1

Introduction to Robust Control Techniques

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1. Introduction

The theory of "Robust" Linear Control Systems has grown remarkably over the past ten years. Its popularity is now spreading over the industrial environment where it is an invaluable tool for analysis and design of servo systems. This rapid penetration is due to two major advantages: its applied nature and its relevance to practical problems of automation engineer.

To appreciate the originality and interest of robust control tools, let us recall that a control has two essential functions:

- shaping the response of the servo system to give it the desired behaviour,
- maintaining this behaviour from the fluctuations that affect the system during operation (wind gusts for aircraft, wear for a mechanical system, configuration change to a robot.).

This second requirement is termed "robustness to uncertainty". It is critical to the reliability of the servo system. Indeed, control is typically designed from an idealized and simplified model of the real system.

To function properly, it must be robust to the imperfections of the model, i.e. the discrepancies between the model and the real system, the excesses of physical parameters and the external disturbances.

The main advantage of robust control techniques is to generate control laws that satisfy the two requirements mentioned above. More specifically, given a specification of desired behaviour and frequency estimates of the magnitude of uncertainty, the theory evaluates the feasibility, produces a suitable control law, and provides a guaranty on the range of validity of this control law (strength). This combined approach is systematic and very general. In particular, it is directly applicable to Multiple-Input Multiple Output systems.

To some extent, the theory of Robust Automatic Control reconciles dominant frequency (Bode, Nyquist, PID) and the Automatic Modern dominated state variables (Linear Quadratic Control, Kalman).

It indeed combines the best of both. From Automatic Classic, it borrows the richness of the frequency analysis systems. This framework is particularly conducive to the specification of performance objectives (quality of monitoring or regulation), of band-width and of robustness. From Automatic Modern, it inherits the simplicity and power of synthesis

methods by the state variables of enslavement. Through these systematic synthesis tools, the engineer can now impose complex frequency specifications and direct access to a diagnostic feasibility and appropriate control law. He can concentrate on finding the best compromise and analyze the limitations of his system.
This chapter is an introduction to the techniques of Robust Control. Since this area is still evolving, we will mainly seek to provide a state of the art with emphasis on methods already proven and the underlying philosophy. For simplicity, we restrict to linear time invariant systems (linear time-invariant, LTI) continuous time. Finally, to remain true to the practice of this theory, we will focus on implementation rather than on mathematical and historical aspects of the theory.

2. Basic concepts

The control theory is concerned with influencing systems to realize that certain output quantities take a desired course. These can be technical systems, like heating a room with output *temperature*, a boat with the output quantities *heading* and *speed*, or a power plant with the output *electrical power*. These systems may well be social, chemical or biological, as, for example, the system of *national economy* with the output *rate of inflation*. The nature of the system does not matter. Only the dynamic behaviour is of great importance to the control engineer. We can describe this behaviour by differential equations, difference equations or other functional equations. In classical control theory, which focuses on technical systems, the system that will be influenced is called the (*controlled*) plant.
In which kinds in manners can we influence the system? Each system is composed not only of output quantities, but as well of input quantities. For the heating of a room, this, for example, will be the position of the valve, for the boat the power of the engine and angle of the rudder. These input variables have to be adjusted in a manner that the output variables take the desired course, and they are called *actuating variables*. In addition to the actuating variables, the *disturbance variables* affect the system, too. For instance, a heating system, where the temperature will be influenced by the number of people in the room or an open window, or a boat, whose course will be affected by water currents.
The desired course of output variables is defined by the *reference variables*. They can be defined by operator, but they can also be defined by another system. For example, the autopilot of an aircraft calculates the reference values for altitude, the course, and the speed of the plane. But we do not discuss the generation of reference variables here. In the following, we take for them for granted. Just take into account that the reference variables do not necessarily have to be constant; they can also be time-varying.
Of which information do have we need to calculate the actuating variables to make the output variables of the system follow the variables of reference? Clearly the reference values for the output quantities, the behavior of the plant and the time-dependent behavior of the disturbance variables must be known. With this information, one can theoretically calculate the values of the actuating variables, which will then affect the system in a way that the output quantities will follow the desired course. This is the principle of a *steering mechanism* (Fig. 1). The input variable of the steering mechanism is the reference variable ω, its output quantity actuating variable u, which again - with disturbance variable w forms the input value of the plant. y represents the output value of the system.

The disadvantage of this method is obvious. If the behavior of the plant is not in accordance with the assumptions which we made about it, or if unforeseen disruptions, then the quantities of output will not continue to follow the desired course. A steering mechanism cannot react to this deviation, because it does not know the output quantity of the plant.

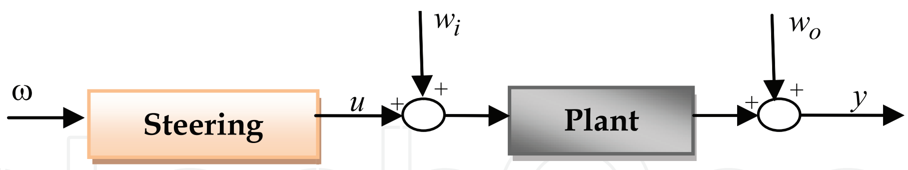


Fig. 1. Principle of a steering mechanism

A improvement which can immediately be made is the principle of an (*automatic*) *control* (Fig. 2). Inside the automatic check, the reference variable ω is compared with the measured output variable of the plant y (*control variable*), and a suitable output quantity of the controller u (*actuating variable*) are calculated inside the control unit of the difference Δy (*control error*).
During old time the control unit itself was called the controller, but the modern controllers, including, between others, the adaptive controllers (Boukhetala et al., 2006), show a structure where the calculation of the difference between the actual and wished output value and the calculations of the control algorithm cannot be distinguished in the way just described. For this reason, the tendency today is towards giving the name *controller* to the section in which the variable of release is obtained starting from the reference variable and the measured control variable.

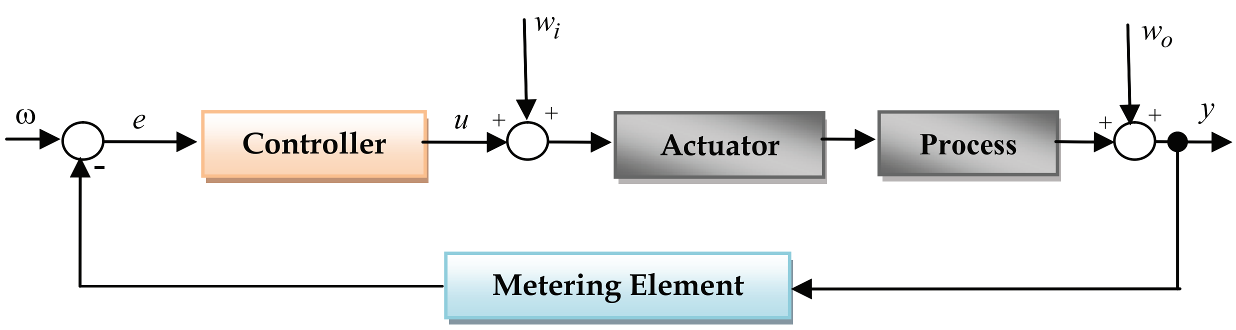


Fig. 2. Elements of a control loop

The quantity u is usually given as low-power signal, for example as a digital signal. But with low power, it is not possible to tack against a physical process. How, for example, could be a boat to change its course by a rudder angle calculated numerically, which means a sequence of zeroes and ones at a voltage of 5 V? Because it's not possible directly, a static inverter and an electric rudder drive are necessary, which may affect the rudder angle and the boat's route. If the position of the rudder is seen as actuating variable of the system, the static inverter, the electric rudder drive and the rudder itself from the *actuator* of the system. The actuator converts the controller output, a signal of low power, into the actuating variable, a signal of high power that can directly affect the plant.
Alternatively, the output of the static inverter, that means the armature voltage of the rudder drive, could be seen as actuating variable. In this case, the actuator would consist only of static converter, whereas the rudder drive and the rudder should be added to the plant. These various views already show that a strict separation between the actuator and the process is not possible. But it is not necessary either, as for the design of the controller;

