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# Numerical Solution of a System of Polynomial Parametric form Fuzzy Linear Equations

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## 1. Introduction

Since fuzzy logic was introduced by Lotfi Zadeh in 1965 (41), it has had many successful applications in all fields that one can imagine. The reason is that many real-world applications problems are involved the systems in which at least some parameters are represented by fuzzy numbers rather than crisp numbers and linguistic labels such as small and large are also associated with the fuzzy sets. On the other hand a system of fuzzy linear equations may appear in a wide variety of problems in various areas such as mathematics, statistics, physics, engineering and social sciences.

The objective of this chapter is to introduce a method to find a good approximate solution to a system of fuzzy linear equations, and first we need to be familiar with some notations on fuzzy numbers in this chapter, however it is assumed that the reader is relatively familiar with the elementary fuzzy logic concepts.

In (14), Chong-Xin and Ming represented a fuzzy number  $\tilde{u}$  by an ordered pair of functions  $(\underline{u}(r), \bar{u}(r))$ ,  $0 \leq r \leq 1$ , which satisfies some requirements.

In papers and books the authors mainly have used linear membership functions as spreads, because they are conceptually the simplest, have a clear interpretation and play a crucial role in many areas of fuzzy applications, and almost every works on this field of study have been done on triangular or trapezoidal fuzzy numbers, but polynomial form fuzzy numbers are simple and have a clear interpretation too, and in order to obtain a richer class of fuzzy numbers we use polynomials of the degree higher than one, as the spreads of membership functions of fuzzy numbers. Thus in (2) we introduced a type of fuzzy numbers in which both left spread function  $\underline{u}(r)$  and right spread function  $\bar{u}(r)$  are polynomials of degree at most  $m$ . We named this type of fuzzy numbers,  $m$ -degree polynomial-form fuzzy numbers.

The main aim of introducing this type of fuzzy numbers is that in many applications of fuzzy logic and fuzzy mathematics we need (or it is better) to work with the same fuzzy numbers. It has been shown in (2) that a fuzzy number  $\tilde{u}$  with continuous left and right spread functions can be approximated by a fuzzy number with  $m$ -degree polynomial-form, where choosing  $m$  depends on the shape of left and right spread functions  $L$  and  $R$ , and the derivation order of them.

Some applications of this approximation in the case  $m = 1$  (trapezoidal fuzzy numbers) are given in (3), and some properties of this approximation operator are recently given in (10).

There are many other literatures which authors tried to approximate a fuzzy number by a simpler one (2; 3; 4; 18; 25; 27; 28; 29; 30; 31; 32; 42). Also there are some distances defined by authors to compare fuzzy numbers (35; 37).

Obviously, if we use a defuzzification rule which replaces a fuzzy set by a single number, we generally lose too much important information. Also, an interval approximation is considered for fuzzy numbers in (25), where a fuzzy computation problem is converted into interval arithmetic problem. But, in this case, we lose the fuzzy central concept. Even in some works such as (4; 31; 32; 42), authors solve an optimization problem to obtain the nearest triangular or trapezoidal fuzzy number which is related to an arbitrary fuzzy number, however in these cases there is not any guarantee to have the same modal value (or interval). But by parametric polynomial approximation we are able to approximate many fuzzy numbers in a good manner.

The problem of finding the nearest parametric approximation of a fuzzy number with respect to the average Euclidean distance is completely solved in (13). Ban point out the wrongs and inadvertence in some recent papers, then correct the results in (12). A parametric fuzzy approximation method based on the decision maker's strategy as an extension of trapezoidal approximation of a fuzzy number offered in (34). An improvement of the nearest trapezoidal approximation operator preserving the expected interval, which is proposed by Grzegorzewski and MrXowka is studied in (39). There are some trapezoidal approximation operators introduced in (26; 29; 38)

But the main aim of this chapter is to give a method to solve linear system of fuzzy equations. There are three categories of a linear system of fuzzy equations

$$\mathbf{Ax} = \mathbf{b}.$$

- In the first category, the coefficient matrix arrays are crisp numbers, the right-hand side column is an arbitrary fuzzy vector and the unknowns are fuzzy numbers.
- In the second category, the coefficient matrix arrays are fuzzy numbers, the right-hand side column is an arbitrary fuzzy vector and the unknowns are crisp numbers.
- In the third category, all the coefficient matrix arrays, the right-hand side arrays and the unknowns, are fuzzy numbers.

There have been few papers of fuzzy linear equations with crisp unknowns, and in this chapter we propose a method for solving an  $n \times n$  linear system from second category based on (9). However there are many reported studies in which researchers tried to solve a system of linear fuzzy equations numerically (1; 5; 6; 7; 11; 15; 21; 22). In most of these studies the problem is considered for a system with fuzzy unknowns and crisp coefficients. Some new work has been done on fully fuzzy linear systems of equations. Some works are done on fully fuzzy linear system of equations (16; 17). Some works have been done on rectangular  $m \times n$  system of equations (11; 24; 43) and some works have been done on blocked matrices (33; 36). A minimal solution for dual fuzzy linear system is given in (8). A class of methods is considered in (40).

The structure of this chapter is as follows: First we represent *source distance* and *m-source distance*; and we present the nearest approximation of a fuzzy number in polynomial parametric form, introduced in (2) and present some properties of it where recently published in (10). Next we present a method for solving a fuzzy system of linear equations with *m*-degree polynomial parametric-form fuzzy coefficients and crisp unknowns by a least squares method. Then we introduce the extension of this method to solve a general fuzzy system of linear equations with *LR* fuzzy coefficients and crisp unknowns.

