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Analytical Solutions of Mass Transfer around a Prolate or an Oblate Spheroid Immersed in a Packed Bed

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1. Introduction

There are several situations of practical interest, both in nature and in man made processes, in which fluid flows through a bed of inert particles, packed around a large solid mass, which is soluble or reacts with the flowing fluid.

In order to predict the rate of mass transfer between the solid and the flowing fluid it is necessary to understand in detail the mechanics of the flow and the associated processes of diffusion and convection.

For many years, intense research on diffusion phenomena in porous bodies using the diffusion model has been applied to different materials (Delgado, 2007 and Delgado and Vázquez da Silva, 2009). Fundamental solutions of the diffusion problems for spheres, cylinders, plates and parallelepipeds have been provided by Crank (1992) and Gebhart (1993), for example. However, in many situations the shape of the particles immersed in a fluid or porous media is not perfectly spherical, and may be classified as prolate or oblate spheroids. Numerical and analytical solutions of the diffusion equation for prolate spheroids have been reported by Coutelieris et al. (2004), Lima et al. (2002), Coutelieris et al. (1995), etc., and for oblate spheroids by Carmo and Lima (2008), Coutelieris et al. (1995), etc.

Fluid flow along buried spheroidal surfaces is an important model situation (e.g. Clift et al., 1978) and in the present work it is investigated analytically. The treatment of irregular shapes could only be done by numerical methods; therefore it was decided to take the prolate and the oblate spheroid as a model of non spherical particle and study the problem of diffusion around it, for two limiting cases: pure diffusion regime and high values of Peclet number.

The main objective of this work is to provide an analytical solution to the considered problem, as it can be very useful in situations such as the analytical models of continuous injection of solute at a point source, in a uniform stream, to estimate the distance from the "contaminant source" beyond which the levels of contaminant are expected to fall below some safe limit, etc.

2. Analytical solutions

In many practical situations it is often required to consider operations in which there are physico-chemical interactions between a solid particle and the fluid flowing around it. In the treatment of these operations it is common practice to assume the soluble particle to be spherical, because the treatment of irregular shapes could only be done by numerical methods. Spheroidal particles can be either prolate or oblate, and using a simple transformation, one obtains the results for an oblate spheroid from the prolate results. If we consider the situation of a prolate spheroid of major and minor axes c and a, respectively, and an oblate spheroid with the major and minor axes, respectively, a and c, the surface of the spheroid ($\theta = \theta_0$) is described by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Since $r^2 = x^2 + y^2$, Eq. (1) can be written as

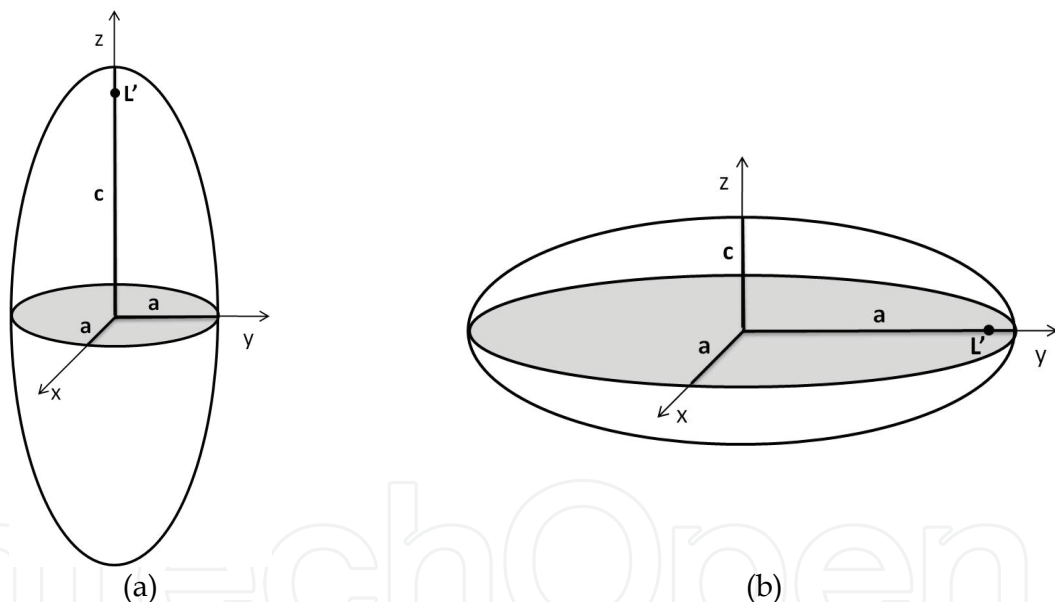$$z = c\sqrt{1 - (r/a)^2} \tag{2}$$

Fig. 1. The coordinates system of: (a)-prolate spheroid and (b)-oblate spheroid.

The surface area S and volume V of a prolate/oblate spheroid are given by

$$S = 2\pi a^2 \left(1 + \frac{c/a}{\sqrt{1 - a^2/c^2}} \sin^{-1}\left(\sqrt{1 - a^2/c^2} \right) \right) \quad \text{Prolate} \tag{3a}$$

$$S = 2\pi a^2 \left(1 + \frac{(c/a)^2}{2\sqrt{1 - c^2/a^2}} \ln\left(\frac{1 + \sqrt{1 - c^2/a^2}}{1 - \sqrt{1 - c^2/a^2}} \right) \right) \quad \text{Oblate} \tag{3b}$$

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$$V = \frac{4}{3}\pi a^2 c \tag{4}$$

where $e = \sqrt{1 - a^2/c^2}$ is the eccentricity for a prolate spheroid and $e = \sqrt{1 - c^2/a^2}$ for an oblate spheroid; where $e = 0$ corresponds to a sphere. Figure 1 describes the prolate/oblate spheroidal coordinate system. The dimensional Cartesian coordinates (x, y, z) are related to the prolate spheroidal ones (θ, η, β) through the equations (see Moon and Spencer, 1971)

$$x = L'\sinh\theta\sin\eta\cos\beta \tag{5a}$$

$$y = L'\sinh\theta\sin\eta\sin\beta \tag{5b}$$

$$z = L'\cosh\theta\cos\eta \tag{5c}$$

and for the case of an oblate spheroid by

$$x = L'\cosh\theta\sin\eta\cos\beta \tag{6a}$$

$$y = L'\cosh\theta\sin\eta\sin\beta \tag{6b}$$

$$z = L'\sinh\theta\cos\eta \tag{6c}$$

where L' is the focal distance ($L' = \sqrt{c^2 - a^2}$, for a prolate and $L' = \sqrt{a^2 - c^2}$, for an oblate spheroid) and the coordinates range are: $0 \le \theta < \infty$, $0 \le \eta \le \pi$ and $0 \le \beta \le 2\pi$.

