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Synthesis of a Robust $\mathcal{H}_\infty$ Fuzzy Controller for Uncertain Nonlinear Dynamical Systems

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1. Introduction

Over the past two decades, there has been rapidly growing interest in application of fuzzy logic to control problem. Researches have been focused on its application to industrial processes and a number of successful results have been reported in the literature. In spite of these successes, there are many basic issues remain to be addressed. One of them is how to achieve a systematic design that guarantees closed-loop stability and performance. Recently, a great amount of effort has been devoted to describing a nonlinear system using a Takagi-Sugeno fuzzy model (see [1-28]). The Takagi-sugeno fuzzy model represents a nonlinear system by a family of local linear models which smoothly blended together through fuzzy membership functions. Unlike conventional modelling techniques which uses a single model to describe the global behavior of a nonlinear system, fuzzy modelling is essentially a multi-model approach in which simple sub-models (typically linear models) are fuzzily combined to described the global behavior of a nonlinear system. Based on this fuzzy model, a number of systematic model-based fuzzy control design methodologies have been developed. The aim of this paper is to study the problem of designing robust $\mathcal{H}_\infty$ fuzzy controller for a class of uncertain fuzzy systems. First, we approximate this class of uncertain nonlinear systems by a Takagi-Sugeno fuzzy model. Then based on an LMI approach, we develop a technique for designing robust $\mathcal{H}_\infty$ fuzzy state-feedback and output feedback controllers such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value.

This paper is organized as follows. In Section 2, system descriptions and definition are presented. In Section 3 and Section 4, based on an LMI approach, we respectively develop a technique for designing robust $\mathcal{H}_\infty$ fuzzy state-feedback and output-feedback controllers such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value for the system described in Section 2. The validity of this approach is demonstrated by an example from a literature in Section 5. Finally, conclusions are given in Section 6.

2. System descriptions and definitions

In this chapter, we generalize the TS fuzzy system to represent a TS fuzzy system with parametric uncertainties as follows:

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$$\begin{array}{rcl} \dot{x}(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[A_i+\Delta A_i]x(t)+[B_{1_i}+\Delta B_{1_i}]w(t) \\ & & +[B_{2_i}+\Delta B_{2_i}]u(t)\Big], \quad x(0)=0 \\ z(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_{1_i}+\Delta C_{1_i}]x(t)+[D_{12_i}+\Delta D_{12_i}]u(t)\Big] \\ y(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_{2_i}+\Delta C_{2_i}]x(t)+[D_{21_i}+\Delta D_{21_i}]w(t)\Big] \end{array} \tag{1}$$

where $\nu(t) = [\nu_1(t) \cdots \nu_\vartheta(t)]$ is the premise variable vector that may depend on states in many cases, $\mu_i(\nu(t))$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\mu_i(\nu(t)) \geq 0$ and $\sum_{i=1}^{r}\mu_i(\nu(t)) = 1$), ϑ is the number of fuzzy sets, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input, $w(t) \in \Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, $y(t) \in \Re^\ell$ is the measurement, $z(t) \in \Re^s$ is the controlled output, the matrices $A_i, B_{1_i}, B_{2_i}, C_{1_i}, C_{2_i}, D_{12_i}$ and D_{21_i} are of appropriate dimensions, and r is the number of IF-THEN rules. The matrices $\Delta A_i, \Delta B_{1_i}, \Delta B_{2_i}, \Delta C_{1_i}, \Delta C_{2_i}, \Delta D_{12_i}$ and ΔD_{21_i} represent the uncertainties in the system and satisfy the following assumption.

Assumption 1

$$\begin{gathered} \Delta A_i = F(x(t),t)H_{1_i}, \\ \Delta B_{1_i} = F(x(t),t)H_{2_i},\ \ \Delta B_{2_i} = F(x(t),t)H_{3_i}, \\ \Delta C_{1_i} = F(x(t),t)H_{4_i},\ \ \Delta C_{2_i} = F(x(t),t)H_{5_i}, \\ \Delta D_{12_i} = F(x(t),t)H_{6_i} \ \textit{and} \ \Delta D_{21_i} = F(x(t),t)H_{7_i} \end{gathered}$$

where H_{j_i}, $j = 1,2,\cdots,7$ are known matrix functions which characterize the structure of the uncertainties. Furthermore, the following inequality holds:

$$\|F(x(t),t)\| \leq \rho \tag{2}$$

for any known positive constant ρ.

Next, let us recall the following definition.

Definition 1 *Suppose γ is a given positive number. A system (1) is said to have an $\mathcal{L}_2$-gain less than or equal to γ if*

$$\int_0^{T_f} z^T(t)z(t)dt \leq \gamma^2 \left[\int_0^{T_f} w^T(t)w(t)dt\right],\ x(0) = 0 \tag{3}$$

for all $T_f \geq 0$ and $w(t) \in \mathcal{L}_2[0, T_f]$.

Note that for the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

3. Robust $\mathcal{H}_\infty$ state-feedback control design

The aim of this section is to design a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller of the form

$$u(t) = \sum_{j=1}^{r}\mu_j K_j x(t) \tag{4}$$

where K_j is the controller gain, such that the inequality (3) holds. The state space form of the fuzzy system model (1) with the controller (4) is given by

$$\begin{aligned}\dot{x}(t) \quad &= \quad \textstyle\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \Big[[(A_i + B_{2_i} K_j) \\ &\quad +(\Delta A_i + \Delta B_{2_i} K_j)] x(t) + [B_{1_i} + \Delta B_{1_i}] w(t) \Big], \quad x(0) = 0.\end{aligned} \tag{5}$$

The following theorem provides sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller. These sufficient conditions can be derived by the Lyapunov approach.

Theorem 1 *Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist a matrix $P = P^T$ and matrices Y_j, $j = 1, 2, \cdots, r$, satisfying the following linear matrix inequalities:*

$$P > 0 \tag{6}$$

$$\Omega_{ii} < 0, \quad i = 1, 2, \cdots, r \tag{7}$$

$$\Omega_{ij} + \Omega_{ji} < 0, \quad i < j \leq r \tag{8}$$

where

$$\Omega_{ij} = \begin{pmatrix} \begin{pmatrix} A_i P + P A_i^T \\ + B_{2_i} Y_j + Y_j^T B_{2_i}^T \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T & -\gamma I & (*)^T \\ \tilde{C}_{1_i} P + \tilde{D}_{12_i} Y_j & 0 & -\gamma I \end{pmatrix} \tag{9}$$

with

$$\tilde{B}_{1_i} = \left[\begin{array}{cccc} \delta I & I & \delta I & B_{1_i} \end{array} \right],$$

$$\tilde{C}_{1_i} = \left[\begin{array}{cccc} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{array} \right]^T,$$

$$\tilde{D}_{12_i} = \left[\begin{array}{cccc} 0 & \frac{\gamma\rho}{\delta} H_{3_i}^T & \sqrt{2}\lambda\rho H_{6_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{array} \right]^T,$$

$$\lambda = \left(1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r \left[\| H_{2_i}^T H_{2_j} \| \right] \right)^{\frac{1}{2}},$$

then the inequality (3) holds. Furthermore, a suitable choice of the fuzzy controller is

$$u(t) = \sum_{j=1}^r \mu_j K_j x(t) \tag{10}$$

where

$$K_j = Y_j P^{-1}. \tag{11}$$

Proof: Using Assumption 1, the closed-loop fuzzy system (5) can be expressed as follows:

$$\dot{x}(t) \quad = \quad \textstyle\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \Big([A_i + B_{2_i} K_j] x(t) + \tilde{B}_{1_i} \tilde{w}(t) \Big) \tag{12}$$

where

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & \delta I & B_{1_i} \end{bmatrix},$$

and the disturbance $\tilde{w}(t)$ is

$$\tilde{w}(t) = \begin{bmatrix} \frac{1}{\delta}F(x(t),t)H_{1_i}x(t) \\ F(x(t),t)H_{2_i}w(t) \\ \frac{1}{\delta}F(x(t),t)H_{3_i}K_j x(t) \\ w(t) \end{bmatrix}. \tag{13}$$

Let consider a Lyapunov function

$$V(x(t)) = \gamma x^T(t)Qx(t)$$

where $Q = P^{-1}$. Differentiate $V(x(t))$ along the closed-loop system (12) yields

$$\begin{aligned} \dot{V}(x(t)) &= \gamma \dot{x}^T(t)Qx(t) + \gamma x^T(t)Q\dot{x}(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(\gamma x^T(t)(A_i + B_{2_i}K_j)^T Qx(t) \\ &\quad + \gamma x^T(t)Q(A_i + B_{2_i}K_j)x(t) \\ &\quad + \gamma\tilde{w}^T(t)\tilde{B}_{1_i}^T Qx(t) + \gamma x^T(t)Q\tilde{B}_{1_i}\tilde{w}(t)\Big). \end{aligned} \tag{14}$$

