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Reasoning about Resource-Sensitive Multi-Agents

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1. Introduction

Aims. Formalizing common knowledge reasoning in multi-agent systems is of growing importance in Computer Science, Artificial Intelligence, Economics, Philosophy and Psychology. Obtaining a concrete logical foundation for common knowledge reasoning plays an important role for formal treatment and verification of multi-agent systems. For this reason, formalizing common knowledge reasoning is also a traditional issue for *multi-agent epistemic logics* (Fagin et al., 1995; Halpern & Moses, 1992; Lismont & Mongin, 1994; Meyer & van der Hoek, 1995). The aim of this paper is to formalize more fine-grained common knowledge reasoning by a new logical foundation based on Girard's linear logics.

Common knowledge. The notion of common knowledge was probably first introduced by Lewis (Lewis, 1969). This notion is briefly explained below. Let A be a fixed set of agents and α be an idea. Suppose that α belongs to the common knowledge of A, and i and j are some members of A. Then, we have the facts "both i and j know α", "i knows that j knows α" and "j knows that i knows α". Moreover, we also have the facts "i knows that j knows that i knows α", and so on. Then, these nesting structures develop an infinite hierarchy as a result.

Iterative interpretation. Suppose that the underlying multi-agent logic has the knowledge operators $\heartsuit_1, \heartsuit_2, ..., \heartsuit_n$, in which a formula $\heartsuit_i\alpha$ means "the agent i knows α." The common knowledge of a formula α is defined below. For any $m \geq 0$, an expression K^m means the set

$$\{\heartsuit_{i_1}\heartsuit_{i_2}\cdots\heartsuit_{i_m} \mid each\ \heartsuit_{i_t}\ is\ one\ of\ \heartsuit_1, ..., \heartsuit_n\ and\ i_t \neq i_{t+1}\ for\ all\ t = 1, ..., m-1\}.$$

When $m = 0$, $\heartsuit_{i_1}\heartsuit_{i_2}\cdots\heartsuit_{i_m}$ is interpreted as the null symbol. The common knowledge $\heartsuit_c\alpha$ of α is defined by using an infinitary conjunction $\bigwedge$ as the so-called *iterative interpretation of common knowledge*: $\heartsuit_c\alpha := \bigwedge\{\heartsuit\alpha \mid \heartsuit \in \bigcup_{m\in\omega} K^m\}$. Then, the formula $\heartsuit_c\alpha$ means "α is common knowledge of agents."

Common knowledge logics. *Common knowledge logics* (CKLs) are multi-agent epistemic logics with some knowledge and common knowledge operators (Fagin et al., 1995; Halpern & Moses, 1992; Lismont & Mongin, 1994; Meyer & van der Hoek, 1995). So far, CKLs have been studied based on classical logic (CL). On the other hand, CL is not so appropriate for expressing more fine-grained reasoning such as resource-sensitive, concurrency-centric and constructive reasoning. Thus, CKLs based on non-classical logics have been required for expressing such fine-grained reasoning.

Linear logics. Girard's *linear logics* (LLs) (Girard, 1987), which are most promising and useful non-classical logics in Computer Science, are logics that can naturally represent the concepts

of "resource consumption" and "parallel execution" in concurrent systems. Applications of linear logics to programming languages have successfully been studied by many researchers (see e.g., (Miller, 2004) and the references therein). Combining LLs with some knowledge and common knowledge operators is thus a natural candidate for realizing an expressive and useful common knowledge logic. Indeed, *intuitionistic linear logic* (ILL) and *classical linear logic* (CLL) (Girard, 1987), which were introduced as refinements of intuitionistic logic (IL) and CL, respectively, are more expressive than IL and CL, respectively.

Multi-agent linear logics. A multi-agent epistemic linear logic with a common knowledge operator has not yet been proposed. A reason may be that to prove the cut-elimination and completeness theorems for such an extended multi-agent linear logic is difficult because of the complexity of the traditional setting of a common knowledge operator in sequent calculus. This paper is trying to overcome such a difficulty by introducing a new simple formulation of a *fixpoint operator*, which can be used as a common knowledge operator, and by using a phase semantic proof method. *Phase semantics*, which was originally introduced by Girard (Girard, 1987), is known to be a very useful Tarskian semantics for linear and other substructural logics. It was shown by Okada that the cut-elimination theorems for CLL and ILL can be proved by using the phase semantics (Okada, 1999; 2002). This paper uses Okada's method effectively to obtain the cut-elimination and completeness theorems for the proposed multi-agent linear logics.

New fixpoint operator. In the following, we explain the proposed formulation of fixpoint operator. The symbol ω is used to represent the set of natural numbers, and the symbol N is used to represent a fixed nonempty subset of ω. The symbol K is used to represent the set $\{\heartsuit_i \mid i \in N\}$ of modal operators, and the symbol K^* is used to represent the set of all words of finite length of the alphabet K. For example, $\{\iota\alpha \mid \iota \in K^*\}$ denotes the set $\{\heartsuit_{i_1} \cdots \heartsuit_{i_k}\alpha \mid i_1, ..., i_k \in N, k \in \omega\}$. Remark that K^* includes $\emptyset$ and hence $\{\iota\alpha \mid \iota \in K^*\}$ includes α. Greek lower-case letters ι and κ are used to represent any members of K^*. The characteristic inference rules for a fixpoint operator $\heartsuit_{\mathrm{F}}$ are as follows:

$$\frac{\iota\kappa\alpha, \Gamma \Rightarrow \gamma}{\iota\heartsuit_{\mathrm{F}}\alpha, \Gamma \Rightarrow \gamma}\ (\heartsuit_{\mathrm{F}}\mathrm{left}) \qquad \frac{\{\, \Gamma \Rightarrow \iota\kappa\alpha \mid \kappa \in K^* \,\}}{\Gamma \Rightarrow \iota\heartsuit_{\mathrm{F}}\alpha}\ (\heartsuit_{\mathrm{F}}\mathrm{right}).$$

These inference rules imply the following axiom scheme: $\heartsuit_{\mathrm{F}}\alpha \leftrightarrow \bigwedge\{\iota\alpha \mid \iota \in K^*\}$. Suppose that for any formula α, f_α is a mapping on the set of formulas such that $f_\alpha(x) := \bigwedge\{\heartsuit_i(x \wedge \alpha) \mid i \in \omega\}$. Then, $\heartsuit_{\mathrm{F}}\alpha$ becomes a fixpoint (or fixed point) of f_α.

Interpretations of new fixpoint operator. The axiom scheme presented above just corresponds to the iterative interpretation of common knowledge. On the other hand, if we take the singleton $K := \{\heartsuit_0\}$, then we can understand $\heartsuit_0$ and $\heartsuit_{\mathrm{F}}$ as the temporal operators X (next-time) and G (any-time), respectively, which are subsumed in *linear-time temporal logic* (LTL) (Emerson, 1990; Pnueli, 1977). The corresponding axiom scheme for the singleton case just represents the following axiom scheme for LTL: $\mathrm{G}\alpha \leftrightarrow \bigwedge\{\mathrm{X}^i\alpha \mid i \in \omega\}$ where $\mathrm{X}^i\alpha$ is defined inductively by $\mathrm{X}^0\alpha := \alpha$ and $\mathrm{X}^{i+1}\alpha := \mathrm{X}\mathrm{X}^i\alpha$. The fixpoint operator $\heartsuit_{\mathrm{F}}$ is thus regarded as a natural generalization of both the any-time temporal operator and the common knowledge operator.

Present paper's results. The results of this paper are then summarized as follows. Two multi-agent versions MILL and MCLL of ILL and CLL, respectively, are introduced as Gentzen-type sequent calculi. MILL and MCLL have the fixpoint operator $\heartsuit_{\mathrm{F}}$, which is naturally formalized based on the idea of iterative interpretation of common knowledge. The completeness theorems with respect to *modality-indexed phase semantics* for MILL and MCLL

are proved by using Okada's phase semantic method. The cut-elimination theorems for MILL and MCLL are then simultaneously obtained by this method. Some related works are briefly surveyed.

