1

Wave Propagation Inside a Cylindrical, Left Handed, Chiral World

Pierre Hillion

Institut Henri Poincaré

France

1. Introduction

Isotropic metamaterials, with negative permittivity and permeability, also called left-handed [Veselago, 1958] [Pendry, 2000], [Sihvola 2007] have opened the way to new physical properties, different from those obtained with a conventional material. So, it is natural to inquire how these properties transform when these metamaterials are also chiral because of the importance of chirality in Nature. Some left-handed materials [Pendry 2006] and metamaterials [Grbic & Elefthertades, 2006] have been recently manufactured. We consider here a metachiral circular cylindrical medium with negative permittivity and permeability endowed with the Post constitutive relations [Post, 1962]. Using the cylindrical coordinates $r > 0$, θ, z and, assuming fields that do not depend on θ, we analyze the propagation of harmonic Bessel beams inside this medium. Two different modes exist charac-terized by negative refractive indices, function of permittivity and permeability but also of chirality which may be positive or negative with consequences on the Poynting vector carefully analyzed. The more difficult problem of wave propagation in a spherical, left handed, chiral world is succinctly discussed in an appendix.

2. Cylindrical Maxwell's equations and Post's constitutive relations

With the cylindrical coordinates $r > 0$, θ, z, the Maxwell equations in a circular cylindrical medium are for fields that do not depend on θ

$$\begin{aligned} &-\partial_z E_\theta + 1/c\, \partial_t B_r = 0, \\ &\partial_z E_r - \partial_r E_z + 1/c\, \partial_t B_\theta = 0, \\ &(\partial_r + 1/r) E_\theta + 1/c\, \partial_t B_z = 0 \end{aligned} \tag{1a}$$

$$\begin{aligned} &\partial_z H_\theta + 1/c\, \partial_t D_r = 0, \\ &\partial_z H_r - \partial_r H_z - 1/c\, \partial_t D_\theta = 0, \\ &(\partial_r + 1/r) H_\theta - 1/c\, \partial_t D_z = 0, \end{aligned} \tag{1b}$$

$$(\partial_r + 1/r)B_r + \partial_z B_z = 0\,,$$
$$(\partial_r + 1/r)D_r + \partial_z D_z = 0 \tag{2}$$

We write $-|\varepsilon|$, $-|\mu|$ the negative permittivity and permeability [Pendry, 2006] in the meta-chiral cylindrical medium and the Post constitutive relations are [Post, 1962]

$$\mathbf{D} = -|\varepsilon|\ \mathbf{E} + i\xi\,\mathbf{B}\,, \qquad \mathbf{H} = -\mathbf{B}/|\mu| + i\xi\,\mathbf{E}\,,\ i = \sqrt{-1} \tag{4}$$

in which ξ is the chirality parameter assumed to be real.
From (2), (3), we get at once the divergence equation satisfied by the electric field

$$(\partial_r + 1/r)E_r + \partial_z E_z = 0 \tag{4}$$

3. Cylindrical harmonic Bessel modes

3.1 The Bessel solutions of Maxwell's equations

Substituting (3) into (1b) gives

$$\partial_z(-B_\theta/|\mu| + i\xi\,E_\theta) + 1/c\partial_t(-|\varepsilon|\ E_r + i\xi\,B_r) = 0$$
$$\partial_z(-B_r/|\mu| + i\xi\,E_r) - \partial_r(-B_z/|\mu| + i\xi\,E_z) - 1/c\partial_t(-|\varepsilon|\ E_\theta + i\xi\,B_\theta) = 0 \tag{5}$$
$$(\partial_r + 1/r)(-B_\theta/|\mu| + i\xi\,E_\theta) - 1/c\partial_t(-|\varepsilon|\ E_z + i\xi\,B_z) = 0$$

