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# Linear Differential Games and High Precision Attitude Stabilization of Spacecrafts With Large Flexible Elements

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## 1. Introduction

Space mission success depends on adequate attitude control system performance. High precision pointing control design for flexible spacecraft with large appendages requires special care (Yamada & Yoshikawa, 1996; Nagashio, 2010). Many linear control schemes such as LQR, LQG, QFT,  $H_\infty$ , and  $\mu$ -synthesis have been used to guarantee robust stability against both vibrations and parameter uncertainties. Future advanced space missions will involve control system design techniques with novel architectures, technologies, and algorithms. Next generation of microcomputers will allow one to use more sophisticated control algorithms. Pontryagin's theory of linear differential games (Pontryagin, 1981) is a natural tool to solve local stabilization problems under conditions of uncertainty. The corresponding mathematical model involves a linear differential equation

$$\dot{x} = Ax - u + v, \quad (1)$$

with two control parameters  $u \in P$  and  $v \in Q$  belonging to bounded sets. One can choose the control  $u$  so as to bring the point  $x$  into the terminal set  $S$ , while the term  $v$  represents the disturbances caused by the model uncertainties, vibration of flexible elements, and state estimation errors. Differential games of stabilization (Smirnov, 2002) have some characteristic features that distinguish them from other differential games. Namely, the terminal set  $S$  should be invariant (see Sec. 3). Moreover, the zero equilibrium position of the linear differential equation

$$\dot{x} = Ax, \quad (2)$$

is assumed to be asymptotically stable. Equation (2) can be obtained applying standard linear stabilization techniques to the control system under consideration. The trajectories of the closed-loop system

$$\dot{x} = Ax + v, \quad (3)$$

subject to the disturbance  $v$ , approach a limit set  $\Omega$  and not tend to zero. The aim of the differential game approach is to construct a smaller invariant set  $S \subset \Omega$ , to bring the trajectory  $x(t)$  into  $S$ , and to maintain it there.

The idea to use differential game methods to solve stabilization problems with uncertainties is not new (see (Gutman & Leitmann, 1975a; Gutman & Leitmann, 1975b), for example). However, the previous studies are based on Lyapunov functions (Gutman, 1979) or solutions to the Hamilton-Jacobi-Isaacs-Bellman equation (Isaacs, 1965; Kurzanski & Varaiya, 2002; Mitchell et al., 2005). In this chapter we discuss theoretical and computational aspects of differential games of stabilization considered in the framework of Pontryagin's approach. Theoretically both approaches are equivalent (Kurzanski & Melnikov, 2000). But, from the computational point of view, the geometric language of Pontryagin's method (alternating integral, support functions, etc.) proves more efficient in constructing the stabilizing control than the approach based on Lyapunov functions or the solution to Hamilton-Jacobi-Isaacs-Bellman equation. Since a local stabilization problem is considered, linear model (1) suffices to adequately describe the system behaviour. Although the dynamics is linear, the approach itself is based on the usage of nonlinear methods such as convex analysis and nonlinear programming.

This chapter is organized in the following way. First we give a short introduction to the theory of linear differential games and describe numerical methods developed for the alternating integral approximation. We also address the problem of construction of a minimal invariant set. Next, we apply the developed techniques to the problem of high precision attitude stabilization of spacecrafts with large flexible elements.

## 2. Mathematical background

Throughout this chapter we denote the set of real numbers by  $R$  and the usual  $n$ -dimensional space of vectors  $x = (x_1, \dots, x_n)$ , where  $x_i \in R$ ,  $i = \overline{1, n}$ , by  $R^n$ . The inner product of two vectors  $x$  and  $y$  in  $R^n$  is expressed by

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n.$$

The norm of a vector  $x \in R^n$  is defined by  $\|x\| = \langle x, x \rangle^{1/2}$ . Let  $A$  be a linear operator from  $R^n$  to  $R^m$ . If  $A$  is an  $m \times n$  real matrix corresponding to the linear operator  $A$  (we use the same symbol), then the transposed matrix  $A^*$  corresponds to the adjoint operator. The unit linear operator from  $R^n$  to  $R^n$  will be denoted by  $I_n$ . We denote the unit ball in  $R^n$  by  $B_n$ :

$$B_n = \{x \in R^n \mid \|x\| \leq 1\}.$$

Let  $C \subset R^n$ . The distance function  $d(\cdot, C) : R^n \rightarrow R$  is defined by

$$d(x, C) = \inf\{\|x - c\| \mid c \in C\}, \quad x \in R^n.$$

Let  $\lambda \in R$ . Then put by definition

$$\lambda C = \{\lambda c \mid c \in C\}.$$

For two sets  $C_1$  and  $C_2$  in  $R^n$  their sum is defined by

$$C_1 + C_2 = \{c_1 + c_2 \mid c_1 \in C_1, c_2 \in C_2\}.$$

A set  $C \subset R^n$  is said to be convex if  $\lambda x + (1 - \lambda)y \in C$  whenever  $x \in C$ ,  $y \in C$ , and  $\lambda \in [0, 1]$ . From definition it follows that an intersection of any number of convex sets is a convex set, and if  $C_1 \subset R^n$ ,  $C_2 \subset R^n$  are convex, and  $\alpha_1$  and  $\alpha_2$  are real numbers, then the set  $\alpha_1 C_1 + \alpha_2 C_2$  is convex. Let  $C \subset R^n$ . The intersection of all convex sets containing  $C$  is called the convex hull of  $C$  and is denoted by  $\text{co}A$ .

All convex sets  $C \subset R^n$  considered in this chapter are assumed to be symmetric, i.e.,  $C = -C$ . This family is enough for our goals. The function  $S(\cdot, C) : R^n \rightarrow R$  defined by

$$S(\varphi, C) = \sup\{\langle x, \varphi \rangle \mid x \in C\}$$

is called the support function of  $C$ . The distance function can be expressed via support function:

$$d(x, C) = \sup\{\langle x, \varphi \rangle - S(\varphi, C) \mid \varphi \in B_n\}. \tag{4}$$

Moreover, the support function allows one to express the inclusion  $x \in C$  in an analytical form. Namely,  $x \in C$  if and only if  $\langle x, \varphi \rangle \leq S(\varphi, C)$  for all  $\varphi \in B_n$ .

Another description of a compact convex set  $C$  can be obtained in terms of the Minkowski function. A point  $x \in C$  if and only if  $\mu(x, C) \leq 1$ , where

$$\mu(x, C) = \inf\{t > 0 \mid t^{-1}x \in C\}$$

is the Minkowski function of the set  $C$ .

Let  $C_1$  and  $C_2$  be convex sets in  $R^n$ . The Minkowski difference of these sets is defined by

$$C_1 \overset{*}{-} C_2 = \{c \mid c + C_2 \subset C_1\}.$$

If  $C_1$  is convex and closed and  $C_2$  is compact, then  $C_1 \overset{*}{-} C_2$  is closed and convex. It is easy to see that the following relations hold

$$(C_1 \overset{*}{-} C_2) + C_2 \subset C_1,$$

and

$$(C_1 + C_2) \overset{*}{-} C_2 = C_1.$$

Moreover if  $C_3$  is compact and convex, then we have

$$(C_1 \overset{*}{-} C_2) + C_3 \subset (C_1 + C_3) \overset{*}{-} C_2.$$

The Hausdorff distance between two sets  $C_1, C_2 \subset R^n$  is defined as

$$h(C_1, C_2) = \min\{h \geq 0 \mid C_1 \subset C_2 + hB_n, C_2 \subset C_1 + hB_n\}.$$

A set-valued map  $F : R^n \rightarrow R^m$  with compact values is called continuous at  $x_0 \in R^n$  if for any  $\epsilon > 0$  there exists  $\delta > 0$  such that  $h(F(x), F(x_0)) < \epsilon$ , whenever  $x \in x_0 + \delta B_n$ .

Let  $F : [a, b] \rightarrow R^n$  be a set-valued map with compact convex values. Its Riemann integral  $\int_a^b F(t)dt$  is defined as a limit in the sense of the Hausdorff distance of the integral sums  $\sum_k F(\xi_k)(t_{k+1} - t_k)$ , where  $a = t_0 < t_1 < \dots < t_N = b$ , is a partition of the interval  $[a, b]$  and  $\xi_k \in [t_k, t_{k+1}]$ . The integral exists whenever  $F$  is continuous at all points of the interval. Moreover, it coincides with the set of all integrals of integrable selections  $f(t) \in F(t)$

$$\int_a^b F(t)dt = \left\{ \int_a^b f(t)dt \mid f(t) \in F(t), t \in [a, b] \right\}$$

(see (Castaing & Valadier, 1977)) and

$$S\left(\varphi, \int_a^b F(t)dt\right) = \int_a^b S(\varphi, F(t))dt$$

for all  $\varphi \in R^n$ . This equality allows one to compute integrals of set-valued maps.