One of the advantages of the proposed method is that by this method not only one can find a good approximation of the solution of such fuzzy systems with polynomial-form fuzzy arrays

but even a system of fuzzy linear equations with general type of fuzzy number arrays can be solved by this method.

Finally some applied examples are given at the end of this chapter .

## 2. Preliminaries

Let  $\mathcal{F}(\mathbb{R})$  be the set of all normal and convex fuzzy numbers on the real line (44).

**Definition 21** (19), A generalized LR fuzzy number  $\tilde{u}$  with the membership function  $\mu_{\tilde{u}}(x), x \in \mathbb{R}$  can be defined as

$$\mu_{\tilde{u}}(x) = \begin{cases} l_{\tilde{u}}(x) & , \quad a \leq x \leq b, \\ 1 & , \quad b \leq x \leq c, \\ r_{\tilde{u}}(x) & , \quad c \leq x \leq d, \\ 0 & , \quad otherwise, \end{cases} \quad (21)$$

where  $l_{\tilde{u}}(x)$  is the left membership function that is an increasing function on  $[a, b]$  and  $r_{\tilde{u}}(x)$  is the right membership function that is a decreasing function on  $[c, d]$ . Furthermore we want to have  $l_{\tilde{u}}(a) = r_{\tilde{u}}(d) = 0$  and  $l_{\tilde{u}}(b) = r_{\tilde{u}}(c) = 1$ . In addition, if  $l_{\tilde{u}}(x)$  and  $r_{\tilde{u}}(x)$  are linear, then  $\tilde{u}$  is a trapezoidal fuzzy number which is denoted by  $(a, b, c, d)$ . If  $b = c$ , we denoted it by  $(a, c, d)$ , which is a triangular fuzzy number.

For  $0 < \alpha \leq 1$ ;  $\alpha$ -cut of a fuzzy number  $\tilde{u}$  is defined by (20),

$$[\tilde{u}]^\alpha = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) \geq \alpha\}. \quad (22)$$

**Definition 22** (35), A continuous function  $s : [0, 1] \rightarrow [0, 1]$  with the following properties is a regular reducing function :

1.  $s(r)$  is increasing.
2.  $s(0) = 0$ ,
3.  $s(1) = 1$ ,
4.  $\int_0^1 s(r)dr = \frac{1}{2}$ .

In (14), Chong-Xin and Ming represented a fuzzy number  $\tilde{u}$  by an ordered pair of functions  $(\underline{u}(r), \overline{u}(r))$ :

The parametric form of a fuzzy number is shown by  $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ , where functions  $\underline{v}(r)$  and  $\overline{v}(r)$ ;  $0 \leq r \leq 1$  satisfy the following requirements:

1.  $\underline{v}(r)$  is monotonically increasing left continuous function.
2.  $\overline{v}(r)$  is monotonically decreasing left continuous function.
3.  $\underline{v}(r) \leq \overline{v}(r)$  ,  $0 \leq r \leq 1$ .
4.  $\overline{v}(r) = \underline{v}(r) = 0$  for  $r < 0$  or  $r > 1$ .

**Definition 23** (35), The value and ambiguity of a fuzzy number  $\tilde{u}$  are defined by

$$val(\tilde{u}) := \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r)]dr, \quad (23)$$

and

$$amb(\tilde{u}) := \int_0^1 s(r)[\overline{u}(r) - \underline{u}(r)]dr, \quad (24)$$

respectively.

**Definition 24** (2; 9) We say a fuzzy number  $\tilde{u}$  has  $m$ -degree polynomial-form, if there exist two polynomials  $p_m(r)$  and  $q_m(r)$ , of degree at most  $m$ ; such that  $\tilde{u} = (p_m(r), q_m(r))$ .

Let  $\mathcal{PF}_m(\mathbb{R})$  be the set of all  $m$ -degree polynomial-form fuzzy numbers.

**Definition 25** (2), Let for positive integer  $k$ , we have  $\underline{u}, \bar{u} \in C^k[0,1]$ . We define  $k$ -validity and  $k$ -unworthiness of a fuzzy number  $\tilde{u}$  by

$$\mathbf{val}_k(\tilde{u}) := \int_0^1 s(r) [\bar{u}^{(k)}(r) + \underline{u}^{(k)}(r)] dr, \quad (25)$$

and

$$\mathbf{amb}_k(\tilde{u}) := \int_0^1 s(r) [\bar{u}^{(k)}(r) - \underline{u}^{(k)}(r)] dr, \quad (26)$$

respectively.

0-validity and 0-unworthiness of a fuzzy number are its value and ambiguity, respectively. i.e.

$$\mathbf{val}_0 := \mathbf{val}, \quad \mathbf{amb}_0 := \mathbf{amb}.$$

**Proposition 21** For any nonnegative integer  $k$ , the  $k$ -validity and  $k$ -unworthiness have the following properties:

1.  $\mathbf{val}_k(\tilde{u} \pm \tilde{v}) = \mathbf{val}_k(\tilde{u}) \pm \mathbf{val}_k(\tilde{v})$ ,
2.  $\mathbf{amb}_k(\tilde{u} \pm \tilde{v}) = \mathbf{amb}_k(\tilde{u}) \pm \mathbf{amb}_k(\tilde{v})$ .

