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# Linear Analog Circuits Problems by Means of Interval Analysis Techniques

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## 1. Introduction

Inevitable fluctuations in the manufacturing processes and environmental operating conditions of linear analog circuits cause circuit parameters to vary about their nominal target values. The mathematical model of an engineering system evaluated by a transfer function (e.g. of an active and even passive circuit) never describes exactly the system's behavior. The changes in the performance of linear circuit due to the variations in circuit parameters are of great practical importance in engineering analysis and design. The tolerance problem for linear analog circuit have been extensively studied and many results have been published, e.g. (Antreich et al., 1994; Spence & Soin, 1997). Because of uncertainties, the values of the parameters of a given circuit may be treated as belonging to some intervals. In recent years, interval analysis becomes powerful tool for tolerance computations of some design problems (Kolev et al., 1988; Femia & Spagnuolo, 1999). Some results have been reported using algorithms for linear interval equations for solving tolerance problems (Tian et al., 1996; Garczarczyk, 1999; Shi et al., 1999; Tian & Shi, 2000).

The structure of the chapter is the following: section 2 explains an interval analysis techniques for linear analog tolerance problem. In that approach we are interested in calculation tolerances (the range of values) for real and imaginary part of transfer function with respect to change of one parameter of the circuit. Section 3 deals with the problem of computing the frequency response of an uncertain transfer function whose numerator and denominator are interval polynomials. Studying a solution set of corresponding  $2 \times 2$  linear interval equation one can obtain bounds on the frequency response. Using Kharitonov polynomials family and complex interval division it's also possible to evaluate the bounds. In this section we compare results obtained by applying presented approaches. Numerical studies are also reported in order to illustrate presented methods.

## 2. Evaluation of linear circuits tolerances

The objective of this section is to develop the interval analysis techniques for linear analog circuit tolerance problem. In that approach we can compute effectively tolerances for real and imaginary parts of the transfer function with respect to change of one parameter of a circuit.

## 2.1 Bilinear and biquadratic form of a circuit function

The functional dependence of circuit performance on the designable parameters is known implicitly through the circuit transfer function. If the dependence on the  $R$ ,  $L$ ,  $C$  elements and on the controlled sources is investigated, the transfer function is a quotient of two linear polynomials, i.e., a bilinear relation, is arrived at. We have the following well-known result:

$$F(s, x) = \frac{L(s, x)}{M(s, x)} = \frac{A(s) + xB(s)}{C(s) + xD(s)} \quad (1)$$

In the above equation the symbol  $x$  denotes dependence on the network element parameter ( $R$  or  $L$  or  $C$  or gain of the controlled source).  $A(s)$ ,  $B(s)$ ,  $C(s)$  and  $D(s)$  are functions of the complex frequency  $s$ . They depend on kind of transfer function and on the structure of a circuit examined. A similar biquadratic relation was derived for the dependence on the ideal transformer ratio  $n$ , on the ideal gyrator resistance  $r$  and on the conversion factor  $k$  of the ideal negative impedance converter (Geher, 1971). The transfer function has the following form:

$$F(s, x) = \frac{L(s, x)}{M(s, x)} = \frac{A(s) + xB(s) + x^2C(s)}{D(s) + xE(s) + x^2G(s)} \quad (2)$$

$A(s)$ ,  $B(s)$ , etc. are depending on the type of the transfer function and the topology of the circuit. For some fixed frequency transfer function can be represented by its real and imaginary part, i.e.

$$F(x) = F(j\omega, x) = \frac{L_1(\omega, x)}{M_1(\omega, x)} + j \frac{L_2(\omega, x)}{M_1(\omega, x)} \quad (3)$$

Here  $L_1(\omega, x)$ ,  $L_2(\omega, x)$ ,  $M_1(\omega, x)$  denote polynomials in  $x$  of second order and fourth order (maximally) for bilinear and biquadratic transfer functions, respectively. We are interested in calculation tolerance (the range of values) for real and imaginary part of the transfer function caused by some parameter  $x$  ranging in known interval, i.e.  $x \in \mathbf{x} = [\underline{x}, \bar{x}]$ .

This one-parameter tolerance problem can be solved by means of the well-known circle diagram method for bilinear transfer function, unfortunately biquadratic transfer function is more difficult problem. Here we propose a unified approach to tolerance problem for bilinear and biquadratic transfer function based on the range evaluation of a rational function by means of interval analysis techniques.

## 2.2 Range values of a rational function

Let  $L(x)$  be a polynomial of degree  $n$  and  $M(x)$  a polynomial of degree  $m$  so that  $f(x) = L(x)/M(x)$  is a rational function. We want to expand  $f(x)$  into its Taylor series

$$f(x) = \sum_{i=0}^k c_i (x - x_0)^i \quad (4)$$

For computing the first  $k$  Taylor coefficients of  $f(x)$  at some point  $x_0$  where  $M(x_0) \neq 0$ , we start by developing the polynomial  $L(x)$  into its Taylor series about the point  $x_0$

$$L(x) = \sum_{i=0}^n a_i (x - x_0)^i \tag{5}$$

Similarly, let

$$M(x) = \sum_{i=0}^m b_i (x - x_0)^i \tag{6}$$

Note that  $\max(m,n) = 2$  or  $4$ .

Coefficients  $a_i$  and  $b_i$  are obtained directly as

$$a_i = L^{(i)}(x_0)/i! , \quad b_i = M^{(i)}(x_0)/i! , \tag{7}$$

$i = 1, 2, \dots, m(n)$

More effectively we can compute them by using the extended Horner scheme (Elden & Wittmeyer-Koch, 1990).

