ with respect to the emphasis on or the complexity of the continuous and discrete dynamics, and on whether they emphasize analysis and synthesis results or analysis only or simulation only. On one end of the spectrum there are approaches to hybrid systems that represent extensions of system theoretic ideas for systems (with continuous-valued variables and continuous time) that are described by ordinary differential equations to include discrete time and variables that exhibit jumps, or extend results to switching systems. Typically these approaches are able to deal with complex continuous dynamics. Their main emphasis has been on the stability of systems with discontinuities. On the other end of the spectrum there are approaches to hybrid systems embedded in computer science models and methods that represent extensions of verification methodologies from discrete systems to hybrid systems. Several approaches to robustness of asymptotic stability and synthesis of hybrid control systems are represented in this section.

4.1 Hybrid stabilization implies input-to-state stabilization
In the paper (Sontag, 1989) it has been shown, for continuous-time control systems, that smooth stabilization involves smooth input-to-stat stabilization with respect to input additive disturbances. The proof was based on converse Lyapunov theorems for continuous-time systems. According to the indications of (Cai et al., 2006), and (Cai et al. 2007), the result generalizes to hybrid control systems via the converse Lyapunov theorem.

In particular, if we can find a hybrid controller, with the type of regularity used in sections 4.2 and 4.3, to achieve asymptotic stability, then the input-to-state stability with respect to input additive disturbance can also be achieved.

Here, consider the special case where the hybrid controller is a logic-based controller where the variable takes values in the logic of a finite set. Consider the hybrid control system

$$\mathcal{H} := \begin{cases} \dot{\xi} = f_q(\xi) + \eta_q(\xi)(u_q + \upsilon_q d) & \xi \in C_q,\ q \in Q \\ \begin{bmatrix} \xi \\ q \end{bmatrix}^+ \in G_q(\xi) & \xi \in D_q,\ q \in Q \end{cases} \qquad (9)$$

where Q is a finite index set, for each $q \in Q$, f_q, $\eta_q : C_q \to \Re^n$ are continuous functions, C_q and D_q are closed and G_q has a closed graph and is locally bounded. The signal u_q is the control, and d is the disturbance, while υ_q is vector that is independent of the state, input, and disturbance. Suppose $\mathcal{H}$ is stabilizable by logic-based continuous feedback; that is, for the case where $d = 0$, there exist continuous functions k_q defined on C_q such that, with $u_q := k_q(\xi)$, the nonempty and compact set $A = \bigcup_{q \in Q} A_q \times \{q\}$ is pre-stable and globally pre-attractive. Converse Lyapunov theorems can then be used to establish the existence of a logic-based continuous feedback that renders the closed-loop system input-to-state stable with respect to d. The feedback has the form

$$u_q := k_q(\xi) - \varepsilon.\eta_q^T(\xi)\nabla V_q(\xi) \tag{10}$$

where $\varepsilon \succ 0$ and $V_q(\xi)$ is a smooth Lyapunov function that follows from the assumed asymptotic stability when $d \equiv 0$. There exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that, with this feedback control, the following estimate holds:

$$\left|\xi(t,j)\right|_{A(t,j)} \leq \max\left\{\alpha_1^{-1}\left(2.\exp(-t-j).\alpha_2\left(\left|\xi(0,0)\right|_{Aq(0,0)}\right)\right), \alpha_1^{-1}\left(\frac{\max_{q \in Q}\left|\upsilon_q\right|^2}{2.\varepsilon}\|d\|_\infty^2\right)\right\} \tag{11}$$

where $\|d\|_\infty := \sup_{(s,i) \in \text{dom } d}\left|d(s,i)\right|$.

4.2 Control Lyapunov functions

Although the control design using a continuously differentiable control-Lyapunov function is well established for input-affine nonlinear control systems, it is well known that not all controllable input-affine nonlinear control system function admits a continuously differentiable control-Lyapunov function. A well known example in the absence of this control-Lyapunov function is the so-called "Brockett", or "nonholonomic integrator". Although this system does not allow continuously differentiable control Lyapunov function, it has been established recently that admits a good "patchy" control-Lyapunov function.

The concept of control-Lyapunov function, which was presented in (Goebel et al., 2009), is inspired not only by the classical control-Lyapunov function idea, but also by the approach to feedback stabilization based on patchy vector fields proposed in (Ancona & Bressan, 1999). The idea of control-Lyapunov function was designed to overcome a limitation of discontinuous feedbacks, such as those from patchy feedback, which is a lack of robustness to measurement noise. In (Goebel et al., 2009) it has been demonstrated that any asymptotically controllable nonlinear system admits a smooth patchy control-Lyapunov function if we admit the possibility that the number of patches may need to be infinite. In addition, it was shown how to construct a robust stabilizing hybrid feedback from a patchy control-Lyapunov function. Here the idea when the number of patches is finite is outlined and then specialized to the nonholonomic integrator.