2.1 Mass transfer around a prolate spheroid
2.1.1 Pure diffusional regime

The spheroid of slightly soluble solid is assumed to be buried in a packed bed, of "infinite extent", the interstices of the bed being filled with a stagnant fluid that is assumed to be free of solute, at a large distance from the spheroid.

In steady state, a mass balance on the solute, without chemical reaction, leads to

$$\frac{\partial}{\partial\theta}\left(\sinh\theta\frac{\partial C}{\partial\theta}\right) = 0 \tag{7}$$

along coordinate θ. The boundary conditions are

$$C = C^* \quad \theta = \theta_0 \tag{8a}$$

$$C \to C_\infty \quad \theta \to \infty \tag{8b}$$

and the solution is given by

$$\frac{C - C_\infty}{C^* - C_\infty} = \frac{\ln|\tan(\theta/2)|}{\ln|\tanh(\theta/2)|} \tag{9}$$

The mass transfer rate is given by the following expression

$$n = \int\limits_{0}^{\pi} \int\limits_{0}^{2\pi} -D_{\mathrm{m}} \left(\frac{\partial C}{\partial \theta} \right)_{\theta_0} dS \tag{10}$$

with the elemental area, dS, of the prolate surface ($\theta = \theta_0$) given by

$$dS = \left(L' \sqrt{\sinh^2 \theta_0 + \sin^2 \eta} \right) d\eta \times (L' \sinh \theta_0 \sin \eta) d\beta \tag{11}$$

From integration of Eq. (10), the total mass transfer rate from the active prolate spheroid is obtained as

$$n = kS(C^* - C_\infty) = \frac{4\pi L' D_{\mathrm{m}}}{\ln |\tanh(\theta_0 / 2)|} (C^* - C_\infty) \tag{12}$$

After rearranging Eq. (11), with S given by Eq. (3) and using the useful mathematical relations $\sinh \theta_0 = a / L'$ and $\cosh \theta_0 = c / L' = 1 / e$, it is possible to obtain the Sherwood number, Sh, for mass transfer by pure diffusion,

$$\mathrm{Sh} = \frac{k2a}{D_{\mathrm{m}}} = \frac{4}{\left[\sqrt{1/e^2 - 1} + 1/e^2 \sin^{-1}(e) \right] \ln \left| \dfrac{\sqrt{1 - e^2}}{1 + e} \right|} \tag{13}$$

and using the equivalent sphere diameter (*i.e.* a sphere with the same volume), $d_{\mathrm{eq}} = 2(a^2 c)^{1/3}$, the previous expression results in

$$\mathrm{Sh} = \frac{k d_{\mathrm{eq}}}{D_{\mathrm{m}}} = \frac{4(1 - e^2)^{-1/6}}{\left[\sqrt{1/e^2 - 1} + 1/e^2 \sin^{-1}(e) \right] \ln \left| \dfrac{\sqrt{1 - e^2}}{1 + e} \right|} \tag{14}$$

For the special case of $e \approx 0$, the well known result of $\mathrm{Sh} = 2$, , corresponding to pure diffusion around a sphere in an unbounded fluid, is obtained.

2.1.2 High Peclet numbers

The theory is based on the assumption that the inert particles in the bed are packed with uniform voidage, ε , and that the gas flow may be approximated everywhere by Darcy's law, $\mathbf{u} = -K \mathbf{grad}\, p$. Furthermore, if the fluid is treated as incompressible, mass conservation leads to $\mathrm{div}\,\mathbf{u} = 0$, Laplace's equation is obtained $\nabla^2 \phi = 0$. This result is well known to hydrologists (see Scheidegger, 1974) and shows that incompressible Darcy flow through a packed bed obeys to the laws of potential flow. Darcy's law is strictly valid only for laminar flow through the packing, but according to Bear (1988) it is still a good approximation for values of the Reynolds number (based on superficial velocity) up to ~ 10 , which for beds with $\varepsilon \sim 0.4$ is equivalent to $\mathrm{Re} \sim 25$, the upper limit for the validity of this analysis.

When a solid prolate spheroid is immersed in a packed bed of significantly smaller particles, through which fluid flows with uniform interstitial velocity u_0, far from the spheroid, the solution of Laplace's equation, in terms of spheroidal coordinates (θ, η, β), is (see Alassar and Badr, 1997)

$$\phi = -u_0 L' \cos\eta \left[\cosh\theta - \frac{\cosh\theta \coth^{-1}(\cosh\theta) - 1}{\coth^{-1}(\cosh\theta_0) - \dfrac{\cosh\theta_0}{\sinh^2\theta_0}} \right] \qquad (15)$$

and the corresponding stream function is given by

$$\psi = u_0 \frac{L^2}{4} \sinh\theta \left[\sinh\theta - \cos(2\eta)\sinh\theta - (1 - \cos(2\eta)) \frac{\sinh\theta \coth^{-1}(\cosh\theta) - \coth\theta}{\coth^{-1}(\cosh\theta_0) - \dfrac{\cosh\theta_0}{\sinh^2\theta_0}} \right] \quad (16)$$