### 3. Linear differential games of pursuit

Differential games are control problems in a conflict situation. For example, if one aircraft pursues another one we have such a situation. The dynamics of the system is described by a differential equation depending on two control parameters

$$\dot{x} = f(x, u, v), \quad u \in P, \quad v \in Q.$$

One player controls the parameter  $u$  and the other one controls the parameter  $v$ . The aim of the first player is to drive the system to a terminal set  $S$  while the aim of the second player is to avoid this event. The solution to this problem consists in determination of functions  $u = u(x)$  and  $v = v(x)$ , known as strategies (Krasovski & Subbotin, 1987), guaranteeing

1. the fastest arrival to the terminal set  $S$  for the first player;
2. the latest arrival to the terminal set  $S$  for the second player.

In this form the problem is extremely involved because the functions  $u(x)$  and  $v(x)$  can be discontinuous and the differential equation  $\dot{x} = f(x, u(x), v(x))$  may have no solution in the classical sense. This difficulty can be overcome introducing the concept of Pontryagin's  $\epsilon$ -strategy. The differential game is considered as a pursuit game or an evasion game.

In the first case we identify ourselves with the first player. At the initial moment of time  $t_0$  the second player communicates to the first player a number  $\epsilon_0 > 0$  and his control  $v(t)$  defined in the time interval  $[t_0, t_0 + \epsilon_0]$ . The first player uses this information to choose his own control  $u(t)$ ,  $t \in [t_0, t_0 + \epsilon_0]$ . Next, at the moment of time  $t_1 = t_0 + \epsilon_0$  the second player communicates to the first player a number  $\epsilon_1 > 0$  and his control  $v(t)$  defined in the time interval  $[t_1, t_1 + \epsilon_1]$ . The first player uses this information to choose his control  $u(t)$ ,  $t \in [t_1, t_1 + \epsilon_1]$ , and so on. In the case of evasion games we identify ourselves with the second player and the first one communicates us numbers  $\epsilon_k > 0$  and controls  $u(t)$ ,  $t \in [t_k, t_k + \epsilon_k]$ .

Here we study only pursuit games for linear control systems

$$\dot{x} = Ax - u + v \tag{5}$$

and without the objective to reach the terminal set  $S$  in an optimal time. Our aim is to finish the game in a time  $T$  not necessarily optimal. The sets  $P$  and  $Q$  are assumed to be compact and convex. The terminal set  $S$  is closed and convex. Moreover, the number  $\epsilon$  from the definition of the  $\epsilon$ -strategy is assumed to be fixed.

Differential games of stabilization have some characteristic features that distinguish them from other differential games. Namely, the terminal set  $S$  should be invariant, i.e., if  $x_0 \in S$ , then for any control  $v(t) \in Q$  there should exist a control  $u(t) \in P$  such that it maintains the trajectory  $x(t, x_0, u(\cdot), v(\cdot))$  in the set  $S$ . There are many possibilities to formalize the concept of invariance. We say that a set  $S$  is  $\epsilon$ -invariant if the following inclusion holds

$$\Lambda_\epsilon S + Q_\epsilon \subset S + P_\epsilon. \tag{6}$$

Here we use the notations

$$\Lambda_\epsilon = e^{\epsilon A}, \quad P_\epsilon = \int_0^\epsilon e^{tA} P dt, \quad \text{and} \quad Q_\epsilon = \int_0^\epsilon e^{tA} Q dt. \tag{7}$$

Let  $x \in \mathcal{S}$  and  $v(t) \in Q, t \in [0, \epsilon]$ , be an admissible disturbance. After a change of variable in integrals (7) we obtain

$$\mathcal{P}_\epsilon = \int_0^\epsilon e^{(\epsilon-t)A} P dt \text{ and } \mathcal{Q}_\epsilon = \int_0^\epsilon e^{(\epsilon-t)A} Q dt \quad (8)$$

From (6) and (8) we see that there exists an admissible control  $u(t) \in Q, t \in [0, \epsilon]$  such that

$$x(\epsilon) = e^{\epsilon A} x - \int_0^\epsilon e^{(\epsilon-t)A} u(t) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t) dt \in \mathcal{S},$$

i.e., starting at  $\mathcal{S}$  we always return to it after time  $\epsilon$ .

The zero equilibrium position of the linear differential equation

$$\dot{x} = Ax, \quad (9)$$

is assumed to be asymptotically stable. This implies that there exists a positive definite symmetric matrix  $V$  satisfying the Lyapunov equation

$$VA + A^*V = -I_n.$$

If  $\alpha > 0$  is sufficiently large and  $\epsilon$  is sufficiently small, then the ellipsoid  $\alpha E$ , where

$$E = \{x \mid \langle x, Vx \rangle \leq 1\} \quad (10)$$

is  $\epsilon$ -invariant.

Consider the sets  $\mathcal{I}_k \subset R^n$  defined by  $\mathcal{I}_0 = \mathcal{S}$ ,

$$\mathcal{I}_{k+1} = (\mathcal{I}_k + \mathcal{P}_\epsilon) \overset{*}{-} \mathcal{Q}_\epsilon, \quad k = \overline{0, N-1}.$$

The sets  $\mathcal{I}_k$  are known as Pontryagin alternating sums. Fix  $T > 0$  and set  $\epsilon = T/N$ . The limit of the Pontryagin alternating sums as  $N$  goes to infinity,

$$\mathcal{I}_T = \lim_{N \rightarrow \infty} \mathcal{I}_N,$$

is called Pontryagin alternating integral. The inclusion  $\Lambda_\epsilon x \in \mathcal{I}_{k+1}$  implies that for any admissible disturbance  $v(t) \in Q, t \in [0, \epsilon]$ , there exists an admissible control  $u(t) \in Q, t \in [0, \epsilon]$  such that

$$x(\epsilon) = e^{\epsilon A} x - \int_0^\epsilon e^{(\epsilon-t)A} u(t) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t) dt \in \mathcal{I}_k.$$

By induction we see that if  $\Lambda_\epsilon^N x \in \mathcal{I}_N$ , then the game can be finished in time  $N\epsilon$ , i.e., that the first player can choose an  $\epsilon$ -strategy in order to guarantee the inclusion  $x(N\epsilon) \in \mathcal{I}_0 = \mathcal{S}$ . The set

$$\mathcal{F}_T = e^{-TA} \mathcal{I}_T$$

is known as Pontryagin-Pshenichnyj pursuit operator and consists of all initial points  $x_0$  such that the game starting from  $x_0$  can be finished in time less than or equal to  $T$  independently on the  $\epsilon$ -strategy of the second player. Instead of the Pontryagin alternating sums we shall use the Pontryagin-Pshenichnyj pursuit  $\epsilon$ -operators defined by  $\mathcal{F}_0 = \mathcal{S}$ ,

$$\mathcal{F}_k = \mathcal{F}_\epsilon^k(\mathcal{S}), \quad k = \overline{0, N}, \quad (11)$$

where

$$\mathcal{F}_\epsilon(\mathcal{C}) = \Lambda_\epsilon^{-1} \left( (\mathcal{C} + \mathcal{P}_\epsilon) \overset{*}{-} \mathcal{Q}_\epsilon \right).$$

If the set  $\mathcal{F}_k$  is  $\epsilon$ -invariant (see (6)), then the set  $\mathcal{F}_{k+1}$  is also invariant. Indeed, from the inclusion

$$\Lambda_\epsilon \mathcal{F}_k + \mathcal{Q}_\epsilon \subset \mathcal{F}_k + \mathcal{P}_\epsilon,$$

we have

$$\mathcal{F}_k \subset \Lambda_\epsilon^{-1} \left( (\mathcal{F}_k + \mathcal{P}_\epsilon) \overset{*}{-} \mathcal{Q}_\epsilon \right) = \mathcal{F}_{k+1}. \quad (12)$$

This implies

$$\Lambda_\epsilon \mathcal{F}_{k+1} + \mathcal{Q}_\epsilon = \left( (\mathcal{F}_k + \mathcal{P}_\epsilon) \overset{*}{-} \mathcal{Q}_\epsilon \right) + \mathcal{Q}_\epsilon \subset \mathcal{F}_k + \mathcal{P}_\epsilon \subset \mathcal{F}_{k+1} + \mathcal{P}_\epsilon.$$

Therefore the operators  $\mathcal{F}_k$  form a monotone family, provided that the terminal set  $\mathcal{S}$  is  $\epsilon$ -invariant.

To construct an absorbing family of Pontryagin-Pshenichnyj operators, suppose that the terminal set is strictly invariant, i.e.,

$$\Lambda_\epsilon(\mathcal{S} + E) + \mathcal{Q}_\epsilon \subset \mathcal{S} + \mathcal{P}_\epsilon,$$

where the ellipsoid  $E$  satisfies the inclusion  $\Lambda_\epsilon E \subset E$  (see (10)). The strict invariance of  $\mathcal{S}$  implies the inclusion  $\mathcal{S} + E \subset \mathcal{F}_\epsilon(\mathcal{S})$ . Observe that

$$\Lambda_\epsilon(\mathcal{F}_\epsilon(\mathcal{S}) + E) + \mathcal{Q}_\epsilon = ((\mathcal{S} + \mathcal{P}_\epsilon) \overset{*}{-} \mathcal{Q}_\epsilon) + \Lambda_\epsilon E + \mathcal{Q}_\epsilon \subset \mathcal{S} + \mathcal{P}_\epsilon + E \subset \mathcal{F}_\epsilon(\mathcal{S}) + \mathcal{P}_\epsilon.$$

From this we obtain

$$\mathcal{S} + 2E \subset \mathcal{F}_\epsilon(\mathcal{S}) + E \subset \mathcal{F}_\epsilon^2(\mathcal{S}).$$

By induction we have  $\mathcal{S} + kE \subset \mathcal{F}_\epsilon^k(\mathcal{S})$ . Therefore  $\bigcup_{k \geq 0} \mathcal{F}_\epsilon^k(\mathcal{S}) = R^n$ .

