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Introduction to Stability of Quasipolynomials

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1. Introduction

In this Chapter we shall consider a generalization of Hermite-Biehler Theorem[1] given by Pontryagin in the paper Pontryagin (1955). It should be understood that Pontryagin's generalization is a very relevant formal *tool* for the mathematical analysis of *stability* of quasipolynomials. Thus, from this point of view, the determination of the zeros of a quasipolynomial by means of Pontryagin's Theorem can be considered to be a mathematical method for analysis of stabilization of a class of linear time invariant systems with time delay. Section 2 contains an overview of the representation of entire functions as an infinite product by way of Weierstrass' Theorem—as well as Hadamard's Theorem. Section 3 is devoted to an exposition to the Theory of Quasipolynomials via Pontryagin's Theorem in addition to a generalization of Hermite-Biehler Theorem. Section 4 deals with applications of Pontryagin's Theorem to analysis of stabilization for a class of linear time invariant systems with time delays.

2. Representation of the entire functions by means of infinite products

In this Section we will present the mathematical background with respect to the Theory of Complex Analysis and to provide the necessary tools for studying the Hermite-Biehler and Pontryagin's Theorems. At the first let us introduce the basic definitions and general results used in the representation of the entire functions as infinite products[2].

2.1 Preliminaries

Definition 1. *(Zeros of analytic functions) Let $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function in a region Ω—i.e., a nonempty open connected subset of the complex plane. A value $\alpha \in \Omega$ is called a zero of f with multiplicity (or order) $m \geq 1$ if, and only if, there is an analytic function $g : \Omega \longrightarrow \mathbb{C}$ such that $f(z) = (z - \alpha)^m g(z)$, where $g(\alpha) \neq 0$. A zero of order one $(m = 1)$ is called a simple zero.*

Definition 2. *(Isolated singularity) Let $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function in a region Ω. A value $\beta \in \Omega$ is called a isolated singularity of f if, and only if, there exists $R > 0$ such that f is analytic in $\{z \in \mathbb{C} : 0 < |z - \beta| < R\}$ but not in $B(\beta, R) = \{z \in \mathbb{C} : |z - \beta| < R\}$.*

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[1] See Levin (1964) for an analytical treatment about the Hermite-Biehler Theorem and a generalization of this theorem to arbitrary entire functions in an alternative way of the Pontryagin's method.

[2] See Ahlfors (1953) and Titchmarsh (1939) for a detailed exposition.

Definition 3. *(Pole) Let Ω be a region. A value $\beta \in \Omega$ is called a pole of analytic function $f : \Omega \longrightarrow \mathbb{C}$ if, and only if, β is a isolated singularity of f and $\lim_{z \longrightarrow \beta} |f(z)| = \infty$.*

Definition 4. *(Pole of order m) Let $\beta \in \Omega$ be a pole of analytic function $f : \Omega \longrightarrow \mathbb{C}$. We say that β is a pole of order $m \geq 1$ of f if, and only if, $f(z) = \dfrac{A_1}{z - \beta} + \dfrac{A_2}{(z - \beta)^2} + \ldots + \dfrac{A_m}{(z - \beta)^m} + g_1(z)$, where g_1 is analytic in $B(\beta, R)$ and $A_1, A_2, \ldots, A_m \in \mathbb{C}$ with $A_m \neq 0$.*

Definition 5. *(Uniform convergence of infinite products) The infinite product*

$$\prod_{n=1}^{+\infty}(1 + f_n(z)) = (1 + f_1(z))(1 + f_2(z)) \ldots (1 + f_n(z)) \ldots \tag{1}$$

where $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of functions of one variable, real or complex, is said to be uniformly convergent if the sequence of partial product ρ_n defined by

$$\rho_n(z) = \prod_{m=1}^{n}(1 + f_m(z)) = (1 + f_1(z))(1 + f_2(z)) \ldots (1 + f_n(z)) \tag{2}$$

converges uniformly in a certain region of values of z to a limit which is never zero.

Theorem 1. *The infinite product (1) is uniformly convergent in any region where the series $\sum_{n=1}^{+\infty} |f_n(z)|$ is uniformly convergent.*

Definition 6. *(Entire function) A function which is analytic in whole complex plane is said to be entire function.*

2.2 Factorization of the entire functions

In this subsection, it will be discussed an important problem in theory of entire functions, namely, the problem of the *decomposition* of an entire function—under the form of an *infinite* product of its zeros—in pursuit of the mathematical basis in order to explain the distribution of the zeros of quasipolynomials.

2.2.1 The problem of factorization of an entire function

Let $P(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial of degree n, $(a_n \neq 0)$. It follows of the Fundamental Theorem of Algebra that $P(z)$ can be decomposed as a *finite* product of the following form: $P(z) = a_n(z - \alpha_1) \ldots (z - \alpha_n)$, where the $\alpha_1, \alpha_2, \ldots, \alpha_n$ are—not necessarily distinct—zeros of $P(z)$. If exactly k_j of the α_j coincide, then the α_j is called a zero of $P(z)$ of *order* k_j [see Definition (1)]. Furthermore, the factorization is *uniquely* determined except for the order of the factors. Remark that we can also find an equivalent form of a polynomial function with a *finite* product of its zeros, more precisely, $P(z) = Cz^m \prod_{j=1}^{N}(1 - \dfrac{z}{\alpha_j})$, where

$C = a_n \prod_{j=1}^{N}(-\alpha_j)$, m is the multiplicity of the zero at the origin, $\alpha_j \neq 0 (j = 1, \ldots, N)$ and $m + N = n$.