The stream and potential functions are related to the dimensionless velocity components (u_θ, u_η) by the equations (see Batchelor, 1997)

$$u_\theta = \frac{1}{L'\sqrt{\sinh^2\theta + \sin^2\eta}} \frac{\partial\phi}{\partial\theta} = \frac{-1}{L'^2 \sqrt{\sinh^2\theta + \sin^2\eta} \sinh\theta \sin\eta} \frac{\partial\psi}{\partial\eta} \qquad (17)$$

$$u_\eta = \frac{1}{L'\sqrt{\sinh^2\theta + \sin^2\eta}} \frac{\partial\phi}{\partial\eta} = \frac{1}{L'^2 \sqrt{\sinh^2\theta + \sin^2\eta} \sinh\theta \sin\eta} \frac{\partial\psi}{\partial\theta} \qquad (18)$$

resulting in the following velocity components

$$u_\theta = \frac{-u_0 \cos\eta}{\sqrt{\sinh^2\theta + \sin^2\eta}} \left[\sinh\theta - \frac{\sinh\theta \coth^{-1}(\cosh\theta) - \coth(\theta)}{\coth^{-1}(\cosh\theta_0) - \dfrac{\cosh\theta_0}{\sinh^2\theta_0}} \right] \qquad (19)$$

$$u_\eta = \frac{u_0 \sin\eta}{\sqrt{\sinh^2\theta + \sin^2\eta}} \left[\cosh\theta - \frac{\cosh\theta \coth^{-1}(\cosh\theta) - 1}{\coth^{-1}(\cosh\theta_0) - \dfrac{\cosh\theta_0}{\sinh^2\theta_0}} \right] \qquad (20)$$

The tangential velocity at the surface of the prolate spheroid ($\theta = \theta_0$) can be found through

$$u_{\eta 0} = \frac{1}{L'\sqrt{\sinh^2\theta + \sin^2\eta}} \frac{\partial\phi}{\partial\eta}\bigg|_{\theta=\theta 0} \qquad (21)$$

and the resulting expression is

$$u_{\eta 0} = \frac{u_0 \sin\eta}{\left(1/e^2 - 1 + \sin^2\eta\right)^{0.5} \left[1/e - (1/e^2 - 1)\tanh^{-1}e \right]} \qquad (22)$$
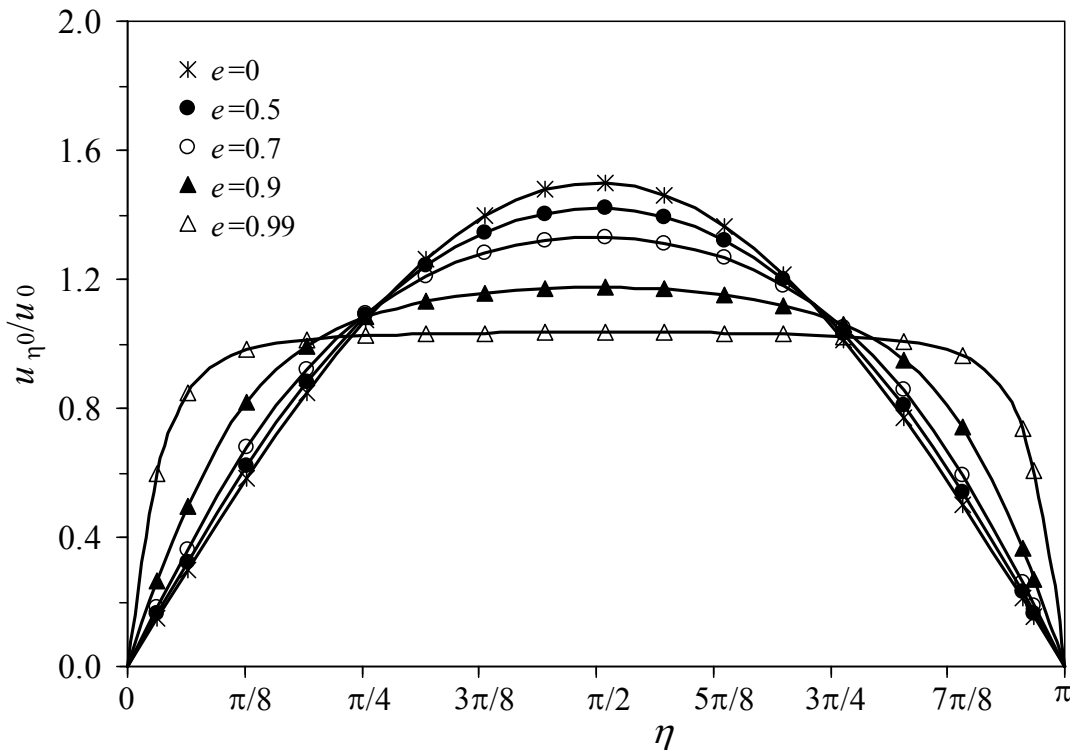
Fig. 2. The dimensionless tangential surface velocity, $u_{\eta 0}/u_0$, of a prolate spheroid as a function of η, for different values of the eccentricity, e.

Figure 2 shows the dimensionless tangential surface velocity, $u_{\eta 0}/u_0$, of a prolate spheroid as a function of η, for different values of the eccentricity, e. Note that, for the case of a sphere, $e \approx 0$, the well-known result of $u_{\eta 0} = 1.5u_0 \sin \eta$ is obtained, for potential flow over the surface of the sphere. On the other hand, for a slender prolate, $e \to 1$, as expected $u_{\eta 0}/u_0 \to 1$.

A convenient way of expressing the differential mass balance on the solute is to take a control volume along a stream tube, between two nearby potential surfaces. The resulting expression, for convection with molecular diffusion, is (see Coelho and Guedes de Carvalho, 1988)

$$\frac{\partial C}{\partial \phi} = \frac{\partial}{\partial \phi}\left(D'_m \frac{\partial C}{\partial \phi}\right) + \frac{\partial}{\partial \psi}\left(D'_m \omega^2 \frac{\partial C}{\partial \psi}\right) \tag{23}$$

For high values of the Peclet number the concentration boundary layer will be thin and the first term on the right hand side of Eq. (23) may be neglected (see Guedes de Carvalho *et al.*, 2004). After some algebraic manipulation and a suitable change of variables, it is then possible to obtain

$$\frac{\partial C}{\partial \xi} = \frac{\partial^2 C}{\partial \psi^2} \tag{24}$$

where ξ is defined by

$$\xi = \int_0^{\eta} L'^3 \sinh^2 \theta_0 \sin^2 \eta' (\sinh^2 \theta_0 + \sin^2 \eta')^{0.5} u_{\eta} D'_m \, d\eta' \tag{25}$$

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with $\omega = L'\sinh\theta_0 \sin\eta$. The boundary conditions for Eq. (24), in our problem, are

$$C = C_0 \quad \xi = 0 \quad \psi > 0 \tag{26a}$$

$$C = C^* \quad \xi > 0 \quad \psi = 0 \tag{26b}$$

$$C \to C_0 \quad \xi > 0 \quad \psi \to \infty \tag{26c}$$

and the corresponding solution is

$$\frac{C - C_0}{C^* - C_0} = 1 - erf\left(\frac{\psi}{2\sqrt{\xi}}\right) \tag{27}$$