we will have to take every transfer characteristic from the controller output to the control variable into account anyway. Thus, we will treat the actuator as an element of the plant, and henceforth we will employ the actuating variable to refer to the output quantity of the controller.
For the feedback of the control variable to the controller the same problem is held, this time only in the opposite direction: a signal of high power must be transformed into a signal of low power. This happens in the measuring element, which again shows dynamic properties that should not be overlooked.
Caused by this feedback, a crucial problem emerges, that we will illustrate by the following example represented in (Fig. 3). We could formulate strategy of a boat's automatic control like this: the larger the deviation from the course is, the more the rudder should be steered in the opposite direction. At a glance, this strategy seems to be reasonable. If for some reason a deviation occurs, the rudder is adjusted. By steering into the opposite direction, the boat receives a rotatory acceleration in the direction of the desired course.
The deviation is reduced until it disappears finally, but the rotating speed does not disappear with the deviation, it could only be reduced to zero by steering in the other direction. In this example, because of the rotating speed of the boat will receive a deviation in the other direction after getting back to the desired course. This is what happened after the rotating speed will be reduced by counter-steering caused by the new deviation. But as we already have a new deviation, the whole procedure starts again, only the other way round. The new deviation could be even greater than the first.
The boat will begin zigzagging its way, if worst comes to worst, with always increasing deviations. This last case is called *instability*. If the amplitude of vibration remains the same, it is called *borderline of stability*.
Only if the amplitudes decrease the system is *stable*. To receive an acceptable control algorithm for the example given, we should have taken the dynamics of the plant into account when designing the control strategy.
A suitable controller would produce a counter-steering with the rudder right in time to reduce the rotating speed to zero at the same time the boat gets back on course.

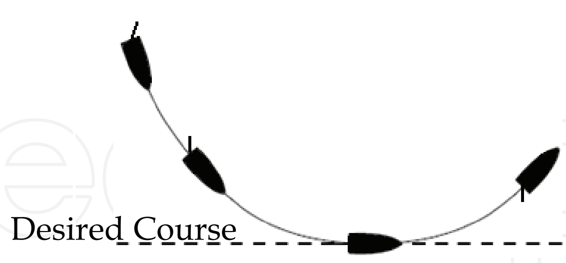


Fig. 3. Automatic cruise control of a boat

This example illustrates the requirements with respect to the controlling devices. A requirement is accuracy, i.e. the control error should be also small as possible once all the initial transients are finished and a stationary state is reached. Another requirement is the speed, i.e. in the case of a changing reference value or a disturbance; the control error should be eliminated as soon as possible. This is called the *response behavior*. The requirement of the third and most important is the stability of the whole system. We will see that these conditions are contradicted, of this fact of forcing each kind of controller (and therefore fuzzy controllers, too) to be a compromise between the three.

3. Frequency response

If we know a plant's transfer function, it is easy to construct a suitable controller using this information. If we cannot develop the transfer function by theoretical considerations, we could as well employ statistical methods on the basis of a sufficient quantity of values measured to determine it. This method requires the use of a computer, a plea which was not available during old time. Consequently, in these days a different method frequently employed in order to describe a plant's dynamic behavior, *frequency response* (Franklin et al., 2002). As we shall see later, the frequency response can easily be measured. Its good graphical representation leads to a clear method in the design process for simple PID controllers. Not to mention only several criteria for the stability, which as well are employed in connection with fuzzy controllers, root in frequency response based characterization of a plant's behavior.
The easiest way would be to define the frequency response to be the transfer function of a linear transfer element with purely imaginary values for s.
Consequently, we only have to replace the complex variable s of the transfer function by a variable purely imaginary. $j\omega$: $G(j\omega) = G(s)|_{s=j\omega}$. The frequency response is thus a complex function of the parameter ω. Due to the restriction of s to purely imaginary values; the frequency response is only part of the transfer function, but a part with the special properties, as the following theorem shows:
Theorem 1 *If a linear transfer element has the frequency response $G(j\omega)$, then its response to the input signal $x(t) = a\sin\omega t$ will be-after all initial transients have settled down-the output signal*

$$y(t) = a|G(j\omega)|\sin(\omega t + \varphi(G(j\omega))) \tag{1}$$

If the following equation holds:

$$\int_0^\infty |g(t)|\,dt \prec \infty \tag{2}$$

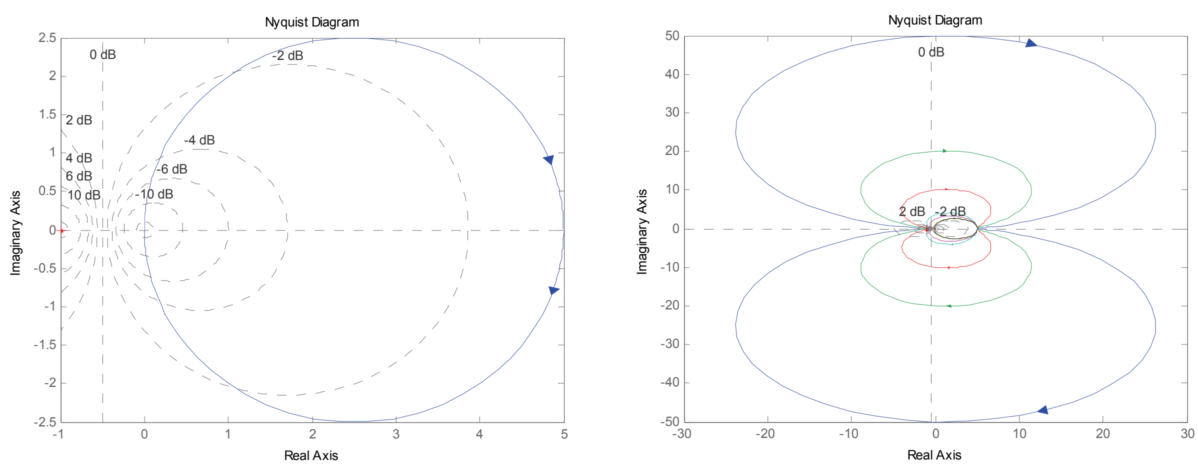
$|G(j\omega)|$ is obviously the ratio of the output sine amplitude to the input sine amplitude ((transmission) gain or amplification). $\phi(G(j\omega)$ is the phase of the complex quantity $G(j\omega)$ and shows the delay of the output sine in relation to the input sine (phase lag). $g(t)$ is the impulse response of the plant. In case the integral given in (2) does not converge, we have to add the term $r(t)$ to the right hand side of (1), which will, even for $t \to \infty$, not vanish.
The examination of this theorem shows clearly what kind of information about the plant the frequency response gives: Frequency response characterizes the system's behavior for any frequency of the input signal. Due to the linearity of the transfer element, the effects caused by single frequencies of the input signal do not interfere with each other. In this way, we are now able to predict the resulting effects at the system output for each single signal component separately, and we can finally superimpose these effects to predict the overall system output.
Unlike the coefficients of a transfer function, we can measure the amplitude and phase shift of the frequency response directly: The plant is excited by a sinusoidal input signal of a certain frequency and amplitude. After all initial transients are installed we obtain a sinusoidal signal at the output plant, whose phase position and amplitude differ from the input signal. The quantities can be measured, and depending to (1), this will also instantly

provide the amplitude and phase lag of the frequency response $G(j\omega)$. In this way, we can construct a table for different input frequencies that give the principle curve of the frequency response. Take of measurements for negative values of ω, i.e. for negative frequencies, which is obviously not possible, but it is not necessary either, delay elements for the transfer functions rational with real coefficients and for $G(j\omega)$ will be conjugate complex to $G(-j\omega)$. Now, knowing that the function $G(j\omega)$ for $\omega \geq 0$ already contains all the information needed, we can omit an examination of negative values of ω.