#### 4. Computational aspects

To numerically compute the pursuit operator and the stabilizing control, the considered sets should be approximated by polyhedrons. In this section we briefly present the computational geometry tools necessary for this purpose.

Fix two sets of unit vectors  $\{\varphi_m\}_{m=1}^M$  and  $\{\xi_l\}_{l=1}^L$ . An exterior polyhedral approximation,  $\bar{C}$ , of a convex compact set  $C \subset R^n$  is given by

$$C \subset \bar{C} = \{x \mid \langle \varphi_m, x \rangle \leq S(\varphi_m, C), m = \overline{1, M}\},$$

and an interior polyhedral approximation,  $\underline{C}$ , of a convex compact set  $C \subset R^n$  is given by

$$C \supset \underline{C} = \text{co} \left\{ (\mu(\xi_l, C))^{-1} \xi_l \mid l = \overline{1, L} \right\}.$$

We shall use the notations  $\sigma_m = S(\varphi_m, C)$  and  $\mu_l = \mu(\xi_l, C)$  for the values of the support function and of the Minkowski function, respectively. The vectors  $\sigma = (\sigma_1, \dots, \sigma_M)$  and  $\mu = (\mu_1, \dots, \mu_L)$  define the exterior and interior approximations of a compact convex set  $C$ . We say that the exterior and interior approximations are consistent if the following conditions are satisfied:

1.  $\langle \xi_l, \varphi_m \rangle \leq \mu_l \sigma_m$ , for all  $l = \overline{1, L}$  and  $m = \overline{1, M}$ ,
2. for any  $l = \overline{1, L}$  there exists  $m(l)$  such that  $\langle \xi_l, \varphi_{m(l)} \rangle = \mu_l \sigma_{m(l)}$ ,
3. for any  $m = \overline{1, M}$  there exists  $l(m)$  such that  $\langle \xi_{l(m)}, \varphi_m \rangle = \mu_{l(m)} \sigma_m$ .

If exterior and interior descriptions  $\sigma = (\sigma_1, \dots, \sigma_M)$  and  $\mu = (\mu_1, \dots, \mu_L)$  are not consistent, they can be made consistent using one of adjustment operators  $\mu \rightarrow \mathcal{A}^\sigma(\mu)$  and  $\sigma \rightarrow \mathcal{A}^\mu(\sigma)$  defined by

$$\mathcal{A}^\sigma(\mu) = (\sigma_1(\mu), \dots, \sigma_M(\mu)), \quad \sigma_m(\mu) = \max_{l=1, L} \mu_l^{-1} \langle \xi_l, \varphi_m \rangle$$

and

$$\mathcal{A}^\mu(\sigma) = (\mu_1(\sigma), \dots, \mu_L(\sigma)), \quad \mu_l(\sigma) = \left( \min_{\substack{m=1, M \\ \langle \xi_l, \varphi_m \rangle > 0}} \frac{\sigma_m}{\langle \xi_l, \varphi_m \rangle} \right)^{-1}.$$

Let  $C_1$  and  $C_2$  be two convex compact sets, and let  $\sigma(C_1)$  and  $\sigma(C_2)$  be the vectors defining their exterior approximations. Since  $S(\varphi, C_1 + C_2) = S(\varphi, C_1) + S(\varphi, C_2)$ , it is natural to define the exterior approximation vector  $\sigma(C_1 + C_2)$  for the sum as

$$\sigma(C_1 + C_2) = \sigma(C_1) + \sigma(C_2).$$

The evaluation of the approximation for the Minkowski difference  $C_1 -^* C_2$  is more involved. The point is that the difference of support functions  $S(\varphi, C_1) - S(\varphi, C_2)$  may be not a support function of a convex set and some correction is needed. This correction is done using the interior description. Namely, we set

$$\sigma(C_1 -^* C_2) = \mathcal{A}^\sigma(\mathcal{A}^\mu(\sigma(C_1) - \sigma(C_2))).$$

If the vectors  $\{\varphi_m\}_{m=1}^M$  and  $\{\xi_l\}_{l=1}^L$  form rather fine meshes in the unite sphere, the above exterior approximations of the sum and the Minkowski difference given by

$$\{x \mid \langle x, \varphi_m \rangle \leq \sigma_m(C_1 + C_2), m = \overline{1, M}\}$$

and

$$\{x \mid \langle x, \varphi_m \rangle \leq \sigma_m(C_1 -^* C_2), m = \overline{1, M}\}$$

tend to  $C_1 + C_2$  and  $C_1 -^* C_2$ , respectively, as  $M$  and  $L$  go to infinity. Some estimates for the precision of the approximations can be found in (Polovinkin et al., 2001).

The approximation of the set  $\Lambda C$ , where  $\Lambda : R^n \rightarrow R^n$  is a linear operator, is based on the following property of support functions:

$$S(\varphi, \Lambda C) = S(\Lambda^* \varphi, C) = \|\Lambda^* \varphi\| S\left(\frac{\Lambda^* \varphi}{\|\Lambda^* \varphi\|}, C\right)$$

and is computed as

$$S(\varphi_m, \Lambda C) = \|\Lambda^* \varphi_m\| S(\varphi_{\lambda(m)}, C),$$

where the vector  $\varphi_{\lambda(m)}$  satisfies the condition

$$\left\| \varphi_{\lambda(m)} - \frac{\Lambda^* \varphi_m}{\|\Lambda^* \varphi_m\|} \right\| = \min_{m'=\overline{1, M}} \left\| \varphi_{m'} - \frac{\Lambda^* \varphi_m}{\|\Lambda^* \varphi_m\|} \right\|.$$

Now, consider the problem of a minimal invariant set construction. Let  $\mathcal{P} \subset R^n$  and  $\mathcal{Q} \subset R^n$  be convex compact sets, and let  $\Lambda : R^n \rightarrow R^n$  be a linear operator. The condition of a convex set  $\mathcal{S}$  invariance,

$$\Lambda\mathcal{S} + \mathcal{Q} \subset \mathcal{S} + \mathcal{P}, \quad (13)$$

in terms of support functions takes the form

$$S(\varphi, \Lambda\mathcal{S}) + S(\varphi, \mathcal{Q}) \leq S(\varphi, \mathcal{S}) + S(\varphi, \mathcal{P}), \text{ for all } \varphi, \|\varphi\| = 1. \quad (14)$$

We say that an invariant set  $\mathcal{S}$  is minimal, if for any  $\mathcal{S}' \subset \mathcal{S}$ ,  $\mathcal{S}' \neq \mathcal{S}$ , we have  $\Lambda\mathcal{S}' + \mathcal{Q} \not\subset \mathcal{S}' + \mathcal{P}$ . Note that the minimal invariant set may be not unique and that the intersection of two invariant sets may be not invariant. Indeed, consider the following example in  $R^2$ . Let  $\Lambda = \frac{1}{2}I_2$ ,  $\mathcal{P} = \text{co}\{(0,2), (0,-2)\}$ , and  $\mathcal{Q} = \text{co}\{(1,1), (1,-1), (-1,1), (-1,-1)\}$ . It is easy to see that any set  $\mathcal{S}_a = \{(x, ax) \mid x \in [-2, 2]\}$ ,  $a \in [-1, 1]$ , is minimal invariant. The intersection  $\mathcal{S}_{a_1} \cap \mathcal{S}_{a_2} = \{0\}$ ,  $a_1 \neq a_2$ , is not invariant.

To restrict the set of invariant sets, we introduce the following definition. Put

$$r(\mathcal{S}) = \min\{r > 0 \mid \mathcal{S} \subset rB_n\}.$$

An invariant set  $\mathcal{S}$  is said to be  $r$ -minimal, if for any  $\mathcal{S}'$  satisfying  $r(\mathcal{S}') < r(\mathcal{S})$ , we have  $\Lambda\mathcal{S}' + \mathcal{Q} \not\subset \mathcal{S}' + \mathcal{P}$ . In the previous example a unique  $r$ -minimal invariant set is  $\text{co}\{(1,0), (-1,0)\}$ .

Note that in general the  $r$ -minimality does not define a unique invariant set, as it is clear from the following example. Set

$$\Lambda = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$\mathcal{P} = \text{co}\{(0,1), (0,-1)\}$ , and  $\mathcal{Q} = \text{co}\{(1,1), (1,-1), (-1,1), (-1,-1), (0,2), (0,-2)\}$ . It is easy to see that the sets  $\mathcal{S}_1 = 2B_2$  and  $\mathcal{S}_1 = \text{co}\{(1,0), (-1,0), (0,1), (0,-1)\}$  are both  $r$ -minimal invariant.