The value of ξ varies over the surface of the spheroid. Now, for potential flow, u_η is given by Eq. (22) over the surface of the spheroid ($\theta = \theta_0$) and the integral in Eq. (25) is

$$\xi = u_0 D'_m \frac{c^3 e^3 (1 - e^2)}{e - (1 - e^2)\tanh^{-1} e}\left(\frac{2}{3} - \cos\eta + \frac{1}{3}\cos^3\eta\right) \tag{28}$$

The flux of solute at any point on the surface of the spheroid is

$$N = -D'_m \varepsilon \left(\frac{\partial C}{\partial b}\right) = -\frac{D'_m \varepsilon}{L'\sqrt{\sinh^2\theta_0 + \sin^2\eta}}\left(\frac{\partial C}{\partial\theta}\right)_{\theta=\theta_0}$$

$$= -D'_m \varepsilon u_\eta L'\sinh\theta_0 \sin\eta\left(\frac{\partial C}{\partial\psi}\right)_{\psi=0} \tag{29}$$

and from Eq. (29) it may be shown that $(\partial C/\partial\psi)_{\psi=0} = \left(-1/\sqrt{(\pi\xi)}\right)\left(C^* - C_0\right)$. The rate of dissolution of the spheroid in the region $0 < \eta < \eta_1$ will then be

$$n(\eta_1) = \int_0^{\eta_1} N 2\pi L'^2 \sqrt{\sinh^2\theta_0 + \sin^2\eta} \sinh\theta_0 \sin\eta \, d\eta = \int_0^{\xi} 2\pi\varepsilon\left(C^* - C_0\right)\left(1/\sqrt{\pi z}\right) dz$$

$$= 4\sqrt{\pi} \, \varepsilon \, \xi^{1/2}\left(C^* - C_0\right) \tag{30}$$

with $\xi(\eta_1)$ given by Eq. (28). In particular, the total rate of dissolution of the spheroid, n_T, may be obtained taking $\eta_1 = \pi$. By definition, the average mass transfer coefficient, k, is

$$k = n_T / \left[S\left(C^* - C_0\right)\right] \tag{31}$$

the resulting expression for k (from Eqs. (30) and (31)) is

$$k = \frac{4\varepsilon\sqrt{\pi}}{2\pi a^2\left(1 + \frac{\sqrt{1/(1 - e^2)}}{e}\sin^{-1}(e)\right)}\left(\frac{4}{3}\frac{c^3(1 - e^2)e^3 u_0 D'_m}{e - (1 - e^2)\tanh^{-1} e}\right)^{1/2} \tag{32}$$

It is convenient to express the rate of dissolution in terms of the Sherwood number, $\text{Sh}' = k d_{\text{eq}} / D'_{\text{m}}$, with $d_{\text{eq}} = 2(a^2 c)^{1/3}$, and the expression obtained, after some algebraic manipulation is

$$\frac{\text{Sh}'}{\varepsilon} = \sqrt{\frac{4}{\pi}\text{Pe}'} \left(\frac{2}{3} \frac{e^3}{e - (1-e^2)\tanh^{-1}e} \right)^{1/2} \frac{2}{(1-e^2)^{1/3} + \frac{(1-e^2)^{-1/6}}{e}\sin^{-1}e} \tag{33}$$

where $\text{Pe}' = u_0 d_{\text{eq}} / D'_{\text{m}}$ is the Peclet number.

For the special case of a sphere, $e \approx 0$, the result of $\text{Sh}'/\varepsilon = \left[4\,\text{Pe}'/\pi \right]^{1/2}$ is obtained, which corresponds to the asymptotic behaviour for thin concentration layer (high values of Peclet number) when dispersion is constant and tend to D'_{m} over the surface of the sphere (see Guedes de Carvalho and Alves, 1999).

For moderate values of Peclet number, Eq. (23), without any simplification, only should be solved with an indispensable numerical analyse.

Nevertheless, it is important to bear in mind the results obtained by Guedes de Carvalho and Alves (1999), for mass transfer around a single sphere buried in a granular bed of inert particles (a limiting case, with $e \approx 0$). In this work, the authors showed that the values of Sh'/ε calculated from the expression obtained by the quadratic mean of the two asymptotes, $\text{Sh}'/\varepsilon = \left[4 + 4\,\text{Pe}'/\pi \right]^{1/2}$, differ at most by 10% from the corresponding numerical solution obtained,

$$\frac{\text{Sh}'}{\varepsilon} = \left[4 + \frac{4}{5}(\text{Pe}')^{2/3} + \frac{4}{\pi}\text{Pe}' \right]^{1/2} \tag{34}$$
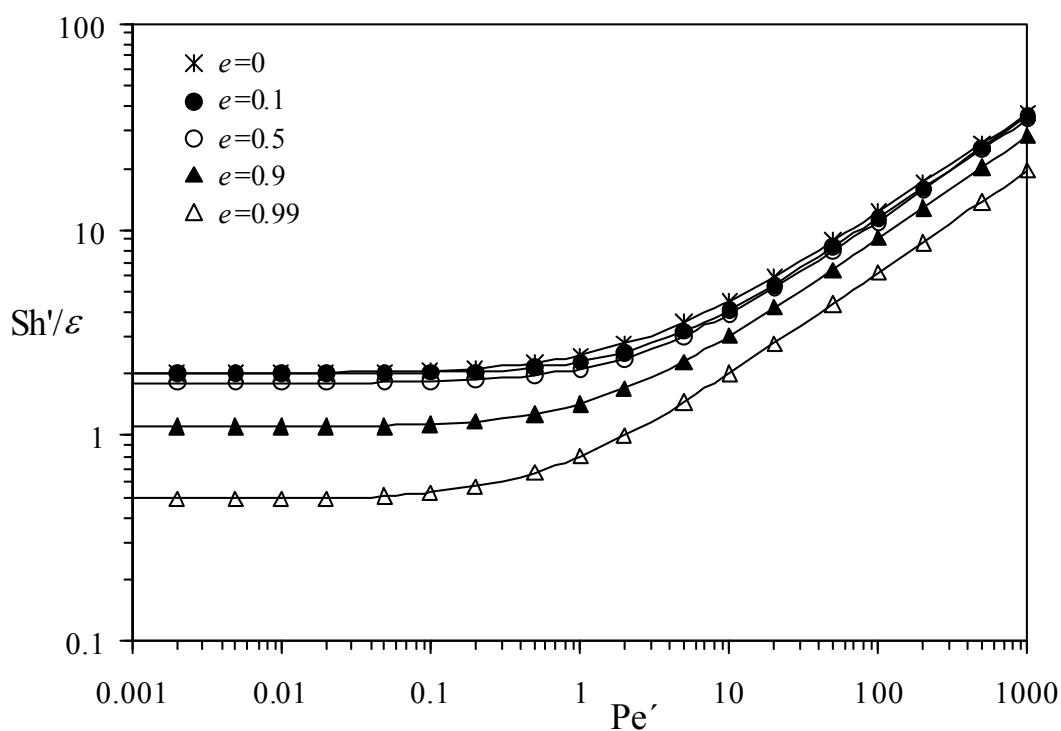