Generally , a global patchy smooth control-Lyapunov function for the origin for the control system $\dot{x} = f(x,u)$ in the case of a finite number of patches is a collection of functions V_q and sets Ω_q and Ω_q' where $q \in Q := \{1,\ldots,m\}$, such as

    a.   for each $q \in Q$, Ω_q and Ω_q' are open and

- $\mathcal{O} := \Re^n \setminus \{0\} = \bigcup_{q \in Q} \Omega_q = \bigcup_{q \in Q} \Omega_q'$
- for each $q \in Q$, the outward unit normal to $\partial\Omega_q$ is continuous on $\left(\partial\Omega_q \setminus \bigcup_{r \succ q} \Omega_r'\right) \bigcap \mathcal{O}$,

- for each $q \in Q$, $\overline{\Omega}_q' \cap \mathcal{O} \subset \Omega_q$;
  b. for each $q \in Q$, V_q is a smooth function defined on a neighborhood (relative to $\mathcal{O}$) of Ω_q.
  c. there exist a continuous positive definite function α and class-$\mathcal{K}_\infty$ functions $\overline{\gamma}$ and $\underline{\gamma}$ such that
- $\underline{\gamma}(|x|) \le V_q(x) \le \overline{\gamma}(|x|)$ $V_q \forall q \in Q$, $x \in \overline{\left(\Omega_q \setminus \bigcup_{r \succ q} \Omega_r'\right)} \cap \mathcal{O}$;
- for each $q \in Q$ and $x \in \left(\Omega_q \setminus \bigcup_{r \succ q} \Omega_r'\right)$ there exists $u_{x,q}$ such that

$$\left\langle \nabla V_q(x), f(x, u_x, q) \right\rangle \le -\alpha(x)$$

- for each $q \in Q$ and $x \in \left(\Omega_q \setminus \bigcup_{r \succ q} \Omega_r'\right) \cap \mathcal{O}$ there exists $u_{x,q}$ such that

$$\left\langle \nabla V_q(x), f(x, u_x, q) \right\rangle \le -\alpha(x)$$
$$\left\langle n_q(x), f(x, u_x, q) \right\rangle \le -\alpha(x)$$

where $x \mapsto n_q(x)$ denotes the outward unit normal to $\partial \Omega_q$.

From this patchy control-Lyapunov function one can construct a robust hybrid feedback stabilizer, at least when the set $\{ u, \upsilon.f(x, u) \le c \}$ is convex for each real number c and every real vector υ, with the following data

$$u_q := k_q(x), C_q = \overline{\left(\Omega_q \setminus \bigcup_{r \succ q} \Omega_r'\right)} \cap \mathcal{O} \tag{12}$$

where k_q is defined on C_q, continuous and such that

$$\begin{aligned} \left\langle \nabla V_q(x), f(x, k_q(x)) \right\rangle &\le -0.5\alpha(x) & \forall x \in C_q \\ \left\langle n_q(x), f(x, k_x(x)) \right\rangle &\le -0.5\alpha(x) & \forall x \in \left(\partial \Omega_q \setminus \bigcup_{r \succ k} \Omega_r'\right) \cap \mathcal{O} \end{aligned} \tag{13}$$

The jump set is given by

$$D_q = \left(\mathcal{O} \setminus \Omega_q\right) \cup \left(\bigcup_{r \succ q} \overline{\Omega_r'} \cap \mathcal{O}\right) \tag{14}$$

and the jump map is

$$G_q(x) = \begin{cases} \left\{ r \in Q : x \in \overline{\Omega_r'} \cap \mathcal{O}, r \succ q \right\} & x \in \left(\bigcup_{r \succ q} \overline{\Omega_r'} \cap \mathcal{O}\right) \cap \Omega_q \\ \left\{ r \in Q : x \in \overline{\Omega_r'} \cap \mathcal{O} \right\} & x \in \mathcal{O} \setminus \Omega_q \end{cases} \tag{15}$$

With this control, the index increases with each jump except probably the first one. Thus, the number of jumps is finite, and the state converges to the origin, which is also stable.