It was derived in (Garczarczyk, 1995) that one can compute the values of the first  $k$  Taylor coefficients of a rational function by solving a  $(k + 1) \times (k + 1)$  lower triangular Toeplitz system of the form:

$$\begin{bmatrix} b_0 & & & & & & 0 \\ b_1 & b_0 & & & & & \\ b_2 & b_1 & b_0 & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ b_k & & & b_2 & b_1 & b_0 & \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{bmatrix} \tag{8}$$

Note that for the case  $k > m(n)$ , the lower triangular Toeplitz system is lower banded. To compute the values of the Taylor coefficients of a rational function the main work is to solve the lower triangular Toeplitz system (8). Special structure of Toeplitz systems leads to the variety of solving algorithms, so they belong to more elaborated linear systems. Because system (8) is lower triangular for a small  $k$ , we can use the usual forward substitution method for its solving. For large  $k$  more efficient method is a variant of Trench algorithm for Toeplitz band matrices (Trench, 1985). Inversion of a nonsingular Toeplitz matrix of the form

$$T = \begin{bmatrix} b_0 & & & & & & 0 \\ b_1 & b_0 & & & & & \\ b_2 & b_1 & b_0 & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ b_k & & & b_2 & b_1 & b_0 & \end{bmatrix} \tag{9}$$

band or not may be computed following:

Let (without loss of generality)  $b_0 = 1$ , then

$$T^{-1} = [h_{rs}]_{r,s=0}^k \quad (10)$$

is the matrix given by

$$h_{rs} = -\psi_{r-s-1}, \quad r = 0, 1, \dots, k, \quad s = 0, 1, \dots, r \quad (11)$$

with  $\psi_j = 0$  if  $j < -1$ ,  $\psi_{-1} = -1$ , and

$$\psi_j = b_{j+1} - \sum_{s=0}^{j-1} b_{j-s} \psi_s, \quad 0 \leq j \leq k-1. \quad (12)$$

Note that matrix  $T^{-1}$  is also lower triangle Toeplitz matrix and is uniquely determined by its first column  $(h_{00}, \dots, h_{k0})^t = (-\psi_{-1}, -\psi_0, \dots, -\psi_{k-1})^t$ . The solution

$$[c_0, c_1, \dots, c_k]^t = T^{-1} [a_0, a_1, \dots, a_k]^t \quad (13)$$

of (8) can be calculated by using the fast Fourier transform.

For any function  $f(x)$  which has an interval arithmetic evaluation the range of values of  $f$  over the interval  $\mathbf{x}$

$$R(f, \mathbf{x}) := \{f(x) \mid x \in \mathbf{x}\} \quad (14)$$

is contained in the interval arithmetic evaluation  $f(\mathbf{x})$ , i.e.

$$R(f, \mathbf{x}) \subseteq f(\mathbf{x}) \quad (15)$$

Additionally, it is strongly dependent on the arithmetic expression which is used for the interval evaluation of the function (Neumeier, 1990; Moore et al., 2009).

Exact Taylor expansion for a rational function  $f(x)$  is following

$$f(x) = p(x) + r(x) \quad (16)$$

where

$$p(x) = \sum_{i=0}^k \alpha_i x^i, \quad \text{with } \alpha_i = \sum_{r=i}^k c_r \binom{r}{i} (-x_0)^{r-i} \quad (17)$$

and

$$r(x) = f^{(k+1)}(x_0 + \xi(x - x_0))(x - x_0)^{k+1} / (k+1)! \quad (18)$$

$$\xi \in [0, 1], \quad x_0 \in \mathbf{x} \text{ (e.g. } x_0 = m(\mathbf{x}) \text{)}.$$

If  $f(x) : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $k+1$  times continuously differentiable, then for all  $\mathbf{x} \subseteq D$  it's fulfilled (Garczarczyk, 1993):

(inclusion)

$$R(f, \mathbf{x}) \subseteq V(f, \mathbf{x}) := R(p, \mathbf{x}) + f^{(k+1)}(\mathbf{x}) w(\mathbf{x})^{k+1} / (k+1)! \quad (19)$$

(distance)

$$q(R(f, \mathbf{x}), V(f, \mathbf{x})) \leq \gamma w(\mathbf{x})^{k+1}, \quad \gamma \geq 0, \quad (20)$$

where  $R(p, \mathbf{x})$  is the exact range of the polynomial  $p(\mathbf{x})$  over  $\mathbf{x}$ , and  $q(R, V) = \max(|\underline{R} - \underline{V}|, |\overline{R} - \overline{V}|)$  means distance between intervals  $R = [\underline{R}, \overline{R}]$  and  $V = [\underline{V}, \overline{V}]$ . Relation (19) gives the way of range values evaluation: we need to calculate the range of polynomial and the range of remainder term. It's seen from (20) that the overestimation of  $R(f, \mathbf{x})$  by  $V(f, \mathbf{x})$  decreases with a power  $k + 1$  of  $w(\mathbf{x})$  (width of  $\mathbf{x}$ ), so if  $f^{(k+1)}(\mathbf{x})$  is bounded we can omit the remainder term in  $V(f, \mathbf{x})$  and then

$$R(f, \mathbf{x}) \approx R(p, \mathbf{x}) \quad (21)$$

### 2.3 Bernstein polynomials

Estimates for the maximum, resp. the minimum, of the polynomial over  $\mathbf{x}$  are obtained by computing Bernstein coefficients.