Fig. 3. Dependence of Sh'/ε on Pe', for different values of the eccentricity, e.

In our study, the expression obtained using the quadratic mean of the Sh'/ε values when $Pe' \to 0$, Eq. (14), and the asymptote for convection with molecular diffusion across a thin boundary layer, Eq. (33), is

$$\frac{Sh'}{\varepsilon} = \left(\frac{16(1-e^2)^{-1/3}}{\left[\left(\frac{1}{e^2} - 1 \right)^{1/2} + \frac{1}{e^2}\sin^{-1}e \right]^2 \left| \ln \left| \frac{\sqrt{1-e^2}}{1+e} \right| \right|^2} + \frac{\frac{4}{\pi}Pe'\frac{8}{3}\frac{e^3}{e-(1-e^2)\tanh^{-1}e}}{\left[(1-e^2)^{1/3} + \frac{(1-e^2)^{-1/6}}{e}\sin^{-1}e \right]^2} \right)^{1/2} \tag{35}$$

and it is expected that Eq. (35) does not differ by more than 10% from the exact solution obtained numerically.

Figure 3 shows the dependence of Sh'/ε on Pe', for different values of the eccentricity, e. Analyzing the Figure, is possible to conclude that the total quantity of material transferred from a prolate spheroid is smaller than that of a soluble sphere (*i.e.* values of Sh'/ε decreasing with eccentricity increase).

2.2 Mass transfer around an oblate spheroid
2.2.1 High peclet numbers
For high values of Peclet numbers, the theory is similar to the case of a prolate spheroid. When a solid oblate spheroid is immersed in a packed bed of significantly smaller particles, through which fluid flows with uniform interstitial velocity u_0, far from the spheroid, the solution of Laplace's equation and the corresponding stream function, in terms of spheroidal coordinates (θ, η, β), are (see Alassar and Badr, 1997)

$$\phi = -u_0 L' \cos\eta \left[\sinh\theta - \frac{\sinh\theta\cot^{-1}(\sinh\theta) - 1}{\cot^{-1}(\sinh\theta_0) - \frac{\sinh\theta_0}{\cosh^2\theta_0}} \right] \tag{36}$$

$$\psi = u_0 \frac{L^2}{4}\cosh\theta \left[\cosh\theta - \cos(2\eta)\cosh\theta - (1-\cos(2\eta))\frac{\cosh\theta\cot^{-1}(\sinh\theta) - \tanh\theta}{\cot^{-1}(\sinh\theta_0) - \frac{\sinh\theta_0}{\cosh^2\theta_0}} \right] \tag{37}$$

The stream and potential functions are related to the dimensionless velocity components (u_θ, u_η) by

$$u_\theta = \frac{1}{L'\sqrt{\cosh^2\theta - \sin^2\eta}}\frac{\partial\phi}{\partial\theta} = \frac{-1}{L'^2\sqrt{\cosh^2\theta - \sin^2\eta}\cosh\theta\sin\eta}\frac{\partial\psi}{\partial\eta} \tag{38}$$

$$u_\eta = \frac{1}{L'\sqrt{\cosh^2\theta - \sin^2\eta}}\frac{\partial\phi}{\partial\eta} = \frac{1}{L'^2\sqrt{\cosh^2\theta - \sin^2\eta}\cosh\theta\sin\eta}\frac{\partial\psi}{\partial\theta} \tag{39}$$

resulting the following velocity components

$$u_\theta = \frac{-u_0 \cos\eta}{\sqrt{\cosh^2\theta - \sin^2\eta}} \left[\cosh\theta - \frac{\cosh\theta \cot^{-1}(\sinh\theta) - \tanh(\theta)}{\cot^{-1}(\sinh\theta_0) - \dfrac{\sinh\theta_0}{\cosh^2\theta_0}} \right] \qquad (40)$$

$$u_\eta = \frac{u_0 \sin\eta}{\sqrt{\cosh^2\theta - \sin^2\eta}} \left[\sinh\theta - \frac{\sinh\theta \cot^{-1}(\sinh\theta) - 1}{\cot^{-1}(\sinh\theta_0) - \dfrac{\sinh\theta_0}{\cosh^2\theta_0}} \right] \qquad (41)$$

The tangential velocity at the surface of the oblate spheroid ($\theta = \theta_0$) can be found through

$$u_{\eta 0} = \frac{1}{L'\sqrt{\cosh^2\theta - \sin^2\eta}} \left.\frac{\partial\phi}{\partial\eta}\right|_{\theta=\theta_0} = \frac{u_0 \sin\eta}{\left(\dfrac{1}{e^2} - \sin^2\eta\right)^{0.5} \left[-\sqrt{\dfrac{1-e^2}{e^2}} + \dfrac{1}{e^2}\cot^{-1}\left(\sqrt{\dfrac{1-e^2}{e^2}}\right) \right]} \qquad (42)$$