4.3 Throw-and-catch control

In (Prieur, 2001), it was shown how to combine local and global state feedback to achieve global stabilization and local performance. The idea, which exploits hysteresis switching (Halbaoui et al., 2009b), is completely simple. Two continuous functions, k_{global} and k_{local} are shown when the feedback $u = k_{global}(x)$ render the origin of the control system $\dot{x} = f(x, u)$ globally asymptotically stable whereas the feedback $u = k_{local}(x)$ makes the

origin of the control system locally asymptotically stable with basin of attraction containing the open set $\mathcal{O}$, which contains the origin. Then we took C_{local} a compact subset of the $\mathcal{O}$ that contains the origin in its interior and one takes D_{global} to be a compact subset of C_{local} , again containing the origin in its interior and such that, when using the controller k_{local} , trajectories starting in D_{global} never reach the boundary of C_{local} (Fig. 6). Finally, the hybrid control which achieves global asymptotic stabilization while using the controller k_q for small signals is as follows

$$
\begin{aligned}
u &:= k_q(x) & C &:= \left\{ (x,q) : x \in C_q \right\} \\
g(q,x) &:= \text{toggle}\,(q) & D &:= \left\{ (x,q) : x \in D_q \right\}
\end{aligned}
\tag{16}
$$

In the problem of uniting of local and global controllers, one can view the global controller as a type of "bootstrap" controller that is guaranteed to bring the system to a region where another controller can control the system adequately.

A prolongation of the idea of combine local and global controllers is to assume the existence of continuous bootstrap controller that is guaranteed to introduce the system, in finite time, in a vicinity of a set of points, not simply a vicinity of the desired final destination (the controller doesn't need to be able to maintain the state in this vicinity); moreover, these sets of points form chains that terminate at the desired final destination and along which controls are known to steer (or "throw") form one point in the chain at the next point in the chain. Moreover, in order to minimize error propagation along a chain, a local stabilizer is known for each point, except perhaps those points at the start of a chain. Those can be employed "to catch" each jet.
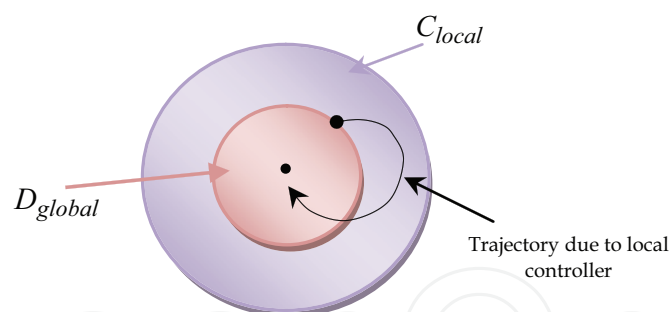

Fig. 6. Combining local and global controllers

4.4 Supervisory control

In this section, we review the supervisory control framework for hybrid systems. One of the main characteristics of this approach is that the plant is approximated by a discrete-event system and the design is carried out in the discrete domain. The hybrid control systems in the supervisory control framework consist of a continuous (state, variable) system to be controlled, also called the plant, and a discrete event controller connected to the plant via an interface in a feedback configuration as shown in (Fig. 7). It is generally assumed that the dynamic behavior of the plant is governed by a set of known nonlinear ordinary differential equations

$$
\dot{x}(t) = f(x(t), r(t))
\tag{17}
$$

where $x \in \mathfrak{R}^n$ is the continuous state of the system and $r \in \mathfrak{R}^m$ is the continuous control input. In the model shown in (Fig. 7), the plant contains all continuous components of the hybrid control system, such as any conventional continuous controllers that may have been developed, a clock if time and synchronous operations are to be modeled, and so on. The controller is an event driven, asynchronous discrete event system (DES), described by a finite state automaton. The hybrid control system also contains an interface that provides the means for communication between the continuous plant and the DES controller.
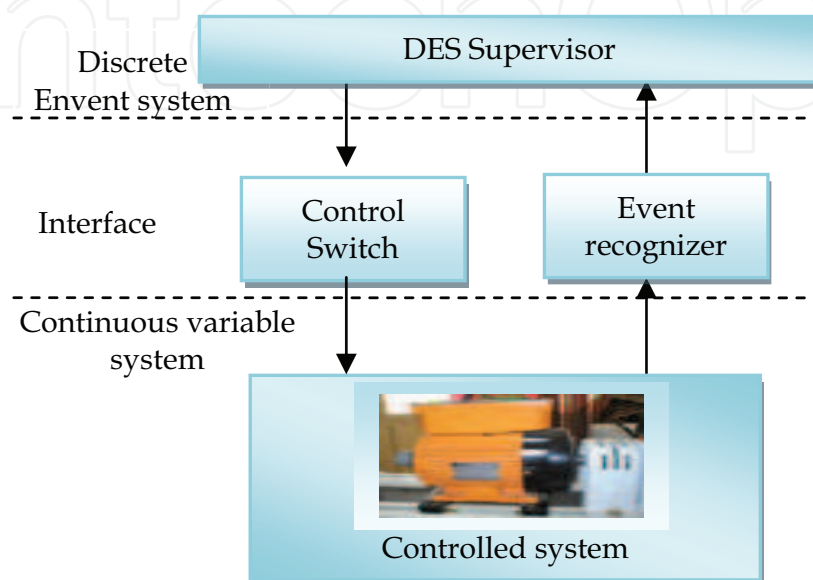
Fig. 7. Hybrid system model in the supervisory control framework.
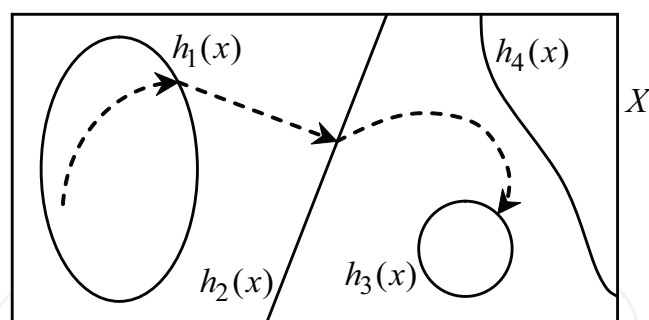
Fig. 8. Partition of the continuous state space.