**Definition 26** Let for positive integer  $m$ , we have  $\underline{u}, \bar{u} \in C^{m-1}[0,1]$ . For  $k = 0, 1, \dots, m-1$ , we define

$$d_k(\tilde{u}, \tilde{v}) := |\mathbf{val}_k(\tilde{u} - \tilde{v})| + |\mathbf{amb}_k(\tilde{u} - \tilde{v})|. \quad (27)$$

**Definition 27** Let  $X$  and  $Y$  be two non-empty subsets of a metric space  $(M, d)$ . We define their Hausdorff distance  $d_H(X, Y)$  by

$$d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}. \quad (28)$$

### 3. Source distance and $m$ -source distance

**Definition 3.1** (2), Let  $\mathcal{AF}(\mathbb{R})$  be a subset of  $\mathcal{F}(\mathbb{R})$ .  $\tilde{v}^* \in \mathcal{AF}(\mathbb{R})$ , is the nearest approximation of an arbitrary fuzzy number  $\tilde{u} \in \mathcal{F}(\mathbb{R})$  out of  $\mathcal{AF}(\mathbb{R})$ , with respect to a meter  $d$  if and only if

$$d(\tilde{u}, \tilde{v}^*) = \min_{\tilde{v} \in \mathcal{AF}(\mathbb{R})} d(\tilde{u}, \tilde{v}). \quad (31)$$

**Definition 3.2** (2), For  $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$ , we define **source distance** of  $\tilde{u}$  and  $\tilde{v}$  by

$$D(\tilde{u}, \tilde{v}) := \frac{1}{2} \{d_0(\tilde{u}, \tilde{v}) + d_H([\tilde{u}]^1, [\tilde{v}]^1)\}, \quad (32)$$

where  $d_H$  is the Hausdorff metric.

In (2) we proved that the source distance,  $D$ , is a metric on the set of all trapezoidal fuzzy numbers.

**Definition 3.3** For a nonnegative integer  $j$ , we define  $j^{\text{th}}$ -source number by

$$I_j := \int_0^1 r^j s(r) dr. \tag{3.3}$$

**Lemma 3.1** Let  $I_j$  be the  $j^{\text{th}}$ -source number, then  $\{I_j\}$  is a positive decreasing sequence, where  $I_0 = \frac{1}{2}$ . i.e.  $0 \leq \dots < I_2 < I_1 < I_0 = \frac{1}{2}$ .

**Proof:** By using the definition of  $j^{\text{th}}$ -source number and regular reducing function we have  $I_0 = \frac{1}{2}$ . By Mean Value Theorem for integrals for any integer  $j \geq 1$ , there exists a  $\psi \in (0, 1)$ , such that

$$I_j = \psi \int_0^1 r^{j-1} s(r) dr < I_{j-1}.$$

□

**Definition 3.4** For  $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$ , we define  $m$ -source distance of  $\tilde{u}$  and  $\tilde{v}$  by

$$D_m(\tilde{u}, \tilde{v}) := \frac{1}{2} d_H([\tilde{u}]^1, [\tilde{v}]^1) + \sum_{k=0}^{m-1} 'd_k(\tilde{u}, \tilde{v}).$$

where  $\sum_{k=0}^{m-1} 'd_k$  is defined by

$$\sum_{i=0}^{m-1} 'a_i = \frac{1}{2} a_0 + \sum_{i=1}^{m-1} a_i.$$

**Theorem 3.2** For  $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z} \in \mathcal{F}(\mathbb{R})$ , the distance,  $D_m$ , satisfies the following properties:

1.  $D_m(\tilde{u}, \tilde{u}) = 0$ ,
2.  $D_m(\tilde{u}, \tilde{v}) = D_m(\tilde{v}, \tilde{u})$ ,
3.  $D_m(\tilde{u}, \tilde{v}) \leq D_m(\tilde{u}, \tilde{w}) + D_m(\tilde{w}, \tilde{v})$ ,
4.  $D_m(k\tilde{u}, k\tilde{v}) = |k| D_m(\tilde{u}, \tilde{v})$  for  $k \in \mathbb{R}$ ,
5.  $D_m(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{z}) \leq D_m(\tilde{u}, \tilde{v}) + D_m(\tilde{w}, \tilde{z})$ .

**Example 3.1** For two crisp real numbers  $a$  and  $b$  we have

$$D_m(a, b) = |a - b|.$$

**Proposition 3.3** The fuzzy number  $\tilde{v}^*$  is a nearest approximation of  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$  if and only if

$$D_m(\tilde{u}, \tilde{v}^*) = \min_{\tilde{v} \in \mathcal{PF}_m(\mathbb{R})} D_m(\tilde{u}, \tilde{v}). \tag{3.4}$$

We denote the set of all the nearest approximations of a fuzzy number  $\tilde{u}$ , out of  $\mathcal{PF}_m(\mathbb{R})$ , by  $N_{\mathcal{PF}_m(\mathbb{R})}^*(\tilde{u})$ .

**Theorem 3.4** Let  $\tilde{u}$  be a fuzzy number. If for positive integer  $m$  we have  $\underline{u}, \bar{u} \in C^{m-1}[0, 1]$ , then  $\tilde{v}$  is the nearest approximation of  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$  if and only if  $D_m(\tilde{u}, \tilde{v}) = 0$ .