We can generalize the problem of factorization of the polynomial function for any entire function expressed likewise as an *infinite* product of its zeros.

Let's supposed that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - \frac{z}{\alpha_n})$$

(3)

where $g(z)$ is an entire function. Hence, the problem can be established in following way: *the representation (3) should be valid if the infinite product converges uniformly on every compact set* [see Definition (5)].

2.2.2 Weierstrass factorization theorem

The problem characterized above was completely resolved by Weierstrass in 1876. As matter of fact, we have the following definitions and theorems.

Definition 7. *(Elementary factors) We can to take*

$$E_0(z) \quad = \quad 1 - z, \; and$$

(4)

$$E_p(z) \quad = \quad (1 - z) \, \exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}), \; for \; all \;\; p = 1, 2, 3, \ldots$$

(5)

These functions are called elementary factors.

Lemma 1. *If $|z| \leq 1$, then $|1 - E_p(z)| \leq |z|^{p+1}$, for $p = 1, 2, 3, \ldots$.*

Theorem 2. *Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers such that $\alpha_n \neq 0$ and $\lim\limits_{n \longrightarrow +\infty} |\alpha_n| = \infty$. If $\{p_n\}_{n \in \mathbb{N}}$ is a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} (\frac{r}{r_n})^{1+p_n} < \infty, \; where \; |\alpha_n| = r_n,$$

(6)

for every positive r, then the infinite product

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{\alpha_n})$$

(7)

define an entire function f which has a zero at each point α_n, $n \in \mathbb{N}$, and has no other zeros in the complex plane.

Remark 1. *The condition (6) is always satisfied if $p_n = n - 1$. Indeed, for every r, it follows that $r_n > 2r$ for all $n > n_0$, since $\lim\limits_{n \longrightarrow +\infty} r_n = \infty$. Therefore, $\dfrac{r}{r_n} < \dfrac{1}{2}$ for all $n > n_0$, then (6) is valid with respect to $1 + p_n = n$.*

Theorem 3. *(Weierstrass Factorization Theorem) Let f be an entire function. Suppose that $f(0) \neq 0$, and let α_1, α_2, ... be the zeros of f, listed according to their multiplicities. Then there exist an entire function g and a sequence $\{p_n\}_{n \in \mathbb{N}}$ of nonnegative integers, such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{\alpha_n}) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\left[\frac{z}{\alpha_n} + \frac{1}{2}(\frac{z}{\alpha_n})^2 + \ldots + \frac{1}{n-1}(\frac{z}{\alpha_n})^{n-1}\right]}$$

(8)

Notice that, by convention, with respect to $n = 1$ the first factor of the infinite product should be
$1 - \dfrac{1}{\alpha_1}$.

Remark 2. *If f has a zero of multiplicity m at $z = 0$, the Theorem (3) can be apply to the function*
$\dfrac{f(z)}{z^m}$.

Remark 3. *The decomposition (8) is not unique.*

Remark 4. *In the Theorem (3), if the sequence $\{p_n\}_{n \in \mathbb{N}}$ of nonnegative integers is constant, i.e., $p_n = \rho$ for all $n \in \mathbb{N}$, then the following infinite product:*

$$e^{g(z)} \prod_{n=1}^{\infty} E_\rho \left(\frac{z}{\alpha_n} \right) \tag{9}$$

*converges and represents an entire function provided that the series $\dfrac{1}{\rho + 1} \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{R}{|\alpha_n|} \right)^{\rho+1}$ converges for all $R > 0$. Suppose that ρ is the smallest integer for which the series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{|\alpha_n|^{\rho+1}}$ converges. In this case, the expression (9) is denominated the **canonical product** associated with the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ and ρ is the **genus** of the canonical product* [3].

With reference to the Remark (4) we can state:

Hadamard Factorization Theorem. *If f is an entire function of finite order ϑ, then it admits a factorization of the following manner: $f(z) = z^m e^{g(z)} \displaystyle\prod_{n=1}^{\infty} E_p \left(\dfrac{z}{\alpha_n} \right)$, where $g(z)$ is a polynomial function of degree q, and $\max\{p, q\} \leq \vartheta$.*

The first example of infinite product representation was given by Euler in 1748, viz., $\sin(\pi z) = \pi z \displaystyle\prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2} \right)$, where $m = 1$, $p = 1$, $q = 0$ $[g(z) \equiv 0]$, and $\vartheta = 1$.

3. Zeros of quasipolynomials due to Pontryagin's theorem

We know that, under the analytic standpoint and a geometric criterion, results concerning the existence and localization of zeros of entire functions like exponential polynomials have received a considerable attention in the area of research in the automation field. In this section the Pontryagin theory is outlined.

Consider the linear difference-differential equation of differential order n and difference order m defined by

$$\sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} x^{(\mu)}(t + \nu) = 0 \tag{10}$$

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[3] See Boas (1954) for analysis of the problem about the connection between the growth of an entire function and the distribution of its zeros.

where m and n are positive integers and $a_{\mu\nu}(\mu = 0,\ldots, n, \nu = 0,\ldots, m)$ are real numbers. The characteristic function associated to (10) is given by:

$$\delta(z) = P(z, e^z), \tag{11}$$

where $P(z,s) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} z^\mu s^\nu$ is a polynomial in two variables.