Taking into account (1a), Eqs.(5) become

$$-1/|\mu|\ \partial_z B_\theta - |\varepsilon|/c\ \partial_t E_r + 2i\xi/c\ \partial_t B_r = 0$$
$$-1/|\mu|\ \partial_z B_r + 1/|\mu|\ \partial_r B_z + |\varepsilon|/c\ \partial_t E_\theta + 2i\xi/c\ \partial_t B_\theta = 0 \tag{5a}$$
$$-1/|\mu|\ (\partial_r + 1/r)B_\theta + |\varepsilon|/c\ \partial_t E_z + 2i\xi/c\ \partial_t B_z = 0$$

Applying the time derivative operator ∂_t to (5a) gives

$$1/c\ \partial_t\partial_z B_\theta + |\varepsilon||\mu|/c^2\ \partial_t^2 E_r - 2i\xi|\mu|/c^2\ \partial_t^2 B_r = 0 \tag{6a}$$

$$1/c\ \partial_t\partial_z B_r - 1/|c\ \partial_t\partial_r B_z - |\varepsilon||\mu|/c^2\ \partial_t^2 E_\theta + 2i\xi|\mu|/c^2\ \partial_t^2 B_\theta = 0 \tag{6b}$$

$$1/c\ \partial_t(\partial_r + 1/r)B_\theta - |\varepsilon||\mu|/c^2\ \partial_t^2 E_z + 2i\xi|\mu|/c^2\ \partial_t^2 B_z = 0 \tag{6c}$$

Using (1a) and the divergence equation (4), we have in the first and last terms of (6a)

$$1/c\ \partial_t\partial_z B_\theta = \partial_r\partial_z E_z - \partial_z^2 E_r = -(\partial_r^2 + 1/r\partial_r - 1/r^2 + \partial_z^2)E_r$$

$$1/c^2\ \partial_t^2 B_r = 1/c\partial_t\partial_z E_\theta$$

Substituting these two relations into (6a) and introducing the wave operator [Morse & Feshbach, 1953]

$$\Delta_1 = \partial_r^2 + 1/r\partial_r - 1/r^2 + \partial_z^2 - |\varepsilon||\mu|/c^2\partial_t^2 \tag{7}$$

we get the equation

$$\Delta_1 E_r + 2i\xi|\mu|/c\,\partial_z\partial_t E_\theta = 0 \tag{8a}$$

We have similarly in (6b) for the first two terms and for the last one taking into account (1a)

$$1/c\,\partial_t\partial_z B_r - 1/c\,\partial_t\partial_r B_z = (\partial_r^2 + 1/r\partial_r - 1/r^2 + \partial_z^2)E_\theta$$

$$1/c^2\,\partial_t^2 B_\theta = 1/c\partial_t\partial_r E_z - 1/c\partial_t\partial_z E_r$$

so that Eq.(6b) becomes

$$\Delta_1 E_\theta + 2i\xi|\mu|/c\,(\partial_r\partial_t E_z - \partial_z\partial_t E_r) = 0 \tag{8b}$$

Finally in (6c) , the first and third terms are according to (1a)

$$1/c\,\partial_t(\partial_r + 1/r)B_\theta = (\partial_r^2 + 1/r\partial_r + \partial_z^2)E_z$$

$$1/c^2\,\partial_t^2 B_z = -1/c(\partial_r + 1/r)\,\partial_t E_\theta$$

and, taking into account these two relations, we get

$$\Delta_0 E_z - 2i\xi|\mu|/c\,(\partial_r + 1/r)\,\partial_t E_\theta = 0 \tag{8c}$$

in which Δ_0 is the wave operator

$$\Delta_0 = \partial_r^2 + 1/r\partial_r + \partial_z^2 - |\varepsilon||\mu|/c^2\partial_t^2 \tag{9}$$