**Proof:** See (2). □

**Theorem 3.5** Let  $m$  be a positive integer and suppose that  $\tilde{u} \in \mathcal{F}(\mathbb{R})$  be an arbitrary fuzzy number.  $\tilde{v} \in \mathcal{PF}_m(\mathbb{R})$  is the nearest approximation of  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$ , if and only if for  $k = 0, \dots, m - 1$ ;  $\tilde{v}$  and  $\tilde{u}$  have the same  $k$ -validity and  $k$ -unworthiness and furthermore  $[\tilde{u}]^1 = [\tilde{v}]^1$ .

**Proof:** See (2). □

**Corollary 3.6** Let  $\tilde{u}$  be a generalized LR fuzzy number. If for positive integer  $m$  we have  $\underline{u}, \bar{u} \in C^{m-1}[0, 1]$ , then the nearest approximation of  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$  exists.

**Theorem 3.7** (10) The nearest approximation of a  $m$ -degree polynomial-form fuzzy number, out of  $\mathcal{PF}_m(\mathbb{R})$ , is itself.

**Proof:** Let  $\tilde{v} \in \mathcal{PF}_m(\mathbb{R})$  be the nearest approximation of  $\tilde{u} \in \mathcal{PF}_m(\mathbb{R})$ . Also let

$$\underline{v}(r) = \underline{d}_m r^m + \dots + \underline{d}_1 r + \underline{d}_0,$$

and

$$\underline{u}(r) = \underline{c}_m r^m + \dots + \underline{c}_1 r + \underline{c}_0,$$

By defining  $w(r) = \underline{v}(r) - \underline{u}(r) = \sum_{j=0}^m a_j r^j$ , where  $a_j = \underline{c}_j - \underline{d}_j$  for  $j = 0, 1, \dots, m$ ; we have  $w(1) = 0$  and

$$\int_0^1 w^{(k)}(r) s(r) dr = 0, \quad k = 0, 1, \dots, m - 1.$$

Therefore we have a homogenous system of linear equations with the following nonsingular coefficients matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ I_0 & I_1 & I_2 & I_3 & \dots & I_m \\ 0 & I_0 & 2I_1 & 3I_2 & \dots & mI_{m-1} \\ 0 & 0 & 2I_0 & 3!I_1 & \dots & m(m-1)I_{m-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & m!I_1 \end{pmatrix}$$

Therefore  $w \equiv 0$ . i.e.  $\underline{u} \equiv \underline{v}$ . In a similar way  $\bar{u} \equiv \bar{v}$ . Thus  $\tilde{u} = \tilde{v}$ . □

**Corollary 3.8**  $D_m(\cdot, \cdot)$  is a meter on  $\mathcal{PF}_m(\mathbb{R})$ .

**Lemma 3.9** (10) Let  $\tilde{u}$  be a fuzzy number. For all  $m \geq 1$  if  $N_{\mathcal{PF}_m}^*(\tilde{u})$  is not empty, then we have

$$|N_{\mathcal{PF}_m(\mathbb{R})}^*(\tilde{u})| = 1.$$

**Proof:** Let  $\tilde{u}_1^*$  and  $\tilde{u}_2^*$  be two nearest approximations of a fuzzy number  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$ . Thus  $D_m(\tilde{u}_1^*, \tilde{u}) = D_m(\tilde{u}_2^*, \tilde{u}) = 0$ , and we have

$$D_m(\tilde{u}_1^*, \tilde{u}_2^*) \leq D_m(\tilde{u}_1^*, \tilde{u}) + D_m(\tilde{u}, \tilde{u}_2^*) = 0.$$

From lemma 3.7 we have  $\tilde{u}_1^* = \tilde{u}_2^*$ . □

**Corollary 3.10** Let  $\tilde{u}$  be a  $l$ -degree polynomial-form fuzzy number where  $l \leq m$ . If  $\tilde{v}^*$  be the nearest approximations of  $\tilde{u}$  out of  $\mathcal{PF}_m(\mathbb{R})$ , then  $\tilde{v}^* = \tilde{u}$ .

**Lemma 3.11** (10) Let  $\tilde{u}^*$  and  $\tilde{v}^*$  be the nearest approximations of two fuzzy numbers  $\tilde{u}$  and  $\tilde{v}$ , respectively. Then we have

$$D_m(\tilde{u}^*, \tilde{v}^*) = D_m(\tilde{u}, \tilde{v})$$

**Proof:** We have

$$D_m(\tilde{u}^*, \tilde{v}^*) \leq D_m(\tilde{u}^*, \tilde{u}) + D_m(\tilde{u}, \tilde{v}) + D_m(\tilde{v}, \tilde{v}^*) = D_m(\tilde{u}, \tilde{v}).$$

In a similar way  $D_m(\tilde{u}, \tilde{v}) \leq D_m(\tilde{u}^*, \tilde{v}^*)$ . □