Figure 4 shows the dimensionless tangential surface velocity, $u_{\eta 0}/u_0$, of an oblate spheroid as a function of η, for different values of the eccentricity, e. Note that, for the case of a sphere, $e \approx 0$, the well-known result of $u_{\eta 0} = 1.5 u_0 \sin\eta$ is obtained, for potential flow over the surface of the sphere. On the other hand, for a slender oblate, $e \to 1$, no simple expression for the velocity profile can be given, since it must be described by a series with many terms.
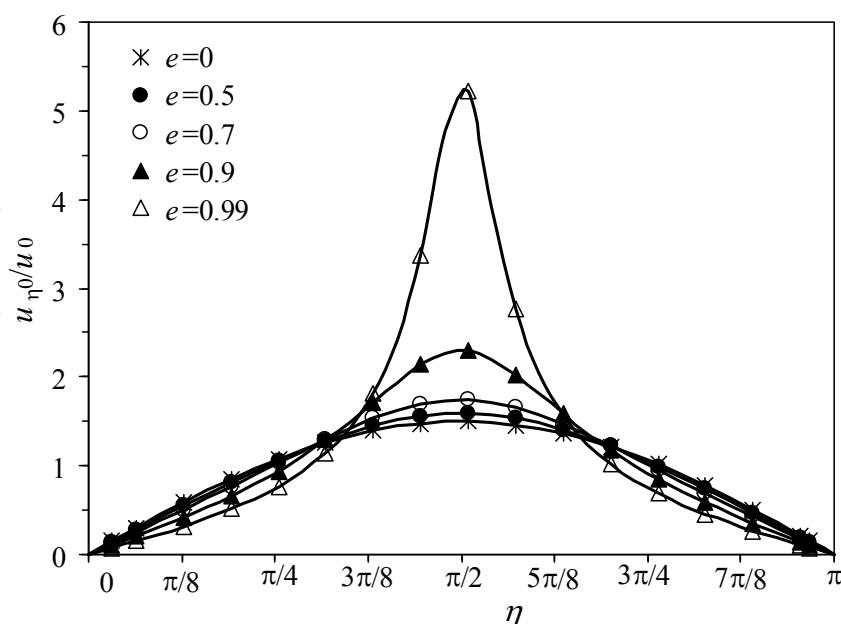

Fig. 4. The dimensionless tangential surface velocity, $u_{\eta 0}/u_0$, as a function of η, for different values of the eccentricity, e.

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A convenient way of expressing the differential mass balance on the solute is to take a control volume along a stream tube, between two nearby potential surfaces. The resulting expression, for convection with molecular diffusion, is given by Eq. (23).

For high values of the Peclet number the concentration boundary layer will be thin and the first term on the right hand side of Eq. (23) may be neglected. After some algebraic manipulation and a suitable change of variables, it is then possible to obtain

$$\frac{\partial C}{\partial \xi} = \frac{\partial^2 C}{\partial \psi^2} \tag{43}$$

where ξ is defined by

$$\xi = \int_0^{\eta} L'^3 \cosh^2 \theta_0 \sin^2 \eta' (\cosh^2 \theta_0 - \sin^2 \eta')^{1/2} u_\eta \, D'_m \, d\eta' \tag{44}$$

with $\omega = L' \cosh \theta_0 \sin \eta$. The boundary conditions for Eq. (43), in our problem, are given by Eqs. (26a) to (26c), and the corresponding solution is

$$\frac{C - C_0}{C^* - C_0} = 1 - erf\left(\frac{\psi}{2\sqrt{\xi}}\right) \tag{45}$$

The value of ξ varies over the surface of the spheroid. Now, for potential flow, u_η is given by Eq. (42) over the surface of the spheroid ($\theta = \theta_0$) and the integral in Eq. (44) is

$$\xi = u_0 D'_m \frac{e^3 c^3/(1-e^2)}{-e + e^3 + \sqrt{1-e^2} \cot^{-1}(\sqrt{1/e^2 - 1})} \left(\frac{2}{3} - \cos\eta + \frac{1}{3}\cos^3\eta\right) \tag{46}$$

The flux of solute at any point on the surface of the spheroid is

$$N = -D'_m \, \varepsilon \left(\frac{\partial C}{\partial b}\right) = -\frac{D'_m \, \varepsilon}{L'\sqrt{\cosh^2\theta_0 - \sin^2\eta}} = -D'_m \, \varepsilon \, u_\eta L' \cosh\theta_0 \sin\eta \left(\frac{\partial C}{\partial \psi}\right)_{\psi=0} \left(\frac{\partial C}{\partial \theta}\right)_{\theta=\theta_0} \tag{47}$$

and from Eq. (47) it may be shown that $(\partial C/\partial \psi)_{\psi=0} = \left(-1/\sqrt{(\pi\xi)}\right)(C^* - C_0)$. The rate of dissolution of the spheroid in the region $0 < \eta < \eta_1$ will then be

$$n(\eta_1) = \int_0^{\eta_1} N 2\pi L'^2 \sqrt{\cosh^2\theta_0 - \sin^2\eta} \cosh\theta_0 \sin\eta \, d\eta = 4\sqrt{\pi} \, \varepsilon \, \xi^{1/2}(C^* - C_0) \tag{48}$$

with $\xi(\eta_1)$ given by Eq. (46). In particular, the total rate of dissolution of the spheroid, n_T, may be obtained taking $\eta_1 = \pi$, and the resulting expression for k is

$$k = \frac{4\varepsilon\sqrt{\pi}}{2\pi m^2 \left(1 + \frac{1-e^2}{2e} \ln\left(\frac{1+e}{1-e}\right)\right)} \left(\frac{2}{3} \frac{e^3 u_0 D'_m c^3/(1-e^2)}{-e + e^3 + \sqrt{1-e^2} \cot^{-1}(\sqrt{1/e^2 - 1})}\right)^{1/2} \tag{49}$$