We look for the solutions of Eqs.(8a,b,c) in the form inside the cylindrical medium:

$$\begin{aligned} E_r(r,z,t) &= E_r\,J_1(k_r r)\exp(i\omega t + ik_z z) \\ E_\theta(r,z,t) &= E_\theta\,J_1(k_r r)\exp(i\omega t + ik_z z) \\ E_z(r,z,t) &= E_z\,J_0(k_r r)\exp(i\omega t + ik_z z) \end{aligned} \tag{10}$$

in which J_0, J_1 are the Bessel functions of the first kind of order zero and one respectively while E_r, E_θ, E_z are arbitrary amplitudes. From now on, we use the two parameters

$$n^2 = |\varepsilon||\mu|/c^2, \qquad \alpha = 2\xi|\mu|/c \tag{11}$$

Let us now substitute (10) into (8a,b,c). The Bessel functions with $k^2 = k_r^2 + k_z^2$ satisfy the following relations with the exponential factor $\exp(i\omega t + ik_z z)$ implicit

$$\Delta_1 J_1(k_r r) = -(k^2 - \omega^2 n^2)\,J_1(k_r r), \qquad \Delta_0 J_0(k_r r) = -(k^2 - \omega^2 n^2)\,J_0(k_r r) \tag{12a}$$

$$\partial_r J_0(k_r r) = -k_r J_1(k_r r), \qquad (\partial_r + 1/r)\,J_1(k_r r) = k_r J_0(k_r r) \tag{12a}$$

then, using (12a,b) we get the homogeneous system of equations on the amplitudes E_r, E_θ, E_z

$$(k^2-\omega^2n^2)E_r + i\omega\alpha k_z E_\theta = 0$$
$$(k^2-\omega^2n^2)E_\theta - \omega\alpha k_r E_z - i\omega\alpha k_z E_r = 0 \qquad (13)$$
$$(k^2-\omega^2n^2)E_z - \omega\alpha k_r E_\theta = 0$$

This system has nontrivial solutions when its determinant is null and a simple calculation gives

$$(k^2-\omega^2n^2)\,[(k^2-\omega^2n^2)^2 - \alpha^2\omega^2k^2] = 0 \qquad (14)$$

Leaving aside $k^2-\omega^2n^2= 0$, the equation (14) implies

$$k^2-\omega^2n^2 \pm \alpha\omega k = 0 \qquad (15)$$

or in terms of refractive index $m = ck/\omega$: $m^2 \pm \alpha cm - cn^2 = 0$. These equations have four solutions, two positive and two negative But, it has been proved [Ziolkowski & Heyman, 2001] that in left handed materials, m must be taken negative: $m = -\,|\alpha c \pm (\alpha^2c^2 + 4n^2c^2)^{1/2}|$ so that introducing the $\gamma > I$ parameter

$$\gamma = (1+4n^2\alpha^{-2})^{1/2} = (1+|\varepsilon|\,|/|\mu|\;\xi^2)^{1/2} \qquad (16)$$

the equation (15) has the two negative roots

$$k_1 = -\omega\,|\alpha|\,(1+\gamma)/2, \qquad k_2 = \omega\,|\alpha|\,(1-\gamma)/2 \qquad (17)$$

So, there exist two modes with respective wave numbers k_1, k_2 able to propagate in the meta-chiral cylindrical medium, with two different negative indices of refraction $m_{1,2}. = ck_{1,2}/\omega$.

3.2 Amplitudes of harmonic Bessel beams

The **B**, **D**, **H** components of the electromagnetic field have the form (10), that is.

$$(B_r,D_r, H_r)(r,z,t) = (B_r, D_r, H_r,)\, J_1(k_r r)\, \exp(i\omega t + ik_z z)$$
$$(B_\theta,D_\theta, H_\theta)(r,z,t) = (B_\theta, D_\theta, H_\theta,)\, J_1(k_r r)\, \exp(i\omega t + ik_z z) \qquad (18)$$
$$(B_z,D_z, H_z)(r,z,t) = (B_z, D_z, H_z,)\, J_0(k_r r)\, \exp(i\omega t + ik_z z)$$

Then, in agreement with (15) and (17), we first assume $k_1^2- \omega^2n^2 = \alpha\omega k_1$. Deleting the exponential factor from (10), (18) and using (12a), we get at once from (13) in terms of $E_\theta \cong E_1$ with $k_r^2+k_z^2 = k_1^2$

$$E_r = -ik_zE_1/k_1, \qquad E_\theta = E_1, \qquad E_z = k_rE_1/k_1 \qquad (19a)$$