Adding and subtracting $-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]$ to and from (14), we get

$$\begin{aligned} \dot{V}(x(t)) &= \gamma\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n\Big(\begin{bmatrix} x^T(t) & \tilde{w}^T(t) \end{bmatrix} \times \\ &\quad \begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +\frac{(\tilde{C}_{1_i}+\tilde{D}_{12_i}K_j)^T(\tilde{C}_{1m}+\tilde{D}_{12m}K_n)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I \end{pmatrix} \begin{bmatrix} x(t) \\ \tilde{w}(t) \end{bmatrix}\Big) \\ &\quad -\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)] \end{aligned} \tag{15}$$

where

$$\tilde{z}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j[\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j]x(t) \tag{16}$$

with

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta}H_{1_i}^T & 0 & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T$$

$$\text{and } \tilde{D}_{12_i} = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta}H_{3_i}^T & \sqrt{2}\lambda\rho H_{6_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{bmatrix}^T.$$

Pre and post multiply (7)-(8) by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ yields

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_i)^T Q \\ +Q(A_i + B_{2_i}K_i) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I & (*)^T \\ \tilde{C}_{1_i} + \tilde{D}_{12_i}K_i & 0 & -\gamma I \end{pmatrix} < 0, \tag{17}$$

$i = 1, 2, \cdots, r$, and

$$\left\{ \begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I & (*)^T \\ \tilde{C}_{1_i} + \tilde{D}_{12_i}K_j & 0 & -\gamma I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} (A_j + B_{2_j}K_i)^T Q \\ +Q(A_j + B_{2_j}K_i) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_{1_j}^T Q & -\gamma I & (*)^T \\ \tilde{C}_{1_j} + \tilde{D}_{12_j}K_i & 0 & -\gamma I \end{pmatrix} \right\} < 0, \tag{18}$$

$i < j \leq r$, respectively. Applying the Schur complement on (17)-(18) and rearranging them, then we have

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_i)^T Q \\ +Q(A_i + B_{2_i}K_i) \\ +\frac{(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_i)^T(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_i)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I \end{pmatrix} < 0, \tag{19}$$

$i = 1, 2, \cdots, r$, and

$$\left\{ \begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +\frac{(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j)^T(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} (A_j + B_{2_j}K_i)^T Q \\ +Q(A_j + B_{2_j}K_i) \\ +\frac{(\tilde{C}_{1_j} + \tilde{D}_{12_j}K_i)^T(\tilde{C}_{1_j} + \tilde{D}_{12_j}K_i)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_{1_j}^T Q & -\gamma I \end{pmatrix} \right\} < 0, \tag{20}$$

$i < j \leq r$, respectively. Using (19)-(20) and the fact that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n M_{ij}^T N_{mn} \leq \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j [M_{ij}^T M_{ij} + N_{ij}N_{ij}^T], \tag{21}$$

it is obvious that we have

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +\frac{(\tilde{C}_{1_i}+\tilde{D}_{12_i}K_j)^T(\tilde{C}_{1_i}+\tilde{D}_{12_i}K_j)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma I \end{pmatrix} < 0 \tag{22}$$

where $i, j = 1, 2, \cdots, r$. Since (22) is less than zero and the fact that $\mu_i \geq 0$ and $\sum_{i=1}^{r} \mu_i = 1$, then (15) becomes

$$\dot{V}(x(t)) \quad \leq \quad -\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]. \tag{23}$$

Integrate both sides of (23) yields

$$\begin{aligned} \int_0^{T_f} \dot{V}(x(t))dt \quad &\leq \quad \int_0^{T_f} \Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big] dt \\ V(x(T_f)) - V(x(0)) \quad &\leq \quad \int_0^{T_f} \Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big] dt. \end{aligned}$$

Using the fact that $x(0) = 0$ and $V(x(T_f)) \geq 0$ for all $T_f \neq 0$, we get

$$\int_0^{T_f} \tilde{z}^T(t)\tilde{z}(t)dt \leq \gamma^2 \left[\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]dt \right]. \tag{24}$$

Putting $\tilde{z}(t)$ and $\tilde{w}(t)$ respectively given in (16) and (13) into (24) and using the fact that $\|F(x(t), t)\| \leq \rho$, $\lambda^2 = \Big(1 + \rho^2 \sum_{i=1}^{r}\sum_{j=1}^{r}[\|H_{2_i}^T H_{2_j}\|]\Big)$ and (21), we have

$$\begin{aligned} &\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(2\lambda^2 x^T(t)[C_{1_i} + D_{12_i}K_j]^T[C_{1_i} + D_{12_i}K_j]x(t) \\ &+2\lambda^2\rho^2 x^T(t)[H_{4_i} + H_{6_i}K_j]^T[H_{4_i} + H_{6_i}K_j]x(t)\Big)\, dt \\ &\leq \gamma^2\lambda^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right]. \end{aligned} \tag{25}$$

Adding and subtracting

$$\begin{aligned} \lambda^2 z^T(t)z(t) \quad = \quad &\lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(x^T(t) \Big[C_{1_i} + F(x(t), t)H_{4_i} + D_{12_i}K_j + F(x(t), t)H_{6_i}K_j\Big]^T \\ &\Big[C_{1_i} + F(x(t), t)H_{4_i} + D_{12_i}K_j + F(x(t), t)H_{6_i}K_j\Big]\, x(t)\Big) \end{aligned}$$

to and from (25), one obtains

$$
\begin{aligned}
&\int_0^{T_f} \Big\{\lambda^2 z^T(t)z(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \times \\
&\Big(2\lambda^2 x^T(t)[C_{1_i} + D_{12_i}K_j]^T[C_{1_i} + D_{12_i}K_j]x(t) \\
&+2\lambda^2\rho^2 x^T(t)[H_{4_i} + H_{6_i}K_j]^T[H_{4_i} + H_{6_i}K_j]x(t) \\
&-\lambda^2 x^T(t)[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{6_i}K_j]^T \\
&[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{6_i}K_j]x(t)\Big)\Big\}\, dt \\
&\le \gamma^2\lambda^2\left[\int_0^{T_f} w^T(t)w(t)\, dt\right].
\end{aligned}
\tag{26}
$$

Using the triangular inequality and the fact that $\|F(x(t),t)\| \le \rho$, we have

$$
\begin{aligned}
&\lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(x^T(t)\left[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{6_i}K_j\right]^T \\
&\left[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{6_i}K_j\right]x(t)\Big) \\
&\le \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(\Big\{2\lambda^2 x^T(t)\left[C_{1_i} + D_{12_i}K_j\right]^T\left[C_{1_i} + D_{12_i}K_j\right]x(t)\Big\} \\
&+2\lambda^2\rho^2 x^T(t)\left[H_{4_i} + H_{6_i}K_j\right]^T\left[H_{4_i} + H_{6_i}K_j\right]x(t)\Big).
\end{aligned}
\tag{27}
$$

Using (27) on (26), we obtain

$$
\int_0^{T_f} z^T(t)z(t) \le \gamma^2\int_0^{T_f} w^T(t)w(t)\, dt. \tag{28}
$$

Hence, the inequality (3) holds. ■

4. Robust $\mathcal{H}_\infty$ output feedback control design

The nature of the information of the state available to the controller has a major effect on the complexity of the designing problem and of the resulting controller. The state-feedback control design problem is an easier problem in which all information are available. However, in most real physical systems, the state is not perfectly known, and so we must estimate it. The process of estimating the system state from the measurement output that are available is called the estimator design. By utilizing the state estimator, the output feedback problem is converted to the state-feedback problem for a new problem. This new problem employs the estimated state as its own state variable and the solution of the new state-feedback problem leads to the solution of the dynamic output feedback control problem. Basically, the dynamic output feedback is a coupling of control and estimation.