Although the property of  $r$ -minimality does not define a unique invariant set, it is quite suitable from the practical point of view.

We developed the following algorithm to compute a minimal invariant set. Let  $\mathcal{S}_0$  be an invariant set. (Recall that in the case of a differential game of stabilization there always exists an invariant ellipsoid (see Sec. 3).) Then we obtain an interior approximation of  $\mathcal{S}_0$  described by a vector  $\mu^{(0)} = (\mu_1^{(0)}, \dots, \mu_L^{(0)})$  and set  $\mathcal{S}_0 = \text{co}\{\pm(\mu_1^{(0)})^{-1}\xi_1, \dots, \pm(\mu_L^{(0)})^{-1}\xi_L\}$ . Let  $\delta > 0$ .

The current invariant set  $\mathcal{S}_k$  is successively shrunk going through the vectors  $\xi_l$ ,  $l = \overline{1, L}$ , and considering the sets

$$\mathcal{S}_k^l = \text{co}\{\pm(\mu_1^{(0)})^{-1}\xi_1, \dots, \pm(\mu_l^{(0)} + \delta)^{-1}\xi_l, \dots, \pm(\mu_L^{(0)})^{-1}\xi_L\}.$$

If the set  $\mathcal{S}_k^l$  is invariant, we put  $\mathcal{S}_{k+1} = \mathcal{S}_k^l$ . After passing through all vectors  $\xi_l$ ,  $l = \overline{1, L}$ , the algorithm turns to the vector  $\xi_1$ . The algorithm stops if none of the modified sets  $\mathcal{S}_k^l$ ,  $l = \overline{1, L}$ , is invariant. This algorithm is very simple and efficient. However, in general, it does not lead to a  $r$ -minimal invariant sets.

The problem of  $r$ -minimal invariant set construction is more involved and can be solved using nonlinear programming techniques. The invariance condition (14) implies that the vector  $\sigma^r = (\sigma_1^r, \dots, \sigma_M^r)$  giving the external description of a  $r$ -minimal invariant set has to be a solution to

the following linear programming problem

$$\begin{aligned} r &\rightarrow \min, \\ \|\Lambda^* \varphi_m\| \sigma_{\lambda(m)} + q_m &\leq \sigma_m + p_m, \quad m = \overline{1, M}, \\ 0 \leq \sigma_m &\leq r, \quad m = \overline{1, M}, \end{aligned}$$

where  $p_m = S(\varphi_m, P)$ ,  $q_m = S(\varphi_m, Q)$ , and  $\sigma_m$ ,  $m = \overline{1, M}$ , and  $r$  are the unknown variables. Unfortunately the solution to this problem is not unique and a vector  $\sigma$ , solving the problem, may be not a vector of a support function values. For this reason it is necessary to use inner approximations for the invariant set and solve the following nonlinear programming problem

$$\begin{aligned} r &\rightarrow \min, \\ \max_{l=\overline{1, L}} \langle \mu_l^{-1} \Lambda \xi_l, \varphi_m \rangle + q_m &\leq \max_{l=\overline{1, L}} \langle \mu_l^{-1} \xi_l, \varphi_m \rangle + p_m, \quad m = \overline{1, M}, \\ 0 \leq \mu_l^{-1} &\leq r, \quad l = \overline{1, L}, \end{aligned}$$

with the variables  $\mu_l$ ,  $l = \overline{1, L}$ , and  $r$ .

A very important issue is the stabilizing control  $u$  construction. Assume that the current position of the system  $x_k$  belongs to the set  $\mathcal{F}_{N-k}$ . To determine the stabilizing control  $u(t)$  defined on the interval  $[k\epsilon, (k+1)\epsilon]$  we numerically solve the optimal control problem

$$\begin{aligned} d \left( e^{\epsilon A} x_k - \int_0^\epsilon e^{(\epsilon-t)A} u(k\epsilon + t) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(k\epsilon + t) dt, \mathcal{F}_{N-k-1} \right) &\rightarrow \min, \\ u(k\epsilon + t) &\in P. \end{aligned}$$

The distance function is calculated using representation (4) and the control  $u(t)$ ,  $t \in [k\epsilon, (k+1)\epsilon]$ , is considered to be a piece-wise constant function,  $u(t) = u_j$ ,  $t \in [(k-j/J)\epsilon, (k-(j+1)/J)\epsilon]$ ,  $j = \overline{0, J-1}$ . Approximating the set  $P$  by a polyhedron, we get the linear programming problem

$$\begin{aligned} r &\rightarrow \min \\ \left\langle e^{\epsilon A} - \frac{\epsilon}{J} \sum_{j=1}^J e^{\epsilon(1-j/J)A} u_j + \int_0^\epsilon e^{(\epsilon-t)A} v(k\epsilon + t) dt, \varphi_m \right\rangle - S(\varphi_m, \mathcal{F}_{N-k-1}) &\leq r, \quad m = \overline{1, M} \\ \langle u_j, \varphi_m \rangle &\leq S(\varphi_m, P), \quad m = \overline{1, M}, \quad j = \overline{1, J}. \end{aligned}$$

Here  $u_j$ ,  $j = \overline{1, J}$ , and  $r$  are the unknown variables. This problem can be solved using the simplex-method or an interior-point method. Since the difference between the problems on the adjacent time intervals is rather small, the solution  $u_j$ ,  $j = \overline{1, J}$ , obtained at the moment  $t = k\epsilon$  can be used as an initial point to solve the linear programming problem on the next time interval.

### 5. Robust Pontryagin-Pshenichnyj operator

At the instant  $t = k\epsilon$  the disturbance  $v(t)$  defined on the interval  $[k\epsilon, (k+1)\epsilon]$ , needed to construct the control  $u(t)$ ,  $t \in [k\epsilon, (k+1)\epsilon]$ , is not available. For this reason we use the disturbance  $v(t)$  defined on the interval  $[(k-1)\epsilon, k\epsilon]$ . It turns out that this can cause serious problems and the construction of the Pontryagin-Pshenichnyj operator should be modified in order to overcome them. To clarify this issue we need some notations. Let  $T(x_0)$  be such that

$x_0 \in \mathcal{F}_{T(x_0)}$  and  $x_0 \notin \mathcal{F}_t, t < T(x_0)$ . By  $u(t, v(t - \epsilon), x_0)$  denote the control  $u(t), t \in [k\epsilon, (k + 1)\epsilon]$ , computed using the disturbance  $v(t)$  defined on the interval  $[(k - 1)\epsilon, k\epsilon]$ , and by  $u(t, v(t), x_0)$  denote the control  $u(t), t \in [k\epsilon, (k + 1)\epsilon]$ , computed using the disturbance  $v(t)$  defined on the interval  $[k\epsilon, (k + 1)\epsilon]$ . The corresponding solutions of system (5) we denote by

$$X_{-\epsilon}(x_0) = e^{\epsilon A} x_0 - \int_0^\epsilon e^{(\epsilon-t)A} u(t, v(t - \epsilon), x_0) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t) dt$$

and

$$X_\epsilon(x_0) = e^{\epsilon A} x_0 - \int_0^\epsilon e^{(\epsilon-t)A} u(t, v(t), x_0) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t) dt.$$

The controls  $u(t, v(t - \epsilon), x_0)$  and  $u(t, v(t), x_0), t \in [k\epsilon, (k + 1)\epsilon]$ , are constructed to minimize the distances  $d(X_{-\epsilon}(x_0), \mathcal{F}_{T(x_0)-\epsilon})$  and  $d(X_\epsilon(x_0), \mathcal{F}_{T(x_0)-\epsilon})$ , respectively. It turns out that, in general, in the first case the trajectory rapidly zigzags in the vicinity of the equilibrium position and in the second case its behaviour is more regular.

Consider the following example. The control system

$$\ddot{x} = -\beta\dot{x} - \alpha x - u + v, \quad |u| \leq u_{\max}, \quad |v| \leq v_{\max} \quad (15)$$

describes the motion of a harmonic oscillator with friction. The control resource of the first player is enough to compensate any disturbance. The control  $v(t)$  takes alternating values  $\pm v_{\max}$  on the intervals  $[k\epsilon, (k + 1)\epsilon]$ . The influence of the delay can be seen comparing Figures 1 and 2. It is clear that the presence of delay causes violent oscillations of the trajectories.