Pontryagin's Theorem, in fact, establishes necessary and sufficient conditions such that the real part of all zeros in (11) may be negative. These conditions transform the problem a real variable one.

Definition 8. *(Quasipolynomials)*[4] *We call the quasipolynomials or exponential polynomials the entire functions of the form:*

$$F(z) = \sum_{\xi=0}^{m} f_\xi(z) e^{\lambda_\xi z} = f_0(z) e^{\lambda_0 z} + f_1(z) e^{\lambda_1 z} + \ldots + f_m(z) e^{\lambda_m z} \tag{12}$$

where $f_\xi(\xi = 0,\ldots, m)$ are polynomial functions with real (or complex) coefficients, and $\lambda_\xi(\xi = 0,\ldots, m)$ are real (or complex) numbers. In particular, if the $\lambda_\xi(\xi = 0,\ldots, m)$ are commensurable real numbers and $0 = \lambda_0 < \lambda_1,\ldots < \lambda_m$, then the quasipolynomial (12) can be written in the form (11) studied by Pontryagin.

Notice that, some trigonometric functions, e.g., sin and cos are quasipolynomials since $\sin(mz) = \frac{1}{2j} e^{jmz} - \frac{1}{2j} e^{-jmz}$ and $\cos(nz) = \frac{1}{2} e^{jnz} + \frac{1}{2} e^{-jnz}$, where $j = \sqrt{-1}$, and $m, n \in \mathbb{N}$.

Remark 5. *If the quasipolynomial $F(z)$ in (12) does not degenerate into a polynomial, then the quasipolynomial $F(z)$ has an **infinite** set of zeros whose unique limit point is infinite. Note that all roots of $F(z)$ are separated from one another by more than some distance $d > 0$, therefore it is possible to determine non-intersecting circles of radius $r < d$ encircling all the roots taken as centers.*

Definition 9. *(Hurwitz Stable) The quasipolynomial $F(z)$ in (12) is said to be a Hurwitz stable if, and only if, all its roots lie in the open left-half of the complex plane.*

Definition 10. *(Interlacing Property) Let $f(\omega)$ and $g(\omega)$ be two real functions of a real variable. The zeros of these two functions interlace (or alternate) if, and only if, we have the following conditions:*

1. *each of the functions has only simple zeros [see Definition 1];*

2. *between every two zeros of one of these functions there exists one and only one zero of the other;*

3. *the functions $f(\omega)$ and $g(\omega)$ have no common zeros.*

We cannot refrain from remark that Cebotarev, in 1942, gave the generalization of the Sturm algorithm to quasipolynomials, therefore we have a general principle for solving that problem for arbitrary quasipolynomials. Notwithstanding, it is of interest to note that Chebotarev's result presuppose a generalization of the Hermite-Biehler Theorem to quasipolynomials.

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[4] See Pontryagin (1969) for a discussion detailed.

Theorem 4. *(Pontryagin's Theorem) Pontryagin (1955) Let $\delta(z) = P(z, e^z)$ be a quasipolynomial, where $P(z, s)$ is a polynomial function in two variables with real coefficients as defined in (11). Suppose that $a_{nm} \neq 0$. Let $\delta(j\omega)$ be the restriction of the quasipolynomial $\delta(z)$ to imaginary axis. We can express $\delta(j\omega) = f(\omega) + jg(\omega)$, where the real functions (of a real variable) $f(\omega)$ and $g(\omega)$ are the real and imaginary parts of $\delta(j\omega)$, respectively. Let us denote by ω_r and ω_i, respectively, the zeros of the function $f(\omega)$ and $g(\omega)$. If all the zeros of the quasipolynomial $\delta(z)$ lie to the left side of the imaginary axis, then the zeros of the functions $f(\omega)$ and $g(\omega)$ are real, alternating, and*

$$g'(\omega)f(\omega) - g(\omega)f'(\omega) > 0. \tag{13}$$

for each $\omega \in \mathbb{R}$. Reciprocally, if one of the following conditions is satisfied:

1. *All the zeros of the functions $f(\omega)$ and $g(\omega)$ are real and alternate and the inequality (13) is satisfied for at least one value ω;*

2. *All the zeros of the function $f(\omega)$ are real , and for each zero of $f(\omega)$ the inequality (13) is satisfied, that is, $g(\omega_r)f'(\omega_r) < 0$;*

3. *All the zeros of the function $g(\omega)$ are real, and for each zero of $g(\omega)$ the inequality (13) is satisfied, that is, $g'(\omega_i)f(\omega_i) > 0$;*

then all the zeros of the quasipolynomial $\delta(z)$ lie to the left side of the imaginary axis.

Remark 6. *Let us note that the above function $\delta(j\omega)$ in Theorem (4) has, also, the following form:*

$$\delta(j\omega) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} \omega^{\mu} \left[\sum_{\rho=0}^{\nu} (j)^{\mu+\nu-\rho} \frac{\nu!}{\rho!(\nu-\rho)!} (\cos\omega)^{\rho} (\sin\omega)^{\nu-\rho} \right]. \tag{14}$$

Consequently, the functions $f(\omega)$ and $g(\omega)$ can be express as $Q(\omega) = q(\omega, \cos(\omega), \sin(\omega))$, where $q(\omega, u, v)$ is a real polynomial function in three variables with real coefficients.