This section aims at designing a full order dynamic $\mathcal{H}_\infty$ fuzzy output feedback controller of the form

$$
\begin{aligned}
\dot{\hat{x}}(t) &= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{\mu}_i\hat{\mu}_j\left[\hat{A}_{ij}\hat{x}(t) + \hat{B}_i y(t)\right] \\
u(t) &= \textstyle\sum_{i=1}^{r}\hat{\mu}_i\hat{C}_i\hat{x}(t)
\end{aligned}
\tag{29}
$$

where $\hat{x}(t) \in \Re^n$ is the controller's state vector, $\hat{A}_{ij}$, $\hat{B}_i$ and $\hat{C}_i$ are parameters of the controller which are to be determined, and $\hat{\mu}_i$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\hat{\mu}_i \geq 0$ and $\sum_{i=1}^{r} \hat{\mu}_i = 1$), such that the inequality (3) holds.
In this section, we consider the designing of the robust $\mathcal{H}_\infty$ output feedback control into two cases as follows. In Subsection A, we consider the case where the premise variable of the fuzzy model μ_i is measurable, while in Subsection B, the premise variable which is assumed to be unmeasurable is considered.

4.1 Case I–$\nu(t)$ is available for feedback

The premise variable of the fuzzy model $\nu(t)$ is available for feedback which implies that μ_i is available for feedback. Thus, we can select our controller that depends on μ_i as follows:

$$\begin{array}{rcl} \dot{\hat{x}}(t) & = & \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \left[\hat{A}_{ij} \hat{x}(t) + \hat{B}_i y(t) \right] \\ u(t) & = & \sum_{i=1}^{r} \mu_i \hat{C}_i \hat{x}(t). \end{array} \tag{30}$$

Before presenting our next results, the following lemma is recalled.

Lemma 1 *Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance γ and a positive constant δ, if there exists a matrix $P = P^T$ satisfying the following linear matrix inequalities:*

$$P > 0 \tag{31}$$

$$\begin{pmatrix} \begin{pmatrix} A_{cl}^{ij} P \\ + P (A_{cl}^{ij})^T \end{pmatrix} & (*)^T & (*)^T \\ (B_{cl}^{ij})^T & -\gamma^2 I & (*)^T \\ C_{cl}^{ij} P & 0 & -I \end{pmatrix} < 0, \tag{32}$$

where $i, j = 1, 2, \cdots, r$

$$A_{cl}^{ij} = \begin{bmatrix} A_i & B_{2_i} \hat{C}_j \\ \hat{B}_i C_{2_j} & \hat{A}_{ij} \end{bmatrix}, \quad B_{cl}^{ij} = \begin{bmatrix} \tilde{B}_{1_i} \\ \hat{B}_i \tilde{D}_{21_j} \end{bmatrix}$$

$$\text{and } \; C_{cl}^{ij} = [\tilde{C}_{1_i} \;\; \tilde{D}_{12_i} \hat{C}_j]$$

with

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & \delta I & 0 & B_{1_i} & 0 \end{bmatrix},$$

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \frac{\gamma\rho}{\delta} H_{5_i}^T & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{12_i} = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta} H_{3_i}^T & 0 & \sqrt{2}\lambda\rho H_{6_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{21_i} = \begin{bmatrix} 0 & 0 & 0 & \delta I & D_{21_i} & I \end{bmatrix}$$

$$\text{and } \; \lambda = \left(1 + \rho^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \left[\| H_{2_i}^T H_{2_j} \| + \| H_{7_i}^T H_{7_j} \| \right] \right)^{\frac{1}{2}},$$

then the inequality (3) is guaranteed.

Proof: The state space form of the fuzzy system model (1) with the controller (30) is given by

$$\begin{aligned} \dot{\breve{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left(A_{cl}^{ij}\breve{x}(t)+B_{cl}^{ij}\tilde{w}(t)\right) \\ \breve{z}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j C_{cl}^{ij}\breve{x}(t) \end{aligned} \tag{33}$$

where $\breve{x}(t)=\left[x^T(t)\ \ \hat{x}^T(t)\right]^T$ and the matrix functions A_{cl}^{ij}, B_{cl}^{ij} and C_{cl}^{ij} are defined in Lemma 1 and the disturbance is

$$\tilde{w}(t)=\begin{bmatrix} \frac{1}{\delta}F(x(t),t)H_{1_i}x(t) \\ F(x(t),t)H_{2_i}w(t) \\ \frac{1}{\delta}F(x(t),t)H_{3_i}\hat{C}_j\hat{x}(t) \\ \frac{1}{\delta}F(x(t),t)H_{5_i}x(t) \\ w(t) \\ F(x(t),t)H_{7_i}w(t) \end{bmatrix}. \tag{34}$$

Let choose a Lyapunov function

$$V(\breve{x}(t))=\breve{x}^T(t)Q\breve{x}(t), \tag{35}$$

where $Q=P^{-1}$. Differentiate $V(\breve{x}(t))$ along the closed-loop system (33) yields

$$\begin{aligned} &\dot{V}(\breve{x}(t))=\dot{\breve{x}}^T(t)Q\breve{x}(t)+\breve{x}^T(t)Q\dot{\breve{x}}(t) \\ &\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(\breve{x}^T(t)(A_{cl}^{ij})^TQ\breve{x}(t)+\breve{x}^T(t)QA_{cl}^{ij}\breve{x}(t) \\ &+\tilde{w}^T(t)(B_{cl}^{ij})^TQ\breve{x}(t)+\breve{x}^T(t)QB_{cl}^{ij}\tilde{w}(t)\Big). \end{aligned} \tag{36}$$

Add and subtract $-\breve{z}^T(t)\breve{z}(t)+\gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}(t)^T\tilde{w}(t)]$ to and from (36) yields

$$\begin{aligned} \dot{V}(\breve{x}(t)) \;=\; &\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n\left[\ \breve{x}^T(t)\quad \tilde{w}^T(t)\ \right] \\ &\left(\begin{array}{cc} \left(\begin{array}{c} (A_{cl}^{ij})^TQ+QA_{cl}^{ij} \\ +(C_{cl}^{ij})^TC_{cl}^{mn} \end{array}\right) & (*)^T \\ QB_{cl}^{ij} & -\gamma^2 I \end{array}\right)\begin{bmatrix} \breve{x}(t) \\ \tilde{w}(t) \end{bmatrix} \\ &-\breve{z}^T(t)\breve{z}(t)+\gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]. \end{aligned} \tag{37}$$

Now suppose there exits a matrix $P>0$ such that (32) holds, i.e.,

$$\begin{pmatrix} A_{cl}^{ij}P+P(A_{cl}^{ij})^T & (*)^T & (*)^T \\ (B_{cl}^{ij})^T & -\gamma^2 I & (*)^T \\ C_{cl}^{ij}P & 0 & -I \end{pmatrix}<0. \tag{38}$$

Pre and post multiply (38) by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ yields

$$\begin{pmatrix} (A_{cl}^{ij})^T Q + Q A_{cl}^{ij} & (*)^T & (*)^T \\ (B_{cl}^{ij})^T Q & -\gamma^2 I & (*)^T \\ C_{cl}^{ij} & 0 & -I \end{pmatrix} < 0. \tag{39}$$

The Schur complement of (39) is

$$\begin{pmatrix} (A_{cl}^{ij})^T Q + Q A_{cl}^{ij} + (C_{cl}^{ij})^T C_{cl}^{ij} & (*)^T \\ (B_{cl}^{ij})^T & -\gamma^2 I \end{pmatrix} < 0. \tag{40}$$

Using (40) and the fact in (21) together with the fact that $\mu_i \geq 0$ and $\sum_{i=1}^{r} \mu_i = 1$, then (37) becomes

$$\dot{V}(\check{x}(t)) \quad \leq \quad -\check{z}^T(t)\check{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n [\tilde{w}^T(t)\tilde{w}(t)]. \tag{41}$$

Integrate both sides of (41) yields

$$\begin{aligned} \int_0^{T_f} \dot{V}(\check{x}(t))dt \quad &\leq \quad \int_0^{T_f} \Big(-\check{z}^T(t)\check{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n [\tilde{w}^T(t)\tilde{w}(t)] \Big) dt \\ V(\check{x}(T_f)) - V(\check{x}(0)) \quad &\leq \quad \int_0^{T_f} \Big(-\check{z}^T(t)\check{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n [\tilde{w}^T(t)\tilde{w}(t)] \Big) dt. \end{aligned}$$

Using the fact that $\check{x}(0) = 0$ and $V(\check{x}(T_f)) > 0$ for all $T_f \neq 0$, we have

$$\int_0^{T_f} \check{z}^T(t)\check{z}(t)dt \leq \gamma^2 \left[\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n [\tilde{w}^T(t)\tilde{w}(t)] \right] dt. \tag{42}$$