It is convenient to express the rate of dissolution in terms of the Sherwood number, $\mathrm{Sh}' = kd_{\mathrm{eq}}/D'_{\mathrm{m}}$, with $d_{\mathrm{eq}} = 2(a^2c)^{1/3}$, and the expression obtained, after some algebraic manipulation, is

$$\frac{\mathrm{Sh}'}{\varepsilon} = \sqrt{\frac{4}{\pi}\mathrm{Pe}'}\left(\frac{4}{3}\frac{e^3}{-e+e^3+\sqrt{1-e^2}\cot^{-1}(\sqrt{1/e^2-1})}\right)^{1/2}\frac{(1-e^2)^{1/3}}{1+\frac{1-e^2}{2e}\ln\left(\frac{1+e}{1-e}\right)} \tag{50}$$

were $\mathrm{Pe}' = u_0 d_{\mathrm{eq}}/D'_{\mathrm{m}}$ is the Peclet number.
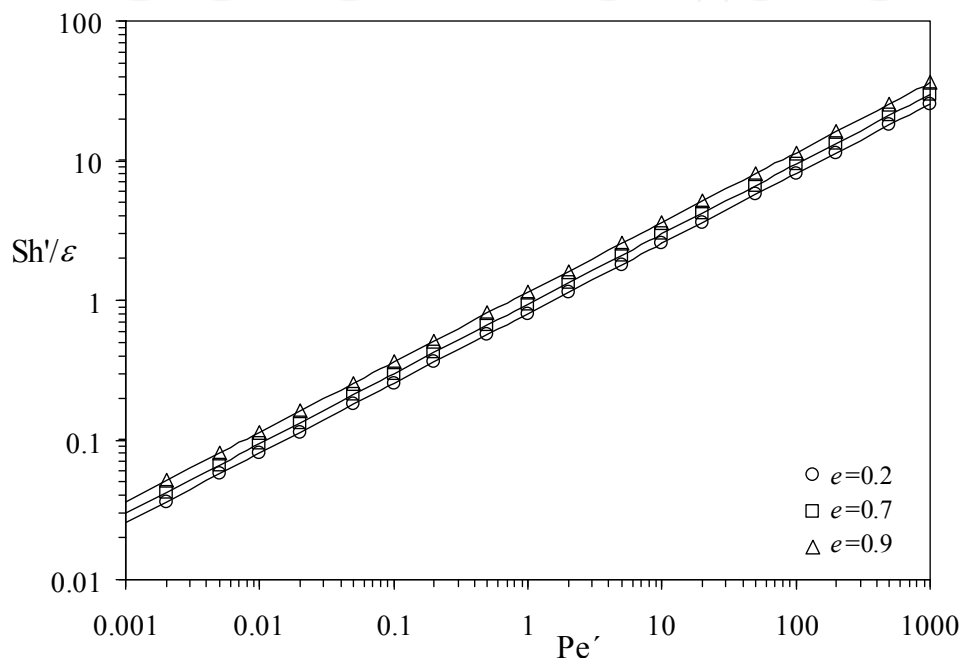

Fig. 5. Dependence of Sh'/ε on Pe', for different values of the eccentricity, e.

For the special case of a sphere, $e \approx 0$, the result of $\mathrm{Sh}'/\varepsilon = \left[4\,\mathrm{Pe}'/\pi\right]^{1/2}$ is obtained, which corresponds to the asymptotic behaviour for thin concentration layer (high values of Peclet number) when dispersion is constant and tends to D'_{m} over the surface of the sphere.
Figure 5 shows the dependence of Sh'/ε on Pe', for different values of the eccentricity, e. The total quantity of material transferred from an oblate spheroid is greater than that of a soluble sphere which is explained by the increasing of the Sh'/ε values with eccentricity.

3. Concentration profiles

3.1 Prolate spheroid buried in a packed bed
The analytical solution for a continuous point source has also been derived by Wexler (1992), solving the three-dimensional solute-transport equation from a point source. The solution is given by

$$c = \frac{n_{\mathrm{T}}}{4\varepsilon\pi(x^2+y^2)^{1/2}\,D'_{\mathrm{m}}}\exp\left[\frac{u_0\left[x-(x^2+y^2)^{1/2}\right]}{2D'_{\mathrm{m}}}\right] \tag{51}$$

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with $D_\text{T} \cong D_\text{L} \cong D'_\text{m}$ and a good estimate for n_T is required. The point source is located at the point $(x, y) = (0,0)$. In our case, the problem of mass transfer around a soluble prolate spheroid immersed in a granular bed of inerts through which fluid flow with uniform interstitial velocity, the mass flux rate is expressed as,

$$n_\text{T} = \varepsilon D'_\text{m} \sqrt{\frac{4}{\pi} \text{Pe}'_\text{ps}} \left(\frac{2}{3} \frac{e^3}{e - (1 - e^2) \tanh^{-1} e} \right)^{1/2} \frac{2\pi a}{(1 - e^2)^{1/6}} (C^* - C_0) \quad (52)$$

with Sherwood number given by Eq. (35). Making use of the dimensionless variables, Eq. (51) results, after re-arrangement, in

$$\frac{C - C_0}{C^* - C_0} = \frac{\frac{1}{2} \left[\frac{4}{\pi} \text{Pe}'_\text{ps} \right]^{1/2} \left(\frac{2}{3} \frac{e^3}{e - (1 - e^2) \tanh^{-1} e} \right)^{1/2}}{4 \left[(x/d_\text{eq})^2 + (y/d_\text{eq})^2 \right]^{1/2}} \times$$
$$\times \exp \left\{ \frac{\text{Pe}'_\text{ps}}{2} \left[x/d_\text{eq} - \left[(x/d_\text{eq})^2 + (y/d_\text{eq})^2 \right]^{1/2} \right] \right\} \quad (53)$$

3.2 Oblate spheroid buried in a packed bed

If a soluble oblate spheroid, buried in a packed bed, is exposed to uniform fluid flow with uniform interstitial velocity u_0, it will then release solute at a rate n, given by

$$n_\text{T} = \varepsilon D'_\text{m} \sqrt{\frac{4}{\pi} \text{Pe}'} \left(\frac{4}{3} \frac{e^3}{-e + e^3 + \sqrt{1 - e^2} \cot^{-1}(\sqrt{1/e^2 - 1})} \right)^{1/2} \frac{2\pi a^2 (1 - e^2)^{1/3}}{d_\text{eq}} (C^* - C_0) \quad (54)$$

with Sherwood number, Sh', given by Eq. (50). Making use of the dimensionless variables, Eq. (51) results, after re-arrangement, in

$$\frac{C - C_0}{C^* - C_0} = \frac{\frac{1}{2} \left[\frac{4}{\pi} \text{Pe}' \right]^{1/2} \left(\frac{4}{3} \frac{e^3}{-e + e^3 + \sqrt{1 - e^2} \cot^{-1}(\sqrt{1/e^2 - 1})} \right)^{1/2}}{4 \left[(x/d_\text{eq})^2 + (y/d_\text{eq})^2 \right]^{1/2}} \times$$
$$\times \exp \left\{ \frac{\text{Pe}'}{2} \left[x/d_\text{eq} - \left[(x/d_\text{eq})^2 + (y/d_\text{eq})^2 \right]^{1/2} \right] \right\} \quad (55)$$