**Lemma 3.12** (10) Let  $\tilde{v}$  and  $\tilde{u}$  be two fuzzy numbers where  $\underline{u}, \underline{v}, \bar{u}, \bar{v} \in C^{m-1}[0, 1]$ . If  $D_m(\tilde{u}, \tilde{v}) = 0$ , then there are two sequences of points  $\{\delta_{i,k}\}_{k=0}^{m-1}$ ,  $i = 1, 2$  such that for  $k = 0, 1, \dots, m - 1$ ,

$$\underline{u}^{(k)}(\delta_{1,k}) = \underline{v}^{(k)}(\delta_{1,k}),$$

and

$$\bar{u}^{(k)}(\delta_{2,k}) = \bar{v}^{(k)}(\delta_{2,k}).$$

**Proof:** Let for two fuzzy numbers  $\tilde{v}$  and  $\tilde{u}$  we have  $D_m(\tilde{u}, \tilde{v}) = 0$ . Thus for  $k = 0, 1, \dots, m - 1$ , we have

$$\begin{aligned} [\tilde{u}]^1 &= [\tilde{v}]^1, \\ \mathbf{val}_k(\tilde{v}) &= \mathbf{val}_k(\tilde{u}), \quad k = 0, \dots, m - 1, \\ \mathbf{amb}_k(\tilde{v}) &= \mathbf{amb}_k(\tilde{u}), \quad k = 0, \dots, m - 1, \end{aligned}$$

Therefore

$$\int_0^1 s(r) [\bar{u}^{(k)}(r) - \bar{v}^{(k)}(r)] dr = \pm \int_0^1 s(r) [\underline{u}^{(k)}(r) - \underline{v}^{(k)}(r)] dr, \quad k = 0, \dots, m - 1,$$

Thus

$$\begin{aligned} \int_0^1 s(r) [\bar{u}^{(k)}(r) - \bar{v}^{(k)}(r)] dr &= 0, \quad k = 0, \dots, m - 1, \\ \int_0^1 s(r) [\underline{u}^{(k)}(r) - \underline{v}^{(k)}(r)] dr &= 0, \quad k = 0, \dots, m - 1. \end{aligned}$$

Thus by Mean Value Theorem for integrals, for any  $k = 0, 1, \dots, m - 1$ , there are two numbers  $\delta_{1,k}$  and  $\delta_{2,k}$  such that

$$\begin{aligned} \bar{u}^{(k)}(\delta_{1,k}) &= \bar{v}^{(k)}(\delta_{1,k}), \\ \underline{u}^{(k)}(\delta_{2,k}) &= \underline{v}^{(k)}(\delta_{2,k}). \end{aligned}$$

□

#### 4. Properties of $m$ -source distance

Some properties of the approximation operators are presented by Grzegorzewski and Mrówka (28). In this section we consider some properties of the approximation operator suggested in Section 3.

Let  $T_m : \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{PF}_m(\mathbb{R})$  be the approximation operator which produces the nearest approximation fuzzy number out of  $\mathcal{PF}_m(\mathbb{R})$  to a given original fuzzy number using Theorem 3.5. Almost all of the theorems of this section are taken out from (10).

**Theorem 4.1** *The nearest approximation operator is 1-cut invariance.*

**Proof:** It is a necessary condition for this approximation that

$$[T_m(\tilde{u})]^1 = [\tilde{u}]^1.$$

□

**Theorem 4.2** *The nearest approximation operator is invariant to translations. i.e. for each  $\tilde{u} \in \mathcal{F}(\mathbb{R})$  and  $a \in \mathbb{R}$ , we have  $T_m(\tilde{u} + a) = T_m(\tilde{u}) + a$ .*

**Proof:** Let  $a$  be a real number. Then  $\underline{u} + a = \underline{u} + a$  and  $\overline{u + a} = \overline{u} + a$ . Therefore

$$\mathbf{val}(T_m(\tilde{u} + a)) = \mathbf{val}(\tilde{u} + a) = \mathbf{val}(\tilde{u}) + a = \mathbf{val}(T_m(\tilde{u})) + a,$$

$$\mathbf{amb}(T_m(\tilde{u} + a)) = \mathbf{amb}(\tilde{u} + a) = \mathbf{amb}(\tilde{u}) = \mathbf{amb}(T_m(\tilde{u})) = \mathbf{amb}(T_m(\tilde{u}) + a),$$

Also for  $k = 1, 2, \dots, m - 1$ , we have

$$\mathbf{val}_k((T_m(\tilde{u}) + a)) = \mathbf{val}_k((\tilde{u} + a)) = \mathbf{val}_k(\tilde{u}) = \mathbf{val}_k(T_m(\tilde{u})),$$

and

$$\mathbf{amb}_k((T_m(\tilde{u}) + a)) = \mathbf{amb}_k((\tilde{u} + a)) = \mathbf{amb}_k(\tilde{u}) = \mathbf{amb}_k(T_m(\tilde{u})).$$

Thus

$$D_m(T_m(\tilde{u} + a), T_m(\tilde{u}) + a) = 0.$$

Since both  $T_m(\tilde{u} + a)$  and  $T_m(\tilde{u}) + a$  have  $m$ -degree polynomial-form, then from Lemma 3.9 we have

$$T_m(\tilde{u} + a) = T_m(\tilde{u}) + a.$$

□

**Theorem 4.3** *The nearest approximation operator is scale invariant. i.e. for each  $\tilde{u} \in \mathcal{F}(\mathbb{R})$  and  $\lambda \in \mathbb{R}$ , we have  $T_m(\lambda\tilde{u}) = \lambda T_m(\tilde{u})$ .*