Putting $\check{z}(t)$ and $\tilde{w}(t)$ respectively given in (33) and (34) into (42) and using the fact that $\|F(x(t),t)\| \leq \rho$, $\lambda^2 = \Big(1 + \rho^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \Big[\|H_{2_i}^T H_{2_j}\| + \|H_{7_i}^T H_{7_j}\| \Big]\Big)$ and (21), we have

$$\begin{aligned} &\int_0^{T_f} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(2\lambda^2 \check{x}^T(t) [C_{1_i} \;\; D_{12_i}\hat{C}_j]^T [C_{1_i} \;\; D_{12_i}\hat{C}_j] \check{x}(t) \\ &+ 2\lambda^2\rho^2 \check{x}^T(t) [H_{4_i} \;\; H_{6_i}\hat{C}_j]^T [H_{4_i} \;\; H_{6_i}\hat{C}_j] \check{x}(t) \Big) \, dt \\ &\leq \gamma^2\lambda^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right]. \end{aligned} \tag{43}$$

Adding and subtracting

$$\begin{aligned} \lambda^2 z^T(t)z(t) \quad = \quad & \lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(\check{x}^T(t) \Big[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j \Big]^T \\ & \Big[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j \Big] \check{x}(t) \Big) \end{aligned}$$

to and from (43), one obtains

$$\begin{aligned}
&\int_0^{T_f} \Big\{ \lambda^2 z^T(t)z(t) + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \times \\
&\Big(2\lambda^2 \breve{x}^T(t)[C_{1_i} \;\; D_{12_i}\hat{C}_j]^T [C_{1_i} \;\; D_{12_i}\hat{C}_j]\breve{x}(t) + 2\lambda^2\rho^2\breve{x}^T(t) \times \\
&[H_{4_i} \;\; H_{6_i}\hat{C}_j]^T[H_{4_i} \;\; H_{6_i}\hat{C}_j]\breve{x}(t) \\
&-\lambda^2\breve{x}^T(t)[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j]^T \\
&[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j]\breve{x}(t) \Big) \Big\}\, dt \\
&\qquad\qquad \leq \gamma^2\lambda^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right]. \qquad (44)
\end{aligned}$$

Using the triangular inequality and the fact that $\|F(x(t),t)\| \leq \rho$, we have

$$\begin{aligned}
&\lambda^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(\breve{x}^T(t) \left[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j \right]^T \\
&\left[C_{1_i} + F(x(t),t)H_{4_i} \;\; D_{12_i}\hat{C}_j + F(x(t),t)H_{6_i}\hat{C}_j \right] \breve{x}(t) \Big) \\
&\leq \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big(2\lambda^2\breve{x}^T(t) \left[C_{1_i} \;\; D_{12_i}\hat{C}_j \right]^T \left[C_{1_i} \;\; D_{12_i}\hat{C}_j \right] \breve{x}(t) \\
&+2\lambda^2\rho^2\breve{x}^T(t) \left[H_{4_i} \;\; H_{6_i}\hat{C}_j \right]^T \left[H_{4_i} \;\; H_{6_i}\hat{C}_j \right] \breve{x}(t) \Big). \qquad (45)
\end{aligned}$$

Using (45) on (44), we obtain

$$\int_0^{T_f} z^T(t)z(t) \leq \gamma^2 \int_0^{T_f} w^T(t)w(t)\, dt. \qquad (46)$$

Hence, the inequality (3) is guaranteed. ■

Knowing that the controller's premise variable is the same as the plant's premise variable, the left hand of (32) can be re-expressed as follows:

$$A_{cl}^{ij}P + P(A_{cl}^{ij})^T + \gamma^{-2}B_{cl}^{ij}(B_{cl}^{ij})^T + P(C_{cl}^{ij})^T C_{cl}^{ij}P. \qquad (47)$$

Before providing LMI-based sufficient conditions for the system (1) to have an $\mathcal{H}_\infty$ performance, let us partition the matrix P as follows:

$$P = \begin{bmatrix} X & Y^{-1} - X \\ Y^{-1} - X & X - Y^{-1} \end{bmatrix} \qquad (48)$$

where $X \in \Re^{n\times n}$ and $Y \in \Re^{n\times n}$. Utilizing the partition above, we define the new controller's input and output matrices as

$$\begin{aligned} \mathcal{B}_i &\triangleq \left[Y^{-1} - X \right] \hat{B}_i \\ \mathcal{C}_i &\triangleq \hat{C}_i Y. \end{aligned} \qquad (49)$$

Using these changes of variable, we have the following theorem.

Theorem 2 *Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist matrices $X = X^T$, $Y = Y^T$, $\mathcal{B}_i$ and $\mathcal{C}_i$, $i = 1,2,\cdots,r$, satisfying the following linear matrix inequalities:*

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{50}$$

$$X > 0 \tag{51}$$

$$Y > 0 \tag{52}$$

$$\Psi_{11_{ii}} < 0, \quad i = 1,2,\cdots,r \tag{53}$$

$$\Psi_{22_{ii}} < 0, \quad i = 1,2,\cdots,r \tag{54}$$

$$\Psi_{11_{ij}} + \Psi_{11_{ji}} < 0, \quad i < j \le r \tag{55}$$

$$\Psi_{22_{ij}} + \Psi_{22_{ji}} < 0, \quad i < j \le r \tag{56}$$

where

$$\Psi_{11_{ij}} = \begin{pmatrix} \begin{pmatrix} A_iY + YA_i^T \\ +B_{2_i}\mathcal{C}_j + \mathcal{C}_i^T B_{2_j}^T \\ +\gamma^{-2}\tilde{B}_{1_i}\tilde{B}_{1_j}^T \end{pmatrix} & (*)^T \\ \left[Y\tilde{C}_{1_i}^T + \mathcal{C}_i^T \tilde{D}_{12_j}^T\right]^T & -I \end{pmatrix} \tag{57}$$

$$\Psi_{22_{ij}} = \begin{pmatrix} \begin{pmatrix} A_i^T X + XA_i \\ +\mathcal{B}_i C_{2_j} + C_{2_i}^T \mathcal{B}_j^T \\ +\tilde{C}_{1_i}^T \tilde{C}_{1_j} \end{pmatrix} & (*)^T \\ \left[X\tilde{B}_{1_i} + \mathcal{B}_i \tilde{D}_{21_j}\right]^T & -\gamma^2 I \end{pmatrix} \tag{58}$$

with

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & \delta I & 0 & B_{1_i} & 0 \end{bmatrix},$$

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \frac{\gamma\rho}{\delta} H_{5_i}^T & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{12_i} = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta} H_{3_i}^T & 0 & \sqrt{2}\lambda\rho H_{6_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{21_i} = \begin{bmatrix} 0 & 0 & 0 & \delta I & D_{21_i} & I \end{bmatrix}$$

$$\text{and } \lambda = \left(1 + \rho^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\left[\|H_{2_i}^T H_{2_j}\| + \|H_{7_i}^T H_{7_j}\|\right]\right)^{\frac{1}{2}},$$

then the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable controller is of the form (30) with

$$\begin{aligned} \hat{A}_{ij} &= \left[Y^{-1} - X\right]^{-1}\mathcal{M}_{ij}Y^{-1} \\ \hat{B}_i &= \left[Y^{-1} - X\right]^{-1}\mathcal{B}_i \\ \hat{C}_i &= \mathcal{C}_i Y^{-1} \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathcal{M}_{ij} = & -A_i^T - XA_iY - XB_{2_i}\hat{C}_jY \\ & -\left[Y^{-1} - X\right]\hat{B}_i C_{2_j} Y - \tilde{C}_{1_i}^T\left[\tilde{C}_{1_j}Y + \tilde{D}_{12_j}\hat{C}_jY\right] \\ & -\gamma^{-2}\left\{X\tilde{B}_{1_i} + \left[Y^{-1} - X\right]\hat{B}_i\tilde{D}_{21_i}\right\}\tilde{B}_{1_j}^T. \end{aligned} \tag{60}$$

Proof: Suppose there exist X and Y such that the inequalities (50) and (51)-(52) hold. The inequality (50) implies that the matrix P defined in (47) is a positive definite matrix. Using the partition (48), the controller (49) and multiplying (47) to the left by $\begin{bmatrix} Y & I \\ Y & 0 \end{bmatrix}$ and to the right by $\begin{bmatrix} Y & Y \\ I & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} \Phi_{11_{ij}} & 0 \\ 0 & \Phi_{22_{ij}} \end{bmatrix} \tag{61}$$

where

$$\begin{aligned} \Phi_{11_{ij}} &= A_iY + YA_i^T + B_{2_i}\mathcal{C}_j + \mathcal{C}_i^T B_{2_j}^T + \gamma^{-2}\tilde{B}_{1_i}\tilde{B}_{1_j}^T \\ &\quad + \left[Y\tilde{C}_{1_i}^T + \mathcal{C}_i^T\tilde{D}_{12_j}^T\right]\left[Y\tilde{C}_{1_i}^T + \mathcal{C}_i^T\tilde{D}_{12_j}^T\right]^T \end{aligned} \tag{62}$$

$$\begin{aligned} \Phi_{22_{ij}} &= A_i^TX + XA_i + \mathcal{B}_iC_{2_j} + C_{2_i}^T\mathcal{B}_j^T + \tilde{C}_{1_i}^T\tilde{C}_{1_j} \\ &\quad + \gamma^{-2}\left[X\tilde{B}_{1_i} + \mathcal{B}_i\tilde{D}_{21_j}\right]\left[X\tilde{B}_{1_i} + \mathcal{B}_i\tilde{D}_{21_j}\right]^T. \end{aligned} \tag{63}$$