2. Intuitionistic case

2.1 Sequent calculus

The language used in this section is introduced below. Let n be a fixed positive integer. Then, the symbol N is used to represent the set $\{1, 2, ..., n\}$ of indexes of modal operators. *Formulas* are constructed from propositional variables, $\mathbf{1}$ (multiplicative constant), $\top$, $\mathbf{0}$ (additive constants), $\rightarrow$ (implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction), ! (of course), $\heartsuit_i$ ($i \in N$) (i-th modality) and $\heartsuit_{\mathrm{F}}$ (fixpoint modality). Remark that the symbols $\rightarrow, \wedge, *$ and $\vee$ are from (Troelstra, 1992), which are different from those in (Girard, 1987). Lower-case letters $p, q, ...$ are used to represent propositional variables, Greek lower-case letters $\alpha, \beta, ...$ are used to represent formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) multisets of formulas. For any $\sharp \in \{!, \heartsuit_i, \heartsuit_{\mathrm{F}}\}$, an expression $\sharp\Gamma$ is used to denote the multiset $\{\sharp\gamma \mid \gamma \in \Gamma\}$. We write $A \equiv B$ to indicate the syntactical identity between A and B. An expression Γ^* means $\Gamma^* \equiv \gamma_1 * \cdots * \gamma_n$ if $\Gamma \equiv \{\gamma_1, \cdots, \gamma_n\}$ $(0 < n)$ and $\Gamma^* \equiv \mathbf{1}$ if $\Gamma \equiv \emptyset$. The symbol ω is used to represent the set of natural numbers. The symbol K is used to represent the set $\{\heartsuit_i \mid i \in N\}$, and the symbol K^* is used to represent the set of all words of finite length of the alphabet K. For example, $\{\iota\alpha \mid \iota \in K^*\}$ denotes the set $\{\heartsuit_{i_1} \cdots \heartsuit_{i_k}\alpha \mid i_1, ..., i_k \in N, k \in \omega\}$. Remark that K^* includes $\emptyset$ and hence $\{\iota\alpha \mid \iota \in K^*\}$ includes α. Greek lower-case letters $\iota, \iota_1, ..., \iota_n$ and κ are used to denote any members of K^*. A *two-sided intuitionistic sequent*, simply called a sequent, is an expression of the form $\Gamma \Rightarrow \gamma$ (the succedent of the sequent is not empty). It is assumed that the terminological conventions regarding sequents (e.g., antecedent, succedent etc.) are the usual ones. If a sequent S is provable in a sequent system L, then such a fact is denoted as $L \vdash S$ or $\vdash S$. The parentheses for $*$ is omitted since $*$ is associative, i.e., $\vdash \alpha * (\beta * \gamma) \Rightarrow (\alpha * \beta) * \gamma$ and $\vdash (\alpha * \beta) * \gamma \Rightarrow \alpha * (\beta * \gamma)$ for any formulas α, β and γ. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.
An *intuitionistic linear logic with fixpoint operator*, MILL, is introduced below.

Definition 2.1. *The initial sequents of* MILL *are of the form:*

$$\alpha \Rightarrow \alpha \qquad \Rightarrow \iota\mathbf{1} \qquad \Gamma \Rightarrow \iota\top \qquad \iota\mathbf{0}, \Gamma \Rightarrow \gamma.$$

The cut rule of MILL *is of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ (\mathrm{cut}).$$

The logical inference rules of MILL *are of the form:*

$$\frac{\Gamma \Rightarrow \gamma}{\iota\mathbf{1}, \Gamma \Rightarrow \gamma}\ (\mathbf{1}\mathrm{we})$$

$$\frac{\Gamma \Rightarrow \iota\alpha \quad \iota\beta, \Delta \Rightarrow \gamma}{\iota(\alpha{\to}\beta), \Gamma, \Delta \Rightarrow \gamma}\ (\to\text{left}) \quad \frac{\iota\alpha, \Gamma \Rightarrow \iota\beta}{\Gamma \Rightarrow \iota(\alpha{\to}\beta)}\ (\to\text{right})$$

$$\frac{\iota\alpha, \Gamma \Rightarrow \gamma}{\iota(\alpha \wedge \beta), \Gamma \Rightarrow \gamma}\ (\wedge\text{left1}) \quad \frac{\iota\beta, \Gamma \Rightarrow \gamma}{\iota(\alpha \wedge \beta), \Gamma \Rightarrow \gamma}\ (\wedge\text{left2}) \quad \frac{\Gamma \Rightarrow \iota\alpha \quad \Gamma \Rightarrow \iota\beta}{\Gamma \Rightarrow \iota(\alpha \wedge \beta)}\ (\wedge\text{right})$$

$$\frac{\iota\alpha, \iota\beta, \Gamma \Rightarrow \gamma}{\iota(\alpha * \beta), \Gamma \Rightarrow \gamma}\ (*\text{left}) \quad \frac{\Gamma \Rightarrow \iota\alpha \quad \Delta \Rightarrow \iota\beta}{\Gamma, \Delta \Rightarrow \iota(\alpha * \beta)}\ (*\text{right})$$

$$\frac{\iota\alpha, \Gamma \Rightarrow \gamma \quad \iota\beta, \Gamma \Rightarrow \gamma}{\iota(\alpha \vee \beta), \Gamma \Rightarrow \gamma}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \iota\alpha}{\Gamma \Rightarrow \iota(\alpha \vee \beta)}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \iota\beta}{\Gamma \Rightarrow \iota(\alpha \vee \beta)}\ (\vee\text{right2})$$

$$\frac{\iota\alpha, \Gamma \Rightarrow \gamma}{\iota!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{left}) \quad \frac{\iota_1!\gamma_1, \ldots, \iota_n!\gamma_n \Rightarrow \kappa\alpha}{\iota_1!\gamma_1, \ldots, \iota_n!\gamma_n \Rightarrow \kappa!\alpha}\ (!\text{right})$$

$$\frac{\iota!\alpha, \iota!\alpha, \Gamma \Rightarrow \gamma}{\iota!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{co}) \quad \frac{\Gamma \Rightarrow \gamma}{\iota!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{we})$$

$$\frac{\iota\kappa\alpha, \Gamma \Rightarrow \gamma}{\iota\heartsuit_{\mathrm{F}}\alpha, \Gamma \Rightarrow \gamma}\ (\heartsuit_{\mathrm{F}}\text{left}) \quad \frac{\{\, \Gamma \Rightarrow \iota\kappa\alpha \,\}_{\kappa \in K^*}}{\Gamma \Rightarrow \iota\heartsuit_{\mathrm{F}}\alpha}\ (\heartsuit_{\mathrm{F}}\text{right}).$$

Remark that ($\heartsuit_{\mathrm{F}}$right) has infinite premises, and that the cases for $\iota = \emptyset$ in MILL derive the usual inference rules for the intuitionistic linear logic. The rules ($\heartsuit_{\mathrm{F}}$left) and ($\heartsuit_{\mathrm{F}}$right) are intended to formalize an informal axiom scheme: $\heartsuit_{\mathrm{F}}\alpha \leftrightarrow \bigwedge\{\iota\alpha \mid \iota \in K^*\}$.
The following proposition is needed in the completeness proof.

Proposition 2.2. *The following rules are admissible in cut-free* MILL*: for any $i \in N$,*

$$\frac{\Gamma \Rightarrow \gamma}{\heartsuit_i\Gamma \Rightarrow \heartsuit_i\gamma}\ (\heartsuit_i\text{regu}) \quad \frac{\Gamma \Rightarrow \iota\heartsuit_{\mathrm{F}}\alpha}{\Gamma \Rightarrow \iota\kappa\alpha}\ (\heartsuit_{\mathrm{F}}\text{right}^{-1})$$

$$\frac{\Gamma \Rightarrow \iota(\alpha{\to}\beta)}{\Gamma, \iota\alpha \Rightarrow \iota\beta}\ (\to\text{right}^{-1}) \quad \frac{\iota(\alpha * \beta), \Gamma \Rightarrow \gamma}{\iota\alpha, \iota\beta, \Gamma \Rightarrow \gamma}\ (*\text{left}^{-1}).$$

An expression $\alpha \Leftrightarrow \beta$ is an abbreviation of $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.3. *The following sequents are provable in cut-free* MILL*: for any formulas α, β and any $i \in N$,*

1. $\heartsuit_i(\alpha \circ \beta) \Leftrightarrow \heartsuit_i\alpha \circ \heartsuit_i\beta$ *where* $\circ \in \{\to, \wedge, *, \vee\}$,
2. $\heartsuit_i!\alpha \Leftrightarrow !\heartsuit_i\alpha$,
3. $\heartsuit_{\mathrm{F}}\alpha \Rightarrow \heartsuit_i\alpha$,
4. $\heartsuit_{\mathrm{F}}\alpha \Rightarrow \alpha$,
5. $\heartsuit_{\mathrm{F}}\alpha \Rightarrow \heartsuit_i\heartsuit_{\mathrm{F}}\alpha$,
6. $\heartsuit_{\mathrm{F}}\alpha \Rightarrow \heartsuit_{\mathrm{F}}\heartsuit_{\mathrm{F}}\alpha$.

Note that a proof of MILL provides both an infinite width and an unbounded depth. Such a fact implies that obtaining a direct cut-elimination proof for MILL may be very difficult. Thus, to prove the cut-elimination theorem effectively, we need the phase semantic cut-elimination method proposed by Okada.