Substituting (18) into (1a), taking into account (19a) and using (12a) give

$$B_r = ck_zE_1/\omega, \qquad B_\theta = ick_1E_1/\omega, \qquad B_z = ick_rE_1/\omega \qquad (19b)$$

and, with (19a,b) substituted into the Post constitutive relations (3), we get

$$D_r = -ik_zD_1^\dagger E_1,\quad D_\theta = -k_1D_1^\dagger E_1,\quad D_z = -k_rD_1^\dagger E_1,\quad D_1^\dagger = |\varepsilon|/k_1 - c\xi/\omega \tag{19c}$$

$$H_r = -k_zH_1^\dagger E_1,\quad H_\theta = -ik_1H_1^\dagger E_1,\quad H_z = -ik_r H_1^\dagger E_1,\quad H_1^\dagger = c/\omega|\mu| - \xi/k_1 \tag{19d}$$

Similarly, with $k_2^2-\omega^2n^2 = -\alpha\omega k_2$, $k_r^2+k_z^2 = k_2^2$, $E_\theta \cong E_2$, we get from (13 and (12a)

$$E_r = ik_zE_2/k_2,\qquad E_\theta = E_2,\qquad E_z = -k_rE_2/k_2 \tag{20a}$$

$$B_r = ck_zE_2/\omega,\qquad B_\theta = -ick_2E_2/\omega,\qquad B_z = ick_rE_2/\omega \tag{20b}$$

and, substituting (20a,b) into (3)

$$D_r = -ik_zD_2^\dagger E_2,\quad D_\theta = -k_2 D_2^\dagger E_2,\quad D_z = k_r D_2^\dagger E_2,\quad D_2^\dagger = |\varepsilon|/k_2 - c\xi/\omega \tag{20c}$$

$$H_r = -k_zH_2^\dagger E_2,\quad H_\theta = ik_2 H_2^\dagger E_2,\quad H_z = -ik_r H_2^\dagger E_2,\quad H_2^\dagger = c/\omega|\mu| + \xi/k_2 \tag{20d}$$

The expressions (19), (20) give in terms of E_1, E_2 the amplitudes of the two modes propagating in metachiral un-bounded cylindrical worlds.

3.3 Energy flow of Bessel waves

Using (10), (18) the Poynting vector $\mathbf{S} = c/8\pi\ (\mathbf{E}\wedge\mathbf{H}^*)$ where the asterisk denotes the complex conjugation, gives for the first mode

$$\begin{aligned}
S_{1,r}(r,z,t) &= c/8\pi(E_\theta H_z^* - E_zH_\theta^*)(r,z,t)\\
&= 0\\
S_{1,\theta}(r,z,t) &= c/8\pi(E_zH_r^* - E_rH_z^*)(r,z,t)\\
&= -ck_rk_z\, H_1^\dagger/4\pi k_1\, J_0(k_rr)\, J_1(k_rr)\ |E_1]^2\\
S_{1,z}(r,z,t) &= c/8\pi(E_rH_\theta^* - E_\theta H_r^*)(r,z,t)\\
&= ck_z\, H_1^\dagger/4\pi\ J_1^2(k_rr)\ |E_1]^2
\end{aligned} \tag{21}$$

Now, according to (11) and (17):

$$\xi/k_1 = -2\xi[\omega|\alpha|(1+\gamma)]^{-1} = -c(\xi/|\xi|\ \omega|\mu|)\ (1+\gamma)^{-1} \tag{22}$$

so that since according to (19d), $H_1^\dagger = c/\omega|\mu| - \xi/k_1$, we get

$$H_1^\dagger = c/\omega|\mu|\ [1+\xi/\ |\xi|(1+\gamma)^{-1}] \tag{22a}$$

and $H_1^\dagger > 0$ whatever the sign of $\xi/|\xi|$ is. So for $k_z > 0$ (resp. $k_z < 0$) the z-component of the energy flow runs in the direction of the positive (resp.negative) z axis while according to (10) and (18), Bessel waves propagate in the opposite direction with the phase velocity $v_z = -\omega/k_z$. Consequently S_z and v_z are antiparallel, but, because $S_{1,\theta}$ is not null, the phase velocity is not strictly antiparallel to the energy flow.
A similar calculation for the second mode gives $S_{2,r}(r,z,t) = 0$ and

$$S_{2,\theta}(r,z,t) = ck_r k_z\, H_2\dagger/4\pi k_2\, J_0(k_r r)\, J_1(k_r r)\ |E_2]^2$$
$$S_{2,z}(r,z,t) = ck_z\, H_2\dagger/4\pi\ J_1^2(k_r r)\ |E_2]^2 \tag{23}$$