4. Tools for analysis of controls

4.1 Nyquist plot

A Nyquist plot is used in automatic control and signal processing for assessing the stability of a system with feedback. It is represented by a graph in polar coordinates in which the gain and phase of a frequency response are plotted. The plot of these phasor quantities shows the phase as the angle and the magnitude as the distance from the origin (see. Fig.4). The Nyquist plot is named after Harry Nyquist, a former engineer at Bell Laboratories.



First-order system | Second-order systems

Fig. 4. Nyquist plots of linear transfer elements

Assessment of the stability of a closed-loop negative feedback system is done by applying the Nyquist stability criterion to the Nyquist plot of the open-loop system (i.e. the same system without its feedback loop). This method is easily applicable even for systems with delays which may appear difficult to analyze by means of other methods.

Nyquist Criterion: We consider a system whose open loop transfer function (OLTF) is $G(s)$; when placed in a closed loop with feedback $H(s)$, the closed loop transfer function (CLTF) then becomes $\frac{G}{1+G.H}$. The case where $H=1$ is usually taken, when investigating stability, and then the *characteristic equation*, used to predict stability, becomes $G+1=0$.

We first construct *The Nyquist Contour*, a contour that encompasses the right-half of the complex plane:

- a path traveling up the $j\omega$ axis, from $0 \ -j\infty$ to $0 \ +j\infty$.
- a semicircular arc, with radius $r \to \infty$, that starts at $0 \ +j\infty$ and travels clock-wise to $0 \ -j\infty$

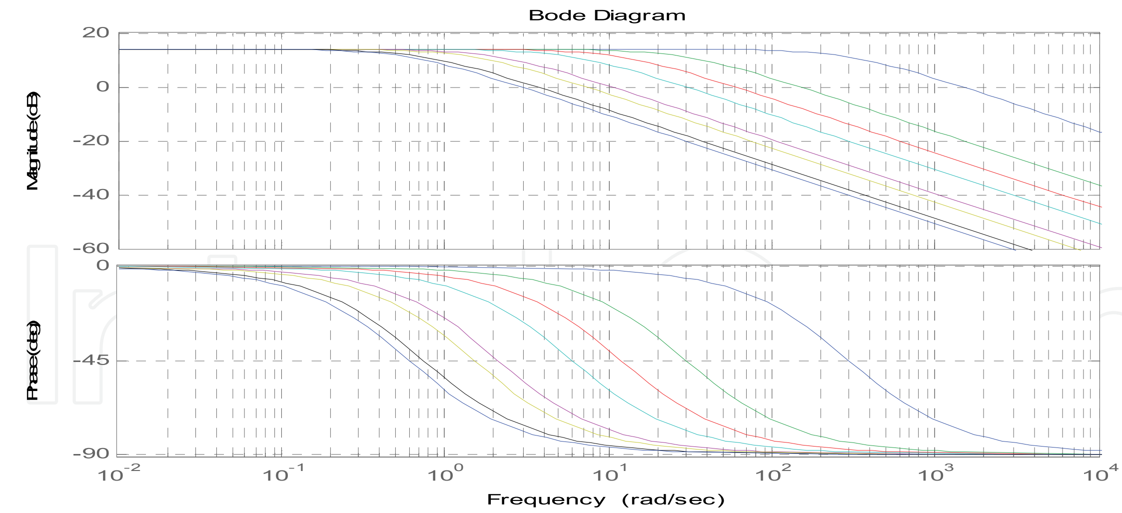
The Nyquist Contour mapped through the function $1+G(s)$ yields a plot of $1+G(s)$ in the complex plane. By the Argument Principle, the number of clock-wise encirclements of the origin must be the number of zeros of $1+G(s)$ in the right-half complex plane minus the poles of $1+G(s)$ in the right-half complex plane. If instead, the contour is mapped through the open-loop transfer function $G(s)$, the result is the Nyquist plot of $G(s)$. By counting the resulting contour's encirclements of -1, we find the difference between the number of poles and zeros in the right-half complex plane of $1+G(s)$. Recalling that the zeros of $1+G(s)$ are the poles of the closed-loop system, and noting that the poles of $1+G(s)$ are same as the poles of $G(s)$, we now state *The Nyquist Criterion*:
Given a Nyquist contour Γ_s, let P be the number of poles of $G(s)$ encircled by Γ_s and Z be the number of zeros of $1+G(s)$ encircled by Γ_s. Alternatively, and more importantly, Z is the number of poles of the closed loop system in the right half plane. The resultant contour in the $G(s)$-plane, $\Gamma_{G(s)}$ shall encircle (clock-wise) the point $(-1+j0)$ N times such that $N=Z-P$. For stability of a system, we must have $Z=0$, i.e. the number of closed loop poles in the right half of the s-plane must be zero. Hence, the number of counterclockwise encirclements about $(-1+j0)$ must be equal to P, the number of open loop poles in the right half plane (Faulkner, 1969), (Franklin, 2002).

4.2 Bode diagram

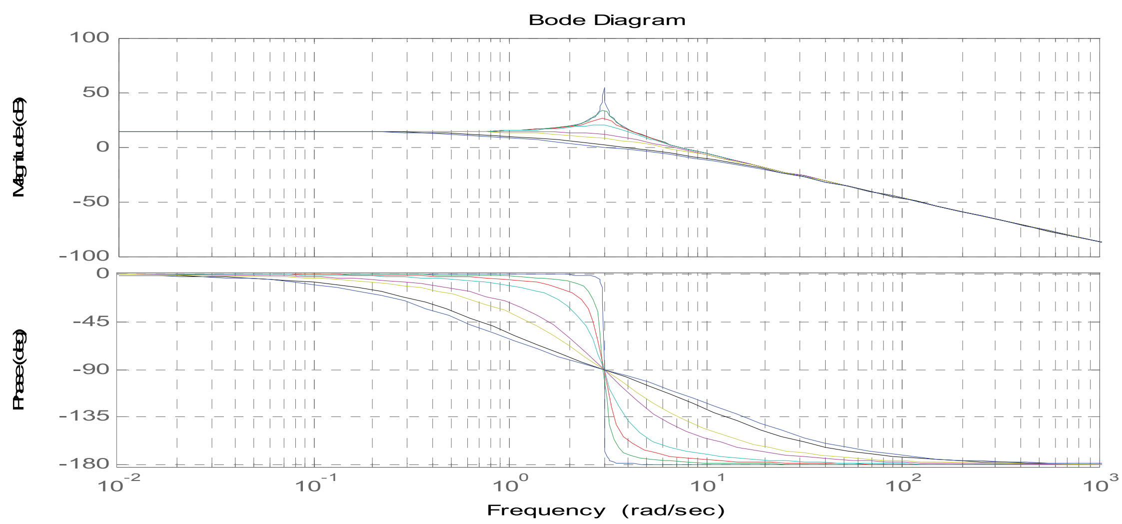
A Bode plot is a plot of either the magnitude or the phase of a transfer function $T(j\omega)$ as a function of ω. The magnitude plot is the more common plot because it represents the gain of the system. Therefore, the term "Bode plot" usually refers to the magnitude plot (Thomas, 2004),(William, 1996),(Willy, 2006). The rules for making Bode plots can be derived from the following transfer function:

$$T(s)=K\left(\frac{s}{\omega_0}\right)^{\pm n}$$

where n is a positive integer. For $+n$ as the exponent, the function has n zeros at $s=0$. For $-n$, it has n poles at $s=0$. With $s=j\omega$, it follows that $T(j\omega)=Kj^{\pm n}(\omega/\omega_0)^{\pm n}$, $|T(j\omega)|=Kj(\omega/\omega_0)^{\pm n}$ and $\angle T(j\omega)=\pm n\times 90^\circ$. If ω is increased by a factor of 10, $|T(j\omega)|$ changes by a factor of $10^{\pm n}$. Thus a plot of $|T(j\omega)|$ versus ω on $\log-\log$ scales has a slope of $\log\left(10^{\pm n}\right)=\pm n\ decades/decade$. There are $20dBs$ in a $decade$, so the slope can also be expressed as $\pm 20n\ dB/decade$.
In order to give an example, (Fig. 5) shows the Bode diagrams of the first order and second order lag. Initial and final values of the phase lag courses can be seen clearly. The same holds for the initial values of the gain courses. Zero, the final value of these courses, lies at negative infinity, because of the logarithmic representation. Furthermore, for the second order lag the resonance magnification for smaller dampings can be see at the resonance frequency ω_0.
Even with a transfer function being given, a graphical analysis using these two diagrams might be clearer, and of course it can be tested more easily than, for example, a numerical analysis done by a computer. It will almost always be easier to estimate the effects of changes in the values of the parameters of the system, if we use a graphical approach instead of a numerical one. For this reason, today every control design software tool provides the possibility of computing the Nyquist plot or the Bode diagram for a given transfer function by merely clicking on a button.



First-order systems



Second-order systems

Fig. 5. Bode diagram of first and second-order systems

4.3 Evans root locus

In addition to determining the stability of the system, the root locus can be used to design for the damping ratio and natural frequency of a feedback system (Franklin et al., 2002). Lines of constant damping ratio can be drawn radially from the origin and lines of constant natural frequency can be drawn as arcs whose center points coincide with the origin (see. Fig. 6). By selecting a point along the root locus that coincides with a desired damping ratio and natural frequency a gain, *K*, can be calculated and implemented in the controller. More elaborate techniques of controller design using the root locus are available in most control textbooks: for instance, lag, lead, PI, PD and PID controllers can be designed approximately with this technique.

The definition of the damping ratio and natural frequency presumes that the overall feedback system is well approximated by a second order system, that is, the system has a dominant pair of poles. This often doesn't happen and so it's good practice to simulate the final design to check if the project goals are satisfied.

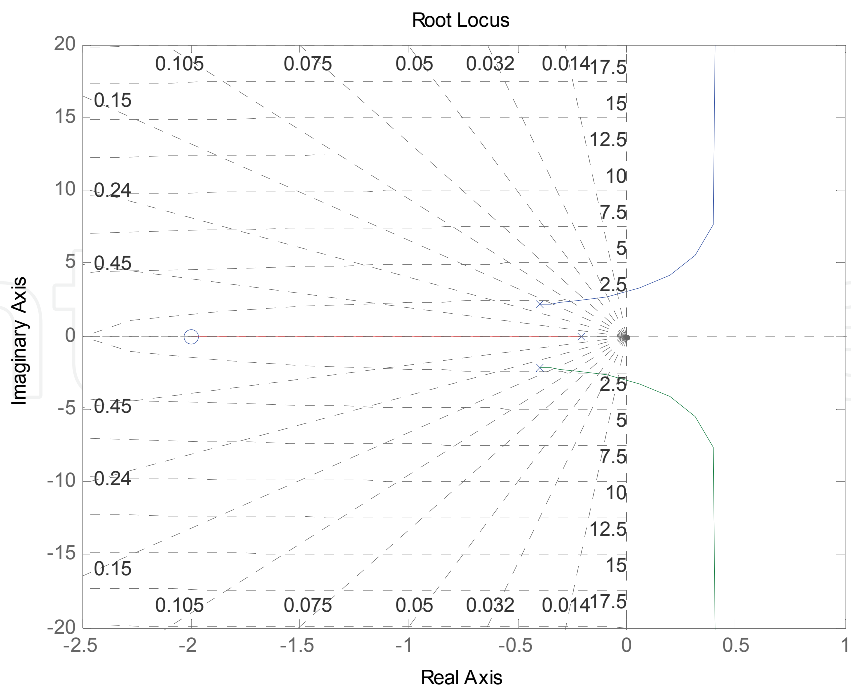


Fig. 6. Evans root locus of a second-order system

Suppose there is a plant (process) with a transfer function expression $P(s)$, and a forward controller with both an adjustable gain K and a transfer function expression $C(s)$. A unity feedback loop is constructed to complete this feedback system. For this system, the overall transfer function is given by:

$$T(s) = \frac{K.C(s).P(s)}{1 + K.C(s).P(s)} \tag{3}$$

Thus the closed-loop poles of the transfer function are the solutions to the equation $1 + K.C(s).P(s) = 0$. The principal feature of this equation is that roots may be found wherever $K.C.P = -1$. The variability of K, the gain for the controller, removes amplitude from the equation, meaning the complex valued evaluation of the polynomial in s $C(s).P(s)$ needs to have net phase of 180 deg, wherever there is a closed loop pole. The geometrical construction adds angle contributions from the vectors extending from each of the poles of KC to a prospective closed loop root (pole) and subtracts the angle contributions from similar vectors extending from the zeros, requiring the sum be 180. The vector formulation arises from the fact that each polynomial term in the factored CP, $(s-a)$ for example, represents the vector from a which is one of the roots, to s which is the prospective closed loop pole we are seeking. Thus the entire polynomial is the product of these terms, and according to vector mathematics the angles add (or subtract, for terms in the denominator) and lengths multiply (or divide). So to test a point for inclusion on the root locus, all you do is add the angles to all the open loop poles and zeros. Indeed a form of protractor, the "spirule" was once used to draw exact root loci.
From the function $T(s)$, we can also see that the zeros of the open loop system (CP) are also the zeros of the closed loop system. It is important to note that the root locus only gives the location of closed loop poles as the gain K is varied, given the open loop transfer function. The zeros of a system cannot be moved.

Using a few basic rules, the root locus method can plot the overall shape of the path (locus) traversed by the roots as the value of K varies. The plot of the root locus then gives an idea of the stability and dynamics of this feedback system for different values of K.