With respect to the Remark (6), it should be pointed out, the polynomial $q(\omega, u, v)$ may be represented in the form:

$$q(\omega, u, v) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} \omega^{\mu} \phi_{\mu}^{(\nu)}(u, v), \tag{15}$$

where $\phi_{\mu}^{(\nu)}(u, v)$ is a real homogeneous polynomial of degree ν in two real variables u and v. The formula $\omega^n \phi_n^{(m)}(u, v)$ is denominated the *principal term* of the polynomial in (15). From (15), we can define $\phi_n^*(u, v)$ as follows

$$\phi_n^*(u, v) = \sum_{\nu=0}^{m} \phi_n^{(\nu)}(u, v). \tag{16}$$

And by substituting $u = \cos(\omega)$ and $v = \sin(\omega)$ in (16) we can express

$$\Phi_n^*(\omega) = \phi_n^*(\cos(\omega), \sin(\omega)). \tag{17}$$

Now, let us consider the above formalization in *complex* field, that is, $\Phi_n^*(z) = \phi_n^*(\cos(z), \sin(z))$, where $z \in \mathbb{C}$.

Theorem 5. *Pontryagin (1955) Let $q(z, u, v)$ be a polynomial function, as given in (15), with three complex variables and real coefficients, in which the principal term is $z^n \phi_n^{(m)}(u, v)$. If ϵ is such that $\Phi_n^*(\epsilon + j\varrho)$ does not take the value zero for every real ϱ, then the function $Q(\omega + j\varrho)$ has exactly $4kn + m$ zeros—for some sufficiently large value of k— for $(\omega, \varrho) \in [-2k\pi + \epsilon, 2k\pi + \epsilon] \times \mathbb{R}$. Hence, in order that the restriction of the function Q to $\mathbb{R}$, denoted by $Q(\omega)$, have only real roots, it is necessary and sufficient that $Q(\omega)$ have exactly $4kn + m$ zeros in the interval $-2k\pi + \epsilon \leq \omega \leq 2k\pi + \epsilon$ for sufficiently large k.*

4. Applications of Pontryagin's theorem to analysis of stabilization for a class of linear time invariant systems with time delay

In this Section we will explain some relevant applications concerning the Hermite-Biehler Theorem and Pontryagin's Theorems in the framework of *Control Theory*. Apropos to the several methodological approaches about the subject of the **Section 3**, we have in technical literature some significant publications, viz., Bellman & Cooke (1963), Bhattacharyya et al. (2009) and Oliveira et al. (2009). These methods constitute a set of analytic tools for mathematical modeling and general criteria for studying of *stability* of the dynamic systems with time delays, that is, for setting a characterization of all stabilizing P, PI or PID controllers for a given plant. It should be pointed out that the definition of the formal concept of *signature*—introduced in the reference Oliveira et al. (2003)—allows to extend results of the polynomial case to quasipolynomial case via property of interlacing in high frequencies of the class of time delay systems considered [5].

The dynamic behavior of many industrial plants may be mathematically modeled by a linear time invariant system with time delay. The problem of stability of linear time invariant systems with time delay make necessary a method for localization of the roots of analytic functions. These systems are described by the linear differential equations with constant coefficients and constant delays of the argument of the following manner

$$\sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} u^{(\mu)}(t - \tau_\nu) = h(t) \tag{18}$$

where the coefficients are denoted by $a_{\mu\nu} \in \mathbb{R} (\mu = 0, \ldots, n, \nu = 0, \ldots, m)$ and the constant delays are symbolized by $\tau_\nu \in \mathbb{R}(\nu = 0, \ldots, m)$ with $0 = \tau_0 < \tau_1, \ldots < \tau_m$.

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[5] The Hermite-Biehler Theorem provides necessary and sufficient conditions for Hurwitz stability of real polynomials in terms of an interlacing property. Notice that, if a given real polynomial is not Hurwitz, the Hermite-Biehler Theorem does not provide information on its roots distribution. A generalization of Hermite-Biehler Theorem with respect to real polynomials was first derived in a report by Özgüler & Koçan (1994) in which was given a formula for a signature of polynomial—not necessarily Hurwitz—applicable to real polynomials without zeros on the imaginary axis except possibly a single root at the origin. This formula was employed to solve the constant gain stabilization problem. It may be mentioned that, in Ho et al. (1999), a different formula applicable to arbitrary real polynomials—but without restrictions on root localizations—was derived and used in the problem of stabilizing PID controllers. In addition, as a result of Ho et al. (2000), a generalization of the Hermite-Biehler Theorem for real polynomials—not necessarily Hurwitz—to the polynomials with complex coefficients was derived and, as a consequence of that extension, we have a technical application to a problem of stabilization in area of Control Theory.