Taking into account (11), (17), we have

$$\xi/k_2 = 2(\xi/\omega\ |\alpha|\ )\,(1-\gamma)^{-1} = c(\xi/|\xi|\,\omega|\mu|)(1-\gamma)^{-1} \tag{24}$$

and, since according to (20d) $H_2\dagger = c/\omega|\mu| + \xi/k_2$, we get taking into account (24)

$$H_2\dagger = c/\omega|\mu|\ [1+ \xi/|\xi|\ (1-\gamma)^{-1}] \tag{25}$$

$H_2\dagger$ is positive for $\xi/|\xi| = -1$ and for $\xi/|\xi| = 1$ with $\gamma > 2$ leading to the same conclusion as for the first mode while for $\xi/|\xi| = 1$ and $1 < \gamma < 2$ Bessel waves propagate in the same direction [Hu & Chui, 2002]. So, the harmonic Bessel waves may be considered as partially left-handed.

3.4 Evanescent waves

It is implicitly assumed in the previous sections that the wave numbers k_r, k_z are real which implies k_r^2, k_z^2 smaller than k_1^2, k_2^2 with $|k_2| < |k_1|$ according to (17). Suppose first $k_r^2 > k_1^2$, then

$$k_{1,z} = \pm\, i(k_r^2 - k_1^2)^{1/2}\ , \qquad k_{2,z} = \pm\, i(k_r^2 - k_2^2)^{1/2} \tag{27}$$

with the plus (minus) sign in the $z > 0$ ($z < 0$) region to make $\exp(ik_z z)$ exponentially decreasing, the only solution physically acceptable. Both modes are evanescent but only the second mode if $k_1^2 > k_r^2 > k_2^2$.
Suppose now $k_z^2 > k_1^2$ then

$$k_{r(1,2)} = \pm ik_{s(1,2)}, \qquad k_{s(1,2)} = (k_z^2 - k_{1,2}^2)^{1/2} \tag{28}$$

and

$$J_0(\pm ik_s r) = I_0\,(k_s r)\ , \qquad J_1(\pm ik_s r) = \pm I_1\,(k_s r) \tag{28a}$$

in which I_0, I_1 are the Bessel functions of second kind of order zero, one respectively.These functions are expo-nentially growing with r and physically unacceptable in unbounded media.. Of course, if $k_1^2 > k_z^2 > k_2^2$ the first mode can exist.

4. Discussion

Wave propagation in chiral materials is made easy for media equipped with Post's constitutive relations because as electromagnetism, they are covariant under the Lorentz group. In a metachiral material, the refractive index m depends not only on ε, μ but also on the chirality ξ and in cylindrical geometry m may have four different expressions among which only the two negative ones are physically convenient. But, the Poynting vector **S** depends on the sign of ξ so that **S** and the phase velocity **v** may be parallel or antiparallel but not strictly because, as easily shown, the Poynting vector **S** is orthogonal to **E** but not to **H**, So that **E**, **H**, **S** do not form a cartesian frame. So, metachiral cylindrical media have some particular features. Wave propagationin uniaxially anisotropic left-handed materials is discussed in [Hu & Chui, 2002]. Incidentally, a cylindrical world has been envisged by Einstein [Eddington, 1957].

Appendix A: Wave propagation in spherical, left handed, chiral media.