The interface consists of the generator and the actuator as shown in (Fig. 7). The generator has been chosen to be a partitioning of the state space (see Fig. 8). The piecewise continuous command signal issued by the actuator is a staircase signal as shown in (Fig. 9), not unlike the output of a zero-order hold in a digital control system. The interface plays a key role in determining the dynamic behavior of the hybrid control system. Many times the partition of the state space is determined by physical constraints and it is fixed and given. Methodologies for the computation of the partition based on the specifications have also been developed.

In such a hybrid control system, the plant taken together with the actuator and generator, behaves like a discrete event system; it accepts symbolic inputs via the actuator and produces symbolic outputs via the generator. This situation is somewhat analogous to the
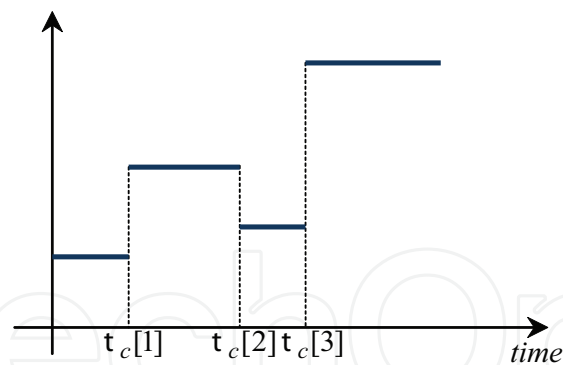
Fig. 9. Command signal issued by the interface.

way a continuous time plant, equipped with a zero-order hold and a sampler, "looks" like a discrete-time plant. The DES which models the plant, actuator, and generator is called the DES plant model. From the DES controller's point of view, it is the DES plant model which is controlled.

The DES plant model is an approximation of the actual system and its behavior is an abstraction of the system's behavior. As a result, the future behavior of the actual continuous system cannot be determined uniquely, in general, from knowledge of the DES plant state and input. The approach taken in the supervisory control framework is to incorporate all the possible future behaviors of the continuous plant into the DES plant model. A conservative approximation of the behavior of the continuous plant is constructed and realized by a finite state machine. From a control point of view this means that if undesirable behaviors can be eliminated from the DES plant (through appropriate control policies) then these behaviors will be eliminated from the actual system. On the other hand, just because a control policy permits a given behavior in the DES plant, is no guarantee that that behavior will occur in the actual system.

We briefly discuss the issues related to the approximation of the plant by a DES plant model.

A *dynamical system* Σ can be described as a triple $T;W;B$ with $T \subseteq \Re$ the *time axis*, W the *signal space*, and $B \subset W^T$ (the set of all functions $f : T \to W$) the *behavior*. The behavior of the DES plant model consists of all the pairs of plant and control symbols that it can generate. The time axis T represents here the occurrences of events. A necessary condition for the DES plant model to be a valid approximation of the continuous plant is that the behavior of the continuous plant model B_c is contained in the behavior of the DES plant model, i.e. $B_c \subseteq B_d$.