We can denominate the equation (18) as an equation with **delayed** argument, if the coefficient $a_{n0} \neq 0$ and the remaining coefficients $a_{nv} = 0(v = 1, \ldots, m)$, that is, $a_{n0}u^{(n)}(t) + \sum_{\mu=0}^{n-1}\sum_{v=0}^{m} a_{\mu v}u^{(\mu)}(t - \tau_v) = h(t)$; analogously, the equation (18) is denominated an equation with **advanced** argument, if the coefficient $a_{n0} = 0$ and, if only for one $v > 0$, $a_{nv} \neq 0$, that is, $a_{nv_0}u^{(n)}(t - \tau_{v_0}) + \sum_{\mu=0}^{n-1}\sum_{v=0}^{m} a_{\mu v}u^{(\mu)}(t - \tau_v) = h(t)$, for only one $v_0 \in \{1, \ldots, m\}$ and, finally, the equation (18) is denominated an equation of **neutral** type, if the coefficient $a_{n0} \neq 0$ and, if only for one $v > 0$, $a_{nv} \neq 0$, that is, $a_{n0}u^{(n)}(t) + a_{nv_0}u^{(n)}(t - \tau_{v_0}) + \sum_{\mu=0}^{n-1}\sum_{v=0}^{m} a_{\mu v}u^{(\mu)}(t - \tau_v) = h(t)$, for only one $v_0 \in \{1, \ldots, m\}$.

Let us consider $h(t) = 0$ in equation (18), we obtain the *homogeneous linear* equation with constant coefficients and constant delays of the argument like

$$\sum_{\mu=0}^{n}\sum_{v=0}^{m} a_{\mu v}u^{(\mu)}(t - \tau_v) = 0. \tag{19}$$

Assuming that $u(t) = e^{zt}$, where z denotes a complex constant, is a particular solution of the equation (19) and, by substituting in (19) we obtain the so-called *characteristic* equation

$$\sum_{\mu=0}^{n}\sum_{v=0}^{m} a_{\mu v}z^{\mu}e^{-\tau_v z} = 0. \tag{20}$$

From the equation (20) we can define the *characteristic quasipolynomial* in the following form

$$\delta^*(z) = \sum_{\mu=0}^{n}\sum_{v=0}^{m} a_{\mu v}z^{\mu}e^{-\tau_v z}. \tag{21}$$

Note that the equation (20) has an infinite set of roots, therefore to every root z_k corresponds a solution $u(t) = e^{z_k t}$ of the equation (19). And, if the sums of infinite series $\sum_{k=0}^{\infty} C_k e^{z_k t}$ of solutions converge and admit n − fold term-by-term differentiation, then those sums are also solutions of the equation (19).

Multiplying the equation (21) by $e^{\tau_m z}$, it follows that

$$\delta(z) = e^{\tau_m z}\delta^*(z) = \sum_{\mu=0}^{n}\sum_{v=0}^{m} a_{\mu v}z^{\mu}e^{(\tau_m - \tau_v)z} = \sum_{v=0}^{m} p_v(z)e^{(\tau_m - \tau_v)z}, \tag{22}$$

where $p_v(z) = \sum_{\mu=0}^{n} a_{\mu v}z^{\mu} (v = 0, \ldots, m)$. For $m \neq 0$, the function (22) belongs to a general class of quasipolynomials [see Definition (8)]. It is evident that $\delta(z) = e^{\tau_m z}\delta^*(z)$ and $\delta^*(z)$ have the same zeros [6]. Thus, from this point of view, the zeros of the function $\delta(z)$ can be obtained from the Theorems (4) and (5).

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[6] see El'sgol'ts (1966) for a fully discussion.

Now, consider a special class of quasipolynomials (with *one* delay) given by

$$\delta^*(z) = p_0(z) + e^{-Lz}p_1(z), \tag{23}$$

where $\quad p_0(z) = z^n + \sum_{\mu=0}^{n-1} a_{\mu 0}z^\mu \quad$ with $\quad a_{\mu 0} \in \mathbb{R}(\mu = 0, \ldots, n-1), \quad p_1(z) = \sum_{\mu=0}^{n} a_{\mu 1}z^\mu \quad$ with

$a_{\mu 1} \in \mathbb{R}(\mu = 0, \ldots, n)$ and $L > 0$. Multiplying the (23) by e^{Lz}, it follows that

$$\delta(z) = e^{Lz}\delta^*(z) = e^{Lz}p_0(z) + p_1(z). \tag{24}$$

We consider the following *Assumptions*:

Hypothesis 1. $\partial(p_1) < n$ *[retarded type]*

Hypothesis 2. $\partial(p_1) = n$ *and* $0 < |a_{n1}| < 1$ *[neutral type]*

where $\partial(p_1)$ stands for the degree of polynomial p_1. Notice that, Hypothesis (1) implies that $a_{n1} = 0$ and $a_{\mu 1} \neq 0$ for some $\mu = 0, \ldots, n-1$.

Firstly, in what follows, we will state the Lemma (2) and Hypothesis (3) to establish the definition of *signature* of the quasipolynomials.