For the special case of a sphere, $e \approx 0$, the result given by Eq. (56) is obtained. It is important to bear in mind that this result was obtained by Guedes de Carvalho *et al.* (2004) for mass transfer around a single sphere buried in a granular bed of inert particles,

$$\frac{C - C_0}{C^* - C_0} = \frac{\left[\frac{4}{\pi} \text{Pe} \right]^{1/2}}{4 \left[(x/d_1)^2 + (y/d_1)^2 \right]^{1/2}} \exp \left\{ \frac{\text{Pe}'}{2} \left[\frac{x}{d_1} - \left[\left(\frac{x}{d_1} \right)^2 + \left(\frac{y}{d_1} \right)^2 \right]^{1/2} \right] \right\} \quad (56)$$

where $\mathrm{Pe}' = u_0 d_1 / D'_{\mathrm{m}}$ is the Peclet number and d_1 is the diameter of the sphere ($d_1 = d_{\mathrm{eq}}$). Figure 6 show the concentration contour plots obtained, taking $\mathrm{Pe}' = 1000$ and $e = 0.5$ as an example, for low values of the dimensionless concentration. As the value of C decreases, the distance of the contour surfaces to the solid soluble particle increases and the solution for the "continuous point source" approach to the "exact" solution, possible to obtain numerically (*i.e.* if a correct value of n_{T} is used, true coincidence is observed).
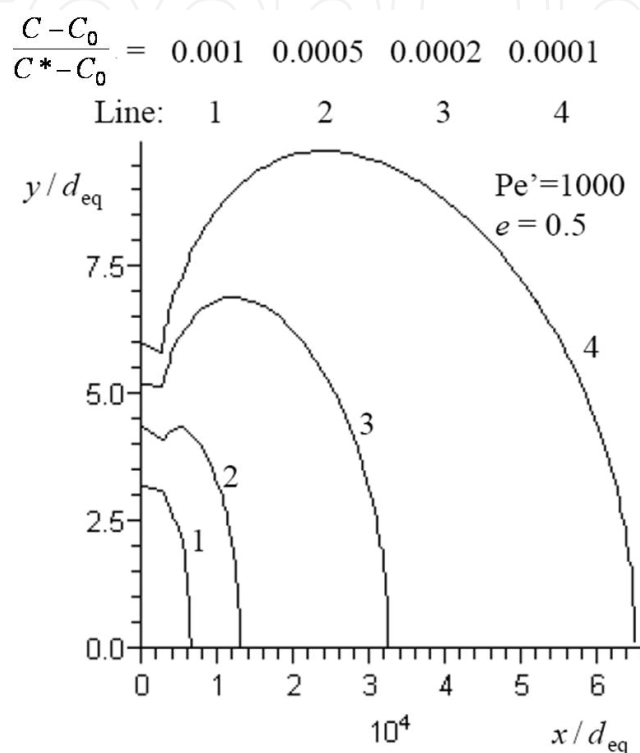
Fig. 6. Dimensionless concentration contour plots obtained with the solution for the "continuous point source", at long distances from the spheroid.

4. Conclusions

The problem of mass transfer around a spheroid buried in a granular bed (be it packed or incipiently fluidised) lends itself to a simple full theoretical analysis, under an appropriate set of conditions. If Darcy flow is considered in the packing, the differential equation describing mass transfer may be obtained analytically considering two asymptotes: one for $\mathrm{Pe} \to 0$ and the other for convection with molecular diffusion across a thin boundary layer, being the results described by Eq. (35), for the case of a prolate spheroid and Eq. (50), for the case of an oblate spheroid.

Results of the analytical solutions were also used to predict the solute migration from an active prolate or oblate spheroid buried in a packed bed of inert particles, through which fluid flows with uniform velocity. The concentration contour surfaces were obtained using an analytical solution of continuous injection of solute at a point source in a uniform stream and the proposed correlations for the mass transfer rate developed.

5. Nomenclature

a, c	Semi-axis of the spheroid
C	Solute concentration
C_0	Bulk concentration of solute
C^*	Saturation concentration of solute
d_{eq}	Equivalent diameter
D_m	Molecular diffusion coefficient
D'_m	Effective molecular diffusion coefficient ($= D_m / \tau$)
e	Eccentricity
K	Permeability in Darcy's law
k	Average mass transfer coefficient
L'	Focus distance
n	Mass transfer rate
n_T	Total mass transfer rate
N	Local flux of solute
p	Pressure
Pe'	Peclet number ($= u_0 d_{eq} / D'_m$)
S	Surface area
Sh'	Sherwood number ($= k d_{eq} / D'_m$)
u	Interstitial velocity (vector)
u_0	Absolute value of interstitial velocity far from the active spheroid
u_η, u_θ	Components of fluid interstitial velocity
V	Volume
x, y, z	Cartesian coordinates

5.1 Greek letters

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β	Spheroidal coordinate
ε	Bed voidage
ϕ	Potential function (defined in Eq. (15) and Eq. (36))
θ	Spheroidal coordinate
η	Spheroidal coordinate
τ	Tortuosity
ω	Cylindrical radial coordinate, distance to the axis ($= L' \sinh \theta_0 \sin \eta$)
ξ	Variable (defined in Eq. (25) and Eq. (44))
ψ	Stream function (defined in Eq. (16) and Eq. (37))

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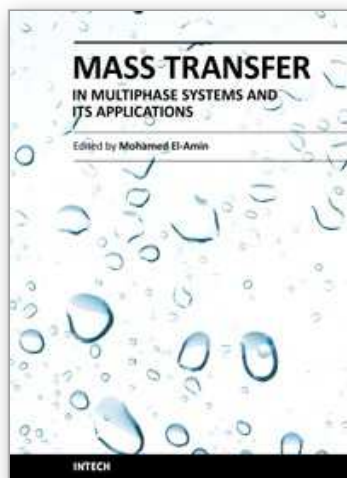
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