**Proof:** Let  $\lambda \neq 0$  be a real number. Thus for  $k = 0, 1, \dots, m - 1$ , we have

$$\mathbf{val}_k(\lambda\tilde{u}) = \lambda\mathbf{val}_k(\tilde{u}) \quad , \quad \mathbf{amb}_k(\lambda\tilde{u}) = \lambda\mathbf{amb}_k(\tilde{u}).$$

and

$$D_m(T_m(\lambda\tilde{u}), \lambda T_m(\tilde{u})) = 0.$$

Therefore from Lemma 3.9 we have

$$T_m(\lambda\tilde{u}) = \lambda T_m(\tilde{u}).$$

□

**Theorem 4.4** *The nearest approximation operator fulfills the nearness criterion with respect to  $m$ -source metric  $D_m$  defined in Definition 3.4, on the set of all  $m$ -degree polynomial-form fuzzy numbers.*

**Proof:** By Lemma 3.3, we have

$$D_m(\tilde{u}, T_m(\tilde{u})) = \min_{\tilde{v} \in \mathcal{PF}_m(\mathbb{R})} D_m(\tilde{u}, \tilde{v}),$$

therefore

$$D_m(\tilde{u}, T_m(\tilde{u})) \leq D_m(\tilde{u}, \tilde{v}), \quad \forall \tilde{v} \in \mathcal{PF}_m(\mathbb{R}).$$

□

**Theorem 45** *The nearest approximation operator is continuous.*

**Proof:** An approximation operator  $T$  is continuous if for any  $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$  we have

$$\forall \epsilon > 0, \exists \delta > 0, D_m(\tilde{u}, \tilde{v}) < \delta \implies D_m(T(\tilde{u}), T(\tilde{v})) < \epsilon.$$

Let  $D_m(\tilde{u}, \tilde{v}) < \delta$ . By Theorem 3.2 we have

$$D_m(T_m(\tilde{u}), T_m(\tilde{v})) \leq D_m(T_m(\tilde{u}), \tilde{u}) + D_m(\tilde{u}, \tilde{v}) + D_m(\tilde{v}, T_m(\tilde{v}))$$

and by Theorem 3.4 we have  $D_m(T_m(\tilde{u}), \tilde{u}) = D_m(\tilde{v}, T_m(\tilde{v})) = 0$ . Thus

$$D_m(T_m(\tilde{u}), T_m(\tilde{v})) \leq D_m(\tilde{u}, \tilde{v}) < \delta.$$

Therefore it suffices to take  $\delta \leq \epsilon$ .

□

**Theorem 46** *The nearest trapezoidal approximation operator (case  $m=1$ ) is monotonic on any set of fuzzy numbers with equal cores.*

**Proof:** See (2).

□

**Theorem 47** *The nearest approximation operator is order invariant with respect to value function.*

**Proof:** The proof is trivial, because  $\mathbf{val}(T_m(\tilde{u})) = \mathbf{val}(\tilde{u})$  and  $\mathbf{val}(T_m(\tilde{v})) = \mathbf{val}(\tilde{v})$ .

□

**Theorem 48** *The nearest approximation operator does not change the distance of fuzzy numbers by  $m$ -source distance. i.e.*

$$D_m(T_m(\tilde{u}), T_m(\tilde{v})) = T_m(D_m(\tilde{u}, \tilde{v})) = D_m(\tilde{u}, \tilde{v}).$$

**Proof:** The proof is trivial by Lemma 3.11.

□

By the following Theorem we show that the nearest approximation operator is a linear operator:

**Theorem 49** *The nearest approximation operator is a linear operator on the set of all fuzzy numbers. i.e. for a real number  $\lambda$  and two fuzzy numbers  $\tilde{u}$  and  $\tilde{v}$  we have*

$$T_m(\lambda\tilde{u} + \tilde{v}) = \lambda T_m(\tilde{u}) + T_m(\tilde{v}).$$

**Proof:** From Lemma 3.2 we have

$$\begin{aligned} D_m(\lambda\tilde{u} + \tilde{v}, \lambda T_m(\tilde{u}) + T_m(\tilde{v})) &\leq D_m(\lambda\tilde{u}, \lambda T_m(\tilde{u})) + D_m(\tilde{v}, T_m(\tilde{v})) \\ &= D_m(\lambda\tilde{u}, \lambda T_m(\tilde{u})) \\ &= |\lambda| D_m(\tilde{u}, T_m(\tilde{u})) = 0. \end{aligned}$$

Also we have

$$D_m(\lambda\tilde{u} + \tilde{v}, T_m(\lambda\tilde{u} + \tilde{v})) = 0,$$

therefore

$$D_m(T_m(\lambda\tilde{u} + \tilde{v}), \lambda T_m(\tilde{u}) + T_m(\tilde{v})) = 0.$$

Since both fuzzy numbers  $T_m(\lambda\tilde{u} + \tilde{v})$  and  $\lambda T_m(\tilde{u}) + T_m(\tilde{v})$  belong to  $\mathcal{PF}_m(\mathbb{R})$ , we have

$$T_m(\lambda\tilde{u} + \tilde{v}) = \lambda T_m(\tilde{u}) + T_m(\tilde{v}).$$

□

## 5. Linear system of fuzzy equations

In this section we introduce a method for solving fuzzy linear system of equations, and this method is taken from (9).