2.2 Phase semantics

We now define a phase semantics for MILL. The difference between such a semantics and the original semantics for the intuitionistic linear logic is only the definition of the valuations: whereas the original semantics for the intuitionistic linear logic has a valuation v, our semantics has an infinite number of modality-indexed valuations v^ι ($\iota \in K^*$), where $v^\emptyset$ is the same as v.

Definition 2.4. *An* intuitionistic phase space *is a structure* $\langle \mathbf{M}, cl, \mathrm{I} \rangle$ *satisfying the following conditions:*

1. $\mathbf{M} := \langle M, \cdot, 1 \rangle$ *is a commutative monoid with the identity* 1,
2. *cl is an operation on the powerset* $P(M)$ *of M such that, for any* $X, Y \in P(M)$,
   C1: $X \subseteq cl(X)$,
   C2: $clcl(X) \subseteq cl(X)$,
   C3: $X \subseteq Y$ *implies* $cl(X) \subseteq cl(Y)$,
   C4: $cl(X) \circ cl(Y) \subseteq cl(X \circ Y)$
   where the operation $\circ$ *is defined as* $X \circ Y := \{x \cdot y \mid x \in X \text{ and } y \in Y\}$ *for any* $X, Y \in P(M)$ *(the operation cl is called here* closure operation*),*
3. I *is a submonoid of M such that* $cl\{x\} \subseteq cl\{x \cdot x\}$ *for any* $x \in \mathrm{I}$.

In order to obtain an interpretation of the logical constants and connectives, the corresponding constants and operations on $P(M)$ are defined below.

Definition 2.5. *Constants and operations on* $P(M)$ *are defined as follows: for any* $X, Y \in P(M)$,

1. $\dot{\mathbf{1}} := cl\{1\}$,
2. $\dot{\top} := M$,
3. $\dot{\mathbf{0}} := cl(\emptyset)$,
4. $X \dot{\to} Y := \{y \mid \forall x \in X\ (x \cdot y \in Y)\}$,
5. $X \dot{\wedge} Y := X \cap Y$,
6. $X \dot{\vee} Y := cl(X \cup Y)$,
7. $X \dot{*} Y := cl(X \circ Y)$,
8. $\dot{!}\, X := cl(X \cap \mathrm{I} \cap \dot{\mathbf{1}})$.

We define $D := \{X \in P(M) \mid X = cl(X)\}$. *Then,*

$\mathbf{D} := \langle D, \dot{\to}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{!}, \dot{\mathbf{1}}, \dot{\top}, \dot{\mathbf{0}} \rangle$

is called an intuitionistic phase structure.

Remark that the following hold: for any $X, X', Y, Y', Z \in P(M)$,

1. $X \subseteq Y \dot{\to} Z$ iff $X \circ Y \subseteq Z$,
2. $X \subseteq X'$ and $Y \subseteq Y'$ imply $X \circ Y \subseteq X' \circ Y'$ and $X' \dot{\to} Y \subseteq X \dot{\to} Y'$.

Remark that D is closed under the operations $\dot{\to}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{!}$ and $\bigcap$ (infinite meet), and that $\dot{\mathbf{1}}, \dot{\top}, \dot{\mathbf{0}} \in D$.

Definition 2.6. Modality-indexed valuations v^ι *for all* $\iota \in K^*$ *on an intuitionistic phase structure* $\mathbf{D} := \langle D, \dot{\to}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{!}, \dot{\mathbf{1}}, \dot{\top}, \dot{\mathbf{0}} \rangle$ *are mappings from the set of all propositional variables to D. Then,* v^ι *for all* $\iota \in K^*$ *are extended to mappings from the set* Φ *of all formulas to D by:*

1. $v^\iota(\mathbf{1}) := \dot{\mathbf{1}}$,
2. $v^\iota(\top) := \dot{\top}$,
3. $v^\iota(\mathbf{0}) := \dot{\mathbf{0}}$,
4. $v^\iota(\alpha \wedge \beta) := v^\iota(\alpha) \dot{\wedge} v^\iota(\beta)$,
5. $v^\iota(\alpha \vee \beta) := v^\iota(\alpha) \dot{\vee} v^\iota(\beta)$,
6. $v^\iota(\alpha * \beta) := v^\iota(\alpha) \dot{*} v^\iota(\beta)$,
7. $v^\iota(\alpha \to \beta) := v^\iota(\alpha) \dot{\to} v^\iota(\beta)$,
8. $v^\iota(!\alpha) := \dot{!} v^\iota(\alpha)$,
9. $v^\iota(\heartsuit_i \alpha) := v^{\iota \heartsuit_i}(\alpha)$,
10. $v^\iota(\heartsuit_{\mathrm{F}} \alpha) := \bigcap_{\kappa \in K^*} v^{\iota\kappa}(\alpha)$.

Definition 2.7. *An* intuitionistic modality-indexed phase model *is a structure* $\langle \mathbf{D}, \{v^\iota\}_{\iota \in K^*} \rangle$ *such that* $\mathbf{D}$ *is an intuitionistic phase structure, and* $\{v^\iota\}_{\iota \in K^*}$ *is a class of modality-indexed valuations. A formula* α *is* true *in an intuitionistic modality-indexed phase model* $\langle \mathbf{D}, \{v^\iota\}_{\iota \in K^*} \rangle$ *if* $\dot{\mathbf{1}} \subseteq v^\emptyset(\alpha)$ *(or equivalently* $1 \in v^\emptyset(\alpha)$*) holds, and* valid *in an intuitionistic phase structure* $\mathbf{D}$ *if it is true for any modality-indexed valuations* $\{v^\iota\}_{\iota \in K^*}$ *on the intuitionistic phase structure. A sequent* $\alpha_1, \cdots, \alpha_n \Rightarrow \beta$ *(or* $\Rightarrow \beta$*) is* true *in an intuitionistic modality-indexed phase model* $\langle \mathbf{D}, \{v^\iota\}_{\iota \in K^*} \rangle$ *if the formula* $\alpha_1 * \cdots * \alpha_n \to \beta$ *(or* β*) is true in it, and valid in an intuitionistic phase structure if so is* $\alpha_1 * \cdots * \alpha_n \to \beta$ *(or* β*).*

The proof of the following theorem is straightforward.

Theorem 2.8 (Soundness)**.** *If a sequent* S *is provable in* MILL*, then* S *is valid for any intuitionistic phase structures.*

2.3 Completeness and cut-elimination

In order to prove the strong completeness theorem, we will construct a canonical model. For the sake of clarity for the completeness proof, an expression $[\Gamma]$ is used to explicitly represent a multiset of formulas, i.e., $[\Gamma]$ and Γ are identical, but only the expressions are different.

Definition 2.9. *We define a commutative monoid* $\langle M, \cdot, 1 \rangle$ *as follows:*

1. $M := \{[\Gamma] \mid [\Gamma]$ *is a finite multiset of formulas*$\}$,
2. $[\Gamma] \cdot [\Delta] := [\Gamma, \Delta]$ *(the multiset union),*
3. $1 := [\,]$ *(the empty multiset).*

We define the following: for any $\iota \in K^*$ *and any formula* α*,*

$$\|\alpha\|^\iota := \{[\Gamma] \mid \vdash_{cf} \Gamma \Rightarrow \iota\alpha\}$$

where $\vdash_{cf}$ *means "provable in cut-free* MILL*".*
We then define

$$D := \{X \mid X = \bigcap_{i \in I} \|\alpha_i\|^\emptyset\}$$

for an arbitrary (non-empty) indexing set I *and an arbitrary formula* α_i*.*
Then we define

$$cl(X) := \bigcap \{Y \in D \mid X \subseteq Y\}.$$

We define the following constants and operations on $P(M)$: for any $X, Y \in P(M)$,

1. $\dot{\mathbf{I}} := cl\{1\}$,
2. $\dot{\top} := M$,
3. $\dot{\mathbf{0}} := cl(\emptyset)$,
4. $X \dot{\to} Y := \{[\Delta] \mid \forall\, [\Gamma] \in X\ ([\Gamma, \Delta] \in Y)\}$,
5. $X \dot{\wedge} Y := X \cap Y$,
6. $X \dot{\vee} Y := cl(X \cup Y)$,
7. $X \dot{*} Y := cl(X \circ Y)$ *where* $X \circ Y := \{[\Gamma, \Delta] \mid [\Gamma] \in X$ *and* $[\Delta] \in Y\}$,
8. $\dot{!}X := cl(X \cap \mathrm{I} \cap \mathbf{1})$ *where*
   $\mathrm{I} := \{[\iota_1!\gamma_1, ..., \iota_n!\gamma_n] \mid \iota_1, ..., \iota_n \in K^*$ *and* $\gamma_1, ..., \gamma_n$ *: formulas*$\}$.