Note that $\Phi_{11_{ij}}$ and $\Phi_{22_{ij}}$ are the Schur complements of $\Psi_{11_{ij}}$ and $\Psi_{22_{ij}}$, Using (53)-(56), we have (61) less than zero. Hence, by Theorem 2, we learn that the inequality (3) holds. ■

4.2 Case II–$\nu(t)$ is unavailable for feedback

The output feedback fuzzy controller is assumed to be the same as the premise variables of the fuzzy system model. This actually means that the premise variables of fuzzy system model are assumed to be measurable. However, in general, it is extremely difficult to derive an accurate fuzzy system model by imposing that all premise variables are measurable. In this subsection, we do not impose that condition, we choose the premise variables of the controller to be different from the premise variables of fuzzy system model of the plant. In here, the premise variables of the controller are selected to be the estimated premise variables of the plant. In the other words, the premise variable of the fuzzy model $\nu(t)$ is unavailable for feedback which implies μ_i is unavailable for feedback. Hence, we cannot select our controller which depends on μ_i. Thus, we select our controller as follows:

$$\begin{aligned} \dot{\hat{x}}(t) &= \textstyle\sum_{i=1}^r\sum_{j=1}^r \hat{\mu}_i\hat{\mu}_j\left[\hat{A}_{ij}\hat{x}(t) + \hat{B}_iy(t)\right] \\ u(t) &= \textstyle\sum_{i=1}^r \hat{\mu}_i\hat{C}_i\hat{x}(t). \end{aligned} \tag{64}$$

where $\hat{\mu}_i$ depends on the premise variable of the controller which is different from μ_i.
Let us re-express the system (1) in terms of $\hat{\mu}_i$, thus the plant's premise variable becomes the same as the controller's premise variable. By doing so, the result given in the previous case can then be applied here. First, let us rewrite (1) as follows:

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=1}^r \mu_i\left[[A_i + \Delta A_i]x(t) + [B_{1_i} + \Delta B_{1_i}]w(t) + [B_{2_i} + \Delta B_{2_i}]u(t)\right] \\ &\quad + \textstyle\sum_{i=1}^r \hat{\mu}_i\left[[A_i + \Delta A_i]x(t) + [B_{1_i} + \Delta B_{1_i}]w(t) + [B_{2_i} + \Delta B_{2_i}]u(t)\right] \\ &\quad - \textstyle\sum_{i=1}^r \hat{\mu}_i\left[[A_i + \Delta A_i]x(t) + [B_{1_i} + \Delta B_{1_i}]w(t) + [B_{2_i} + \Delta B_{2_i}]u(t)\right] \end{aligned}$$

$$
\begin{array}{rcl}
z(t) & = & \sum_{i=1}^{r}\mu_i\Big[[C_{1_i}+\Delta C_{1_i}]x(t)+[D_{12_i}+\Delta D_{12_i}]u(t)\Big] \\
 & & +\sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{1_i}+\Delta C_{1_i}]x(t)+[D_{12_i}+\Delta D_{12_i}]u(t)\Big] \\
 & & -\sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{1_i}+\Delta C_{1_i}]x(t)+[D_{12_i}+\Delta D_{12_i}]u(t)\Big] \\
y(t) & = & \sum_{i=1}^{r}\mu_i\Big[[C_{2_i}+\Delta C_{2_i}]x(t)+[D_{21_i}+\Delta D_{21_i}]w(t)\Big] \\
 & & +\sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{2_i}+\Delta C_{2_i}]x(t)+[D_{21_i}+\Delta D_{21_i}]w(t)\Big] \\
 & & -\sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{2_i}+\Delta C_{2_i}]x(t)+[D_{21_i}+\Delta D_{21_i}]w(t)\Big].
\end{array}
\tag{65}
$$

Rearranging (65) together with employing Assumption 1, we obtain

$$
\begin{array}{l}
\dot{x}(t)=\sum_{i=1}^{r}\hat{\mu}_i\Big([A_i+F(x(t),t)H_{1_i}+(\mu_1-\hat{\mu}_1)A_1+\cdots+(\mu_r-\hat{\mu}_r)A_r \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{1_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{1_r}]x(t) \\
+[B_{1_i}+F(x(t),t)H_{2_i}+(\mu_1-\hat{\mu}_1)B_{1_1}+\cdots+(\mu_r-\hat{\mu}_r)B_{1_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{2_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{2_r}]w(t) \\
+[B_{2_i}+F(x(t),t)H_{3_i}+(\mu_1-\hat{\mu}_1)B_{2_1}+\cdots+(\mu_r-\hat{\mu}_r)B_{2_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{3_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{3_r}]u(t)\Big) \\
z(t)=\sum_{i=1}^{r}\hat{\mu}_i\times\Big([C_{1_i}+F(x(t),t)H_{4_i}+(\mu_1-\hat{\mu}_1)C_{1_1}+\cdots+(\mu_r-\hat{\mu}_r)C_{1_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{4_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{4_r}]x(t) \\
+[D_{12_i}+F(x(t),t)H_{5_i}+(\mu_1-\hat{\mu}_1)D_{12_1}+\cdots+(\mu_r-\hat{\mu}_r)D_{12_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{5_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{5_r}]u(t)\Big) \\
y(t)=\sum_{i=1}^{r}\hat{\mu}_i\Big([C_{2_i}+F(x(t),t)H_{6_i}+(\mu_1-\hat{\mu}_1)C_{2_1}+\cdots+(\mu_r-\hat{\mu}_r)C_{2_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{6_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{6_r}]x(t) \\
+[D_{21_i}+F(x(t),t)H_{7_i}+(\mu_1-\hat{\mu}_1)D_{21_1}+\cdots+(\mu_r-\hat{\mu}_r)D_{21_r} \\
+F(x(t),t)(\mu_1-\hat{\mu}_1)H_{7_1}+\cdots+F(x(t),t)(\mu_r-\hat{\mu}_r)H_{7_r}]w(t)\Big)
\end{array}
\tag{66}
$$

Then, from (66), we get

$$
\begin{array}{rcl}
\dot{x}(t) & = & \sum_{i=1}^{r}\hat{\mu}_i\Big[[A_i+\Delta\bar{A}_i]x(t)+[B_{1_i}+\Delta\bar{B}_{1_i}]w(t) \\
 & & +[B_{2_i}+\Delta\bar{B}_{2_i}]u(t)\Big],\ x(0)=0 \\
z(t) & = & \sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{1_i}+\Delta\bar{C}_{1_i}]x(t) \\
 & & +[D_{12_i}+\Delta\bar{D}_{12_i}]u(t)\Big] \\
y(t) & = & \sum_{i=1}^{r}\hat{\mu}_i\Big[[C_{2_i}+\Delta\bar{C}_{2_i}]x(t) \\
 & & +[D_{21_i}+\Delta\bar{D}_{21_i}]w(t)\Big]
\end{array}
\tag{67}
$$

where

$$
\begin{gathered}
\Delta\bar{A}_i=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{1_i}, \\
\Delta\bar{B}_{1_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{2_i},\ \Delta\bar{B}_{2_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{3_i}, \\
\Delta\bar{C}_{1_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{4_i},\ \Delta\bar{C}_{2_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{5_i}, \\
\Delta\bar{D}_{12_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{6_i} \\
\text{and }\ \Delta\bar{D}_{21_i}=\bar{F}(x(t),\hat{x}(t),t)\bar{H}_{7_i}
\end{gathered}
$$