5. Ingredients for a robust control

The design of a control consists in adjusting the transfer function of the compensator so as to obtain the properties and the behavior wished in closed loop. In addition to the constraint of stability, we look typically the best possible performance. This task is complicated by two principal difficulties. On the one hand, the design is carried out on a idealized model of the system. We must therefore ensure the robustness to imperfections in the model, i.e. to ensure that the desired properties for a family of systems around the reference model. On the other hand, it faces inherent limitations like the compromise between performances and robustness.
This section shows how these objectives and constraints can be formulated and quantified in a consistent framework favorable to their taking into systematic account.

5.1 Robustness to uncertainty

The design of a control is carried out starting from a model of the real system often called **nominal model** or **reference model**. This model may come from the equations of physics or a process identification. In any case, this model is only one approximation of reality. Its deficiencies can be multiple: dynamic nonlinearities neglected, uncertainty on certain physical parameters, assumptions simplifying, errors of measurement to the identification, etc.. In addition, some system parameters can vary significantly with time or operating conditions. Finally, from the unforeseeable external factors can come to disturb the operation of the control system.
It is thus insufficient to optimize control compared to the nominal model: it is also necessary to be guarded against the uncertainty of modeling and external risks. Although these factors are poorly known, one has information in general on their maximum amplitude or their statistical nature. For example, the frequency of the oscillation, maximum intensity of the wind, or the terminals min and max on the parameter value. It is from this basic knowledge that one will try to carry out a robust control.
There are two classes of uncertain factors. A first class includes the uncertainty and external disturbances. These are signals or actions randomness that disrupt the controlled system. They are identified according to their point of entry into the loop. Referring again to (Fig. 2) there are basically:

- the disruption of the control w_i which can come from errors of discretization or quantification of the control or parasitic actions on the actuators.
- Disturbances at exit w_o corresponding to external effects on the output or unpredictable on the system, e.g. the wind for a airplane, an air pressure change for a chemical reactor, etc..

It should be noted that these external actions do not modify the dynamic behavior interns system, but only the “trajectory” of its outputs.
A second class of uncertain factors joins together imperfections and variations of the dynamic model of the system. Recall that the robust control techniques applied to finite dimensional linear models, while real systems are generally non-linear and infinite

dimensional. Typically, the model used thus neglects non-linear ties and is valid only in one limited frequency band. It depends of more than physical parameters whose value can fluctuate and is often known only roughly. For practical reasons, one will distinguish:

- **the dynamic uncertainty** which gathers the dynamic ones neglected in the model. There is usually only an upper bound on the amplitude of these dynamics. One must thus assume and guard oneself against worst case in the limit of this marker.
- **the parametric uncertainty** or **structured** which is related to the variations or errors in estimation on certain physical parameters of the system, or with uncertainties of dynamic nature, but entering the loop at different points. Parametric uncertainty intervenes mainly when the model is obtained starting from the equations of physics. The way in which the parameters influential on the behavior of the system determines the "structure" of the uncertainty.

5.2 Representation of the modeling uncertainty

The dynamic uncertainty (unstructured) can encompass of physical phenomena very diverse (linear or nonlinear, static or time-variant, frictions, hysteresis, etc.). The techniques discussed in this chapter are particularly relevant when one does not have any specific information if not an estimate of the maximum amplitude of dynamic uncertainty, In other words, when uncertainty is reasonably modeled by a ball in the space of bounded operators of ℓ_2 in ℓ_2.

Such a model is of course very rough and tends to include configurations with physical sense. If the real system does not comprise important nonlinearities, it is often preferable to be restricted with a stationary purely linear model of dynamic uncertainty. We can then balance the degree of uncertainty according to the frequency and translate the fact that the system is better known into low than in high frequency. Uncertainty is then represented as a disturbing system **LTI** $\Delta G(s)$ which is added to the nominal model $G(s)$ of the real system:

$$G_{true}(s) = G(s) + \Delta G(s) \tag{4}$$

This system must be BIBO-stable (bounded ℓ_2 in ℓ_2), and it usually has an estimate of the maximum amplitude of $\Delta G(j\omega)$ in each frequency band. Typically, this amplitude is small at lower frequencies and grows rapidly in the high frequencies where the dynamics neglected become important. This profile is illustrated in (Fig. 7). It defines a family of systems whose envelope on the Nyquist diagram is shown in (Fig. 8) (case SISO). The radius of the disk of the frequency uncertainty ω is $|\Delta G(j\omega)|$.

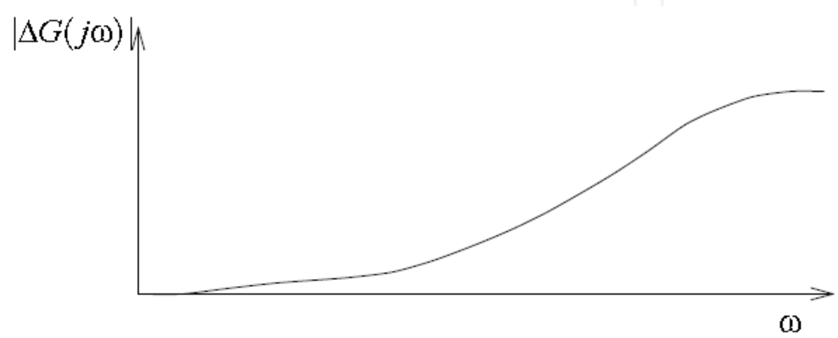


Fig. 7. Standard profile for $|\Delta G(j\omega)|$.

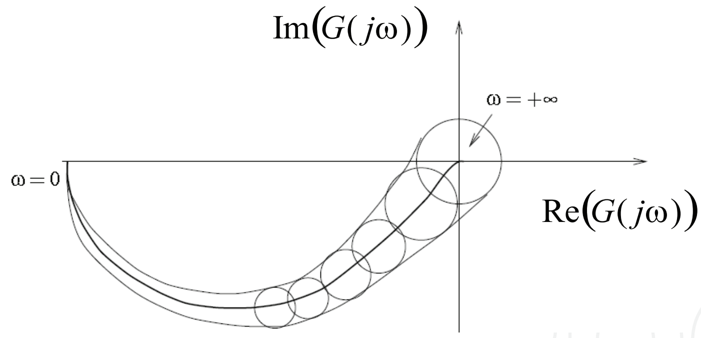


Fig. 8. Family of systems

The information on the amplitude $|\Delta G(j\omega)|$ of the uncertainty can be quantified in several ways:

- **additive uncertainty**: the real system is of the form:

$$G_{true}(s) = G(s) + \Delta(s) \tag{5}$$

Where $\Delta(s)$ is a stable transfer function satisfying:

$$\|W_l(\omega)\Delta(j\omega)W_r(\omega)\|_\infty \prec 1 \tag{6}$$

for certain models $W_l(s)$ and $W_r(s)$. These weighting matrices make it possible to incorporate information on the frequential dependence and directional of the maximum amplitude of $\Delta(s)$ (see singular values).

- **multiplicative uncertainty at the input**: the real system is of the form:

$$G_{true}(s) = G(s).(I + \Delta(s)) \tag{7}$$

where $\Delta(s)$ is like above. This representation models errors or fluctuations on the behavior in input.

- **multiplicative uncertainty at output**: the real system is of the form:

$$G_{true}(s) = (I + \Delta(s)).G(s) \tag{8}$$

This representation is adapted to modeling of the errors or fluctuations in the output behavior. According to the data on the imperfections of the model, one will choose one or the other of these representations. Let us note that multiplicative uncertainty has a relative character.