Lemma 2. *Suppose a quasipolynomial of the form (24) given. Let $f(\omega)$ and $g(\omega)$ be the real and imaginary parts of $\delta(j\omega)$, respectively. Under Hypothesis (1) or (2), there exists $0 < \omega_0 < \infty$ such that in $[\omega_0, \infty)$ the functions $f(\omega)$ and $g(\omega)$ have only real roots and these roots interlace[7].*

Hypothesis 3. *Let $\eta_g + 1$ be the number of zeros of $g(\omega)$ and η_f be the number of zeros of $f(\omega)$ in $(0, \omega_1)$. Suppose that $\omega_1 \in \mathbb{R}^+$, $\eta_g, \eta_f \in \mathbb{N}$ are sufficiently large, such that the zeros of $f(\omega)$ and $g(\omega)$ in $[\omega_0, \infty)$ interlace (with $\omega_0 < \omega_1$). Therefore, if $\eta_f + \eta_g$ is even, then $\omega_0 = \omega_{g_{\eta_g}}$, where $\omega_{g_{\eta_g}}$ denotes the η_g-th (non-null) root of $g(\omega)$, otherwise $\omega_0 = \omega_{f_{\eta_f}}$, where $\omega_{f_{\eta_f}}$ denotes the η_f-th root of $f(\omega)$.*

Note that, the Lemma (2) establishes *only* the condition of *existence* for ω_0 such that $f(\omega)$ and $g(\omega)$ have only real roots and these roots interlace, by another hand the Hypothesis (3) has a *constructive* character, that is, it allows to calculate ω_0.

Definition 11. *(Signature of Quasipolynomials) Let $\delta(z)$ be a given quasipolynomial described as in (24) without real roots in imaginary axis. Under Hypothesis (3), let $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_{\eta_g}} \leq \omega_0$ and $\omega_{f_1} < \ldots < \omega_{f_{\eta_f}} \leq \omega_0$ be real and distinct zeros of $g(\omega)$ and $f(\omega)$, respectively. Therefore, the signature of δ is defined by*

$$\sigma(\delta) = \begin{cases} \left\{ sgn[f(\omega_{g_0})] + 2\left(\sum_{k=1}^{\eta_g - 1}(-1)^k sgn[f(\omega_{g_k})]\right) + (-1)^{\eta_g}sgn[f(\omega_{g_{\eta_g}})]\right\}(-1)^{\eta_g - 1}sgn[g(\omega_{g_{\eta_g-1}}^+)], \\ \quad \text{if } \eta_f + \eta_g \text{ is even;} \\ \left\{ sgn[f(\omega_{g_0})] + 2\left(\sum_{k=1}^{\eta_g}(-1)^k sgn[f(\omega_{g_k})]\right)\right\}(-1)^{\eta_g}sgn[g(\omega_{g_{\eta_g}}^+)], \\ \quad \text{if } \eta_f + \eta_g \text{ is odd;} \end{cases}$$

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[7] The proof of Lemma (2) follows from Theorems (4) - (5); indeed, under Hypothesis (2) the roots of $\delta(z)$ go into the left hand complex plane for $|z|$ sufficiently large. A detailed proof can be find in Oliveira et al. (2003) and Oliveira et al. (2009).

where *sgn* *is* *the* *standard* *signum* *function,* $sgn[g(\omega_\lambda^+)]$ *stands* *for* $\lim_{\omega \longrightarrow \omega_\lambda^+} sgn[g(\omega)]$ *and*
$\omega_\lambda, (\lambda = 0, \ldots, g_{\eta_g})$ *is the* λ*-th zero of* $g(\omega)$.

Now, by means of the Definition of *Signature* the following Lemma can be established.

Lemma 3. *Consider a Hurwitz stable quasipolynomial* $\delta(z)$ *described as in (24) under Hypothesis (1)*
or (2). Let η_f *and* η_g *be given by Hypothesis (3). Then the signature for the quasipolynomial* $\delta(z)$ *is*
given by $\sigma(\delta) = \eta_f + \eta_g$.

Referring to the feedback system with a *proportional controller* $C(z) = k_p$, the resulted
quasipolynomial is given by:

$$\delta(z, k_p) = e^{Lz} p_0(z) + k_p p_1(z) \tag{25}$$

where $p_0(z)$ and $p_1(z)$ are given in (24). In the next Lemma we consider $\delta(z, k_p)$
under Hypothesis (1) or (2). Consequently, we obtain a frequency range signature for
the quasipolynomial given by the product $\delta(z, k_p) p_1(-z)$ which is used to establish the
subsequent Theorem with respect to the *stabilization* problem.

Lemma 4. *For any stabilizing* k_p, *let* $\eta_g + 1$ *and* η_f *be, respectively, the number of real and distinct*
zeros of imaginary and real parts of the quasipolynomial $\delta(j\omega, k_p)$ *given in (25). Suppose* η_g *and*
η_f *sufficiently large, it follows that* $\delta(j\omega, k_p)$ *is Hurwitz stable if, and only if, the signature for*
$\delta(j\omega, k_p) p_1(-j\omega)$ *in* $[0, \omega_0]$ *with* ω_0 *as in Hypothesis (3), is given by* $\eta_g + \eta_f - \sigma(p_1)$, *where* $\sigma(p_1)$
stands for the signature of the polynomial p_1.