Let  $\tilde{\mathbf{A}}$  and  $\tilde{\mathbf{b}}$  be a matrix and a vector with fuzzy number arrays, respectively. A system of fuzzy linear equations with crisp variables is a system with the following form:

$$\tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}. \quad (5.1)$$

Equivalently we have

$$\begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + \dots + \tilde{a}_{1n}x_n &= \tilde{b}_1 \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 + \dots + \tilde{a}_{2n}x_n &= \tilde{b}_2 \\ &\vdots \\ \tilde{a}_{n1}x_1 + \tilde{a}_{n2}x_2 + \dots + \tilde{a}_{nn}x_n &= \tilde{b}_n \end{cases} \quad (5.2)$$

where for  $i, j = 1, \dots, n$ ;  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's are fuzzy numbers.

### Lemma 5.1

Consider two systems of linear equations as follows

$$\tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}. \quad (5.3)$$

and

$$[\tilde{\mathbf{A}}]^{-1}\mathbf{x} = [\tilde{\mathbf{b}}]^{-1}. \quad (5.4)$$

If system (5.3) has a solution then the system (5.4) has a solution. Furthermore if  $\mathbf{x}^*$  be the solution of (5.3) then  $\mathbf{x}^*$  is the solution of (5.4) too.

**Proof:** It is straightforward. □

**Corollary 5.2** *If system (5.4) hasn't any solution then the system (5.3) hasn't any solution too.*

### 6. Solution of fuzzy linear equations

In this Section we try to solve a system of fuzzy linear equations with  $m$ -degree polynomial-form fuzzy number coefficients. Let for positive integer  $m$ , all  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's are fuzzy numbers with  $m$ -degree polynomial-form, for  $i, j = 1, \dots, n$ . Our equations now are as follows:

$$\sum_{j=1}^n \tilde{a}_{ij} x_j = \tilde{b}_i, \quad i = 1, \dots, n. \tag{6.1}$$

where for  $i, j = 1, \dots, n; \tilde{a}_{ij}, \tilde{b}_i \in \mathcal{PF}_m(\mathbb{R})$ , and  $x_j \in \mathbb{R}$ . We can consider (6.1) as

$$\sum_{j=1}^n (\underline{a}_{ij}(r), \overline{a}_{ij}(r)) x_j = (\underline{b}_i(r), \overline{b}_i(r)), \quad i = 1, \dots, n.$$

It means that

$$\sum_{x_j \geq 0} \underline{a}_{ij}(r) x_j + \sum_{x_j < 0} \overline{a}_{ij}(r) x_j = \underline{b}_i(r), \quad i = 1, \dots, n, \tag{6.2}$$

and

$$\sum_{x_j \geq 0} \overline{a}_{ij}(r) x_j + \sum_{x_j < 0} \underline{a}_{ij}(r) x_j = \overline{b}_i(r), \quad i = 1, \dots, n. \tag{6.3}$$

Let  $x_j = x'_j - x''_j$ , where

$$x'_j = \begin{cases} x_j, & x_j \geq 0, \\ 0, & x_j < 0, \end{cases}$$

and

$$x''_j = \begin{cases} 0, & x_j \geq 0, \\ -x_j, & x_j < 0, \end{cases}$$

thus  $x'_j, x''_j \geq 0$ , and we have

$$\sum_{j=1}^n \underline{a}_{ij}(r) x'_j - \sum_{j=1}^n \overline{a}_{ij}(r) x''_j = \underline{b}_i(r), \quad i = 1, \dots, n, \tag{6.4}$$

and

$$\sum_{j=1}^n \overline{a}_{ij}(r) x'_j - \sum_{j=1}^n \underline{a}_{ij}(r) x''_j = \overline{b}_i(r), \quad i = 1, \dots, n. \tag{6.5}$$

Since  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's are fuzzy numbers with  $m$ -degree polynomial-form, there exist coefficients  $c_{ijk}, d_{ijk}, e_{ik}$  and  $f_{ik}$  such that

$$\begin{cases} \underline{a}_{ij}(r) = \sum_{k=0}^m c_{ijk} r^k, & i, j = 1, \dots, n, \\ \overline{a}_{ij}(r) = \sum_{k=0}^m d_{ijk} r^k, & i, j = 1, \dots, n, \\ \underline{b}_i(r) = \sum_{k=0}^m e_{ik} r^k, & i = 1, \dots, n, \\ \overline{b}_i(r) = \sum_{k=0}^m f_{ik} r^k, & i = 1, \dots, n. \end{cases} \tag{6.6}$$

By substituting (6.6) into the equations (6.4) and (6.5), we have

$$\sum_{j=1}^n \sum_{k=0}^m c_{ijk} r^k x'_j - \sum_{j=1}^n \sum_{k=0}^m d_{ijk} r^k x''_j = \sum_{k=0}^m e_{ik} r^k, \quad i = 1, \dots, n,$$

and

$$\sum_{j=1}^n \sum_{k=0}^m d_{ijk} r^k x'_j - \sum_{j=1}^n \sum_{k=0}^m c_{ijk} r^k x''_j = \sum_{k=0}^m f_{ik} r^k, \quad i = 1, \dots, n.$$