5.3 Robust stability

Let the linear system be given by the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} e^{T_L s}$$
$$= \frac{b_m}{a_n} \frac{\prod_{\mu=1}^{m}(s - n_\mu)}{\prod_{\nu=1}^{n}(s - p_\nu)} e^{T_L s} = V \frac{\prod_{\mu=1}^{m}(\frac{-s}{n_\mu} + 1)}{\prod_{\nu=1}^{n}(\frac{-s}{p_\nu} + 1)} e^{-T_L s} \quad \text{where } m \le n \tag{9}$$

with the gain

$$V = \frac{b_0}{a_0} \tag{10}$$

First we must explain what we mean by stability of a system. Several possibilities exist to define the term, two of which we will discuss now. A third definition by the Russian mathematician Lyapunov will be presented later. The first definition is based on the step response of the system:

Definition 1 *A system is said to be stable if, for* $t \to \infty$, *its step response converges to a finite value. Otherwise, it is said to be instable.*

This unit step function has been chosen to stimulate the system does not cause any restrictions, because if the height of the step is modified by the factor *k*, the values to the system output will change by the same factor *k*, too, according to the linearity of the system. Convergence towards a finite value is therefore preserved.

A motivation for this definition can be the idea of following illustration: If a system converges towards a finished value after strong stimulation that a step in the input signal represents, it can suppose that it will not be wedged in permanent oscillations for other kinds of stimulations.

It is obvious to note that according to this definition the first order and second order lag is stable, and that the integrator is instable.

Another definition is attentive to the possibility that the input quantity may be subject to permanent changes:

Definition 2 *A linear system is called stable if for an input signal with limited amplitude, its output signal will also show a limited amplitude. This is the BIBO-Stability (bounded input - bounded output).*

Immediately, the question on the connection between the two definitions arises, that we will now examine briefly. The starting point of discussion is the convolution integral, which gives the relationship between the system's input and the output quantity (the impulse response):

$$y(t) = \int_{\tau=0}^{t} g(t-\tau)x(\tau)d\tau = \int_{\tau=0}^{t} g(\tau)x(t-\tau)d\tau \tag{11}$$

$x(t)$ is bounded if and only if $|x(t)| \le k$ holds (with $k > 0$) for all t. This implies:

$$|y(t)| \le \int_{\tau=0}^{t} |g(\tau)||x(t-\tau)|d\tau \le k \int_{\tau=0}^{t} g(\tau)d\tau \tag{12}$$

Now, with absolute convergence of the integral of the impulse response,

$$\int_{\tau=0}^{\infty} |g(\tau)|d\tau = c < \infty \tag{13}$$

$y(s)$ will be limited by kc, also, and thus the whole system will be BIBO-stable. Similarly it can be shown that the integral (13) converges absolutely for all BIBO-stable systems. BIBO

stability and the absolute convergence of the impulse response integral are the equivalent properties of system.
Now we must find the conditions under which the system will be stable in the sense of a finite step response *(Definition 2)*: Regarding the step response of a system in the frequency domain,

$$y(s) = G(s)\frac{1}{s} \tag{14}$$

If we interpret the factor $1/s$ as an integration (instead of the Laplace transform of the step signal), we obtain

$$y(s) = \int_{\tau=0}^{t} g(\tau)d\tau \tag{15}$$

in the time domain for $y(0) = 0.y(t)$ converge to a finite value only if the integral converges:

$$\int_{\tau=0}^{t} g(\tau)d\tau = c < \infty \tag{16}$$

Convergence is obviously a weaker criterion than absolute convergence. Therefore, each BIBO-stable system will have a finite step response. To treat the stability always in the sense of the BIBO-stability is tempting because this stronger definition makes other differentiations useless. On the other hand, we can simplify the following considerations much if we use the finite-step-response-based definition of stability (Christopher, 2005),(Arnold, 2006). In addition to this, the two definitions are equivalent as regards the transfer functions anyway. Consequently, henceforth we will think of stability as characterized in *(Definition 2).*
Sometimes stability is also defined while requiring that the impulse response to converge towards zero for $t \to \infty$. A glance at the integral (16) shows that this criterion is necessary but not sufficient condition for stability as defined by *(Definition 2)*, while *(Definition 2)* is the stronger definition. If we can prove a finite step response, then the impulse response will certainly converge to zero.

5.3.1 Stability of a transfer function

If we want to avoid having to explicitly calculate the step response of a system in order to prove its stability, then a direct examination of the transfer function of the system's, trying to determine criteria for the stability, seems to suggest itself (Levine, 1996). This is relatively easy concerning all ideas that we developed up to now about the step response of a rational transfer function. The following theorem is valid:
Theorem 2 *A transfer element with a rational transfer function is stable in the sense of (Definition 2) if and only if all poles of the transfer function have a negative real part.*
According to equation (17), the step response of a rational transfer element is given by:

$$y(t) = \sum_{\lambda=1}^{i} h_{\lambda}(t)e^{s_{\lambda}t} \tag{17}$$

For each pole s_λ of multiplicity n_λ, we obtain a corresponding operand $h_\lambda(t)e^{s_\lambda t}$, which $h_\lambda(t)$ is a polynomial of degree $n_\lambda - 1$. For a pole with a negative real part, this summand disappears to increase t, as the exponential function converges more quickly towards zero than the polynomial $h_\lambda(t)$ can increase. If all the poles of the transfer function have a negative real part, then all corresponding terms disappear. Only the summand $h_i(t)e^{s_i t}$ for the simple pole $s_i = 0$ remains, due to the step function. The polynomial $h_i(t)$ is of degree $n_i - 1 = 0$, i.e. a constant, and the exponential function is also reduced to a constant. In this way, this summand form the finite final value of the step function, and the system is stable.

We omit the proof in the opposite direction, i.e. a system is instable if at least one pole has a positive real part because it would not lead to further insights. It is interesting that *(Theorem 2)* holds as well for systems with delay according to (9). The proof of this last statement will be also omitted.

Generally, the form of the initial transients as reaction to the excitations of outside will also be of interest besides that the fact of stability. If a plant has, among others, a complex conjugate pair of poles s_λ, $\overline{s_\lambda}$, the ratio $|\mathrm{Re}(s_\lambda)|\sqrt{\mathrm{Re}(s_\lambda)^2 + \mathrm{Im}(s_\lambda)^2}$ is equal to the damping ratio D and therefore responsible for the form of the initial transient corresponding to this pair of poles. In practical applications one will therefore pay attention not only to that the system's poles have a negative real part, but also to the damping ratio D having a sufficiently high value, i.e. that a complex conjugate pair of poles lies at a reasonable distance to the axis of imaginaries.

5.3.2 Stability of a control loop

The system whose stability must be determined will in the majority of the cases be a closed control loop (Goodwin, 2001), as shown in (Fig. 2). A simplified structure is given in (Fig. 9). Let the transfer function of the control unit is $K(s)$, the plant will be given by $G(s)$ and the metering element by $M(s)$. To keep further derivations simple, we set $M(s)$ to 1, i.e. we neglect the dynamic behavior of the metering element, for simple cases, but it should normally be no problem to take the metering element also into consideration.