with

$$\bar{H}_{1_i} = \left[H_{1_i}^T\, A_1^T\, \cdots A_r^T\, H_{1_1}^T \cdots H_{1_r}^T\right]^T,$$
$$\bar{H}_{2_i} = \left[H_{2_i}^T\, B_{1_1}^T \cdots B_{1_r}^T\, H_{2_1}^T \cdots H_{2_r}^T\right]^T,$$
$$\bar{H}_{3_i} = \left[H_{3_i}^T\, B_{2_1}^T \cdots B_{2_r}^T\, H_{3_1}^T \cdots H_{3_r}^T\right]^T,$$
$$\bar{H}_{4_i} = \left[H_{4_i}^T\, C_{1_1}^T \cdots C_{1_r}^T\, H_{4_1}^T \cdots H_{4_r}^T\right]^T,$$
$$\bar{H}_{5_i} = \left[H_{5_i}^T\, C_{2_1}^T \cdots C_{2_r}^T\, H_{5_1}^T \cdots H_{5_r}^T\right]^T,$$
$$\bar{H}_{6_i} = \left[H_{6_i}^T\, D_{12_1}^T \cdots D_{12_r}^T\, H_{6_1}^T \cdots H_{6_r}^T\right]^T$$
$$\bar{H}_{7_i} = \left[H_{7_i}^T\, D_{21_1}^T \cdots\; D_{21_r}^T\, H_{7_1}^T \cdots H_{7_r}^T\right]^T$$

and $\bar{F}(x(t),\hat{x}(t),t) = \Big[F(x(t),t)\ (\mu_1 - \hat{\mu}_1)\ \cdots\ (\mu_r - \hat{\mu}_r)\ F(x(t),t)(\mu_1 - \hat{\mu}_1)\ \cdots\ F(x(t),t)(\mu_r - \hat{\mu}_r)\Big]$. Note that $\|\bar{F}(x(t),\hat{x}(t),t)\| \leq \bar{\rho}$ where $\bar{\rho} = \{3\rho^2 + 2\}^{\frac{1}{2}}$. $\bar{\rho}$ is derived by utilizing the concept of vector norm in basic system control theory and the fact that $\mu_i \geq 0$, $\hat{\mu}_i \geq 0$, $\sum_{i=1}^r \mu_i = 1$ and $\sum_{i=1}^r \hat{\mu}_i = 1$.
Note that the above technique is basically employed in order to obtain the plant's premise variable to be the same as the controller's premise variable; e.g. (22). Now, the premise variable of the system is the same as the premise variable of the controller, thus we can apply the result given in Case I.

Theorem 3 *Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and a positive constant δ, if there exist matrices X, Y, $\mathcal{B}_i$ and $\mathcal{C}_i$, $i = 1, 2, \cdots, r$, satisfying the following linear matrix inequalities:*

$$\begin{aligned}
\begin{bmatrix} X & I \\ I & Y \end{bmatrix} &> 0 && (68)\\
X &> 0 && (69)\\
Y &> 0 && (70)\\
\Psi_{11_{ii}} &< 0, \quad i = 1,2,\cdots,r && (71)\\
\Psi_{22_{ii}} &< 0, \quad i = 1,2,\cdots,r && (72)\\
\Psi_{11_{ij}} + \Psi_{11_{ji}} &< 0, \quad i < j \leq r && (73)\\
\Psi_{22_{ij}} + \Psi_{22_{ji}} &< 0, \quad i < j \leq r && (74)
\end{aligned}$$

where

$$\Psi_{11_{ij}} = \begin{pmatrix} \begin{pmatrix} A_iY + YA_i^T \\ +B_{2_i}\mathcal{C}_j + \mathcal{C}_i^T B_{2_j}^T \\ +\gamma^{-2}\tilde{\bar{B}}_{1_i}\tilde{\bar{B}}_{1_j}^T \end{pmatrix} & (*)^T \\ \left[Y\tilde{\bar{C}}_{1_i}^T + \mathcal{C}_i^T \tilde{\bar{D}}_{12_j}^T\right]^T & -I \end{pmatrix} \qquad (75)$$

$$\Psi_{22_{ij}} = \left(\begin{array}{cc} \left(\begin{array}{c} A_i^T X + X A_i \\ +\mathcal{B}_i C_{2_j} + C_{2_i}^T \mathcal{B}_j^T \\ +\tilde{\bar{C}}_{1_i}^T \tilde{\bar{C}}_{1_j} \end{array} \right) & (*)^T \\ \left[X\tilde{\bar{B}}_{1_i} + \mathcal{B}_i \tilde{\bar{D}}_{21_j} \right]^T & -\gamma^2 I \end{array} \right) \tag{76}$$

with

$$\tilde{\bar{B}}_{1_i} = \left[\begin{array}{cccccc} \delta I & I & \delta I & 0 & B_{1_i} & 0 \end{array} \right],$$

$$\tilde{\bar{C}}_{1_i} = \left[\begin{array}{ccccc} \frac{\gamma\bar{\rho}}{\delta}\bar{H}_{1_i}^T & 0 & \frac{\gamma\bar{\rho}}{\delta}\bar{H}_{5_i}^T & \sqrt{2}\bar{\lambda}\bar{\rho}\bar{H}_{4_i}^T & \sqrt{2}\bar{\lambda}C_{1_i}^T \end{array} \right]^T,$$

$$\tilde{\bar{D}}_{12_i} = \left[\begin{array}{ccccc} 0 & \frac{\gamma\bar{\rho}}{\delta}\bar{H}_{3_i}^T & 0 & \sqrt{2}\bar{\lambda}\bar{\rho}\bar{H}_{6_i}^T & \sqrt{2}\bar{\lambda}D_{12_i}^T \end{array} \right]^T,$$

$$\tilde{\bar{D}}_{21_i} = \left[\begin{array}{cccccc} 0 & 0 & 0 & \delta I & D_{21_i} & I \end{array} \right]$$

$$\text{and } \bar{\lambda} = \left(1 + \bar{\rho}^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \left[\|\bar{H}_{2_i}^T \bar{H}_{2_j}\| + \|\bar{H}_{7_i}^T \bar{H}_{7_j}\| \right] \right)^{\frac{1}{2}},$$

then the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable controller is of the form (64) with

$$\begin{array}{rcl} \hat{A}_{ij} & = & \left[Y^{-1} - X \right]^{-1} \mathcal{M}_{ij} Y^{-1} \\ \hat{B}_i & = & \left[Y^{-1} - X \right]^{-1} \mathcal{B}_i \\ \hat{C}_i & = & \mathcal{C}_i Y^{-1} \end{array} \tag{77}$$

where

$$\begin{array}{rcl} \mathcal{M}_{ij} & = & -A_i^T - XA_iY - XB_{2_i}\hat{C}_jY \\ & & -\left[Y^{-1} - X\right]\hat{B}_iC_{2_j}Y - \tilde{\bar{C}}_{1_i}^T\left[\tilde{\bar{C}}_{1_j}Y + \tilde{\bar{D}}_{12_j}\hat{C}_jY\right] \\ & & -\gamma^{-2}\left\{X\tilde{\bar{B}}_{1_i} + \left[Y^{-1} - X\right]\hat{B}_i\tilde{\bar{D}}_{21_i}\right\}\tilde{\bar{B}}_{1_j}^T. \end{array} \tag{78}$$

Proof: Since (67) is of the form of (1), it can be shown by employing the proof for Theorem 2. ∎

5. Example

Consider the following problem of the chaotic Lorenz system which is described by the following equations (see [29]).

$$\begin{array}{rcl} \dot{x}_1(t) & = & -\sigma x_1(t) + \sigma x_2(t) + u(t) + 0.1w_1(t) \\ \dot{x}_2(t) & = & rx_1(t) - x_2(t) - x_1(t)x_3(t) + 0.1w_2(t) \\ \dot{x}_3(t) & = & x_1(t)x_2(t) - bx_3(t) + 0.1w_3(t) \\ z(t) & = & \left[\begin{array}{ccc} x_1^T(t) & x_2^T(t) & x_3^T(t) \end{array} \right]^T \\ y(t) & = & Jx(t) + 0.1w_1(t) \end{array} \tag{79}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the state vectors, $u(t)$ is the control input, $w_1(t)$, $w_2(t)$, $w_3(t)$ are the disturbance noise inputs, $y(t)$ is the measurement output, $z(t)$ is the controlled output, J is the sensor matrix and the bounded uncertain parameters σ, r and b are given by $10 \pm 30\%$, $28 \pm 30\%$ and $8/3 \pm 30\%$, respectively. Note that the variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ are treated as the deviation variables (variables deviate from the desired trajectories).