Modality-indexed valuations v^ι for all $\iota \in K^*$ are mappings from the set of all propositional variables to D such that

$v^\iota(p) := \|p\|^\iota$.

We have the following: for any $X, Y, Z \in P(M)$,

$X \circ Y \subseteq Z$ iff $X \subseteq Y \dot{\to} Z$.

Remark that D is closed under arbitrary $\bigcap$. Remark also that I is a monoid.
Moreover, we have to check the following fact.

Proposition 2.10. *For any $[\Sigma] \in \mathrm{I}$, $cl\{[\Sigma]\} \subseteq cl\{[\Sigma, \Sigma]\}$.*

Proof. Let $[\Sigma] \in \mathrm{I}$. Suppose $[\Sigma] \in cl\{[\Sigma]\}$, i.e., $[\Sigma] \in \bigcap\{Y \in D \mid \{[\Sigma]\} \subseteq Y\}$ iff $\forall W\ [W \in D$ and $\{[\Sigma]\} \subseteq W$ imply $[\Delta] \in W]$. We show $[\Delta] \in cl\{[\Sigma, \Sigma]\}$, i.e., $\forall W\ [W \in D$ and $\{[\Sigma, \Sigma]\} \subseteq W$ imply $[\Delta] \in W]$. To show this, suppose $W \in D$ and $\{[\Sigma, \Sigma] \subseteq W\}$, i.e., $\{[\Sigma, \Sigma]\} \subseteq W = \bigcap_{i \in I} \|\alpha_i\|^0 = \{\Pi \mid \forall i \in I\ (\vdash_{cf} \Pi \Rightarrow \alpha_i)\}$. This means $\forall i \in I\ (\vdash_{cf} \Sigma, \Sigma \Rightarrow \alpha_i)$. Moreover, $[\Sigma]$ is of the form $[\iota_1!\gamma_1, ..., \iota_k!\gamma_k]$ since $[\Sigma] \in \mathrm{I}$. Thus, we have:

$\forall i \in I\ (\vdash_{cf} \iota_1!\gamma_1, ..., \iota_k!\gamma_k, \iota_1!\gamma_1, ..., \iota_k!\gamma_k \Rightarrow \alpha_i)$,

and hence obtain:

$\forall i \in I\ (\vdash_{cf} \iota_1!\gamma_1, ..., \iota_k!\gamma_k \Rightarrow \alpha_i)$

by (!co). This means $\{[\iota_1!\gamma_1, ..., \iota_k!\gamma_k]\} \subseteq \bigcap_{i \in I} \|\alpha_i\|^0 = W$, i.e., $\{[\Sigma]\} \subseteq W$. Therefore we obtain $[\Delta] \in W$ by the hypothesis. ∎

Proposition 2.11. *The following hold: for any $\iota \in K^*$ and any formula α,*

1. $\|\heartsuit_i \alpha\|^\iota = \|\alpha\|^{\iota \heartsuit_i}$,
2. $\|\heartsuit_{\mathrm{F}} \alpha\|^\iota = \bigcap_{\kappa \in K^*} \|\alpha\|^{\iota\kappa}$.

Proof. (1) is obvious. (2) can be shown using the rules ($\heartsuit_{\mathrm{F}}$right) and ($\heartsuit_{\mathrm{F}}$right^{-1}), where ($\heartsuit_{\mathrm{F}}$right^{-1}) is admissible in cut-free MILL by Proposition 2.2. ∎

Lemma 2.12. *Let D be $\{X \mid X = \bigcap_{i \in I} \|\alpha_i\|^\emptyset\}$, and D_c be $\{X \in P(M) \mid X = cl(X)\}$. Then: $D = D_c$.*

Proof. First, we show $D_c \subseteq D$. Suppose $X \in D_c$. Then $X = cl(X) = \bigcap\{Y \in D \mid X \subseteq Y\} \in D$. Next, we show $D \subseteq D_c$. Suppose $X \in D$. We show $X \in D_c$, i.e., $X = \bigcap\{Y \in D \mid X \subseteq Y\}$. To show this, it is sufficient to prove that

1. $X \subseteq \{[\Gamma] \mid \forall\, W\, [W \in D \text{ and } X \subseteq W \text{ imply } [\Gamma] \in W]\}$,
2. $\{[\Gamma] \mid \forall\, W\, [W \in D \text{ and } X \subseteq W \text{ imply } [\Gamma] \in W]\} \subseteq X$.

First, we show (1). Suppose $[\Delta] \in X$ and assume $W \in D$ and $X \subseteq W$ for any W. Then we have $[\Delta] \in X \subseteq W$. Next we show (2). Suppose $[\Delta] \in \{[\Gamma] \mid \forall\, W\, [W \in D \text{ and } X \subseteq W \text{ imply } [\Gamma] \in W]\}$. By the assumption $X \in D$ and the fact that $X \subseteq X$, we have $[\Delta] \in X$. ∎

Lemma 2.13. *For any $X \subseteq M$ and any $Y \in D$, we have $X \dot{\to} Y \in D$.*

Proof. By using Proposition 2.2 for the admissibility of ($\to$right^{-1}) and ($*$left^{-1}) in cut-free MILL. ∎

Then, we can show the following.

Proposition 2.14. *The structure $\mathbf{D} := \langle D, \dot{\to}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{!}, \dot{\mathbf{1}}, \dot{\top}, \dot{\mathbf{0}}\rangle$ defined in Definition 2.9 forms an intuitionistic phase structure for* MILL*.*

Proof. We can verify that D is closed under $\dot{\to}, \dot{*}, \dot{\wedge}, \dot{\vee}, \dot{!}$ and $\bigcap$. In particular, for $\dot{\to}$, we use Lemma 2.13. The fact $\dot{\mathbf{1}}, \dot{\top}, \dot{\mathbf{0}} \in D$ is obvious. We can verify that the conditions C1—C4 for closure operation hold for this structure. The conditions C1—C3 are obvious. We only show C4: $cl(X) \circ cl(Y) \subseteq cl(X \circ Y)$ for any $X, Y \in P(M)$. We have $X \circ Y \subseteq cl(X \circ Y)$ by the condition C1, and hence $X \subseteq Y \dot{\to} cl(X \circ Y)$. Moreover, by the condition C3, we have $cl(X) \subseteq cl(Y \dot{\to} cl(X \circ Y))$. Here, $cl(X \circ Y) \in D$ and by Lemma 2.13, we have $Y \dot{\to} cl(X \circ Y) \in D$. Thus, we obtain

$$cl(X) \subseteq cl(Y \dot{\to} cl(X \circ Y)) = Y \dot{\to} cl(X \circ Y)$$

by Lemma 2.12. Therefore we obtain (*): $cl(X) \circ Y \subseteq cl(X \circ Y)$ for any $X, Y \in P(M)$. By applying the fact (*) twice, Lemma 2.12 and the commutativity of $\circ$, we have

$$cl(X) \circ cl(Y) \subseteq cl(cl(X) \circ Y) \subseteq cl(cl(X \circ Y)) = cl(X \circ Y).$$

∎

We then have a modified version of the key lemma of Okada (Okada, 2002).

Lemma 2.15. *For any $\iota \in K^*$ and any formula α,*

$$[\iota\alpha] \in v^\iota(\alpha) \subseteq \|\alpha\|^\iota.$$

Proof. By induction on the complexity of α. We show only some critical cases.
(Case $\alpha \equiv \heartsuit_i\beta$): By induction hypothesis, we have $[\iota\heartsuit_i\beta] \in v^{\iota\heartsuit_i}(\beta) \subseteq \|\beta\|^{\iota\heartsuit_i}$, i.e., $[\iota(\heartsuit_i\beta)] \in v^\iota(\heartsuit_i\beta) \subseteq \|\heartsuit_i\beta\|^\iota$ by Proposition 2.11.
(Case $\alpha \equiv \heartsuit_{\mathrm{F}}\beta$): We show $[\iota\heartsuit_{\mathrm{F}}\beta] \in v^\iota(\heartsuit_{\mathrm{F}}\beta) \subseteq \|\heartsuit_{\mathrm{F}}\beta\|^\iota$. First, we show $[\iota\heartsuit_{\mathrm{F}}\beta] \in v^\iota(\heartsuit_{\mathrm{F}}\beta)$, i.e., $[\iota\heartsuit_{\mathrm{F}}\beta] \in \bigcap_{\kappa\in K^*} v^{\iota\kappa}(\beta)$ iff $\forall\kappa \in K^*$ $([\iota\heartsuit_{\mathrm{F}}\beta] \in v^{\iota\kappa}(\beta))$. Since $v^{\iota\kappa}(\beta) \in D$, we have $v^{\iota\kappa}(\beta) = \bigcap_{k\in I} \|\delta_k\|^{\varnothing} = \{[\Gamma] \mid \forall k \in I(\vdash_{cf} \Gamma \Rightarrow \delta_k)\}$. Thus, $\forall\kappa \in K^*$ $([\iota\heartsuit_{\mathrm{F}}\beta] \in v^{\iota\kappa}(\beta))$ means (*): $\forall k \in I(\vdash_{cf} \iota\heartsuit_{\mathrm{F}}\beta \Rightarrow \delta_k)$. On the other hand, by induction hypothesis, we have $\forall\kappa \in K^*$ $([\iota\kappa\beta] \in v^{\iota\kappa}(\beta))$, i.e., (**): $\forall k \in I\ \forall\kappa \in K^*$ $(\vdash_{cf} \iota\kappa\beta \Rightarrow \delta_k)$. By applying ($\heartsuit_{\mathrm{F}}$left) to (**), we obtain (*).