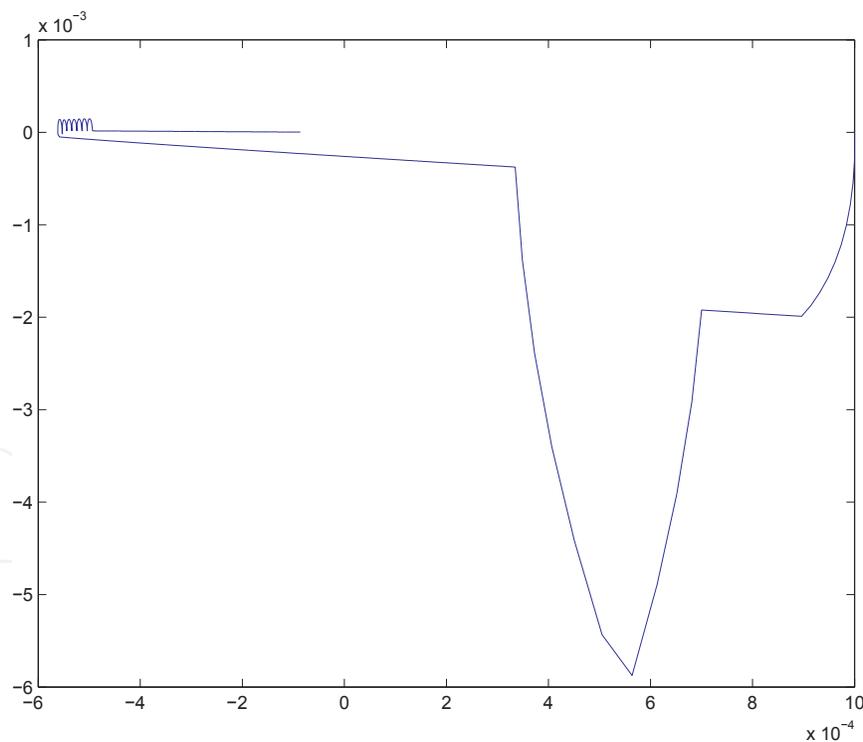


Fig. 1. Trajectory  $(x, \dot{x})$ : motion without delay.

To overcome this difficulty we introduce a robust Pontryagin-Pshenichnyj  $\epsilon$ -operator. The definition of  $\epsilon$ -invariant set also should be revised. We say that a convex set  $\mathcal{S}$  is robustly  $\epsilon$ -invariant if  $\mathcal{S} = \mathcal{S}_0 + 2\mathcal{Q}_\epsilon$  and

$$\Lambda_\epsilon \mathcal{S}_0 + 2\Lambda_\epsilon \mathcal{Q}_\epsilon + \mathcal{Q}_\epsilon \subset \mathcal{S}_0 + \mathcal{P}_\epsilon. \quad (16)$$

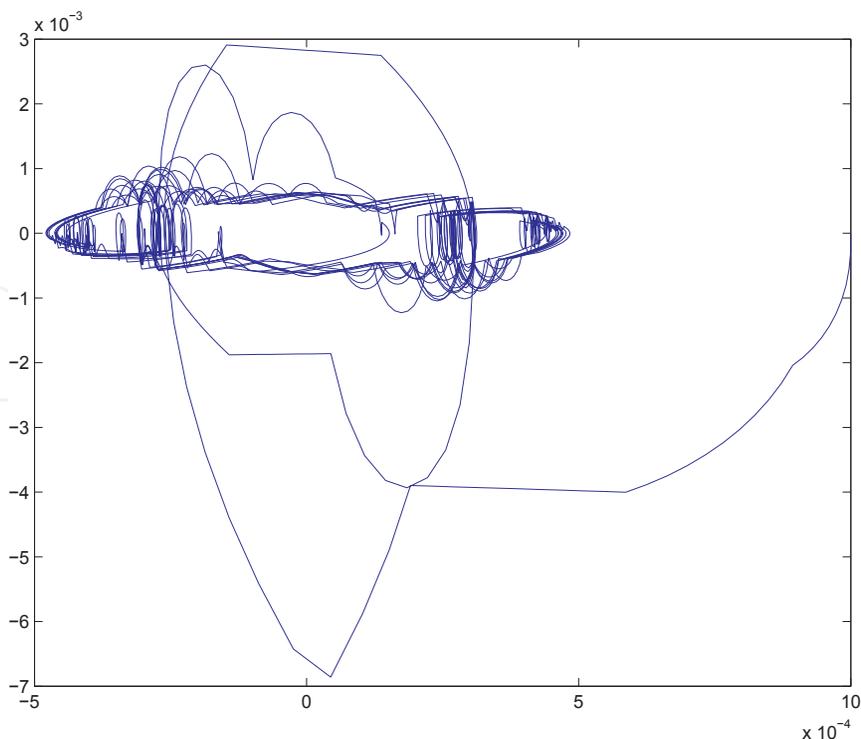


Fig. 2. Trajectory  $(x, \dot{x})$ : motion with delay.

This definition implies the inclusion

$$\Lambda_\epsilon \mathcal{S} + \mathcal{Q}_\epsilon \subset (\mathcal{S}^* - 2\mathcal{Q}_\epsilon) + \mathcal{P}_\epsilon. \tag{17}$$

The robust Pontryagin-Pshenichnyj  $\epsilon$ -operator is defined by  $\mathcal{G}_0 = \mathcal{S}$ ,

$$\mathcal{G}_k = \mathcal{G}_\epsilon^k(\mathcal{S}), \quad k = \overline{0, N}, \tag{18}$$

where

$$\mathcal{G}_\epsilon(\mathcal{C}) = \Lambda_\epsilon^{-1} \left( \left( \left( \mathcal{C}^* - 2\mathcal{Q}_\epsilon \right) + \mathcal{P}_\epsilon \right)^* - \mathcal{Q}_\epsilon \right).$$

If  $x_0 \in \mathcal{G}_{k+1}$  and we choose the control  $u(t, v(t - \epsilon), x_0)$  to guarantee the inclusions  $X_-(\epsilon, x_0) \in \mathcal{G}_k^* - 2\mathcal{Q}_\epsilon$ , then we have

$$\begin{aligned} X_\epsilon(x_0) &= e^{\epsilon A} x_0 - \int_0^\epsilon e^{(\epsilon-t)A} u(t, v(t - \epsilon), x_0) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t - \epsilon) dt \\ &- \int_0^\epsilon e^{(\epsilon-t)A} v(t - \epsilon) dt + \int_0^\epsilon e^{(\epsilon-t)A} v(t) dt \in \left( \mathcal{G}_k^* - 2\mathcal{Q}_\epsilon \right) + 2\mathcal{Q}_\epsilon \subset \mathcal{G}_k. \end{aligned} \tag{19}$$

A trajectory generated by the robust Pontryagin-Pshenichnyj  $\epsilon$ -operator for the above example can be seen in Fig. 3. It is more regular although the limit set is larger. The latter can be reduced diminishing the parameter  $\epsilon$ . From the qualitative point of view, the difference between the behaviours of the trajectories generated by the usual Pontryagin-Pshenichnyj  $\epsilon$ -operator and the robust one can be explained as follows. The inclusion  $x_0 \in \mathcal{F}_{k+1}$  does not imply the inclusion  $X_\epsilon(x_0) \in \mathcal{F}_k$ . In general, we need much time than  $\epsilon$  to achieve the set  $\mathcal{F}_k$  and the search of the way to the set  $\mathcal{F}_k$  results in zigzags of the trajectories. On the other hand, the inclusion  $x_0 \in \mathcal{G}_{k+1}$  always imply (19).

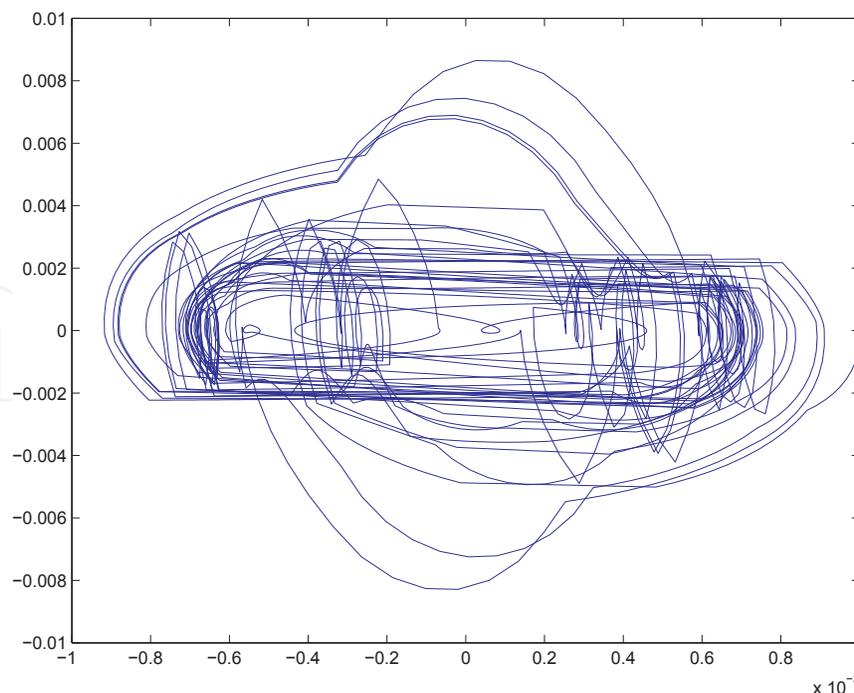


Fig. 3. Trajectory  $(x, \dot{x})$ : motion generated by the robust Pontryagin-Pshenichnyj  $\epsilon$ -operator.