For some order  $v$  of Bernstein polynomial we have (Ratschek & Rokne, 1984)

$$\min B_j \leq \min p(\mathbf{x}) \leq \max p(\mathbf{x}) \leq \max B_j, \quad 0 \leq j \leq v, \mathbf{x} \in \mathbf{x}, \quad (22)$$

where  $v \geq k$  and

$$B_j = \sum_{s=0}^j \sum_{t=s}^k \binom{t}{s} \alpha_t \underline{x}^{t-s} w(\mathbf{x})^s \frac{\binom{j}{s}}{\binom{v}{s}}, \quad j = 0, 1, \dots, v \quad (23)$$

The coefficients  $B_j$  are computed using a following finite difference table

$$\begin{array}{cccccccccc} B_0 & B_1 & B_2 & \cdots & \cdots & \cdots & B_{v-2} & B_{v-1} & B_v \\ & \Delta B_0 & \cdots & \cdots & \cdots & \cdots & \cdots & \Delta B_{v-1} & \\ & & \Delta^2 B_0 & & \cdots & & \Delta^2 B_{v-2} & & \\ & & & \ddots & & \ddots & & & \\ & & & & \Delta^v B_0 & & & & \end{array} \quad (24)$$

The initial slanted entries are generated basing on coefficients of polynomial  $p(\mathbf{x})$  following

$$\Delta^r B_0 = A_r \sum_{l=r}^k \binom{l}{r} \alpha_l \underline{x}^{l-r}, \quad (25)$$

$$\Delta^r B_{v-r} = A_r \sum_{l=r}^k \binom{l}{r} \alpha_l \overline{x}^{l-r} \quad (26)$$

where  $A_r = w(\mathbf{x})^r \binom{v}{r}^{-1}$ ,  $r = 0, 1, \dots, v$ ,  $\mathbf{x} = [\underline{x}, \overline{x}]$ .

The top row of table contains the desired Bernstein coefficients. Finite differences are computed following

$$\Delta^r B_j = \Delta^{r-1} B_{j+1} - \Delta^{r-1} B_j, \quad r > 0, j = 0, 1, \dots, v. \quad (27)$$

For example

$$\Delta B_0 = B_1 - B_0 \Rightarrow B_1 = \Delta B_0 + B_0 \quad (28)$$

and

$$\Delta B_{v-1} = B_v - B_{v-1} \Rightarrow B_{v-1} = B_v - \Delta B_{v-1} . \quad (29)$$

Relations (24) – (29) lead directly to the following scheme of computing of Bernstein coefficients

$$\begin{array}{ccccccc}
 B_0 & \rightarrow (+) & B_1 & \rightarrow (+) & B_2 & \cdots \cdots & B_{v-2} & (+) \leftarrow & B_{v-1} & (+) \leftarrow & B_v \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & (+) & & (+) & & (-) & & (-) & & \\
 \Delta B_0 & \rightarrow (+) & \Delta B_1 & \cdots \cdots & \Delta B_{v-2} & (+) \leftarrow & \Delta B_{v-1} & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & (+) & & (-) & & & & & & \\
 \Delta^2 B_0 & \cdots \cdots & \Delta^2 B_{v-2} & & & & & & & & \\
 & & \vdots & & \vdots & & & & & & 
 \end{array} \quad (30)$$

It's seen we can develop the algorithm of a parallel computation of Bernstein coefficients starting from slanted entries. We note that since  $\alpha_l = 0$  for  $l > k$  there is no need to compute entries  $\Delta^r B_j$  for  $r > k$ ; a triangle table turns into trapezium one. In the trapezium table a bottom row has all entries equal, i.e.

$$\Delta^s B_0 = \Delta^s B_1 = \cdots = \Delta^s B_{v-s}, v > s . \quad (31)$$

Realisation of scheme (30) leads to the three cases of parallel computation slightly different according to the value of  $v$  (Garczarczyk, 2002).

## 2.3 Numerical examples

To illustrate the basic ideas of our approach two examples are considered. The first example refers to the bilinear transfer function and the second to the biquadratic one. Taylor coefficients  $a_i$  and  $b_i$   $i = 0, 1, \dots, k$ ,  $k = 2$  or  $4$ , were computed by means of extended Horner scheme. For example, polynomial  $L(x)$  was developed by the algorithm written in Pascal-like code as:

```

for i = 0, 1, ..., n
    ai = coefficient(L(x));
for k = 0, 1, ..., n
    for i = n-1, n-2, ..., k
        ai = ai+1x0 + ai;

```

In both examples Toeplitz system (8) is banded and was solved using algorithm based on Trench's concept (10) – (13).

EXAMPLE 1. Consider a second-order low-pass filter section of Fig.1, originally proposed by Sallen and Key.

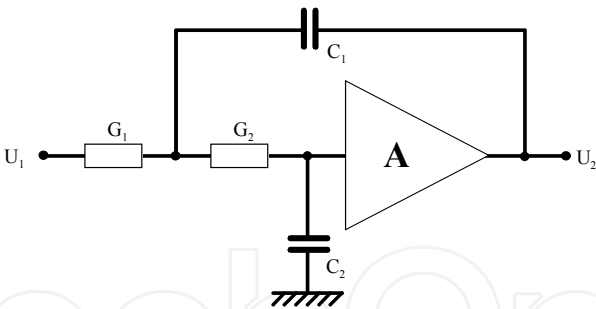


Fig. 1. Second-order low-pass filter section

Bilinear transfer function considered here is following

$$F(s,x) = \frac{U_2}{U_1} = \frac{x \frac{G_1 G_2}{C_1 C_2}}{s^2 + (\frac{G_2}{C_2} (1-x) + \frac{G_1 + G_2}{C_1})s + \frac{G_1 G_2}{C_1 C_2}}$$

where  $x = A$ .