Next we show $v^{\iota}(\heartsuit_{\mathrm{F}}\beta) \subseteq \|\heartsuit_{\mathrm{F}}\beta\|^{\iota}$. Suppose $[\Gamma] \in v^{\iota}(\heartsuit_{\mathrm{F}}\beta)$, i.e., $[\Gamma] \in \bigcap_{\kappa\in K^*} v^{\iota\kappa}(\beta)$. We show $[\Gamma] \in \|\heartsuit_{\mathrm{F}}\beta\|^{\iota}$, i.e., $\vdash_{cf} \Gamma \Rightarrow \iota\heartsuit_{\mathrm{F}}\beta$. By induction hypothesis, we have $v^{\iota\kappa}(\beta) \subseteq \|\beta\|^{\iota\kappa}$. Thus, we obtain $[\Gamma] \in \bigcap_{\kappa\in K^*} v^{\iota\kappa}(\beta) \subseteq \bigcap_{\kappa\in K^*} \|\beta\|^{\iota\kappa}$, and hence $[\Gamma] \in \bigcap_{\kappa\in K^*} \|\beta\|^{\iota\kappa}$, i.e., $\forall\kappa \in K^*$ ($[\Gamma] \in \|\beta\|^{\iota\kappa}$) iff $\forall\kappa \in K^*$ ($\vdash_{cf} \Gamma \Rightarrow \iota\kappa\beta$). By applying ($\heartsuit_{\mathrm{F}}$right) to this, we obtain $\vdash_{cf} \Gamma \Rightarrow \iota\heartsuit_{\mathrm{F}}\beta$. ∎

Theorem 2.16 (Strong completeness)**.** *If a sequent S is valid for any intuitionistic phase structures, then S is provable in cut-free* MILL.

Proof. Using Lemma 2.15, we can obtain this theorem as follows. Let $\Gamma \Rightarrow \gamma$ be S, and α be $\Gamma^* \to \gamma$. If formula α is true, then $[\,] \in v^{\emptyset}(\alpha)$. On the other hand $v^{\iota}(\alpha) \subseteq \|\alpha\|^{\iota}$ for any $\iota \in K^*$, and hence $[\,] \in \|\alpha\|^{\emptyset}$, which means "$\Rightarrow \alpha$ is provable in cut-free MILL". ∎

Theorem 2.17 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* MILL.

Proof. If a sequent S is provable in MILL, then S is valid by Theorem 2.8 (Soundness). By Theorem 2.16 (Strong completeness), S is provable in cut-free MILL. ∎

3. Classical case

3.1 Sequent calculus

The language used in this section is introduced below. *Formulas* are constructed from propositional variables, $\mathbf{1}, \bot$ (multiplicative constants), $\mathbf{0}$, $\top$ (additive constants [*]), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction), $+$ (fission), $\cdot^{\bot}$ (negation), ! (of course), ? (why not), $\heartsuit_i$ ($i \in N$) (i-th modality), $\heartsuit_{\mathrm{F}}$ (fixpoint modality) and $\heartsuit_{\mathrm{D}}$ (co-fixpoint modality). The notational conventions are almost the same as that in the previous section. For example, for any $\sharp \in \{!, ?, \heartsuit_i, \heartsuit_{\mathrm{F}}, \heartsuit_{\mathrm{D}}\}$, an expression $\sharp\Gamma$ is used to denote the multiset $\{\sharp\gamma \mid \gamma \in \Gamma\}$. A *classical one-sided sequent*, simply a sequent, is an expression of the form $\vdash \Gamma$. An expression $\alpha \leftrightarrow \beta$ is used to represent the fact that both $\vdash \alpha^{\bot}, \beta$ and $\vdash \alpha, \beta^{\bot}$ are provable. In the one-sided calculi discussed here, the De Morgan duality is assumed, i.e., the following laws and the replacement (or substitution) theorem are assumed: $\mathbf{1}^{\bot} \leftrightarrow \bot$, $\bot^{\bot} \leftrightarrow \mathbf{1}$, $\top^{\bot} \leftrightarrow \mathbf{0}$, $\mathbf{0}^{\bot} \leftrightarrow \top$, $\alpha^{\bot\bot} \leftrightarrow \alpha$, $(\alpha \wedge \beta)^{\bot} \leftrightarrow \alpha^{\bot} \vee \beta^{\bot}$, $(\alpha \vee \beta)^{\bot} \leftrightarrow \alpha^{\bot} \wedge \beta^{\bot}$, $(\alpha * \beta)^{\bot} \leftrightarrow \alpha^{\bot} + \beta^{\bot}$, $(\alpha + \beta)^{\bot} \leftrightarrow \alpha^{\bot} * \beta^{\bot}$, $(!\alpha)^{\bot} \leftrightarrow ?(\alpha^{\bot})$, $(?\alpha)^{\bot} \leftrightarrow !(\alpha^{\bot})$, $(\heartsuit_i\alpha)^{\bot} \leftrightarrow \heartsuit_i(\alpha^{\bot})$, $(\heartsuit_{\mathrm{F}}\alpha)^{\bot} \leftrightarrow \heartsuit_{\mathrm{D}}(\alpha^{\bot})$ and $(\heartsuit_{\mathrm{D}}\alpha)^{\bot} \leftrightarrow \heartsuit_{\mathrm{F}}(\alpha^{\bot})$.

A *classical linear logic with fixpoint operator*, MCLL, is introduced below.

Definition 3.1. *The initial sequents of* MCLL *are of the form:*

$$\vdash \alpha, \alpha^{\bot} \qquad \vdash \iota\mathbf{1} \qquad \vdash \Gamma, \iota\top.$$

The cut rule of MCLL *is of the form:*

$$\frac{\vdash \Gamma, \iota\alpha \quad \vdash \Delta, \iota(\alpha^{\top})}{\vdash \Gamma, \Delta}\ (\mathrm{cut}).$$

[*] Although $\mathbf{0}$ is not appeared explicitly in the one-sided calculi discussed in this paper, it is used as an abbreviation of $\top^{\bot}$.

The logical inference rules of MCLL *are of the form:*

$$\frac{\vdash \Gamma}{\vdash \Gamma, \iota\bot}\ (\bot) \quad \frac{\vdash \Gamma, \iota\alpha \quad \vdash \Delta, \iota\beta}{\vdash \Gamma, \Delta, \iota(\alpha * \beta)}\ (*) \quad \frac{\vdash \Gamma, \iota\alpha, \iota\beta}{\vdash \Gamma, \iota(\alpha + \beta)}\ (+)$$

$$\frac{\vdash \Gamma, \iota\alpha \quad \vdash \Gamma, \iota\beta}{\vdash \Gamma, \iota(\alpha \wedge \beta)}\ (\wedge) \quad \frac{\vdash \Gamma, \iota\alpha}{\vdash \Gamma, \iota(\alpha \vee \beta)}\ (\vee 1) \quad \frac{\vdash \Gamma, \iota\beta}{\vdash \Gamma, \iota(\alpha \vee \beta)}\ (\vee 2)$$

$$\frac{\vdash \iota_1?\gamma_1, ..., \iota_n?\gamma_n, \kappa\alpha}{\vdash \iota_1?\gamma_1, ..., \iota_n?\gamma_n, \kappa!\alpha}\ (!) \quad \frac{\vdash \Gamma, \iota\alpha}{\vdash \Gamma, \iota?\alpha}\ (?)$$

$$\frac{\vdash \Gamma, \iota?\alpha, \iota?\alpha}{\vdash \Gamma, \iota?\alpha}\ (?\mathrm{co}) \quad \frac{\vdash \Gamma}{\vdash \Gamma, \iota?\alpha}\ (?\mathrm{we})$$

$$\frac{\{\vdash \Gamma, \iota\kappa\alpha\}_{\kappa \in K^*}}{\vdash \Gamma, \iota\heartsuit_{\mathrm{F}}\alpha}\ (\heartsuit_{\mathrm{F}}) \quad \frac{\vdash \Gamma, \iota\kappa\alpha}{\vdash \Gamma, \iota\heartsuit_{\mathrm{D}}\alpha}\ (\heartsuit_{\mathrm{D}}).$$

Note that the following conditions hold for MCLL: for any $i \in N$ and any formulas α and β, $\heartsuit_i(\alpha \circ \beta) \leftrightarrow (\heartsuit_i\alpha) \circ (\heartsuit_i\beta)$ where $\circ \in \{\wedge, \vee, *, +\}$, $(\heartsuit_i\alpha)^\bot \leftrightarrow \heartsuit_i(\alpha^\bot)$ and $\heartsuit_i(\sharp\alpha) \leftrightarrow \sharp(\heartsuit_i\alpha)$ where $\sharp \in \{!, ?\}$.