The main objective of the controller is to restrict the behavior of the DES plant model in order to specify the control specifications. The specifications can be described by a behavior B_{spec} . *Supervisory control of hybrid systems is based on the fact that if undesirable behaviors can be eliminated from the DES plant then these behaviors can likewise be eliminated from the actual system.* This is described formally by the relation

$$B_d \bigcap B_s \subseteq B_{spec} \Rightarrow B_c \bigcap B_s \subseteq B_{spec} \tag{18}$$

and is depicted in (Fig. 10). The challenge is to find a discrete abstraction with behavior B_d which is a approximation of the behavior B_c of the continuous plant and for which is possible to design a supervisor in order to guarantee that the behavior of the closed loop system satisfies the specifications B_{spec}. A more accurate approximation of the plant's behavior can be obtained by considering a finer partitioning of the state space for the extraction of the DES plant.
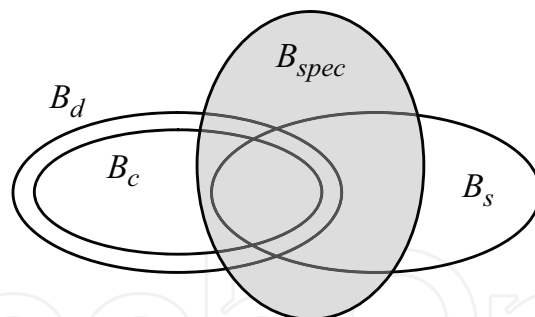
Fig. 10. The DES plant model as an approximation.

An interesting aspect of the DES plant's behavior is that it is distinctly nondeterministic. This fact is illustrated in (Fig.11). The figure shows two different trajectories generated by the same control symbol. Both trajectories originate in the same DES plant state $\tilde{p}_1$. (Fig.11) shows that for a given control symbol, there are at least two possible DES plant states that can be reached from $\tilde{p}_1$. Transitions within a DES plant will usually be nondeterministic unless the boundaries of the partition sets are invariant manifolds with respect to the vector fields that describe the continuous plant.
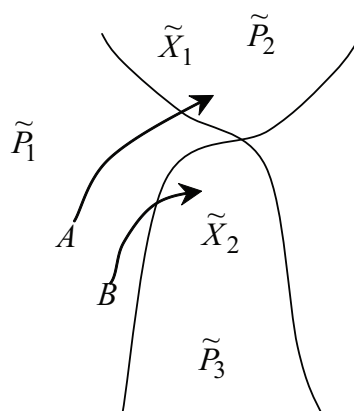
Fig. 11. Nondeterminism of the DES plant model.

There is an advantage to having a hybrid control system in which the DES plant model is deterministic. It allows the controller to drive the plant state through any desired sequence of regions provided, of course, that the corresponding state transitions exist in the DES plant model. If the DES plant model is not deterministic, this will not always be possible. This is because even if the desired sequence of state transitions exists, the sequence of inputs which achieves it may also permit other sequences of state transitions. Unfortunately, given a continuous-time plant, it may be difficult or even impossible to design an interface that leads to a DES plant model which is deterministic. Fortunately, it is not generally necessary to have a deterministic DES plant model in order to control it. The supervisory control problem for hybrid systems can be formulated and solved when the DES plant model is nondeterministic. This work builds upon the frame work of supervisory control theory used in (Halbaoui et al., 2008) and (Halbaoui et al., 2009a).

5. Robustness to perturbations

In control systems, several perturbations can occur and potentially destroy the good behavior for which the controller was designed for. For example, noise in the measurements

of the state taken by controller arises in all implemented systems. It is also common that when a controller is designed, only a simplified model of the system to control exhibiting the most important dynamics is considered. This simplifies the control design in general. However, sensors/actuators that are dynamics unmodelled can substantially affect the behavior of the system when in the loop. In this section, it is desired that the hybrid controller provides a certain degree of robustness to such disturbances. In the following sections, general statements are made in this regard.

5.1 Robustness via filtered measurements

In this section, the case of noise in the measurements of the state of the nonlinear system is considered. Measurement noise in hybrid systems can lead to nonexistence of solutions. This situation can be corrected, at least for the small measurement noise, if under global existence of solutions, C_c and D_c always "overlap" while ensuring that the stability properties still hold. The "overlap" means that for every $\xi \in O$, either $\xi + e \in C_c$ or $\xi + e \in D_c$ all or small e. There exist generally always inflations of C and D that preserve the semiglobal practices asymptotic stability, but they do not guarantee the existence of solutions for small measurement noise.

Moreover, the solutions are guaranteed to exist for any locally bounded measurement noise if the measurement noise does not appear in the flow and jump sets. This can be carried out by filtering measures. (Fig. 12) illustrates this scenario. The state x is corrupted by the noise e and the hybrid controller H_c measures a filtered version of $x + e$.
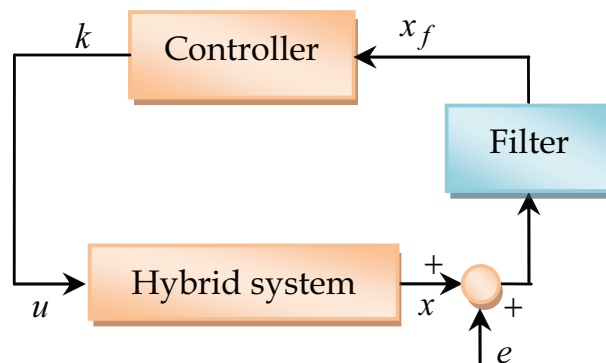

Fig. 12. Closed-loop system with noise and filtered measurements.