Assuming  $G_1 = G_2 = 1$  and  $C_1 = C_2 = 1$  for fixed frequency we obtain

$$F(x) = F(j\omega, x) = \frac{(1 - \omega^2)x}{M(\omega, x)} + j \frac{\omega x - 3\omega}{M(\omega, x)}$$

where  $M(\omega, x) = 1 + 7\omega^2 + \omega^4 - 6\omega^2 x + \omega^2 x^2$ .

We have applied relation (21) for Taylor expansion of degree  $k = 5$  and Bernstein coefficients of degree  $v = 10$  were used. For  $x \in \mathbf{x} = A_0[1-\varepsilon, 1+\varepsilon]$  with  $A_0 = 1$ ,  $\varepsilon = 0.01$  we obtained results presented in the Table 1. In the second column there are values of the ranges for real and imaginary part of the transfer function, the third column contains their nominal values.

$\omega$	$\mathbf{x} \in \mathbf{x}$	$X = A_0$
0.2	$[0.878692, 0.899103] + j[-0.372772, -0.367971]$	$0.888889 - j0.370370$
2.0	$[-0.127097, -0.122930] + j[-0.168624, -0.164735]$	$-0.125005 - j0.166667$
20.0	$[-0.024009, -0.023489] + j[-0.002395, -0.002367]$	$-0.023749 - j0.002381$

Table 1. Range values of transfer function of Sallen-Key low-pass section

EXAMPLE 2. Consider the gyrator circuit with feedback shown in Fig.2.

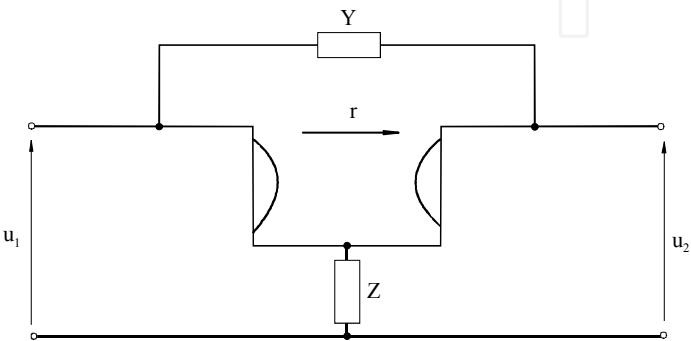


Fig. 2. Gyrator circuit



Let  $Y = sC_1$  and  $Z = 1/sC_2$ . Biquadratic transfer function is of the form

$$F(s,x) = \frac{U_2}{U_1} = 1 + \frac{xsC_2}{x^2s^2C_1C_2 + 1}$$

where  $x = r$  is the gyration resistance. This circuit appropriately loaded can realize a transfer function of phase equalizer.  
For fixed frequency we have

$$F(x) = F(j\omega,x) = \frac{U_2}{U_1} = 1 - j \frac{\omega C_2 x}{M(\omega,x)}$$

where  $M(\omega,x) = \omega^2 C_1 C_2 x^2 - 1$ .  
It was assumed for simplicity  $C_1 = C_2 = 1$ . For  $x \in \mathbf{x} = r_0[1-\varepsilon, 1+\varepsilon]$  with  $r_0 = 2$ ,  $\varepsilon = 0.05$  we have obtained following results

$\omega$	$x \in \mathbf{x}$	$x = r_0$
0.1	$1 + j[0.197086, 0.219623]$	$1 + j0.208333$
1.0	$1 - j[0.615803, 0.727913]$	$1 - j0.666667$
10.0	$1 - j[0.047712, 0.052745]$	$1 - j0.050125$

Table 2. Range values of transfer function for circuit with gyrator  
Degrees of Taylor and Bernstein coefficients were analogous to previous example.

3. Frequency response envelopes of interval systems

The computation of the frequency responses of uncertain transfer functions plays a major role in the application of frequency domain methods for the analysis and design of robust systems. There is a rich resource of prior works on this subject, e.g. (Bartlett et al., 1993; Chen & Hwang, 1998a, 1998b; Tan & Atherton, 2000; Hwang & Yang, 2002; Tan, 2002; Nataraj & Barve, 2003).  
In this section we consider continuous-time systems characterized by rational transfer functions. Motivated by the above we incorporate uncertainties into the transfer function. We assume that the system’s performance is governed by the interval transfer function

$$K(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1s + \cdots + a_ms^m}{b_0 + b_1s + \cdots b_ns^n} \tag{32}$$

where coefficients of numerator and denominator are not known exactly, but are given in prescribed real intervals

$$\begin{aligned} \underline{a}_i &\leq a_i \leq \overline{a}_i, \quad i = 0, \cdots, m \\ \underline{b}_j &\leq b_j \leq \overline{b}_j, \quad j = 0, \cdots, n. \end{aligned} \tag{33}$$

A problem of major importance and significance is to be able to determine the envelopes of the amplitude and phase of  $K(j\omega)$  of the above family of transfer functions. Phase and

amplitude bounds have a simple geometric interpretation: they represent envelopes of the Nyquist plot.