3.2 Phase semantics

We now define a phase semantics for MCLL. The difference between such semantics and the original semantics is only the definition of the valuations.

Definition 3.2. *Let $\langle M, \cdot, 1\rangle$ be a commutative monoid with the unit 1. If $X, Y \subseteq M$, we define $X \circ Y := \{x \cdot y \mid x \in X \text{ and } y \in Y\}$. A* phase spase *is a structure $\langle M, \hat{\bot}, \hat{I}\rangle$ where $\hat{\bot}$ is a fixed subset of M, and $\hat{I} := \{x \in M \mid x \cdot x = x\} \cap \hat{\bot}^{\hat{\bot}}$. For $X \subseteq M$, we define $X^{\hat{\bot}} := \{y \mid \forall x \in X\ (x \cdot y \in \hat{\bot})\}$. $X\ (\subseteq M)$ is called a* fact *if $X^{\hat{\bot}\hat{\bot}} = X$. The set of facts is denoted by D_M.*

Remark that the operation $\circ$ is commutative and assosiative, and has the monotonicity property w.r.t. $\circ$: $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ imply $X_1 \circ X_2 \subseteq Y_1 \circ Y_2$ for any $X_1, X_2, Y_1, Y_2\ (\subseteq M)$.

Proposition 3.3. *Let $X, Y \subseteq M$. Then:*

1. $X \subseteq Y^{\hat{\bot}}$ *iff* $X \circ Y \subseteq \hat{\bot}$,
2. *if* $X \subseteq Y$ *then* $X \circ Y^{\hat{\bot}} \subseteq \hat{\bot}$,
3. $X \circ X^{\hat{\bot}} \subseteq \hat{\bot}$,
4. *if* $X \subseteq Y$ *then* $Y^{\hat{\bot}} \subseteq X^{\hat{\bot}}$,
5. *if* $X \subseteq Y$ *then* $X^{\hat{\bot}\hat{\bot}} \subseteq Y^{\hat{\bot}\hat{\bot}}$,
6. $X \subseteq X^{\hat{\bot}\hat{\bot}}$,
7. $(X^{\hat{\bot}\hat{\bot}})^{\hat{\bot}\hat{\bot}} \subseteq X^{\hat{\bot}\hat{\bot}}$,
8. $X^{\hat{\bot}\hat{\bot}} \circ Y^{\hat{\bot}\hat{\bot}} \subseteq (X \circ Y)^{\hat{\bot}\hat{\bot}}$,
9. $x \in X^{\hat{\bot}}$ *iff* $\{x\} \circ X \subseteq \hat{\bot}$,
10. *if* $X \circ Y \subseteq \hat{\bot}$ *then* $X \circ Y^{\hat{\bot}\hat{\bot}} \subseteq \hat{\bot}$.

Note that $\cdot^{\hat{\bot}\hat{\bot}}$ is a closure operator similar to *cl* discussed in the previous section.

Proposition 3.4. *Let $X, Y \subseteq M$. Then:*

1. $X^{\hat{\perp}}$ *is a fact,*
2. $X^{\hat{\perp}\hat{\perp}}$ *is the smallest fact that includes* X*,*
3. *if* X *and* Y *are facts, then so is* $X \cap Y$*,*
4. *if* X_i *for all* $i \in \omega$ *are facts, then so is* $\bigcap_{i\in\omega} X_i$*.*

Definition 3.5. *Let $A, B \subseteq M$. We define the following operators and constants:*

1. $\hat{\perp} := \{1\}^{\hat{\perp}}$,
2. $\hat{\mathbf{1}} := \hat{\perp}^{\hat{\perp}} = \{1\}^{\hat{\perp}\hat{\perp}}$,
3. $\hat{\top} := M = \emptyset^{\hat{\perp}}$,
4. $\hat{\mathbf{0}} := \hat{\top}^{\hat{\perp}} = M^{\hat{\perp}} = \emptyset^{\hat{\perp}\hat{\perp}}$,
5. $A \mathbin{\hat{\wedge}} B := A \cap B$,
6. $A \mathbin{\hat{\vee}} B := (A \cup B)^{\hat{\perp}\hat{\perp}}$,
7. $A \mathbin{\hat{*}} B := (A \circ B)^{\hat{\perp}\hat{\perp}}$,
8. $A \mathbin{\hat{+}} B := (A^{\hat{\perp}} \circ B^{\hat{\perp}})^{\hat{\perp}}$,
9. $\hat{!}A := (A \cap \hat{I})^{\hat{\perp}\hat{\perp}}$,
10. $\hat{?}A := (A^{\hat{\perp}} \cap \hat{I})^{\hat{\perp}}$.

We can show that, by Proposition 3.4, the constants defined above are facts and the operators defined above are closed under D_M.

Definition 3.6. Modality-indexed valuations ϕ^ι *for all* $\iota \in K^*$ *on a phase space* $\langle M, \hat{\perp}, \hat{I}\rangle$ *are mappings which assign a fact to each propositional variables. Each modality-indexed valuation* ϕ^ι *(*$\iota \in K^*$*) can be extended to a mapping* $\cdot^\iota$ *(*$\iota \in K^*$*) from the set* Φ *of all formulas to* D_M *by:*

1. $p^\iota := \phi^\iota(p)$ *for any propositional variable* p*,*
2. $\perp^\iota := \hat{\perp}$,
3. $\mathbf{1}^\iota := \hat{\mathbf{1}}$,
4. $\top^\iota := \hat{\top}$,
5. $\mathbf{0}^\iota := \hat{\mathbf{0}}$,
6. $(\alpha^\perp)^\iota := (\alpha^\iota)^{\hat{\perp}}$,
7. $(\alpha \wedge \beta)^\iota := \alpha^\iota \mathbin{\hat{\wedge}} \beta^\iota$,
8. $(\alpha \vee \beta)^\iota := \alpha^\iota \mathbin{\hat{\vee}} \beta^\iota$,
9. $(\alpha * \beta)^\iota := \alpha^\iota \mathbin{\hat{*}} \beta^\iota$,
10. $(\alpha + \beta)^\iota := \alpha^\iota \mathbin{\hat{+}} \beta^\iota$,
11. $(!\alpha)^\iota := \hat{!}(\alpha^\iota)$,
12. $(?\alpha)^\iota := \hat{?}(\alpha^\iota)$,
13. $(\heartsuit_i\alpha)^\iota := \alpha^{\iota\heartsuit_i}$,

14. $(\heartsuit_{\mathrm{F}}\alpha)^{\iota} := \bigcap_{\kappa\in K^*} \alpha^{\iota\kappa}$,

15. $(\heartsuit_{\mathrm{D}}\alpha)^{\iota} := (\bigcup_{\kappa\in K^*} \alpha^{\iota\kappa})^{\hat{\perp}\hat{\perp}}$.

We call the values α^{ι} $(\iota \in K^*)$ *the* inner-values *of* α $(\in \Phi)$.