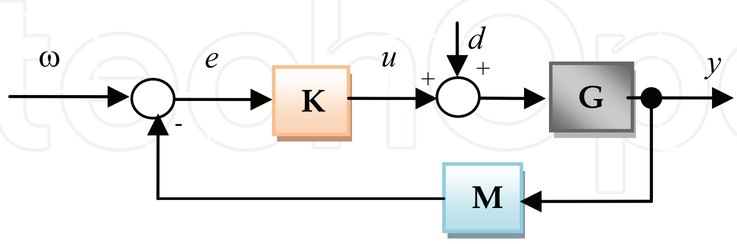


Fig. 9. Closed-loop system

We summarize the disturbances that could affect the closed loop system to virtually any point, into a single disturbance load that we impressed at the plant input. This step simplifies the theory without the situation for the controller easier than it would be in practical applications. Choose the plant input as the point where the disturbance affects the plant is most unfavorable: The disturbance can affect plants and no countermeasure can be applied, as the controller can only counteract after the changes at the system output.

To be able to apply the criteria of stability to this system we must first calculate the transfer function that describes the transfer characteristic of the entire system between the input quantity ω and the output quantity y. This is the transfer function of the closed loop, which is sometimes called the *reference (signal) transfer function*. To calculate it, we first set d to zero. In the frequency domain we get

$$y(s) = G(s)u(s) = G(s)K(s)(\omega(s) - y(s)) \tag{18}$$

$$T(s) = \frac{y(s)}{\omega(s)} = \frac{G(s)K(s)}{G(s)K(s)+1} \tag{19}$$

In a similar way, we can calculate a *disturbance transfer function,* which describes the transfer characteristic between the disturbance d and the output quantity y:

$$S(s) = \frac{y(s)}{d(s)} = \frac{G(s)K(s)}{G(s)K(s)+1} \tag{20}$$

The term $G(s)K(s)$ has a special meaning: if we remove the feedback loop, so this term represents the transfer function of the resulting open circuit. Consequently, $G(s)K(s)$ is sometimes called the *open-loop transfer function*. The gain of this function (see (9)) is called *open-loop gain*.

We can see that the reference transfer function and the disturbance transfer function have the same denominator $G(s)K(s)+1$. On the other hand, by *(Theorem 2)*, it is the denominator of the transfer function that determines the stability. It follows that only the open-loop transfer function affects the stability of a system, but not the point of application of an input quantity. We can therefore restrict an analysis of the stability to a consideration of the term $G(s)K(s)+1$.

However, since both the numerator and denominator of the two transfer functions $T(s)$ and $S(s)$ are obviously relatively prime to each other, the zeros of $G(s)K(s)+1$ are the poles of these functions, and as a direct consequence of *(Theorem 2)* we can state:

Theorem 3 *A closed-loop system with the open-loop transfer function* $G(s)K(s)$ *is stable if and only if all solutions of the characteristic equation have a negative real part.*

$$G(s)K(s)+1=0 \tag{21}$$

Computing these zeros in an analytic way will no longer be possible if the degree of the plant is greater than two, or if an exponential function forms a part of the open-loop transfer function. Exact positions of the zeros, though, are not necessary in the analysis of stability. Only the fact whether the solutions have a positive or negative real part is of importance. For this reason, in the history of the control theory criteria of stability have been developed that could be used to determine precisely without having to make complicated calculations (Christopher, 2005), (Franklin, 2002).

5.3.3 Lyapunov's stability theorem

We state below a variant of Lyapunov's direct method that establishes global asymptotic stability.

Theorem 4 Consider the dynamical system $\dot{x}(t) = f(x(t))$ and let $x = 0$ be its unique equilibrium point. If there exists a continuously differentiable function $V : \Re^n \to \Re$ such that

$$V(0) = 0 \tag{22}$$

$$V(x) \succ 0 \quad \forall x \neq 0 \tag{23}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{24}$$

$$\dot{V}(x) \prec 0 \quad \forall x \neq 0, \tag{25}$$

then $x = 0$ is globally asymptotically stable.
Condition (25) is what we refer to as the *monotonicity requirement* of Lyapunov's theorem. In the condition, $\dot{V}(x)$ denotes the derivation of $V(x)$ along the trajectories of $\dot{x}(t)$ and is given by

$$\dot{V}(x) = < \frac{\partial V(x)}{\partial x}, f(x) >,$$

where <.,.> denotes the standard inner product in $\Re^n$ and $\frac{\partial V(x)}{\partial x} \in \Re^n$ is the gradient of $V(x)$. As far as the first two conditions are concerned, it is only needed to assume that $V(x)$ is lower bounded and achieves its global minimum at $x = 0$. There is no conservatism, however, in requiring (22) and (23). A function satisfying condition (24) is called *radially unbounded*. We refer the reader to (Khalil, 1992) for a formal proof of this theorem and for an example that shows condition (24) cannot be removed. Here, we give the geometric intuition of Lyapunov's theorem, which essentially carries all of the ideas behind the proof.

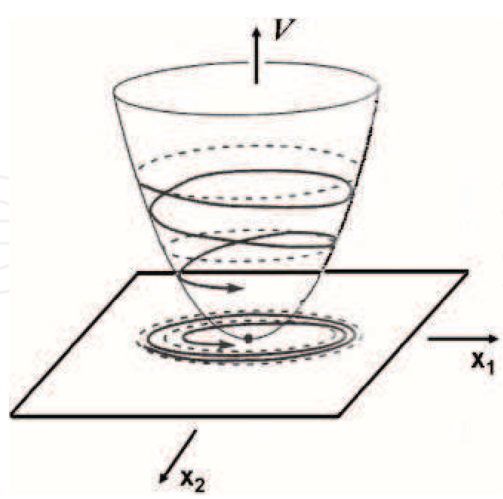


Fig. 10. Geometric interpretation of Lyapunov's theorem.

(Fig. 10) shows a hypothetical dynamical system in $\Re^2$. The trajectory is moving in the (x_1, x_2) plane but we have no knowledge of where the trajectory is as a function of time. On the other hand, we have a scalar valued function $V(x)$, plotted on the z-axis, which has the guaranteed property that as the trajectory moves the value of this function along the trajectories strictly decreases. Since $V(x(t))$ is lower bounded by zero and is strictly decreasing, it must converge to a nonnegative limit as time goes to infinity. It takes a relatively straightforward argument appealing to continuity of $V(x)$ and $\dot{V}(x)$) to show that the limit of $V(x(t))$ cannot be strictly positive and indeed conditions (22)-(25) imply

$$V(x(t)) \rightarrow 0 \text{ as } t \rightarrow \infty$$

Since $x = 0$ is the only point in space where $V(x)$ vanishes, we can conclude that $x(t)$ goes to the origin as time goes to infinity.
It is also insightful to think about the geometry in the (x_1, x_2) plane. The level sets of $V(x)$ are plotted in (Fig. 10) with dashed lines. Since $V(x(t))$ decreases monotonically along trajectories, we can conclude that once a trajectory enters one of the level sets, say given by $V(x) = c$, it can never leave the set $\Omega_c := \left\{ x \in \Re^n \middle| V_x \leq c \right\}$.This property is known as *invariance of sub-level sets*.
Once again we emphasize that the significance of Lyapunov's theorem is that it allows stability of the system to be verified without explicitly solving the differential equation. Lyapunov's theorem, in effect, turns the question of determining stability into a search for a so-called Lyapunov function, a positive definite function of the state that decreases monotonically along trajectories. There are two natural questions that immediately arise. First, do we even know that Lyapunov functions always exist?
Second, if they do in fact exist, how would one go about finding one? In many situations, the answer to the first question is positive. The type of theorems that prove existence of Lyapunov functions for every stable system are called *converse theorems*. One of the well known converse theorems is a theorem due to Kurzweil that states if *f* in *(Theorem 4)* is continuous and the origin is globally asymptotically stable, then there exists an infinitely differentiable Lyapunov function satisfying conditions of *(Theorem 4).* We refer the reader to (Khalil, 1992) and (Bacciotti & Rosier,2005) for more details on converse theorems. Unfortunately, converse theorems are often proven by assuming knowledge of the solutions of *(Theorem 4)* and are therefore useless in practice. By this we mean that they offer no systematic way of finding the Lyapunov function. Moreover, little is known about the connection of the dynamics *f* to the Lyapunov function *V*. Among the few results in this direction, the case of linear systems is well settled since a stable linear system always admits a quadratic Lyapunov function. It is also known that stable and smooth homogeneous systems always have a homogeneous Lyapunov function (Rosier, 1992).