1. Maxwell's equations in spherical metachiral media

With the spherical polar coordinates r, θ, ϕ, the Maxwell equations in a spherical medium are for fields that do not depend on ϕ

$$\begin{gathered}(1/r\sin\theta)\,\partial_\theta(E_\phi \sin\theta) + c^{-1}\partial_t B_r = 0\\ -1/r\partial_r(rE_\phi)] + c^{-1}\partial_t B_\theta = 0\\ 1/r\,[\partial_r(rE_\theta) - \partial_\theta E_r] + c^{-1}\partial_t B_\phi = 0\end{gathered} \tag{A.1}$$

And

$$\begin{gathered}(1/r\sin\theta)\,\partial_\theta(H_\phi \sin\theta) - c^{-1}\partial_t D_r = 0\\ 1/r\partial_r(rH_\phi)] + c^{-1}\partial_t D_\theta = 0\\ 1/r\,[\partial_r(rH_\theta) - \partial_\theta H_r] - c^{-1}\partial_t D_\phi = 0\end{gathered} \tag{A.2}$$

with the divergence equations

$$\begin{gathered}(1/r^2)\partial_r(r^2 B_r) + (1/r\sin\theta)\,\partial_\theta(\sin\theta\, B_\theta) = 0,\\ (1/r^2)\partial_r(r^2 D_r) + (1/r\sin\theta)\,\partial_\theta(\sin\theta\, D_\theta) = 0\end{gathered} \tag{A.3}$$

We look for the solutions of these equations in a metachiral material endowed with the constitutive relations (3) that is

$$\mathbf{D} = -|\varepsilon|\,\mathbf{E} + i\xi\,\mathbf{B}, \qquad \mathbf{H} = -\mathbf{B}/|\mu| + i\xi\,\mathbf{E}, \qquad i = \sqrt{-1} \tag{A.4}$$

Substituting (A.4) into (A.2) gives a set of equations depending only on **E** and **B**:

$$\begin{gathered}(1/r\sin\theta)\,\partial_\theta[\sin\theta\,(-B_\phi/|\mu| + i\xi E_\phi)] - c^{-1}\partial_t[-|\varepsilon|E_r + i\xi B_r] = 0\\ 1/r\partial_r[r(-B_\phi/|\mu| + i\xi E_\phi)] - c^{-1}\partial_t[-|\varepsilon|E_\theta + i\xi B_\theta] = 0\\ 1/r\partial_r[r(-B_\theta/|\mu| + i\xi E_\theta)] - 1/r\partial_\theta[-B_r/|\mu|) + i\xi E_r) - c^{-1}\partial_t[-|\varepsilon|E_\phi + i\xi B_\phi] = 0\end{gathered} \tag{A.5}$$

while, taking into account (A,3), (A.4), the divergence equation for **E** is

$$(1/r^2)\partial_r(r^2 E_r) + (1/r\sin\theta)\,\partial_\theta(\sin\theta\, E_\theta) = 0 \tag{A.6}$$

Substituting (A.1) into (A.5), the Maxwell equations become

$$
\begin{aligned}
&(-1/|\mu|r\sin\theta)\,\partial_\theta(\sin\theta\, B_\phi) + |\varepsilon|c^{-1}\partial_t\,|E_r - 2i\xi c^{-1}\partial_t\, B_r = 0\\
&(-1/|\mu|r)\,\partial_r(rB_\phi) - |\varepsilon|\, c^{-1}\partial_t\, E_\theta + 2i\xi\, c^{-1}\partial_t B_\theta = 0\\
&(-1/|\mu|r)\,[\partial_r(rB_\theta) - \partial_\theta B_r) + |\varepsilon|c^{-1}\partial_t|E_\phi - 2i\xi\, c^{-1}\partial_t B_\phi = 0
\end{aligned}
\qquad (A.7)
$$

To look for the solutions of Eqs.(A.7) taking into account (A.1) is a challenge imposing simplifying assumptions, as for instance $B_\phi = 0$, which seems to be the most evident.