Definition 3.7. $\langle M, \hat{\perp}, \hat{I}, \{\phi^{\iota}\}_{\iota\in K^*}\rangle$ *is a* modality-indexed phase model *if* $\langle M, \hat{\perp}, \hat{I}\rangle$ *is a phase space and* ϕ^{ι} $(\iota \in K^*)$ *are modality-indexed valuations on* $\langle M, \hat{\perp}, \hat{I}\rangle$. *A sequent* $\vdash \alpha$ *is* true *in a modality-indexed phase model* $\langle M, \hat{\perp}, \hat{I}, \{\phi^{\iota}\}_{\iota\in K^*}\rangle$ *if* $\alpha^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$ *(or equivalently* $1 \in \alpha^{\emptyset}$*), and* valid *in a phase space* $\langle M, \hat{\perp}, \hat{I}\rangle$ *if it is true for any modality-indexed valuations* ϕ^{ι} $(\iota \in K^*)$ *on the phase space. A sequent* $\vdash \alpha_1, \cdots, \alpha_n$ *is* true *in a modality-indexed phase model* $\langle M, \hat{\perp}, \hat{I}, \{\phi^{\iota}\}_{\iota\in K^*}\rangle$ *if* $\vdash \alpha_1 + \cdots + \alpha_n$ *is true in the model, and* valid *in a phase space* $\langle M, \hat{\perp}, \hat{I}\rangle$ *if it is true for any modality-indexed valuations* ϕ^{ι} $(\iota \in K^*)$ *on the phase space.*

Theorem 3.8 (Soundness)**.** *If a sequent S is provable in* MCLL*, then S is valid for any phase space.*

Proof. By induction on the length of the proof P of S. For example, if the last rule of inference in P is of the form:

$$\frac{\vdash \Gamma'}{\vdash \Gamma}$$

where $\Gamma \equiv \{\alpha_1, \cdots, \alpha_n\}$ and $\Gamma' \equiv \{\alpha'_1, \cdots, \alpha'_n\}$, then we show that $(\alpha'_1 + \cdots + \alpha'_n)^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$ implies $(\alpha_1 + \cdots + \alpha_n)^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$, i.e. $(\alpha_1'^{\emptyset\hat{\perp}} \circ \cdots \circ \alpha_n'^{\emptyset\hat{\perp}})^{\hat{\perp}\hat{\perp}} \subseteq \hat{\perp}$ implies $(\alpha_1^{\emptyset\hat{\perp}} \circ \cdots \circ \alpha_n^{\emptyset\hat{\perp}})^{\hat{\perp}\hat{\perp}} \subseteq \hat{\perp}$. To show this, it is enough to prove that $\alpha_1'^{\emptyset\hat{\perp}} \circ \cdots \circ \alpha_n'^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$ implies $\alpha_1^{\emptyset\hat{\perp}} \circ \cdots \circ \alpha_n^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$, since we have Proposition 3.3 (10) and (6). $\Gamma^{\emptyset\hat{\perp}}$ denotes $\hat{\perp}$ if Γ is empty, and $\Gamma^{\emptyset\hat{\perp}}$ denotes $\gamma_1^{\emptyset\hat{\perp}} \circ \cdots \circ \gamma_n^{\emptyset\hat{\perp}}$ if $\Gamma \equiv \{\gamma_1, \cdots, \gamma_n\}$. In the proof, we will sometimes use the properties in Propositions 3.3 and 3.4 implicitly. Here we show only the following case.
Case $(\heartsuit_{\mathrm{F}})$: The last inference of P is of the form:

$$\frac{\{\vdash \Gamma, \iota\kappa\alpha\}_{\kappa\in K^*}}{\vdash \Gamma, \iota\heartsuit_{\mathrm{F}}\alpha}\ (\heartsuit_{\mathrm{F}}).$$

Suppose $\forall\kappa \in K^*\ [\Gamma^{\emptyset\hat{\perp}} \circ (\iota\kappa\alpha)^{\emptyset\hat{\perp}} \subseteq \hat{\perp}]$, i.e., $\forall\kappa \in K^*\ [\Gamma^{\emptyset\hat{\perp}} \circ (\alpha^{\iota\kappa})^{\hat{\perp}} \subseteq \hat{\perp}]$. Then we obtain (*): $\forall\kappa \in K^*\ [\Gamma^{\emptyset\hat{\perp}} \subseteq \alpha^{\iota\kappa}]$ by Proposition 3.3 (1). We show $\Gamma^{\emptyset\hat{\perp}} \circ (\iota\heartsuit_{\mathrm{F}}\alpha)^{\emptyset\hat{\perp}} \subseteq \hat{\perp}$, i.e., $\Gamma^{\emptyset\hat{\perp}} \circ (\bigcap_{\kappa\in K^*} \alpha^{\iota\kappa})^{\hat{\perp}} \subseteq \hat{\perp}$. This is equivalent to $\Gamma^{\emptyset\hat{\perp}} \subseteq \bigcap_{\kappa\in K^*} \alpha^{\iota\kappa}$ by Proposition 3.3 (1), and hence we show this below. Suppose $x \in \Gamma^{\emptyset\hat{\perp}}$. Then we obtain $\forall\kappa \in K^*\ [x \in \alpha^{\iota\kappa}]$ by (*). This means $x \in \bigcap_{\kappa\in K^*} \alpha^{\iota\kappa}$. ∎

3.3 Completeness and cut-elimination

Next, we consider the strong completeness theorem for MCLL. In order to prove this theorem, we have to construct a canonical model.

Definition 3.9. *We construct a canonical modality-indexed phase model* $\langle M, \hat{\perp}, \hat{I}, \{\phi^{\iota}\}_{\iota\in K^*}\rangle$. *Here M is the set of all multisets of formulas where multiple occurrence of a formula of the form* $\iota?\alpha$ *in the multisets counts only once.* $\langle M, \cdot, 1\rangle$ *is a commutavive monoid where* $\Delta \cdot \Gamma := \Delta \cup \Gamma$ *(the multiset union) for all* $\Delta, \Gamma \in M$*, and* 1 $(\in M)$ *is* $\emptyset$ *(the empty multiset). For any formula* α*, we define*

$[\alpha]_\iota := \{\Delta \mid \vdash_{cf} \Delta, \iota\alpha\}$ *where* $\vdash_{cf} \Delta, \iota\alpha$ *means that* $\vdash \Delta, \iota\alpha$ *is cut-free provable. We call* $[\alpha]_\iota$ $(\iota \in K^*)$ *the* outer-values *of* α $(\in \Phi)$. *We define* $\hat{\perp} := [\perp]_\emptyset = \{\Delta \mid \vdash_{cf} \Delta\}$. $\hat{I}$ *is defined as* $\{[\iota_1?\gamma_1, ..., \iota_n?\gamma_n] \mid [\gamma_1, ..., \gamma_n] \in M, \iota_1, ..., \iota_n \in K^*\}$. *The modality-indexed valuations* ϕ^ι $(\iota \in K^*)$ *are defined as* $\phi^\iota(p) := [p]_\iota$ *for any propositional variable* p.

Proposition 3.10. *Let* $\hat{I}$ *be* $\{[\iota_1?\gamma_1, ..., \iota_n?\gamma_n] \mid [\gamma_1, ..., \gamma_n] \in M, \iota_1, ..., \iota_n \in K^*\}$, *and* $\dot{I}$ *be* $\{\Delta \in M \mid \Delta \cup \Delta = \Delta\} \cap \hat{\perp}^{\hat{\perp}}$. *Then:* $\hat{I} = \dot{I}$.

We then have the following.

Proposition 3.11. $\langle M, \hat{\perp}, \hat{I}\rangle$ *defined in Definition 3.9 is a phase space.*

To prove the completeness theorem, we must prove some lemmas which are analogous to the lemmas established by Okada in (Okada, 1999).

Lemma 3.12. *For any* $\iota \in K^*$ *and any formula* α, *if* $\alpha^\iota \subseteq [\alpha]_\iota$ *then* $\{\iota\alpha\} \in \alpha^{\iota\hat{\perp}}$.

Lemma 3.13. *Let* α *be any formula. Then:*

1. $[\alpha]_\iota^{\hat{\perp}\hat{\perp}} = [\alpha]_\iota$,
2. $[\heartsuit_i\alpha]_\iota = [\alpha]_{\iota\heartsuit_i}$,
3. $[\heartsuit_{\mathrm{F}}\alpha]_\iota = \bigcap_{\kappa \in K^*} [\alpha]_{\iota\kappa}$,
4. $[\heartsuit_{\mathrm{D}}\alpha]_\iota = \bigcup_{\kappa \in K^*} [\alpha]_{\iota\kappa}$.

Proof. We show only (4).
(4): First, we show $[\heartsuit_{\mathrm{D}}\alpha]_\iota \subseteq \bigcup_{\kappa \in K^*} [\alpha]_{\iota\kappa}$. To show this, we use the fact that the following rule is admissible in cut-free MCLL:

$$\frac{\vdash \Gamma, \iota\heartsuit_{\mathrm{D}}\alpha}{\vdash \Gamma, \iota\alpha}\ (\heartsuit_{\mathrm{D}}^{-1}).$$

Suppose $\Gamma \in [\heartsuit_{\mathrm{D}}\alpha]_\iota$, i.e., $\vdash_{cf} \Gamma, \iota\heartsuit_{\mathrm{D}}\alpha$. We show $\Gamma \in \bigcup_{\kappa \in K^*} [\alpha]_{\iota\kappa}$, i.e., $\exists\kappa \in K^*$ $(\vdash_{cf} \Gamma, \iota\kappa\alpha)$. By applying the rule $(\heartsuit_{\mathrm{D}}^{-1})$ to the hypothesis $\vdash_{cf} \Gamma, \iota\heartsuit_{\mathrm{D}}\alpha$, we obtain $\vdash_{cf} \Gamma, \iota\alpha$. Next, we show $\bigcup_{\kappa \in K^*} [\alpha]_{\iota\kappa} \subseteq [\heartsuit_{\mathrm{D}}\alpha]_\iota$. Suppose $\Gamma \in \bigcup_{\kappa \in K^*} [\alpha]_{\iota\kappa}$, i.e., $\exists\kappa \in K^*$ $(\vdash_{cf} \Gamma, \iota\kappa\alpha)$. By applying the rule $(\heartsuit_{\mathrm{D}})$ to this, we obtain $\vdash_{cf} \Gamma, \iota\heartsuit_{\mathrm{D}}\alpha$, i.e., $\Gamma \in [\heartsuit_{\mathrm{D}}\alpha]_\iota$. ∎

Using Lemmas 3.12 and 3.13, we can prove the following main lemma.