## 6. High precision attitude stabilization of spacecrafts with large flexible elements

Satellites with flexible appendages are modelled by hybrid systems of differential equations

$$\dot{x} = f(x, g(y, \dot{y}, \ddot{y}), u), \quad (20)$$

$$\ddot{y} = G(x, \dot{x}, \ddot{x}, y), \quad (21)$$

where  $x \in R^n$ ,  $y \in Y$  is vector in a Hilbert space, and  $g : Y^3 \rightarrow R^m$  is an integral operator (Junkins & Kim, 1993). Equation (20) is an ordinary differential equation describing the motion of the satellite and depending on the control  $u \in U$ , while (21) is a partial differential equation modelling the dynamics of flexible appendages. We illustrate the stabilization techniques based on the differential game approach by a model example.

Consider a spacecraft composed of a rigid body with a flexible appendage (a beam, see Fig. 4). The satellite is modelled as a cylinder. The distance between its longitudinal axis and the point  $c$  where the beam is cantilevered is denoted by  $r_0$ . The length of the beam is denoted by  $l$ . We use two systems of coordinates: the inertial one denoted by  $OXYZ$  and the system  $oxyz$  rigidly connected to the satellite. The axis  $oz$  is directed along the satellite longitudinal axis, and the axis  $ox$  passes through the point  $c$ . The position of the point  $o$  is described by the coordinates  $(X_0, Y_0)$ , and the position of the axis  $ox$  relatively to the inertial coordinate system is defined by the angle  $\theta$ . The deflection of the beam from the axis  $ox$  is described by the function  $y(t, x)$  (see Fig. 5). We assume that the oscillations of the flexible appendage are small and can be described in the framework of linear theory of elasticity. We consider only a rotation of the satellite around its longitudinal axis.

To obtain the Lagrange equations for this system we write down the Lagrangian function

$$L = \frac{1}{2}m(\dot{X}_0^2 + \dot{Y}_0^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{\rho}{2} \int_{r_0}^{r_0+l} \left( (\dot{X}_0 - (\dot{y} + x\dot{\theta}) \sin\theta)^2 + (\dot{Y}_0 + (\dot{y} + x\dot{\theta}) \cos\theta)^2 \right) dx$$

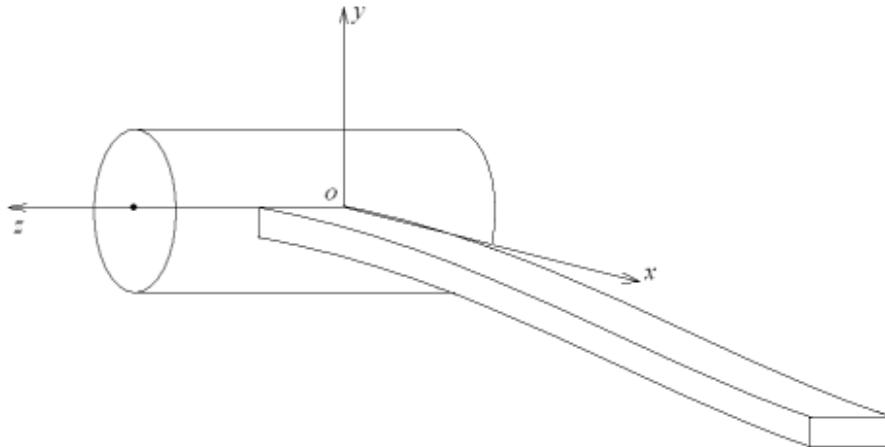


Fig. 4. Satellite with a flexible appendage.

$$-\frac{1}{2}EI \int_{r_0}^{r_0+l} (y'')^2 dx.$$

Here  $m$  is the mass of the satellite,  $I$  is its moment of inertia about the longitudinal axis,  $\rho$  is the mass/unit length of the beam,  $EI$  is the bending stiffness of the beam. 'Dot' is used to denote the derivatives with respect to time, while 'prime' stands for the derivative with respect to  $x$ . The Lagrangian equations of free oscillations of the system have the form

$$\begin{aligned} &(m + l\rho)\ddot{X}_0 - \frac{\rho}{2}((r_0 + l)^2 - r_0^2)(\ddot{\theta} \sin \theta + \dot{\theta} \cos \theta) \\ &- \rho \int_{r_0}^{r_0+l} (\ddot{y} \sin \theta + \dot{\theta} \dot{y} \cos \theta) dx = 0, \\ &(m + l\rho)\ddot{Y}_0 + \frac{\rho}{2}((r_0 + l)^2 - r_0^2)(\ddot{\theta} \cos \theta - \dot{\theta} \sin \theta) \\ &+ \rho \int_{r_0}^{r_0+l} (\ddot{y} \cos \theta - \dot{\theta} \dot{y} \sin \theta) dx = 0, \\ &I\ddot{\theta} - \frac{\rho}{2}((r_0 + l)^2 - r_0^2)(\ddot{X}_0 \sin \theta - \ddot{Y}_0 \cos \theta) \\ &+ \rho \int_{r_0}^{r_0+l} x(\ddot{y} + x\ddot{\theta}) dx + \rho \int_{r_0}^{r_0+l} (\dot{X}_0 \dot{y} \cos \theta + \dot{Y}_0 \dot{y} \sin \theta) dx = 0, \\ &\rho(-\ddot{X}_0 \sin \theta + \ddot{Y}_0 \cos \theta - \ddot{X}_0 \dot{\theta} \cos \theta - \dot{Y}_0 \dot{\theta} \sin \theta + \ddot{y} + x\ddot{\theta}) + EIy'''' = 0. \end{aligned}$$

Linearizing these equations in the vicinity of the zero equilibrium position  $X_0 = Y_0 = \theta = 0$ ,  $y(\cdot, \cdot) \equiv 0$ , we get

$$\begin{aligned} &(m + l\rho)\ddot{X}_0 = 0, \\ &(m + l\rho)\ddot{Y}_0 + \frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{\theta} + \rho \int_{r_0}^{r_0+l} \ddot{y} dx = 0, \\ &\left(I + \frac{\rho}{3}((r_0 + l)^3 - r_0^3)\right)\ddot{\theta} + \frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{Y}_0 + \rho \int_{r_0}^{r_0+l} x\ddot{y} dx = 0, \\ &\rho x\ddot{\theta} + \rho \ddot{y} + \rho \ddot{Y}_0 + EIy'''' = 0. \end{aligned}$$

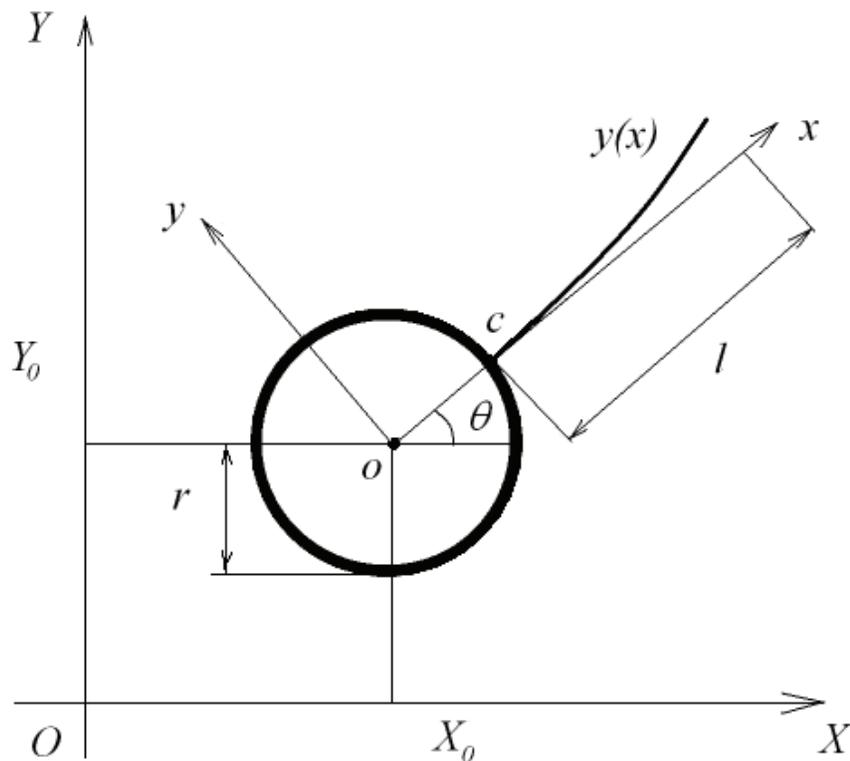


Fig. 5. Transversal section of the satellite with a flexible appendage.