The objective of this section is to develop the interval analysis techniques to the problem presented above. Focusing on this specific class of uncertain systems we compare two approaches to computation of Nyquist plot collections.

### 3.1 Linear interval equations approach

In this section we collect some known results on the linear interval equations and their use to the problem explained in the previous section. This approach was explicitly presented in (Garczarczyk, 1999).

Let  $G(s)$  be the inverse of interval transfer function  $K(s)$ . Introducing input signal  $x(j\omega)$  and output signal  $y(j\omega)$  the input-output relationship for linear continuous-time system, can be written as

$$x_1(\omega) + jx_2(\omega) = (\operatorname{Re}\{G(j\omega, p)\} + j\operatorname{Im}\{G(j\omega, p)\})(y_1(\omega) + jy_2(\omega)) \quad (34)$$

where

$$x_1(\omega) = \operatorname{Re}\{x(j\omega)\}, x_2(\omega) = \operatorname{Im}\{x(j\omega)\}, \\ \text{and } y_1(\omega) = \operatorname{Re}\{y(j\omega)\}, y_2(\omega) = \operatorname{Im}\{y(j\omega)\}.$$

Assuming  $x_1(\omega)=1$ ,  $x_2(\omega)=0$  (sinusoidal input  $x(t) = \cos(\omega t)$  is applied) we can rewrite eq.(34) as the system of two linear equations

$$\begin{bmatrix} \operatorname{Re}\{G(j\omega)\} & -\operatorname{Im}\{G(j\omega)\} \\ \operatorname{Im}\{G(j\omega)\} & \operatorname{Re}\{G(j\omega)\} \end{bmatrix} \begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (35)$$

For a fixed frequency, we obtain following equation

$$\begin{bmatrix} [a, b] & -[c, d] \\ [c, d] & [a, b] \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (36)$$

Here the ranges of values of  $\operatorname{Re}\{G(j\omega)\}$  and  $\operatorname{Im}\{G(j\omega)\}$  are represented by intervals  $[a, b]$  and  $[c, d]$ , respectively.

Equation (36) forms a system of linear interval equations. It can be denoted as

$$\mathbf{A}\mathbf{y} = \mathbf{b} \quad (37)$$

Such a system represents a family of ordinary linear systems which can be obtained from it by fixing coefficients values in the prescribed intervals. Every of these systems, under the assumption that each  $A \in \mathbf{A}$  is nonsingular, has a unique solution, and all these solutions constitute a so-called solution set  $S$ .

The solution set of eq. (37) can be expressed as

$$S = \{y : Ay = b, A \in \mathbf{A}, b \in \mathbf{b}\} \quad (38)$$

It forms some two-dimensional region of output values of a system in the sinusoidal steady-state.

If interval matrix  $\mathbf{A}$  is regular i.e. if  $\det \mathbf{A} \neq 0$  for each  $\mathbf{A} \in \mathbf{A}$ , the solution set of a linear interval equation is described by Oettli and Prager in their famous equivalence (Oettli & Prager, 1964; Neumeier, 1990)

$$y \in S \Leftrightarrow |Ay - b| \leq \Delta|y| + \delta \quad (39)$$

where  $\mathbf{A} = m(\mathbf{A})$ ,  $\mathbf{b} = m(\mathbf{b})$  and  $\Delta = w(\mathbf{A}/2)$ ,  $\delta = w(\mathbf{b})/2$ .

Applying Oettli-Prager formula to the equation (36) we obtain following inequality

$$\begin{bmatrix} m_1 & -m_2 \\ m_2 & m_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leq \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (40)$$

where  $m_1 = (a+b)/2$ ,  $m_2 = (c+d)/2$

and  $\rho_1 = (b-a)/2$ ,  $\rho_2 = (d-c)/2$ .

Computation of the regions of values of  $y_1$  and  $y_2$  for which inequality (40) is true gives us the full information about changes of frequency response caused by variations some of system parameters. To obtain this information we solve inequality (40) for whole complex plane. In Fig. 3 region of solutions (region of uncertainty) in the fourth quadrant is represented by the tetragon ABCD. The straight lines  $l_1$  and  $l_2$  are here defined following

$$l_1: y_2 = -\frac{c}{b}y_1, \quad l_2: y_2 = -\frac{d}{a}y_1, \quad (41)$$

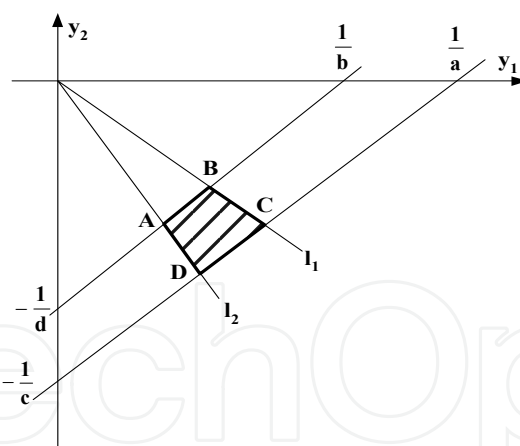


Fig. 3. Region of uncertainty in the fourth quadrant

Calculation coordinates of the points of intersections in each quadrant leads to the bounds of a frequency response.

At the border of two quadrants structure for the solution set is quite different. In Fig.4 is shown a region at the border of III and IV quadrants, i.e. if  $m_1=0$  ( $a=-b$ ) and  $m_2>0$ .