Lemma 3.14. *For any formula* α *and any* $\iota \in K^*$, $\alpha^\iota \subseteq [\alpha]_\iota$.

Proof. By induction on the complexity of α.
• Base step: Obvious by the definitions.
• Induction step: We show some cases. Other cases are almost the same as those in (Okada, 1999).
(Case $\alpha \equiv \heartsuit_i\beta$): Suppose $\Gamma \in (\heartsuit_i\beta)^\iota$, i.e., $\Gamma \in \beta^{\iota\heartsuit_i}$. Then we have $\Gamma \in \beta^{\iota\heartsuit_i} \subseteq [\beta]_{\iota\heartsuit_i}$ by induction hypothesis, and hence obtain $\vdash_{cf} \Gamma, \iota\heartsuit_i\beta$, i.e., $\vdash_{cf} \Gamma, \iota(\heartsuit_i\beta)$. Therefore $\Gamma \in [\heartsuit_i\beta]_\iota$.

(Case $\alpha \equiv \heartsuit_{\mathrm{F}}\beta$): Suppose $\Gamma \in (\heartsuit_{\mathrm{F}}\beta)^{\iota}$, i.e., $\Gamma \in \bigcap_{\kappa\in K^*} \beta^{\iota\kappa}$. Then we have $\Gamma \in \bigcap_{\kappa\in K^*} \beta^{\iota\kappa} \subseteq \bigcap_{\kappa\in K^*} [\beta]_{\iota\kappa}$ by induction hypothesis, and hence obtain $\forall\kappa \in K^*$ $(\Gamma \in [\beta]_{\iota\kappa})$, i.e., $\{\vdash_{cf} \Gamma, \iota\kappa\beta\}_{\kappa\in K^*}$. By applying the rule $(\heartsuit_{\mathrm{F}})$ to this, we obtain $\vdash_{cf} \Gamma, \iota\heartsuit_{\mathrm{F}}\beta$. Therefore $\Gamma \in [\heartsuit_{\mathrm{F}}\beta]_{\iota}$.
(Case $\alpha \equiv \heartsuit_{\mathrm{D}}\beta$): We will show:

$$(\heartsuit_{\mathrm{D}}\beta)^{\iota} =_{df} (\bigcup_{\kappa\in K^*} \beta^{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq (\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq [\heartsuit_{\mathrm{D}}\beta]_{\iota}.$$

For this, $(\bigcup_{\kappa\in K^*} \beta^{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq (\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa})^{\hat{\perp}\hat{\perp}}$ can be proved by the induction hypothesis $\beta^{\iota\kappa} \subseteq [\beta]_{\iota\kappa}$ and Proposition 3.3 (5), i.e., the monotonicity of $\cdot^{\hat{\perp}\hat{\perp}}$. Next, we prove $(\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq [\heartsuit_{\mathrm{D}}\beta]_{\iota}$. By Lemma 3.13 (4), we have $\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa} \subseteq [\heartsuit_{\mathrm{D}}\beta]_{\iota}$. Moreover, by Proposition 3.3 (5), i.e., the monotonicity of $\cdot^{\hat{\perp}\hat{\perp}}$, we obtain $(\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq [\heartsuit_{\mathrm{D}}\beta]_{\iota}^{\hat{\perp}\hat{\perp}}$. By Lemma 3.13 (1), we have $[\heartsuit_{\mathrm{D}}\beta]_{\iota}^{\hat{\perp}\hat{\perp}} = [\heartsuit_{\mathrm{D}}\beta]_{\iota}$. Therefore $(\bigcup_{\kappa\in K^*} [\beta]_{\iota\kappa})^{\hat{\perp}\hat{\perp}} \subseteq [\heartsuit_{\mathrm{D}}\beta]_{\iota}$. ∎

Theorem 3.15 (Strong completeness)**.** *If a sequent S is valid for any phase space, then S is provable in cut-free* MCLL*.*

Proof. Lemma 3.14 implies this theorem as follows. Let $\vdash \alpha_1, ..., \alpha_n$ be S, and $\vdash \alpha$ be $\vdash \alpha_0 + \cdots + \alpha_n$. If $\vdash \alpha$ is true, then $\emptyset \in \alpha^{\emptyset}$. On the other hand, we have $\alpha^{\iota} \subseteq [\alpha]_{\iota}$ for any $\iota \in K^*$, and hence obtain $\emptyset \in [\alpha]_{\emptyset}$. This means that $\vdash \alpha$ (i.e., S) is provable in cut-free MCLL. ∎

Theorem 3.16 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* MCLL*.*

Proof. If a sequent S is provable in MCLL, then S is valid by Theorem 3.8 (Soundness). By Theorem 3.15 (Strong completeness), S is provable in cut-free MCLL. ∎

4. Related works

It is known that LLs are useful for formalizing and analyzing multi-agent systems (Harland & Winikoff, 2001; 2002; Pham & Harland, 2007). In (Harland & Winikoff, 2001), LLs were used by Harland and Winikoff as a basis for BDI (Belief, Desire, Intention)-style agent systems. In (Harland & Winikoff, 2002), the notion of negotiation in multi-agent systems was discussed by Harland and Winikoff based on LLs. As mentioned in (Harland & Winikoff, 2002), the resource-sensitive character of LLs is more appropriate for handling agent negotiation. In (Pham & Harland, 2007), a temporal version of LLs was used by Pham and Harland as a basis for flexible agent interactions.
In order to directly express agents' knowledge, some *epistemic linear logics*, which have some knowledge operators, have been introduced and studied by several researchers (Baltag et al., 2007; Kamide, 2006). Epistemic linear and affine logics, which are formulated as Hilbert-style axiomatizations, were proposed by Kamide (Kamide, 2006). The completeness theorems with respect to Kripke semantics for these epistemic logics were shown by him. A resource-sensitive dynamic epistemic logic, which is based on a sequent calculus for a non-commutative ILL with some program operators, was introduced by Baltag et al. (Baltag et al., 2007). The completeness theorem with respect to epistemic quantales for this logic was proved by them.

A fixed point linear logic μMALL$^{=}$ which has some least and greatest fixed point operators was introduced and studied by Baelde and Miller (Baelde, 2009; Baelde & Miller, 2007). The logic μMALL$^{=}$ enjoys cut-elimination and has a complete focused proof system. μMALL$^{=}$, also called μMALL, was motivated to offer a natural framework for reasoning about automata (Baelde, 2009). The least fixed point operator μ in μMALL$^{=}$ is formalized using the following inference rule:

$$\frac{\vdash \Gamma, B(\mu B)\mathbf{t}}{\vdash \Gamma, \mu B\mathbf{t}}\ (\mu)$$

where B represents a formula abstracted over a predicate and terms, and $\mathbf{t}$ represents a vector of terms. Compared with $\heartsuit_{\mathrm{F}}$ in MCLL, the operator μ in μMALL$^{=}$ does not use an infinitary rule.

Some linear logics with some additional modal operators have been proposed by some researchers. For example, *(linear-time) temporal linear logics*, which are roughly regarded as special cases of fixpoint linear logics, were studied by Kanovich and Ito (Kanovich, 1997) and by Kamide (Kamide, 2010).

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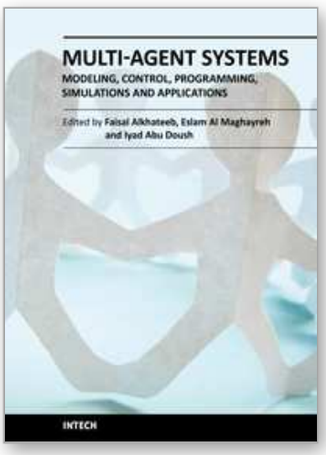
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A multi-agent system (MAS) is a system composed of multiple interacting intelligent agents. Multi-agent systems can be used to solve problems which are difficult or impossible for an individual agent or monolithic system to solve. Agent systems are open and extensible systems that allow for the deployment of autonomous and proactive software components. Multi-agent systems have been brought up and used in several application domains.

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