2. 2D-electromagnetic harmonic field

For a time harmonic field $\partial_t \Rightarrow i\omega$ and if $B_\phi = 0$, Eqs.(A.7) reduce to

$$
\begin{aligned}
&|\varepsilon|\, E_r - 2i\xi B_r = 0\,, \qquad |\varepsilon|\, E_\theta - 2i\xi B_\theta = 0\\
&-1/|\mu|r\,[\partial_r(rB_\theta) - \partial_\theta B_r) - i\omega|\varepsilon|\, c^{-1}E_\phi = 0
\end{aligned}
\qquad (A.8)
$$

Now let $\mathbf{B}(r,\theta) = \nabla\Phi(r,\theta)$ be the gradient of a magnetic scalar potential Φ

$$
B_r = \partial_r\Phi, \qquad B_\theta = 1/r\,\partial_\theta\Phi \qquad (A.9)
$$

Substituting (A.9) into the third relation (A.8) gives $E_\phi = 0$ so that since $B_\phi = 0$, we have according to (A.4) $D_\phi = H_\phi = 0$. So, all the ϕ-components of the electromagnetic field are null and consequently, we have to deal with a 2D-field.
With the first two relations (A.8) substituted into (A.4), we get

$$
\{H_r, H_\theta\} = -\{B_r, B_\theta)(1/|\mu| + 2\xi/|\varepsilon|), \qquad \{D_r, D_\theta\} = -|\varepsilon|/2\,\{E_r, E_\theta) \qquad (A.10)
$$

So, according to (A.9), we have just to determine the potential Φ. Then, using the equations fulfilled by the spherical Bessel functions $j_n(kr)$ and by the Legendre polynomials $P_n(\theta)$ where n is a positive integer

$$
\begin{aligned}
&\partial_r^2 j_n(kr) + 2/r\, j_n(kr) + [k^2 - n(n+1)/r^2]\, j_n(kr = 0\\
&\partial_\theta^2 P_n(\theta) + \cos\theta/\sin\theta\,\partial_\theta P_n(\theta) + n(n+1)\, P_n(\theta) = 0
\end{aligned}
\qquad (A.8)
$$

the divergence equation (A.3) is satisfied with $\phi_n(r,\theta) = j_n(kr)\, P_n(\theta)$ and $k^2 = \omega^2|\varepsilon|\,|\mu|c^{-2}$ since

$$
(\Delta + k^2)\phi_n = 0, \qquad \Delta = 1/r^2\partial_r(r^2\partial_r) + 1/r^2\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta) \qquad (A.12)
$$

So, the potential $\Phi(r,\theta,)$ with the complex amplitudes A_n) is

$$
\Phi(r,\theta) = \textstyle\sum_0^\infty A_n\, j_n(kr)\, P_n(\theta) \qquad (A.13)
$$

which achieves to determine the 2D-electromagnetic field.

In this situation where $E_\phi = H_\phi = 0$, the Poynting vector has only a non null component $S_r = S_\theta = 0$ and

$$S_\phi = c/8\pi \ \mathrm{Re}\ (E_r H_\theta^* - E_\theta H_r^*) \tag{A.14}$$

in which the asterisk denotes the complex conjugation. But, acording to (A.4) and (A.8):

$$\begin{aligned} E_r H_\theta^* &= -2i\ (\xi/|\varepsilon|\ \ |\mu|)\ (1+2\xi^2|\mu|/|\varepsilon|) B_r B_\theta^* \\ E_\theta H_r^* &= -2i\ (\xi/|\varepsilon|\ \ |\mu|)\ (1+2\xi^2|\mu|/|\varepsilon|) B_\theta B_r^* \end{aligned} \tag{A.15}$$

Substituting (A.15) into (A.14) gives

$$S_\phi = (c/8\pi/\ |\varepsilon|\ \ |\mu|)\ (1+2\xi^2|\mu|/|\varepsilon|)\ \mathrm{Im}\{\ B_r B_\theta^* - B_\theta B_r^*\} \tag{A.16}$$

Now, according to (A.4), (A.8), (A.10) we have with $B^2 = |B_r|^2 + |B_\theta|^2$

$$\mathbf{E}.\mathbf{D}^* = -2\xi^2 B^2/|\varepsilon|, \qquad \mathbf{BH}^* = -\ B^2\ (1/|\mu| +\ 2\xi^2/|\varepsilon|) \tag{A.17}$$