The filter used for the noisy output $y = x + e$ is considered to be linear and defined by the matrices A_f, B_f, and L_f, and an additional parameter $\varepsilon_f > 0$. It is designed to be asymptotically stable. Its state is denoted by x_f which takes value in R^{n_f}. At the jumps, x_f is given to the current value of y. Then, the filter has flows given by

$$\varepsilon_f \dot{x}_f = A_f x_f + B_f y, \tag{19}$$

and jumps given by

$$x_f^+ = A_f^{-1} B_f x_f + B_f y. \tag{20}$$

The output of the filter replaces the state x in the feedback law. The resulting closed-loop system can be interpreted as family of hybrid systems which depends on the parameter ε_f. It is denoted by $H_{cl}^{\varepsilon_f}$ and is given by

$$H_{cl}^{\varepsilon_f} : \quad \begin{cases} \begin{rcases} \dot{x} = f_p(x + \kappa(L_f x_f, x_c)) \\ \dot{x}_c = f_c(L_f x_f, x_c) \\ \varepsilon_f \dot{x}_f = A_f x_f + B_f(x + e) \end{rcases} & (L_f x_f, x_c) \in C_c \\[2em] \begin{rcases} x^+ = x \\ x_c^+ \in G_c(L_f x_f, x_c) \\ x_f^+ = -A_f^{-1} B_f(x + e) \end{rcases} & (L_f x_f, x_c) \in D_c \end{cases} \qquad (21)$$

5.2 Robustness to sensor and actuator dynamics

This section reviews the robustness of the closed-loop H_{cl} when additional dynamics, coming from sensors and actuators, are incorporated. (Fig. 13) shows the closed loop H_{cl} with two additional blocks: a model for the sensor and a model for the actuator. Generally, to simplify the controller design procedure, these dynamics are not included in the model of the system $\dot{x} = f_p(x,u)$ when the hybrid controller H_c is conceived. Consequently, it is important to know whether the stability properties of the closed-loop system are preserved, at least semiglobally and practically, when those dynamics are incorporated in the closed loop.

The sensor and actuator dynamics are modeled as stable filters. The state of the filter which models the sensor dynamics is given by $x_s \in R^{n_s}$ with matrices (A_s, B_s, L_s), the state of the filter that models the actuator dynamics is given by $x_a \in R^{n_a}$ with matrices (A_a, B_a, L_a), and $\varepsilon_d > 0$ is common to both filters.

Augmenting H_{cl} by adding filters and temporal regularization leads to a family $H_{cl}^{\varepsilon_d}$ given as follows

$$H_{cl}^{\varepsilon_d} : \quad \begin{cases} \begin{rcases} \dot{x} = f_p(x, L_a x_a) \\ \dot{x}_c = f_c(L_s x_s, x_c) \\ \dot{\tau} = -\tau + \tau^* \\ \varepsilon_d \dot{x}_s = A_s x_s + B_s(x + e) \\ \varepsilon_d \dot{x}_a = A_a x_a + B_a \kappa(L_s x_s, x_c) \end{rcases} & (L_s x_s, x_c) \in C_c \ \ \text{or} \ \ \tau \leq \bar{\tau} \\[3em] \begin{rcases} x^+ = x \\ x_c^+ \in G_c(L_s x_s, x_c) \\ x_s^+ = x_s \\ x_a^+ = x_a \\ \tau^+ = 0 \end{rcases} & (L_s x_s, x_c) \in D_c \ \ \text{and} \ \ \tau \geq \bar{\tau} \end{cases} \qquad (22)$$

where τ^* is a constant satisfying $\tau^* > \bar{\tau}$.

The following result states that for fast enough sensors and actuators, and small enough temporal regularization parameter, the compact set A is semiglobally practically asymptotically stable.
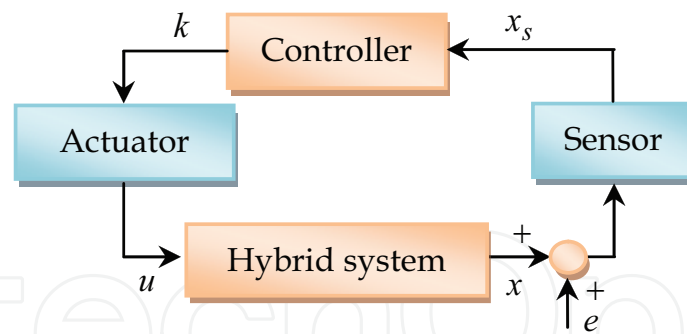
Fig. 13. Closed-loop system with sensor and actuator dynamics.