Definition 12. *(Set of strings) Let* $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_k} \leq \omega_0$ *be real and distinct zeros of*
$g(\omega)$. *Then the set of strings* $\mathcal{A}_k$ *in the range determined by frequency* ω_0 *is defined as*

$$\mathcal{A}_k = \{s_0, \ldots, s_k : s_0 \in \{-1, 0, 1\}; s_l \in \{-1, 1\}; l = 1, \ldots, k\} \tag{26}$$

with s_l *identified as* $sgn[f(\omega_{g_l})]$ *in the Definition (11).*

Theorem 6. *Let* $\delta(z, k_p)$ *be the quasipolynomial given in (25). Consider*
$f(\omega, k_p) = f_1(\omega) + k_p f_2(\omega)$ *and* $g(\omega)$ *as the real and imaginary parts of the quasipolynomial*
$\delta(j\omega, k_p) p_1(-j\omega)$, *respectively. Suppose there exists a stabilizing* k_p *of the quasipolynomial*
$\delta(z, k_p)$, *and by taking* ω_0 *as given in Hypothesis (3) associated to the quasipolynomial* $\delta(z, k_p)$. *Let*
$0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_l} \leq \omega_0$ *be the real and distinct zeros of* $g(\omega)$ *in* $[0, \omega_0]$. *Assume that*
the polynomial $p_1(z)$ *has no zeros at the origin. Then the set of all* k_p—*denoted by* $\mathcal{I}$—*such that*
$\delta(z, k_p)$ *is Hurwitz stable may be obtained using the signature of the quasipolynomial* $\delta(z, k_p) p_1(-z)$.
In addition, if $\mathcal{I}_\iota = (\max_{s_t \in \mathcal{A}_\iota^+} [-\frac{1}{G(j\omega_{g_t})}], \min_{s_t \in \mathcal{A}_\iota^-} [-\frac{1}{G(j\omega_{g_t})}])$, *where* $\frac{1}{G(j\omega)} = \frac{f_1(\omega) - jg(\omega)}{f_2(\omega)}$,
$\mathcal{A}_\iota$ *is a set of string as in Definition (12)* , $\mathcal{A}_\iota^+ = \{s_t \in \mathcal{A}_\iota : s_t.sgn[f_2(\omega_{g_t})] = 1\}$ *and*
$\mathcal{A}_\iota^- = \{s_t \in \mathcal{A}_\iota : s_t.sgn[f_2(\omega_{g_t})] = -1\}$, *such that* $\max_{s_t \in \mathcal{A}_\iota^+} [-\frac{1}{G(j\omega_{g_t})}] < \min_{s_t \in \mathcal{A}_\iota^-} [-\frac{1}{G(j\omega_{g_t})}]$,
then $\mathcal{I} = \bigcup \mathcal{I}_\iota$, *with* ι *the number of feasible strings.*

4.1 Stabilization using a PID Controller

In the preceding section we take into account statements *introduced* in Oliveira et al. (2003), namely, Hypothesis (3), Definition (11), Lemma (2), Lemma (3), Lemma (4), and Theorem (6). Now, we shall regard a technical application of these results.

In this subsection we consider the *problem of stabilizing* a first order system with time delay using a PID controller. We will utilize the standard notations of Control Theory, namely, $G(z)$ stands for the plant to be controller and $C(z)$ stands for the PID controller to be designed. Let $G(z)$ be given by

$$G(z) = \frac{k}{1 + Tz}e^{-Lz} \tag{27}$$

and $C(z)$ is given by

$$C(z) = k_p + \frac{k_i}{z} + k_d z,$$

where k_p is the proportional gain, k_i is the integral gain, and k_d is the derivative gain.

The *main* problem is to analytically determine the set of controller parameters (k_p, k_i, k_d) for which the closed-loop system is stable. The closed-loop characteristic equation of the system with PID controller is express by means of the quasipolynomial in the following *general* form

$$\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega) = f(\omega, k_i, k_d) + jg(\omega, k_p) \tag{28}$$

where

$$f(\omega, k_i, k_d) = f_1(\omega) + (k_i - k_d\omega^2)f_2(\omega)$$
$$g(\omega, k_p) = g_1(\omega) + k_p g_2(\omega)$$

with

$f_1(\omega) = -\omega[\omega^2 p_0^o(-\omega^2)p_1^o(-\omega^2) + p_0^e(-\omega^2)p_1^e(-\omega^2)]\sin(L\omega) + \omega^2[\omega^2 p_1^o(-\omega^2)p_0^e(-\omega^2) - p_0^o(-\omega^2)p_1^e(-\omega^2)]\cos(L\omega)$

$f_2(\omega) = p_1^e(-\omega^2)p_1^e(-\omega^2) + \omega^2 p_1^o(-\omega^2)p_1^o(-\omega^2)$

$g_1(\omega) = \omega[\omega^2 p_0^o(-\omega^2)p_1^o(-\omega^2) + p_0^e(-\omega^2)p_1^e(-\omega^2)]\cos(L\omega) + \omega^2[\omega^2 p_1^o(-\omega^2)p_0^e(-\omega^2) - p_0^o(-\omega^2)p_1^e(-\omega^2)]\sin(L\omega)$

$g_2(\omega) = \omega f_2(\omega) = \omega[p_1^e(-\omega^2)p_1^e(-\omega^2) + \omega^2 p_1^o(-\omega^2)p_1^o(-\omega^2)]$

where p_0^e and p_0^o stand for the even and odd parts of the decomposition $p_0(\omega) = p_0^e(\omega^2) + \omega p_0^o(\omega^2)$, and analogously for $p_1(\omega) = p_1^e(\omega^2) + \omega p_1^o(\omega^2)$. Notice that for a fixed k_p the polynomial $g(\omega, k_p)$ does not depend on k_i and k_d, therefore we can obtain the stabilizing k_i and k_d values by solving a linear programming problem for each $g(\omega, k_d)$, which is establish in the next Lemma.