The straight lines  $l_1$  and  $l_2$  are following

$$l_1: y_2 = -\frac{c}{b}y_1, \quad l_2: y_2 = \frac{c}{b}y_1 \quad (42)$$

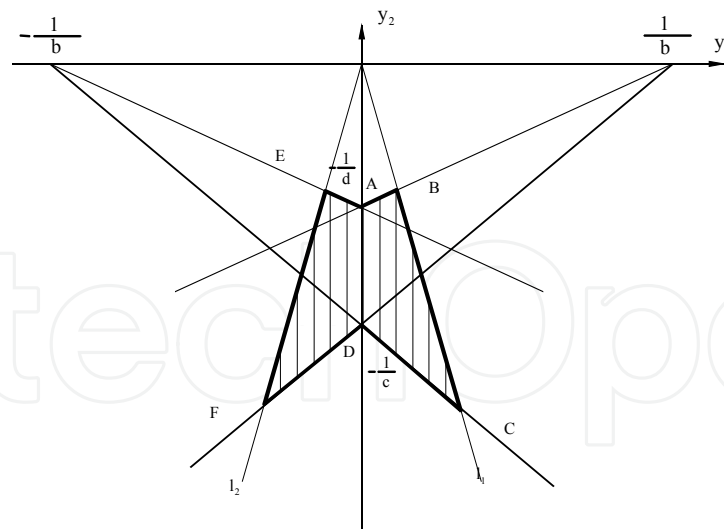


Fig. 4. Structure of the solution set at the border of two quadrants

### 3.2 Kharitonov polynomials method

Problem of evaluating the frequency response envelopes can be treated as the task of finding the maximum and minimum of  $|P(j\omega)|$  and of  $\text{Arg}[P(j\omega)]$  of a family of polynomials

$$P(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \dots + \alpha_k s^k \quad (43)$$

$$\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i, i = 0, \dots, k$$

The value set of a polynomial with uncertain coefficients at a frequency  $\omega$  denote the region in the complex plane occupied by all the values of the polynomial over all allowable coefficients values.

From (43) we have

$$P(j\omega) = \text{Re}\{P(j\omega)\} + j\text{Im}\{P(j\omega)\} \quad (44)$$

Formula (44) defines for every  $\omega \in \mathbb{R}$ , a linear transformation from the  $(k+1)$ -dimensional real coefficient set to the complex plane. Assuming that the intervals of the coefficients are independent, the  $(k+1)$ -dimensional interval vector (box) is mapped into a complex rectangular interval (rectangle with edges parallel to the axes of the complex plane).

It has been observed in (Dasgupta, 1988) that the corners of that rectangular interval clearly correspond to the four Kharitonov polynomials (Kharitonov, 1979)

$$\begin{aligned} P_1(j\omega) &= \bar{\alpha}_0 + \bar{\alpha}_1 s + \underline{\alpha}_2 s^2 + \underline{\alpha}_3 s^3 + \dots \Big| s = j\omega \\ P_2(j\omega) &= \underline{\alpha}_0 + \underline{\alpha}_1 s + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s^3 + \dots \Big| s = j\omega \\ P_3(j\omega) &= \underline{\alpha}_0 + \bar{\alpha}_1 s + \bar{\alpha}_2 s^2 + \underline{\alpha}_3 s^3 + \dots \Big| s = j\omega \\ P_4(j\omega) &= \bar{\alpha}_0 + \underline{\alpha}_1 s + \underline{\alpha}_2 s^2 + \bar{\alpha}_3 s^3 + \dots \Big| s = j\omega \end{aligned} \quad (45)$$

From (45) it's seen that the value sets of  $N(s)$  and  $D(s)$  are the members of the set of complex rectangular intervals (is denoted here by  $R(C)$ ).

They have the form

$$N(j\omega) = N = N_1 + jN_2 = [\underline{n}_1, \bar{n}_1] + j[\underline{n}_2, \bar{n}_2], \quad (46)$$

and

$$D(j\omega) = D = D_1 + jD_2 = [\underline{d}_1, \bar{d}_1] + j[\underline{d}_2, \bar{d}_2] \quad (47)$$

To calculate value set of interval transfer function we need to divide those two complex intervals. Complex interval operations should deliver the closest inclusion of the set of all possible values, i.e.

$$\{a : b \mid a \in N, b \in D\} \subseteq N : D \quad (48)$$

For rectangular complex arithmetic addition, subtraction and multiplication are optimal, whereas division is not. We apply here an improved version of division (in the sense of inclusion), namely (Rokne & Lancaster, 1971; Petkovic & Petkovic, 1998)

$$N : D = N \cdot \frac{1}{D} \quad (49)$$

where

$$\frac{1}{D} = \inf \left\{ X \in R(C) \mid \left\{ \frac{1}{b} \mid b \in D \right\} \subseteq X \right\}. \quad (50)$$

Relation (50) is illustrated in Fig. 5. for the interval D from the first quadrant.

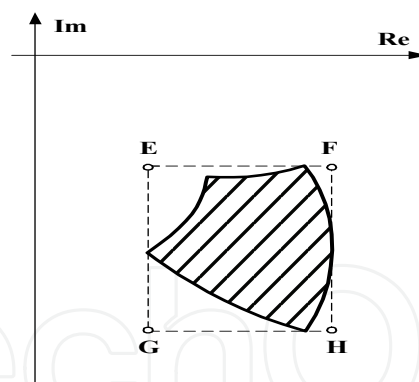


Fig. 5. Optimal rectangular enclosure

Optimal enclosure has the form of rectangle EFGH. Curvilinear hatched region which was generated by conformal mapping corresponds to the exact range of  $D^{-1}$ . The shape of the exact region and adequate enclosure depend on the position of interval D on the complex plane.