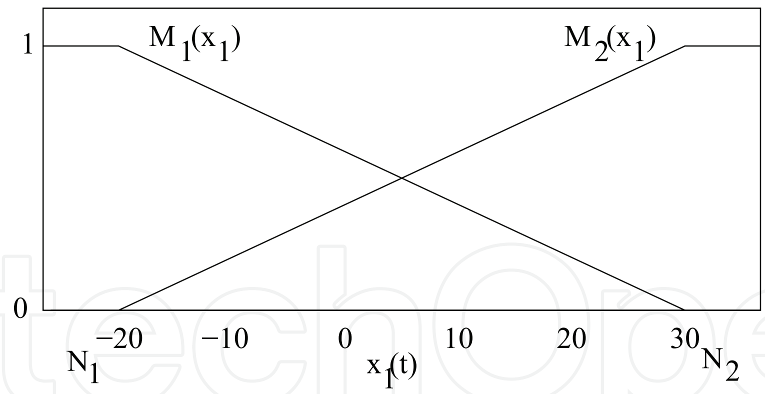

Fig. 1. Membership functions for the two fuzzy set.

Since the nonlinear terms in (79) can be viewed as a function of $x_1(t)$, we can re-expressed (79) as

$$\begin{array}{rcl} \dot{x}_1(t) & = & -\sigma x_1(t) + \sigma x_2(t) + u(t) + 0.1w_1(t) \\ \dot{x}_2(t) & = & rx_1(t) - x_2(t) - (x_1(t)) \cdot x_3(t) + 0.1w_2(t) \\ \dot{x}_3(t) & = & (x_1(t)) \cdot x_2(t) - bx_3(t) + 0.1w_3(t) \\ z(t) & = & \begin{bmatrix} x_1^T(t) & x_2^T(t) & x_3^T(t) \end{bmatrix}^T \\ y(t) & = & Jx(t) + 0.1w_1(t). \end{array} \tag{80}$$

The control objective is to control the state variable $x_1(t)$ for the range $x_1(t) \in [N_1\ N_2]$. For the sake of simplicity, we will use as few rules as possible. Note that Figure 1 shows the plot of the membership functions represented by

$$M_1(x_1(t)) = \frac{-x_1(t) + N_2}{N_2 - N_1} \text{ and } M_2(x_1(t)) = \frac{x_1(t) - N_1}{N_2 - N_1}.$$

Knowing that $x_1(t) \in [N_1\ N_2]$, the nonlinear system (80) can be approximated by the following two rules TS model:

Plant Rule 1: IF $x_1(t)$ is $M_1(x_1(t))$ THEN

$$\begin{array}{rcl} \dot{x}(t) & = & [A_1 + \Delta A_1]x(t) + B_{1_1}w(t) + B_{2_1}u(t),\ x(0) = 0, \\ z(t) & = & C_{1_1}x(t), \\ y(t) & = & C_{2_1}x(t) + D_{21_1}w(t). \end{array}$$

Plant Rule 2: IF $x_1(t)$ is $M_2(x_1(t))$ THEN

$$\begin{array}{rcl} \dot{x}(t) & = & [A_2 + \Delta A_2]x(t) + B_{1_2}w(t) + B_{2_2}u(t),\ x(0) = 0, \\ z(t) & = & C_{1_2}x(t), \\ y(t) & = & C_{2_2}x(t) + D_{21_2}w(t) \end{array}$$

where

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -N_1 \\ 0 & N_1 & -8/3 \end{bmatrix},\ A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -N_2 \\ 0 & N_2 & -8/3 \end{bmatrix},$$

$$B_{1_1} = B_{1_2} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \; B_{2_1} = B_{2_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{1_1} = C_{1_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C_{2_1} = C_{2_2} = J,$$

$$D_{21_1} = D_{21_2} = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}, \; \Delta A_1 = F(x(t),t)H_{1_1}, \; \Delta A_2 = F(x(t),t)H_{1_2},$$

$$x(t) = [x_1^T(t)\; x_2^T(t)\; x_3^T(t)]^T \text{ and } w(t) = [w_1^T(t)\; w_2^T(t)\; w_3^T(t)]^T.$$

Let us choose the value of $[N_1 \;\; N_2]$ in the membership function as $[-20 \; 30]$. Now, by assuming that in (2), $\|F(x(t),t)\| \leq \rho = 1$ and since the values of σ, r, b are uncertain but bounded within 30% of their nominal values given in (79), we have

$$H_{1_1} = H_{1_2} = \begin{bmatrix} -0.3\sigma & 0.3\sigma & 0 \\ 0.3r & 0 & 0 \\ 0 & 0 & -0.3b \end{bmatrix}.$$

State-feedback controller design

Using the LMI optimization algorithm and Theorem 1 with $\gamma = 1$ and $\delta = 1$, we obtain

$$P = \begin{bmatrix} 104.7498 & -8.1629 & -1.1823 \\ -8.1629 & 5.1783 & 0.9345 \\ -1.1823 & 0.9345 & 6.7383 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -38.8875 & -816.1115 & -3.9273 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -37.4290 & -815.5695 & 4.1287 \end{bmatrix}.$$

The resulting fuzzy controller is

$$u(t) = \sum_{j=1}^{2} \mu_j K_j x(t)$$

where

$$\mu_1 = M_1(x_1(t)) \text{ and } \mu_2 = M_2(x_1(t)).$$

Output feedback controller design

Case I: $v(t)$ are available for feedback

In this case, $x_1(t) = v(t)$ is assumed to be available for feedback; for instance, $J = [1 \; 0 \; 0]$. This implies that μ_i is available for feedback. Using the LMI optimization algorithm and Theorem 2 with $\gamma = 1$ and $\delta = 1$, we obtain the following results:

$$X = \begin{bmatrix} 40.9617 & -0.3001 & 0.0003 \\ -0.3001 & 0.0326 & -0.0020 \\ 0.0003 & -0.0020 & 0.0529 \end{bmatrix}, \qquad Y = \begin{bmatrix} 64.0418 & -6.6279 & -0.0180 \\ -6.6279 & 0.7784 & 0.0345 \\ -0.0180 & 0.0345 & 0.8385 \end{bmatrix},$$

$$\hat{A}_{11} = \begin{bmatrix} -52.6459 & 913.0329 & 11.1683 \\ 0.4211 & -93.8119 & -1.1292 \\ 2.3239 & -0.4233 & 0.0865 \end{bmatrix}, \; \hat{A}_{12} = \begin{bmatrix} -52.9740 & 909.6351 & 0.8313 \\ 0.5070 & -93.0535 & -0.2157 \\ 2.3414 & -0.2540 & 0.1024 \end{bmatrix},$$

$$\hat{A}_{21} = \begin{bmatrix} -54.8390 & 912.4579 & -6.7553 \\ 1.4467 & -93.6196 & 0.6829 \\ -3.5367 & -0.1599 & 0.2080 \end{bmatrix}, \hat{A}_{22} = \begin{bmatrix} -54.7676 & 913.4610 & -17.1638 \\ 1.3897 & -94.0748 & 1.5985 \\ -3.5229 & -0.0374 & 0.1865 \end{bmatrix},$$

$$\hat{B}_1 = \begin{bmatrix} -110.4306 \\ 4.8589 \\ 2.9909 \end{bmatrix}, \qquad \hat{B}_2 = \begin{bmatrix} 113.2188 \\ 6.1387 \\ -4.5464 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} -36.1488 & -710.9845 & -3.2817 \end{bmatrix}, \quad \hat{C}_2 = \begin{bmatrix} -35.9847 & -709.7215 & 5.1803 \end{bmatrix}.$$