5.3 Robustness to sensor dynamics and smoothing

In many hybrid control applications, the state of the controller is explicitly given as a continuous state ξ and a discrete state $q \in Q := \{1,...,n\}$, that is, $x_c := [\xi \; q]^T$. Where this is the case and the discrete state q chooses a different control law to be applied to the system for for various values of q, then the control law generated by the hybrid controller H_c can have jumps when q changes. In many scenarios, it is not possible for the actuator to switch between control laws instantly. In addition, particularly when the control law $\kappa(\cdot,\cdot,q)$ is continuous for each $q \in Q$, it is desired to have a smooth transition between them when q changes.
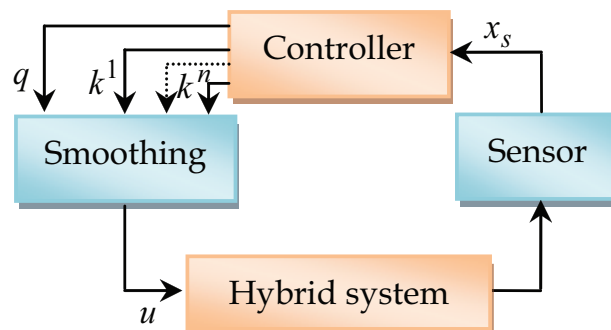
Fig. 14. Closed-loop system with sensor dynamics and control smoothing.

(Fig. 14) shows the closed-loop system, noted that $H_{cl}^{\varepsilon_d}$, resulting from adding a block that makes the smooth transition between control laws indexed by q and indicated by κ^q. The smoothing control block is modeled as a linear filter for the variable q. It is defined by the parameter ε_u and the matrices (A_u, B_u, L_u).

The output of the control smoothing block is given by

$$\alpha(x, x_c, L_u x_u) = \sum_{q \in Q} \lambda_q (L_u x_u) \kappa(x, x_c, q) \tag{23}$$

where for each $q \in Q, \lambda_q : R \rightarrow [0,1]$, is continuous and $\lambda_q(q) = 1$. Note that the output is such that the control laws are smoothly "blended" by the function λ_q.

In addition to this block, a filter modeling the sensor dynamics is also incorporated as in section 5.2. The closed loop $H_{cl}^{\varepsilon_f}$ can be written as

$$
H_{cl}^{\varepsilon_f} : \begin{cases}
\begin{cases}
\dot{x} = f_p(x + \alpha(x, x_c, L_u x_u)) \\
\dot{x}_c = f_c(L_s x_s, x_c) \\
\dot{q} = 0 \\
\dot{\tau} = -\tau + \bar{\tau}^* \\
\varepsilon_u \dot{x}_s = A_s x_s + B_s(x) \\
\varepsilon_u \dot{x}_u = A_u x_u + B_u q
\end{cases} & (L_s x_s, x_c) \in C_c \ \text{ or } \ \tau \le \bar{\tau} \\[2em]
\begin{cases}
x^+ = x \\
\begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix}^+ \in G_c(L_s x_s, x_c) \\
x_s^+ = x_s \\
x_u^+ = x_u \\
\tau^+ = 0
\end{cases} & (L_s x_s, x_c) \in D_c \ \text{ and } \ \tau \ge \bar{\tau}
\end{cases}
\tag{24}
$$

6. Conclusion

In this chapter, a dynamic systems approach to analysis and design of hybrid systems has been continued from a robust control point of view. Stability and convergence tools for hybrid systems presented include hybrid versions of the traditional Lyapunov stability theorem and of LaSalle's invariance principle.

The robustness of asymptotic stability for classes of closed-loop systems resulting from hybrid control was presented. Results for perturbations arising from the presence of measurement noise, unmodeled sensor and actuator dynamics, control smoothing.

It is very important to have good software tools for the simulation, analysis and design of hybrid systems, which by their nature are complex systems. Researchers have recognized this need and several software packages have been developed.

7. References

Rajeev, A.; Thomas, A. & Pei-Hsin, H.(1993). *Automatic symbolic verification of embedded systems*, In IEEE Real-Time Systems Symposium, 2-11, DOI: 10.1109/REAL.1993.393520 .

Dang, T. (2000). *Vérification et Synthèse des Systèmes Hybrides*. PhD thesis, Institut National Polytechnique de Grenoble.