### 3.3 Numerical studies

To compare properties of presented approaches two examples are considered. The first example refers to the transfer function of the form (32), the second one to the case represented in the relation (50).

EXAMPLE 3. Let us consider T-bridged circuit depicted in Fig. 3. The frequency response is represented by the transmittance (Chen, 2009)

$$K(s)=\frac{U_2}{U_1}=\frac{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2)s+1}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2+R_2C_1)s+1}$$

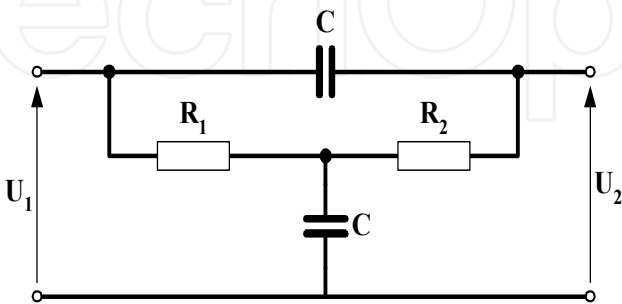


Fig. 6. Bridget-T circuit

Let assume  $R_1C_1 = R_2C_2 = RC = [1-\varepsilon, 1+\varepsilon]$ ,  $\varepsilon = 0.05$ .  
Then the interval transmittance is done as

$$K(s)=\frac{U_2}{U_1}=\frac{[0.9025,1.1025]s^2+[1.9,2.1]s+1}{[0.9025,1.1025]s^2+[2.85,3.15]s+1}.$$

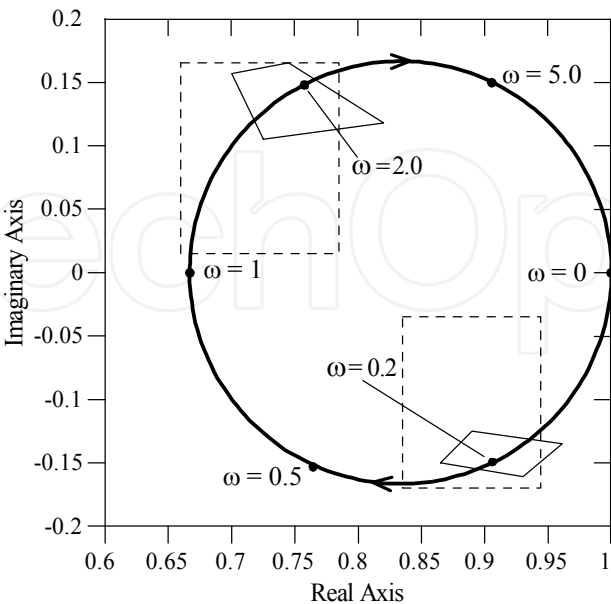


Fig. 7. Regions of uncertainty against a background of Nyquist plot

The ranges of values of  $\text{Re}\{G(j\omega)\}$  and  $\text{Im}\{G(j\omega)\}$  are computed with use of Taylor and Bernstein representations.

$$\text{Re}\{G(j\omega)\} \in 1 + \frac{2x}{(1+x^2)^2}, \quad \text{for } x = [0.95\omega, 1.05\omega]$$

$$\text{Im}\{G(j\omega)\} \in \frac{(1-x^2)x}{(1+x^2)^2}, \quad \text{for } x = [0.95\omega, 1.05\omega]$$

In Fig. 7 are presented the Nyquist plot for nominal value  $RC = 1$  and the regions ABCD (tetragon) and EFGH (rectangle) for two frequencies  $\omega = 0.2$  and  $\omega = 2.0$ . It gives us the possibility to evaluate the envelope of Nyquist plot for these frequencies. It's seen that Kharitonov polynomials approach (rectangle) gives some overestimation compared with linear interval equations method.

EXAMPLE 4. Consider a second-order low-pass Sallen - Key section of Fig.1

Let denote  $R_1 = 1/G_1$  and  $R_2 = 1/G_2$ .

We have now a transmittance of the form

$$K(s) = \frac{U_2}{U_1} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 + R_2) C_2 s + 1}.$$

Assuming  $R_1 C_1 = R_2 C_2 / 2 = RC = [1-\varepsilon, 1+\varepsilon]$ ,  $\varepsilon = 0.1$ , we have

$$K(s) = \frac{U_2}{U_1} = \frac{1}{[1.62, 2.42]s^2 + [2.7, 3.3]s + 1}$$

$$\text{Re}\{G(j\omega)\} \in 1 - x, \quad \text{for } x = [1.62\omega^2, 2.42\omega^2]$$

$$\text{Im}\{G(j\omega)\} = x, \quad \text{for } x = [2.7\omega, 3.3\omega]$$

In Fig. 8a and 8b are drawn fragments of Nyquist plot for nominal value  $RC = 1.0$  and appropriate regions for  $\omega = 0.2$  and  $\omega = 1.0$ .

Although uncertainties in the Example 4 are greater then in previous one both methods produce smaller regions. There are two reasons of such results: Firstly, the different coefficients of the transfer function are sometimes dependent; secondly, improved division defined by (49) is not optimal whereas relation (50) leads to the optimal enclosure.