So, the energy density $w = 1/8\pi\ (\mathbf{E}.\mathbf{D}^* + \mathbf{BH}^*)$ is

$$w = -1/8\pi\ ((1/|\mu| +\ 4\xi^2/|\varepsilon|) \tag{A.18}$$

The energy density is negative which suggests that spherical left handed chiral media behave as a rotating plasma [Ilysionis et al] 2009], [Lashmore-Davies, 2005] and that the solutions obtained with the magnetic potential Φ are of the Alfven wave type. This result could also be a consequence of the approximations imposed here.

5. References

Eddington A.S. (1957) *The Mathematical Theory of Relativity*. Academic Press, Cambridge.

Grbic A. & Eleftheriades G.V. (2002) Experimental verification of backward-wave radiation from a negative refractive index material. *J.Appl.Phys.* Vol.92 (2002) 5930-5935.

Hu L. & Chui S.T.(2002) Characteristics of electromagnetic wave propagation in uniaxially aniotropic left-handed materials, *Phys.Rev.B* Vol.66, 08510.

Ilysionis V.I., Khalzov I.V. & Smolyakov A.I.(2009). Negative energy waves and MHD stability of rotating plasmas. *Nucl.Fusion* Vol. 49, 035008.

Lashmore-Davies C.N (2005). Negative energy waves. *J. of Plasma Phys.*Vol. 71, 101-10.

Morse P.M. &. Feshbach H.(1953). *Methods of Theoretical Physics*. Mac Graw Hill, New York.

Pendry J.B. (2000), Negative refraction makes a perfect lens. *Phys.Rev.Lett*. Vol. 85, 3966-3969.

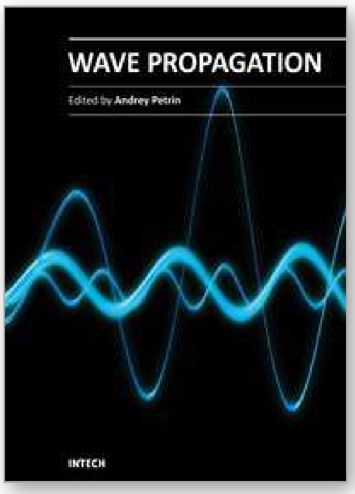
Pendry J.B., Schurig D. & Smith D.R., (2006) Controlling electromagnetic fields, *Science* Vol. 312,1780-1782.

Post E.J., (1962) *Formal Structure of Electromagnetism*, North Holland, Amsterdam.

Sihvola A.A. (2007), Metamaterials in electromagnetism. *Metamaterials* Vol.1, 1-11.

Veselago V. (1968), The electrodynamic of substances with simultaneously negative ε and μ, *Sov.Phys.Usp. Vol.10. 509-514.*

Ziolkowski R.W. & Heyman E., (2001) Wave propagation in media having negative permittivity and permeability. *Phys.Rev.E* Vol.64, 056605.



Wave Propagation
Edited by Dr. Andrey Petrin

ISBN 978-953-307-275-3
Hard cover, 570 pages
Publisher InTech
Published online 16, March, 2011
Published in print edition March, 2011

The book collects original and innovative research studies of the experienced and actively working scientists in the field of wave propagation which produced new methods in this area of research and obtained new and important results. Every chapter of this book is the result of the authors achieved in the particular field of research. The themes of the studies vary from investigation on modern applications such as metamaterials, photonic crystals and nanofocusing of light to the traditional engineering applications of electrodynamics such as antennas, waveguides and radar investigations.

How to reference
In order to correctly reference this scholarly work, feel free to copy and paste the following:

Pierre Hillion (2011). Wave Propagation Inside a Cylindrical, Left Handed, Chiral World, Wave Propagation, Dr. Andrey Petrin (Ed.), ISBN: 978-953-307-275-3, InTech, Available from: http://www.intechopen.com/books/wave-propagation/wave-propagation-inside-a-cylindrical-left-handed-chiral-world