5.3.4 Criterion of Cremer, Leonhard and Michailow

Initially let us discuss a criterion which was developed independently by Cremer , Leonhard and Michailov during the years 1938-1947. The focus of interest is the phase shift of the Nyquist plot of a polynomial with respect to the zeros of the polynomial (Mansour, 1992). Consider a polynomial of the form

$$P(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = \prod_{\nu=1}^{n} (s - s_\nu) \tag{26}$$

be given. Setting $s = j\omega$ and substituting we obtain

$$\begin{aligned} P(j\omega) &= \prod_{\nu=1}^{n}(j\omega - s_\nu) = \prod_{\nu=1}^{n}(|j\omega - s_\nu|)e^{j\varphi_\nu(\omega)}) \\ &= \prod_{\nu=1}^{n}|j\omega - s_\nu|e^{j\sum_{\nu=1}^{n}\varphi_\nu(\omega)} = |P(j\omega)|e^{j\varphi(\omega)} \end{aligned} \tag{27}$$

We can see, that the frequency response $P(j\omega)$ is the product of the vectors $(j\omega - s_\nu)$, where the phase $\varphi(\omega)$ is given by the sum of the angles $\varphi_\nu(\omega)$ of those vectors. (Fig.11) shows the situation corresponding to a pair of complex conjugated zeros with negative real part and one zero with a positive real part.

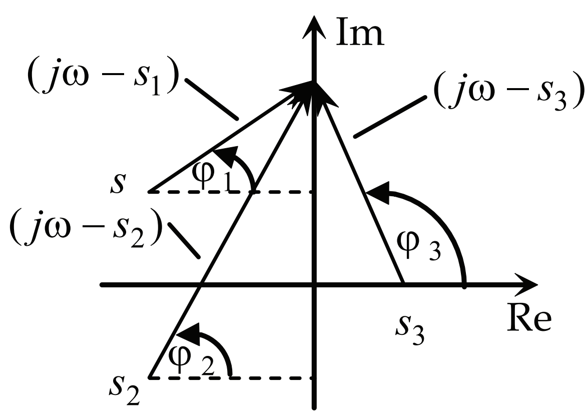


Fig. 11. Illustration to the Cremer-Leonhard-Michailow criterion

If the parameter ω traverses the interval $(-\infty,\infty)$, it causes the end point of the vectors $(j\omega - s_\nu)$ to move along the axis of imaginaries in positive direction. For zeros with negative real part, the corresponding angle φ_ν traverses the interval from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, for zeros with positive real part the interval from $+\frac{3\pi}{2}$ to $+\frac{\pi}{2}$. For zeros lying on the axis of imaginaries the corresponding angle φ_ν initially has the value $-\frac{\pi}{2}$ and switches to the value $+\frac{\pi}{2}$ at $j\omega = s_\nu$.

We will now analyze the phase of frequency response, i.e. the entire course which the angle $\varphi(\omega)$ takes. This angle is just the sum of the angles $\varphi u_\nu(\omega)$. Consequently, each zero with a negative real part contributes an angle of $+\pi$ to the phase shift of the frequency response, and each zero with a positive real part of the angle $-\pi$. Nothing can be said about zeros located on the imaginary axis because of the discontinuous course where the values of the phase to take. But we can immediately decide zeros or not there watching the Nyquist plot of the polynomial $P(s)$. If she got a zero purely imaginary $s = s_\nu$, the corresponding Nyquist plot should pass through the origin to the frequency $\omega = |s_\nu|$. This leads to the following theorem:

Theorem 5 *A polynomial P(s) of degree n with real coefficients will have only zeros with negative real part if and only if the corresponding Nyquist plot does not pass through the origin of the complex plane and the phase shift* $\Delta\varphi$ *of the frequency response is equal to* $n\pi$ *for* $-\infty < \omega < +\infty$. *If ω traverses the interval* $0 \le \omega < +\infty$ *only, then the phase shift needed will be equal to* $\frac{n}{2}\pi$.

We can easily prove the fact that for $0 \le \omega < +\infty$ the phase shift needed is only $\frac{n}{2}\pi$ —only half the value:

For zeros lying on the axis of reals, it is obvious that their contribution to the phase shift will be only half as much if ω traverses only half of the axis of imaginaries (from 0 to ∞). The zeros with an imaginary part different from zero are more interesting. Because of the polynomial's real-valued coefficients, they can only appear as a pair of complex conjugated zeros. (Fig. 12) shows such a pair with $s_1 = \overline{s_2}$ and $\alpha_1 = -\alpha_2$. For $-\infty < \omega < +\infty$ the contribution to the phase shift by this pair is 2π. For $0 \le \omega < +\infty$, the contribution of s_1 is $\frac{\pi}{2} + |\alpha_1|$ and the one for s_2 is $\frac{\pi}{2} - |\alpha_1|$. Therefore, the overall contribution of this pair of poles is π, so also for this case the phase shift is reduced by one half if only the half axis of imaginaries is taken into consideration.

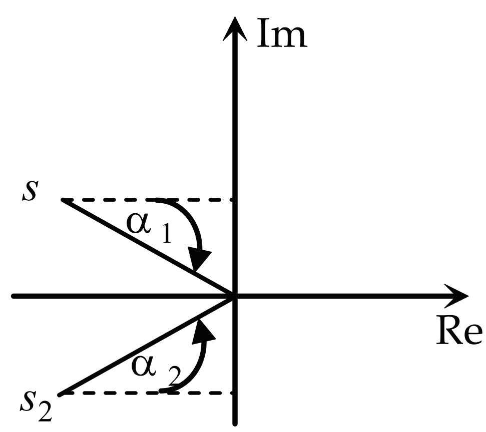


Fig. 12. Illustration to the phase shift for a complex conjugated pair of poles

6. Beyond this introduction

There are many good textbooks on Classical Robust Control. Two popular examples are (Dorf & Bishop, 2004) and (Franklin et al., 2002). A less typical and interesting alternative is the recent textbook (Goodwin et al., 2000). All three of these books have at least one chapter devoted to the Fundamentals of Control Theory. Textbooks devoted to Robust and Optimal Control are less common, but there are some available. The best known is probably (Zhou et al.1995). Other possibilities are (Aström & Wittenmark, 1996),(Robert, 1994)(Joseph et al, 2004). An excellent book about the Theory and Design of Classical Control is the one by Aström and Hägglund (Aström & Hägglund, 1995). Good references on the limitations of control are (Looze & Freudenberg, 1988). Bode's book (Bode, 1975) is still interesting, although the emphasis is on vacuum tube circuits.

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The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.