## 4. Conclusions

An efficient and well motivated approach to the problem linear analog circuit tolerance was described. One-parameter tolerance problem was solved for bilinear and biquadratic transfer function. This unified method was based on the range evaluation of a quotient of two polynomials of second or fourth order. It was done by computing coefficients of Bernstein polynomials generated for some Taylor expansion (form) of a rational function. The Taylor forms together with Bernstein expansions constitute a significant enhancement of the toolkit of interval analysis, see also (Neumaier, 2002).

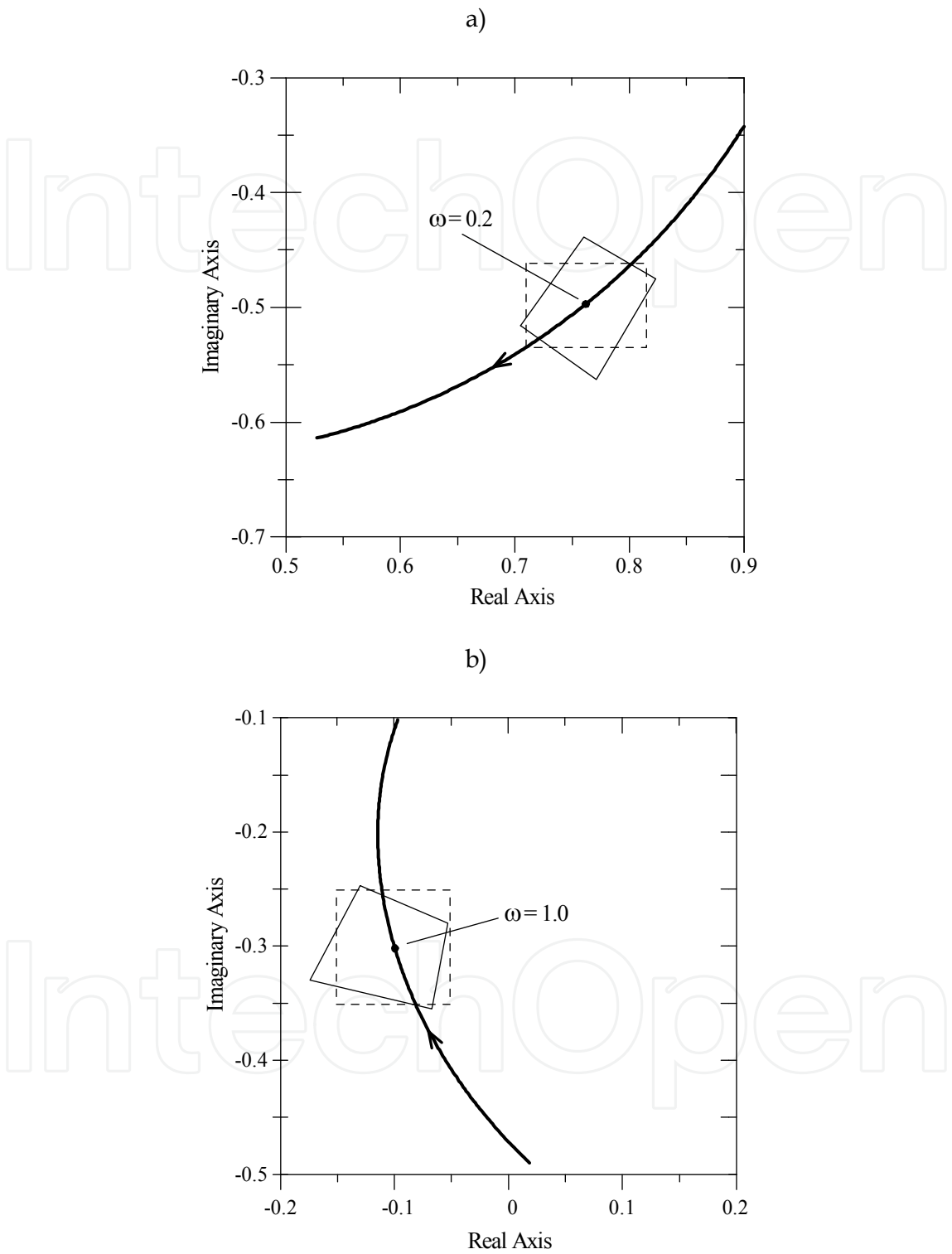


Fig. 8. Regions of uncertainty and Nyquist plot



The results presented in this chapter make it possible, by simple algorithms, to obtain the Nyquist envelope (consequently the amplitude envelope and the phase envelope) of an interval rational transfer function of a continuous-time system. It gives possibility to readily check whether system with such uncertainty comply with frequency response specifications. The results of the numerical calculations are quite satisfactory. It indicates that the interval analysis seems to be a promising tool for robust analysis of linear systems. Numerical studies show that it's necessary next step to "more" optimal complex interval division (Lohner & Wolff von Gudenberg, 1985; Moore et al., 2009).

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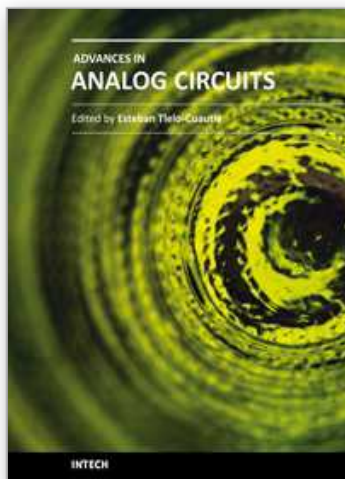
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