Lemma 5. *Consider a stabilizing set (k_p, k_i, k_d) for the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)$ as given in (28). Let $\eta_g + 1$ and η_f be the number of real and distinct zeros, respectively, of the imaginary and real parts of $\delta(j\omega, k_p, k_i, k_d)$ in $[0, \omega_0]$, with a sufficiently large frequency ω_0 as given in the Hypothesis (3). Then, $\delta(j\omega, k_p, k_i, k_d)$ is stable if, and only if, for any stabilizing set (k_p, k_i, k_d) the signature of the*

quasipolynomial $\delta(z, k_p, k_i, k_d)p_1(-z)$ determined by the frequency ω_0 is given by $\eta_g + \eta_f - \sigma(p_1)$, where $\sigma(p_1)$ stands for the signature of the polynomial p_1.

Finally, we make the standing statement to determine the range of stabilizing PID gains.

Theorem 7. *Consider the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega)$ as given in (28). Suppose there exists a stabilizing set (k_p, k_i, k_d) for a given plant $G(z)$ satisfying Hypothesis (1) or (2). Let η_f, η_g and ω_0 be associated to the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)$ be choosen as in Hypothesis (3). For a $\square$xed k_p, let $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_\iota} \leq \omega_0$ be real and distinct zeros of $g(\omega, k_p)$ in the frequency range given by ω_0. Then, the (k_i, k_d) values—such that the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)$ is stable—are obtained by solving the following linear programming problem:*

$$\begin{cases} f_1(\omega_{g_t}) + (k_i - k_d \omega_{g_t}^2) f_2(\omega_{g_t}) > 0, & \text{for } s_t = 1, \\ f_1(\omega_{g_t}) + (k_i - k_d \omega_{g_t}^2) f_2(\omega_{g_t}) < 0, & \text{for } s_t = -1; \end{cases}$$

with $s_t \in \mathcal{A}_\iota (t = 0, 1, \ldots, \iota)$ and, such that the signature for the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega)$ equals $\eta_g + \eta_f - \sigma(p_1)$, where $\sigma(p_1)$ stands for the signature of the polynomial p_1.

Now, we shall formulate an *algorithm* for PID controller by way of the above theorem. The algorithm[8] can be state in following form:

Step 1: Adopt a value for the set (k_p, k_i, k_d) to stabilize the given plant $G(z)$. Select η_f and η_g, and choose ω_0 as in the Hypothesis (3).

Step 2: Enter functions $f_1(\omega)$ and $g_1(\omega)$ as given in (28).

Step 3: In the frequency range determined by ω_0 find the zeros of $g(\omega, k_p)$ as defined in (28) for a fixed k_p.

Step 4: Using the Definition(11) for the quasipolynomial $\delta(z, k_p, k_i, k_d)p_1(-z)$, and find the strings $\mathcal{A}_\iota$ that satisfy $\sigma(\delta(z, k_p, k_i, k_d)p_1(-z)) = \eta_g + \eta_f - \sigma(p_1)$.

Step 5: Apply Theorem (7) to obtain the inequalities of the above *linear programming* problem.

5. Conclusion

In view of the following fact concerning the bibliographic references (in this Chapter): all the quasipolynomials have only one delay, it follows that we can express $\delta(z) = P(z, e^z)$ as in (24), where $P(z, s) = p_0(z)s + p_1(z)$ with $\partial(p_0) = 1$, $\partial(p_1) = 0$ and $\partial(p_0) = 2$, $\partial(p_1) = 1$ in Silva et al. (2000), $\partial(p_0) = 2$, $\partial(p_1) = 0$ in Silva et al. (2001), $\partial(p_0) = 2$, $\partial(p_1) = 2$ in Silva et al. (2002), $\partial(p_0) = 2$, $\partial(p_1) = 2$ in Capyrin (1948), $\partial(p_0) = 5$, $\partial(p_1) = 5$ in Capyrin (1953), and $\partial(p_0) = 1$, $\partial(p_1) = 0$ [Hayes' equation] and $\partial(p_0) = 2$, $\partial(p_1) = 0, 1, 2$ [particular cases] in Bellman & Cooke (1963), respectively. Similarly, in the cases studied in Oliveira et al. (2003) and Oliveira et al. (2009)—and described in this Chapter—the Hypothesis (3) and Definition (11) take into account Pontryagin's Theorem. In addition, if we have particularly the following form $F(z) = f_1(z)e^{\lambda_1 z} + f_2(z)e^{\lambda_2 z}$, with $\lambda_1, \lambda_2 \in \mathbb{R}$ (noncommensurable) and $0 < \lambda_1 < \lambda_2$, we can write $F(z) = e^{\lambda_1 z}\delta(z)$, where $\delta(z) = f_1(z) + f_2(z)e^{(\lambda_2 - \lambda_1)z}$ with $\partial(f_2) > \partial(f_1)$, therefore $\delta(z)$ can be studied by Pontryagin's Theorem.

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[8] See Oliveira et al. (2009) for an example of PID application with the graphical representation.

It should be observed that, in the state-of-the-art, we do *not* have a *general* mathematical analysis via an extension of Pontryagin's Theorem for the cases in which the quasipolynomials $\delta(z) = P(z, e^z)$ have two or more real (noncommensurable) delays .

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Time-Delay Systems

Edited by Prof. Dragutin Debeljkovic

Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.

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