Girard, A. (2006). *Analyse algorithmique des systèmes hybrides*. PhD thesis, Universitè Joseph Fourier (Grenoble-I).

Ancona, F. & Bressan, A. (1999). *Patchy vector fields and asymptotic stabilization*, ESAIM: Control, Optimisation and Calculus of Variations, 4:445–471, DOI: 10.1051/cocv:2004003.

Byrnes, C. I. & Martin, C. F. (1995). *An integral-invariance principle for nonlinear systems*, IEEE Transactions on Automatic Control, 983–994, ISSN: 0018-9286.

Cai, C.; Teel, A. R. & Goebel, R. (2007). *Results on existence of smooth Lyapunov functions for asymptotically stable hybrid systems with nonopen basin of attraction*, submitted to the 2007 American Control Conference, 3456 – 3461, ISSN: 0743-1619.

Cai, C.; Teel, A. R. & Goebel, R. (2006). *Smooth Lyapunov functions for hybrid systems* Part I: *Existence is equivalent to robustness* & Part II: *(Pre-)asymptotically stable compact sets*, 1264-1277, ISSN 0018-9286.

Cai, C.; Teel, A. R. & Goebel, R. (2005). *Converse Lyapunov theorems and robust asymptotic stability for hybrid systems*, Proceedings of 24th American Control Conference, 12–17, ISSN: 0743-1619.

Chellaboina, V.; Bhat, S. P. & HaddadWH. (2002). *An invariance principle for nonlinear hybrid and impulsive dynamical systems.* Nonlinear Analysis, Chicago, IL, USA, 3116 – 3122,ISBN: 0-7803-5519-9.

Goebel, R.; Prieur, C. & Teel, A. R. (2009). *smooth patchy control Lyapunov functions.* Automatica (Oxford) Y, 675-683 ISSN : 0005-1098.

Goebel, R. & Teel, A. R. (2006). *Solutions to hybrid inclusions via set and graphical convergence with stability theory applications.* Automatica, 573–587, DOI: 10.1016/j.automatica.2005.12.019.

LaSalle, J. P. (1967). *An invariance principle in the theory of stability, in Differential equations and dynamical systems.* Academic Press, New York.

LaSalle, J. P. (1976) *The stability of dynamical systems.* Regional Conference Series in Applied Mathematics, SIAM ISBN-13: 978-0-898710-22-9.

Lygeros, J.; Johansson, K. H., Simi´c, S. N.; Zhang, J. & Sastry, S. S. (2003). *Dynamical properties of hybrid automata.* IEEE Transactions on Automatic Control, 2–17 ,ISSN: 0018-9286.

Prieur, C. (2001). *Uniting local and global controllers with robustness to vanishing noise*, Mathematics Control, Signals, and Systems, 143–172, DOI: 10.1007/PL00009880

Ryan, E. P. (1998). *An integral invariance principle for differential inclusions with applications in adaptive control.* SIAM Journal on Control and Optimization, 960–980, ISSN 0363-0129.

Sanfelice, R. G.; Goebel, R. & Teel, A. R. (2005). *Results on convergence in hybrid systems via detectability and an invariance principle.* Proceedings of 2005 American Control Conference, 551–556, ISSN: 0743-1619.

Sontag, E. (1989). *Smooth stabilization implies coprime factorization.* IEEE Transactions on Automatic Control, 435–443, ISSN: 0018-9286.

DeCarlo, R.A.; Branicky, M.S.; Pettersson, S. & Lennartson, B.(2000). *Perspectives and results on the stability and stabilizability of hybrid systems.* Proc. of IEEE, 1069–1082, ISSN: 0018-9219.

Michel, A.N.(1999). *Recent trends in the stability analysis of hybrid dynamical systems.* IEEE Trans. Circuits Syst. – I. Fund. Theory Appl., 120–134,ISSN: 1057-7122.

Halbaoui, K.; Boukhetala, D. and Boudjema, F.(2008). *New robust model reference adaptive control for induction motor drives using a hybrid controller.*International Symposium on Power Electronics, Electrical Drives, Automation and Motion, Italy, 1109 - 1113 ISBN: 978-1-4244-1663-9.

Halbaoui, K.; Boukhetala, D. and Boudjema, F.(2009a). *Speed Control of Induction Motor Drives Using a New Robust Hybrid Model Reference Adaptive Controller.* Journal of Applied Sciences, 2753-2761, ISSN:18125654.

Halbaoui, K.; Boukhetala, D. and Boudjema, F.(2009b). *Hybrid adaptive control for speed regulation of an induction motor drive,* Archives of Control Sciences,V2.

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