The resulting fuzzy controller is

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{2}\sum_{j=1}^{2} \mu_i \mu_j \hat{A}_{ij}\hat{x}(t) + \sum_{i=1}^{2} \mu_i \hat{B}_i y(t) \\ u(t) &= \sum_{i=1}^{2} \mu_i \hat{C}_i \hat{x}(t) \end{aligned}$$

where

$$\mu_1 = M_1(x_1(t)) \text{ and } \mu_2 = M_2(x_1(t)).$$

Case II: $\nu(t)$ are unavailable for feedback

In this case, $x_1(t) = \nu(t)$ is assumed to be unavailable for feedback; for instance, $J = [0\ \ 0\ \ 1]$. This implies that μ_i is unavailable for feedback. Using the LMI optimization algorithm and Theorem 3 with $\gamma = 1$ and $\delta = 1$, we obtain the following results:

$$X = \begin{bmatrix} 15.3866 & -0.0454 & 0.0001 \\ -0.0454 & 0.0086 & -0.0005 \\ 0.0001 & -0.0005 & 0.0121 \end{bmatrix}, \qquad Y = \begin{bmatrix} 195.0825 & -19.8577 & -0.0836 \\ -19.8577 & 2.3203 & 0.1018 \\ -0.0836 & 0.1018 & 2.5038 \end{bmatrix},$$

$$\hat{A}_{11} = \begin{bmatrix} -72.5111 & 1594.5334 & 6.34563 \\ 5.0232 & -162.6656 & -0.6001 \\ 1.2000 & -0.7556 & 0.1000 \end{bmatrix}, \hat{A}_{12} = \begin{bmatrix} -72.9233 & 1603.7455 & -9.7233 \\ 5.1345 & -162.8555 & 0.9974 \\ 1.2000 & -0.5689 & 0.1000 \end{bmatrix},$$

$$\hat{A}_{21} = \begin{bmatrix} -74.5456 & 1595.2543 & -5.6743 \\ 5.5411 & -162.1785 & 0.5609 \\ -1.7009 & -0.9421 & 0.2000 \end{bmatrix}, \hat{A}_{22} = \begin{bmatrix} -74.5290 & 1595.2231 & -5.6744 \\ 5.5411 & -162.1323 & 0.5966 \\ -1.7008 & -0.9432 & 0.2000 \end{bmatrix},$$

$$\hat{B}_1 = \begin{bmatrix} -166.7783 \\ 7.4682 \\ 4.5048 \end{bmatrix}, \qquad \hat{B}_2 = \begin{bmatrix} -173.8473 \\ 9.1193 \\ -6.8346 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} 14.1938 & -410.5257 & -0.3593 \end{bmatrix}, \quad \hat{C}_2 = \begin{bmatrix} 14.2366 & -412.9750 & 3.8984 \end{bmatrix}.$$

The resulting fuzzy controller is

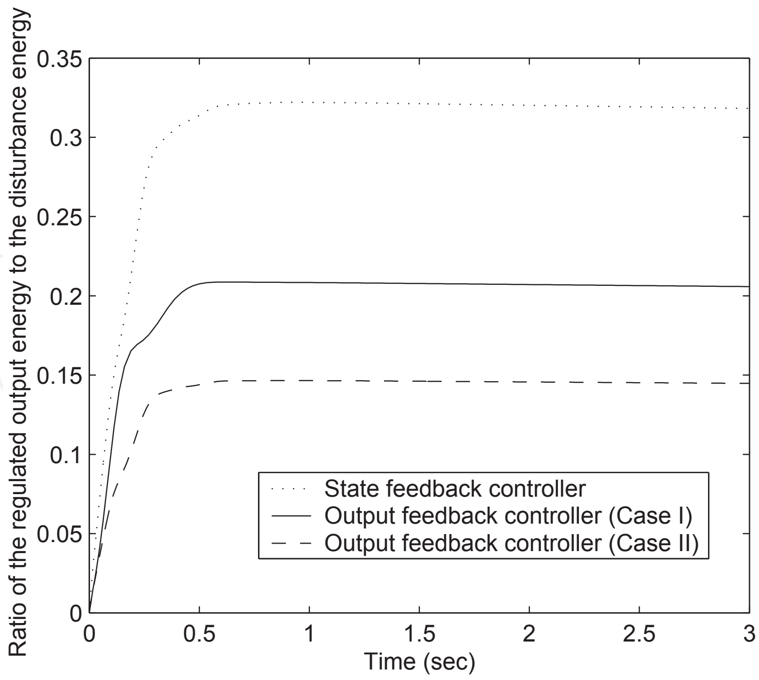

Fig. 2. The ratio of the regulated output energy to the disturbance noise energy: $\left(\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}\right)$.

$$\begin{aligned}\dot{\hat{x}}(t) &= \sum_{i=1}^{2}\sum_{j=1}^{2}\hat{\mu}_i\hat{\mu}_j\hat{A}_{ij}\hat{x}(t) + \sum_{i=1}^{2}\hat{\mu}_i\hat{B}_i y(t) \\ u(t) &= \sum_{i=1}^{2}\hat{\mu}_i\hat{C}_i\hat{x}(t)\end{aligned}$$

where

$$\hat{\mu}_1 = M_1(\hat{x}_1(t)) \text{ and } \hat{\mu}_2 = M_2(\hat{x}_1(t)).$$

Remark 1 *Both robust fuzzy state and output controllers guarantee that the $\mathcal{L}_2$-gain, γ, is less than the prescribed value. The ratio of the regulated output energy to the disturbance input noise energy which is obtained by using the $\mathcal{H}_\infty$ fuzzy controllers is depicted in Figure 2. The disturbance input signals, $w_1(t)$, $w_2(t)$ and $w_3(t)$, which were used during the simulation is given in Figure 3. After 3 seconds, the ratio of the regulated output energy to the disturbance input noise energy tends to a constant value which is about* 0.32 *for the state-feedback controller, and* 0.21 *for the output feedback controller in Case I and* 0.14 *in Case II. Thus, for the state-feedback controller where $\gamma = \sqrt{0.32} = 0.566$, for output feedback controller in Case I where $\gamma = \sqrt{0.21} = 0.458$ and in Case II where $\gamma = \sqrt{0.14} = 0.374$, all are less than the prescribed value* 1. □

6. Conclusion

This chapter has investigated the problem of designing a robust fuzzy controller for a TS fuzzy system with parametric uncertainties that guarantees the $\mathcal{L}_2$-gain from an exogenous input to a regulated output being less than or equal to the prescribed value. An LMI-based approach has been employed to derive sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ fuzzy controller in terms of a family of LMIs. Finally, a numerical simulation example has been presented to illustrate the effectiveness of the designs.

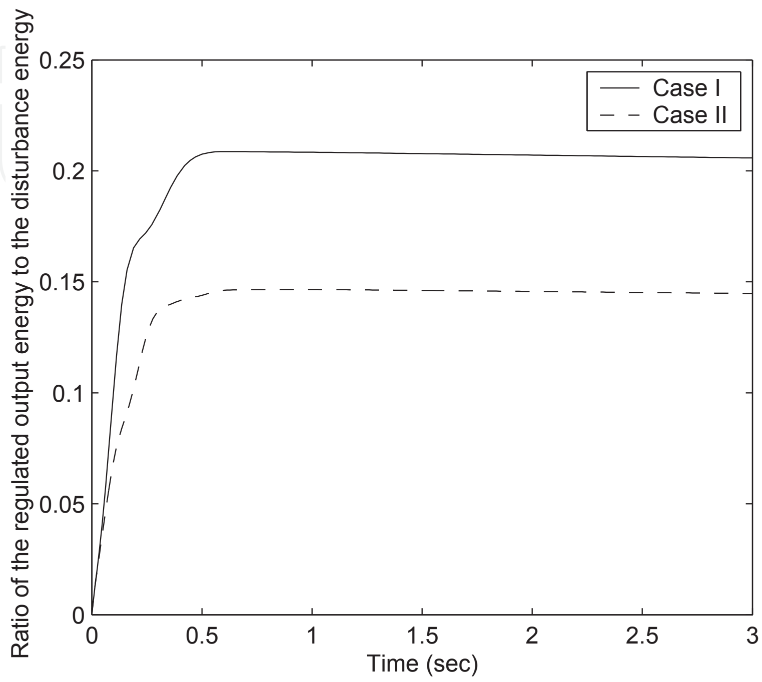

Fig. 3. The disturbance input signals, $w_1(t)$, $w_2(t)$ and $w_3(t)$.

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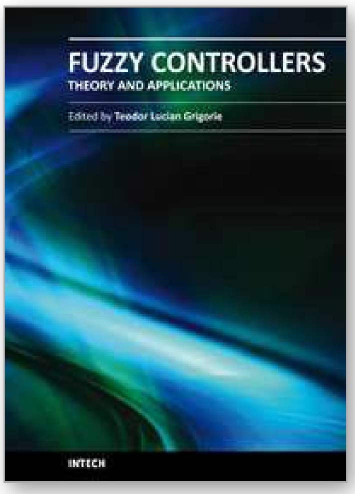

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Trying to meet the requirements in the field, present book treats different fuzzy control architectures both in terms of the theoretical design and in terms of comparative validation studies in various applications, numerically simulated or experimentally developed. Through the subject matter and through the inter and multidisciplinary content, this book is addressed mainly to the researchers, doctoral students and students interested in developing new applications of intelligent control, but also to the people who want to become familiar with the control concepts based on fuzzy techniques. Bibliographic resources used to perform the work includes books and articles of present interest in the field, published in prestigious journals and publishing houses, and websites dedicated to various applications of fuzzy control. Its structure and the presented studies include the book in the category of those who make a direct connection between theoretical developments and practical applications, thereby constituting a real support for the specialists in artificial intelligence, modelling and control fields.

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