The function  $y = y(t, x)$  satisfies the following boundary conditions:

$$y(r_0, t) = y'(r_0, t) = y''(r_0 + l, t) = y'''(r_0 + l, t) = 0.$$

Adding the control moment  $M$ ,  $|M| \leq M_{\max}$ , and the internal viscous friction, we obtain the following system of differential equations:

$$(m + l\rho)\ddot{X}_0 = 0, \quad (22)$$

$$(m + l\rho)\ddot{Y}_0 + \frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{\theta} + \rho \int_{r_0}^{r_0+l} \ddot{y} dx = 0, \quad (23)$$

$$\frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{Y}_0 + \left(I + \frac{\rho}{3}((r_0 + l)^3 - r_0^3)\right)\ddot{\theta} + \rho \int_{r_0}^{r_0+l} x \ddot{y} dx = M, \quad (24)$$

$$\rho \ddot{Y}_0 + \rho x \ddot{\theta} + \rho \ddot{y} + EI y'''' + EI \chi \dot{y}'''' = 0, \quad (25)$$

where  $\chi$  is the coefficient of internal viscous friction.

Using the Galerkin method we approximate  $y(t, x)$  by a linear combination

$$y(t, x) = \sum q_i(t) \Phi_i(x - r_0) \quad (26)$$

of eigenfunctions  $\Phi_i(x)$  of the differential operator  $d^4/dx^4$  with the boundary conditions  $\Phi(0) = \Phi'(0) = \Phi''(l) = \Phi'''(l) = 0$ . Substituting (26) to system (23) - (25), multiplying (25) by

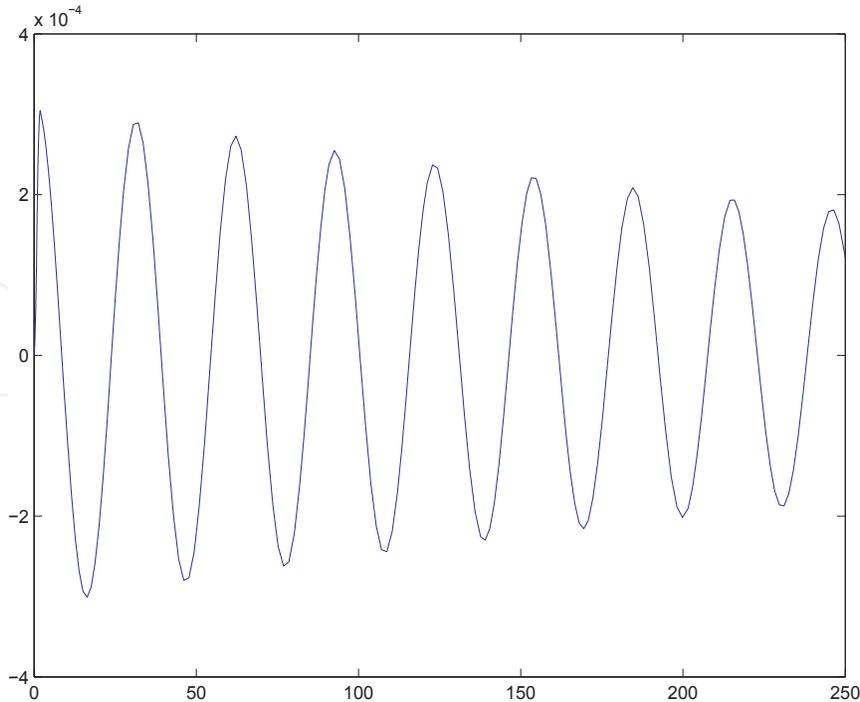


Fig. 6. The disturbance  $v(t)$  caused by the flexible appendage.

$\Phi_i(x)$  and integrating in  $x \in [r_0, r_0 + l]$ , we get a system of ordinary differential equations for the variables  $X_0, Y_0, \theta$ , and  $q_i$ .

For simplicity consider the approximation involving the first natural mode only:

$$y(t) = q(t)\Phi(x - r_0),$$

where

$$\Phi(x) = \cosh(\beta x) - \cos(\beta x) - \frac{\cosh(\beta l) + \cos(\beta l)}{\sinh(\beta l) + \sin(\beta l)} (\sinh(\beta x) - \sin(\beta x)),$$

and  $\beta \approx 1.875/l$ . Then from system (22) - (25) we obtain

$$(m + l\rho)\ddot{X}_0 = 0, \tag{27}$$

$$(m + l\rho)\ddot{Y}_0 + \frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{\theta} + \rho J_1 l \ddot{q} = 0, \tag{28}$$

$$\frac{\rho}{2}((r_0 + l)^2 - r_0^2)\ddot{Y}_0 + \left(I + \frac{\rho}{3}((r_0 + l)^3 - r_0^3)\right)\ddot{\theta} + \rho(J_2 l^2 + J_1 l r_0)\ddot{q} = M, \tag{29}$$

$$\rho J_1 l \ddot{Y}_0 + \rho(J_2 l^2 + J_1 l r_0)\ddot{\theta} + \rho J_3 l \ddot{q} + EI\beta^4 J_3 l q + EI\chi\beta^4 J_3 l \dot{q} = 0, \tag{30}$$

where  $J_1 = 0.7829, J_2 = 0.5688, J_3 = 0.9998$ . System (28) - (30) can be written in the matrix form as

$$\frac{d^2}{dt^2} \begin{pmatrix} Y_0 \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ M \\ -EI\beta^4 J_3 l (q + \chi \dot{q}) \end{pmatrix},$$

where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} (m + l\rho) & \frac{\rho}{2}((r_0 + l)^2 - r_0^2) & \rho J_1 l \\ \frac{\rho}{2}((r_0 + l)^2 - r_0^2) & \left(I + \frac{\rho}{3}((r_0 + l)^3 - r_0^3)\right) & \rho(J_2 l^2 + J_1 l r_0) \\ \rho J_1 l & \rho(J_2 l^2 + J_1 l r_0) & \rho J_3 l \end{pmatrix}.$$

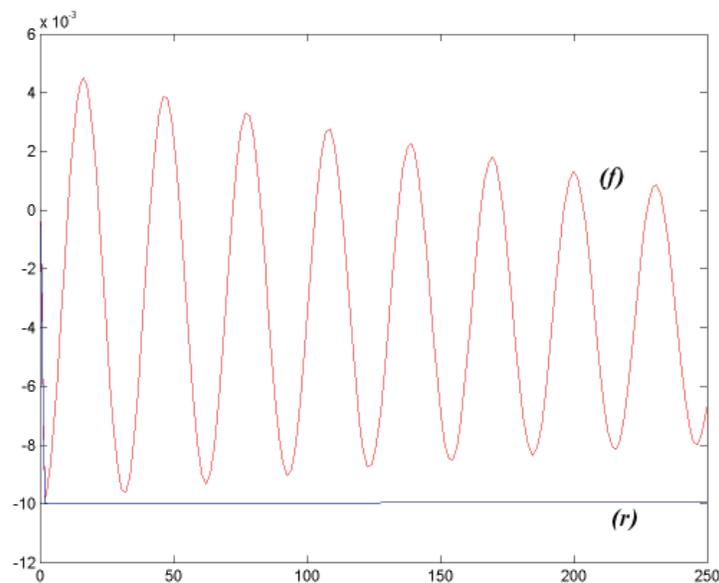


Fig. 7. Rest-to-rest manoeuvre: the discrepancy of the attitudes  $\theta_r(t)$  and  $\theta_f(t)$  between the motion of the rigid system and the flexible one.

Denote this matrix by  $\mathcal{A}$ . Thus, the angular dynamics of the satellite near the zero equilibrium position is described by the linear control system

$$\ddot{\theta} = \left(\mathcal{A}^{-1}\right)_{22} M - D \left(\mathcal{A}^{-1}\right)_{23} (q + \chi\dot{q}), \quad (31)$$

$$\ddot{q} = \left(\mathcal{A}^{-1}\right)_{32} M - D \left(\mathcal{A}^{-1}\right)_{33} (q + \chi\dot{q}), \quad (32)$$

where  $D = EI\beta^4 J_3 l$ .

In our numerical simulations we use a model example with the following values of parameters:  $m = 10$ ,  $l = 10$ ,  $r_0 = 3$ ,  $\rho = 0.5$ ,  $I = 45$ ,  $EI = 3.5$ , and  $\chi = 0.1$  (SI units).

The influence of the flexible appendages can be rather significant. Consider a rest-to-rest manoeuvre for the model under study. We apply the moment  $+M$  and then  $-M$  during the same time and compare the motion of the configuration considering the appendage as flexible with low stiffness ( $EI = 3.5$ ) and as rigid. The disturbance caused by the appendage is shown in Fig. 6, while the difference between the angular positions of the satellite with flexible and rigid appendages is shown in Fig. 7. It is quite large. Therefore a high-precision attitude stabilization system should take into account the flexibility.

To stabilize system (31) and (32) we use the linear stabilizer

$$M = a\theta + b\dot{\theta} \quad (33)$$

with the coefficients  $a$  and  $b$  determined from the condition of the maximum degree of stability of the closed-loop system

$$\ddot{\theta} = \left(\mathcal{A}^{-1}\right)_{22} (a\theta + b\dot{\theta}) - D \left(\mathcal{A}^{-1}\right)_{23} (q + \chi\dot{q}),$$

$$\ddot{q} = \left(\mathcal{A}^{-1}\right)_{32} (a\theta + b\dot{\theta}) - D \left(\mathcal{A}^{-1}\right)_{33} (q + \chi\dot{q}).$$

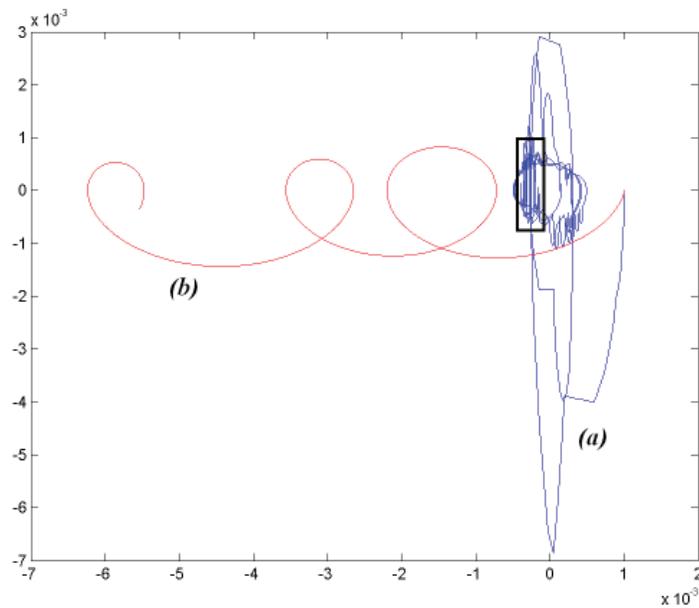


Fig. 8. Trajectories  $(\theta, \dot{\theta})$  with high precision stabilizer (a) and without it (b).

If the angular position  $\theta$  and the angular velocity  $\dot{\theta}$  are known with some errors  $\delta\theta$  and  $\delta\dot{\theta}$ , respectively, we obtain the following differential game:

$$\ddot{\Theta} = (\mathcal{A}^{-1})_{22} (a\Theta + b\dot{\Theta}) - u + v,$$

where  $\Theta = \theta + \delta\theta$ ,  $v = -(\mathcal{A}^{-1})_{22} (a\delta\theta + b\delta\dot{\theta}) - D(\mathcal{A}^{-1})_{23} (q + \chi\dot{q})$ , and  $q$  is governed by the differential equation

$$\ddot{q} = (\mathcal{A}^{-1})_{32} \left( (a\theta + b\dot{\theta}) - (\mathcal{A}^{-1})_{22}^{-1} u \right) - D(\mathcal{A}^{-1})_{33} (q + \chi\dot{q}).$$

The differential game approach to the stabilizer design problem presumes that the controls  $u$  and  $v$  satisfy the restrictions  $|u| \leq u_{\max}$  and  $|v| \leq v_{\max}$ , respectively. To determine the values  $u_{\max}$  and  $v_{\max}$  and the neighborhood of the equilibrium position where the differential game stabilizer works we use the following method. Consider the vector  $x = (\theta, \dot{\theta}, q, \dot{q})^T$ . Its behaviour is described by the differential equation

$$\dot{x} = \mathcal{B}x + b, \tag{34}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ (\mathcal{A}^{-1})_{22} a & (\mathcal{A}^{-1})_{22} b & -D(\mathcal{A}^{-1})_{23} & (\mathcal{A}^{-1})_{23} \chi \\ 0 & 0 & 0 & 1 \\ (\mathcal{A}^{-1})_{32} a & (\mathcal{A}^{-1})_{32} b & -D(\mathcal{A}^{-1})_{33} & (\mathcal{A}^{-1})_{33} \chi \end{pmatrix}$$

and

$$b = (0, -u + (\mathcal{A}^{-1})_{22} a\delta\theta + (\mathcal{A}^{-1})_{22} b\delta\dot{\theta}, 0, -(\mathcal{A}^{-1})_{32} u_{\max} / (\mathcal{A}^{-1})_{22})^T.$$

If  $|\delta\theta| \leq \delta\theta_{\max}$  and  $|\delta\dot{\theta}| \leq \delta\dot{\theta}_{\max}$ , then the estimate

$$\|b\| \leq b(u_{\max})$$

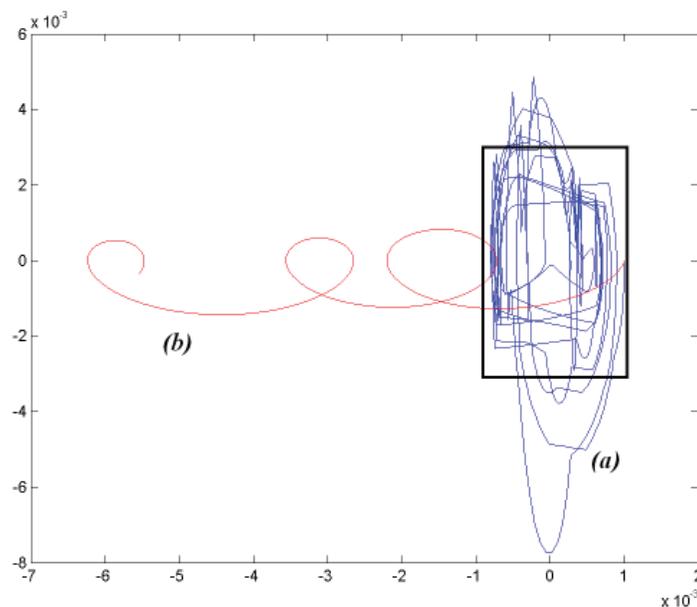


Fig. 9. Trajectories  $(\theta, \dot{\theta})$  with high precision stabilizer (a) and without it (b): the robust Pontryagin-Pshenichnyj stabilizer.

$$= \left( u_{\max} + \left| (\mathcal{A}^{-1})_{22} \right| (|a|\delta\theta_{\max} + |b|\delta\dot{\theta}_{\max}) \right)^2 + \left( (\mathcal{A}^{-1})_{32} (\mathcal{A}^{-1})_{22}^{-1} u_{\max} \right)^2 \Big)^{1/2}$$

holds. Let  $V$  be a symmetric  $(4 \times 4)$  matrix satisfying the Lyapunov equation

$$\mathcal{B}^T V + V \mathcal{B} = -I_4.$$

The solution  $x(t)$  to differential equation (34) satisfies the following differential inequality:

$$\frac{d}{dt} \langle x(t), Vx(t) \rangle = -\|x(t)\|^2 + 2 \langle Vx(t), b(t) \rangle \leq -\|x(t)\|^2 + 2\|V\| \|x(t)\| b(u_{\max}) \leq 0,$$

whenever  $2\|V\| b(u_{\max}) \leq \|x(t)\|$ . Define the family of ellipsoids

$$E_c = \{x \mid \langle x, Vx \rangle \leq c\}$$

and put

$$c(u_{\max}) = \min\{c \mid 2\|V\| b(u_{\max}) B_4 \subset E_c\}.$$

Obviously  $x(t) \in E_{c(u_{\max})}$ , whenever  $t$  is large enough. Thus, the differential game of stabilization is playable if the Pontryagin-Pshenichnyj  $\epsilon$ -operators are contained in  $E_{c(u_{\max})}$  and the following conditions are satisfied:

$$v_{\max} < u_{\max},$$

$$v_{\max} \leq \left| (\mathcal{A}^{-1})_{22} \right| (|a|\delta\theta_{\max} + |b|\delta\dot{\theta}_{\max}) + \max \left\{ \left| D (\mathcal{A}^{-1})_{23} \right| |q + \chi\dot{q}| \mid (\theta, \dot{\theta}, q, \dot{q}) \in E_{c(u_{\max})} \right\},$$

and

$$\max\{|a\theta + b\dot{\theta}| \mid (\theta, \dot{\theta}, q, \dot{q}) \in E_{c(u_{\max})}\} + u_{\max} \leq M_{\max}.$$

Typical trajectories generated by a linear feed-back and by a high-precision stabilizer with the Pontryagin-Pshenichnyj  $\epsilon$ -operator and the robust Pontryagin-Pshenichnyj  $\epsilon$ -operator, respectively, are shown in Fig. 8 and 9. (The disturbance  $v$  in the numerical simulations is a periodic function.) The differential game method of stabilization yields significantly smaller limit set (marked by box) than the simple linear stabilizer (33).

## 7. Conclusion

We present here a new approach to stabilization of mechanical systems with uncertainties in parameters and/or state data. This approach considers the perturbations caused by these uncertainties as an evader control in a linear pursuit differential game. We describe the general theoretical basis and the numerical algorithms for implementation of the described differential game stabilizer. Estimates for the amplitude of the evader control should be obtained for any specific case of control system using its mechanical properties.

We consider here an application of the suggested method to the stabilization problem for a satellite with large flexible appendages. The estimates for the evader control caused by uncertainties are deduced applying the method of Lyapunov functions. We construct a high-precision stabilizer using the differential game approach and the above estimates for the evader control.

The principal advantage of the suggested method is that, to achieve a high-precision stabilization, it requires only the satellite attitude data and does not need any estimation for the flexible elements' state and/or unknown system parameters.

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