Consequently for all  $i = 1, 2, \dots, n$ , and  $k = 0, 1, \dots, m$ , we have

$$\sum_{j=1}^n c_{ijk} x'_j - \sum_{j=1}^n d_{ijk} x''_j = e_{ik}, \quad (6.7)$$

and

$$\sum_{j=1}^n d_{ijk} x'_j - \sum_{j=1}^n c_{ijk} x''_j = f_{ik}. \quad (6.8)$$

Considering

$$\mathbf{C} = \begin{pmatrix} c_{110} & \dots & c_{1n0} \\ \vdots & & \vdots \\ c_{11m} & \dots & c_{1nm} \\ \vdots & & \vdots \\ c_{n10} & \dots & c_{nm0} \\ \vdots & & \vdots \\ c_{n1m} & \dots & c_{nmm} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_{110} & \dots & d_{1n0} \\ \vdots & & \vdots \\ d_{11m} & \dots & d_{1nm} \\ \vdots & & \vdots \\ d_{n10} & \dots & d_{nm0} \\ \vdots & & \vdots \\ d_{n1m} & \dots & d_{nmm} \end{pmatrix}$$

and

$$\mathbf{e} = \begin{pmatrix} e_{10} \\ \vdots \\ e_{1m} \\ \vdots \\ e_{n0} \\ \vdots \\ e_{nm} \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_{10} \\ \vdots \\ f_{1m} \\ \vdots \\ f_{n0} \\ \vdots \\ f_{nm} \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}, \quad \mathbf{x}'' = \begin{pmatrix} x''_1 \\ \vdots \\ x''_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \mathbf{x}' \\ \mathbf{x}'' \end{pmatrix}$$

one should solve the following system of linear equations,

$$\mathbf{S}\mathbf{y} = \mathbf{d}, \quad (6.9)$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{C} & -\mathbf{D} \\ \mathbf{D} & -\mathbf{C} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}' \\ \mathbf{x}'' \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

and we have  $\mathbf{x} = \mathbf{x}' - \mathbf{x}''$ .

$\mathbf{S}$  is a  $2n(m+1) \times 2n$  matrix, and since  $m > 0$ , we try to solve the following least squares problem:

$$\min_{\mathbf{y} \in \mathbb{R}^{2n}} \|\mathbf{S}\mathbf{y} - \mathbf{d}\|_2. \quad (6.10)$$

It is well known that the solution of (6.10) is equal to the solution of its corresponding normal system of equations as follows:

$$S^T S y = S^T d. \tag{6.11}$$

**Lemma 6.1**

The system (6.11) always has a solution even when  $S^T S$  is singular.

**Proof:** See (23). □

Let  $x^*$  and  $\bar{x}$  be the solution of systems (5.3) and (5.4), respectively and let  $\tilde{x}$  and  $\hat{x}$  be the obtained solutions corresponding to the systems (6.9) and (6.11), respectively. From the last lemma it is known that  $\hat{x}$  always exists.

**Lemma 6.2**

Let all  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's have  $m$ -degree polynomial-form. If  $x^*$  exists then  $\bar{x}$  and  $\tilde{x}$  exist and  $x^* = \bar{x} = \tilde{x} = \hat{x}$ .

**Proof:** It is straightforward. □

$\hat{x}$  is the least squares solution of the system  $\tilde{A}x = \tilde{b}$ , also if it's corresponding vector  $y$  satisfies in the system  $Sy = d$  then  $\hat{x}$  is the exact solution of the system  $\tilde{A}x = \tilde{b}$ .

If  $S^T S$  is nonsingular then the solutions of two systems (6.9) and (6.11) are the same and  $y^* = (S^T S)^{-1} S^T d$ .

By a simple computations it can be shown that

$$S^T S = \begin{pmatrix} L & -M \\ -M & L \end{pmatrix},$$

where

$$L = C^T C + D^T D, \quad M = C^T D + D^T C.$$

Thus

$$S^T S \rightarrow \begin{pmatrix} L - M & L - M \\ -M & L \end{pmatrix} \rightarrow \begin{pmatrix} L - M & 0 \\ -M & L + M \end{pmatrix},$$

therefore

$$|S^T S| = |L - M| \cdot |L + M|.$$

Also we have

$$L - M = (C - D)^T (C - D), \quad L + M = (C + D)^T (C + D).$$

If  $C - D$  be a full rank matrix then  $L - M$  is nonsingular and if  $C + D$  be a full rank matrix then  $L + M$  is nonsingular. i.e. if  $rank(C \pm D) = n$  then  $S^T S$  is nonsingular.

**7. Numerical solution of fuzzy linear equations**

In this section we consider a system of fuzzy linear equations as

$$\tilde{A}x = \tilde{b}, \tag{7.1}$$

where for  $i, j = 1, \dots, n$ ;  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's, are from LR type fuzzy numbers.

For the purpose of solving (7.1), we consider the following system of equations

$$\tilde{\mathbf{A}}^{(m)} \mathbf{x} = \tilde{\mathbf{b}}^{(m)}, \quad (7.2)$$

where for  $i, j = 1, \dots, n$ ; the quantities  $\tilde{a}_{ij}^{(m)}$ 's and  $\tilde{b}_i^{(m)}$ 's are the nearest approximations of the given numbers  $\tilde{a}_{ij}$ 's and  $\tilde{b}_i$ 's out of  $\mathcal{PF}_m(\mathbb{R})$ , respectively. Existence of the nearest approximations of the given fuzzy numbers is known from Lemma 3.6.

From Lemma 6.1 we know that this system has a least squares solution. Using the process in the Section 6 we obtain a least squares solution  $\mathbf{x}^{(m)}$  for this system. We name this solution, the  $m$ -degree nearest least squares ( $m$ -DNLS) solution of the system (5.3). Thus we have

$$\tilde{\mathbf{A}}^{(m)} \mathbf{x}^{(m)} = \tilde{\mathbf{b}}^{(m)}, \quad (7.3)$$

Choosing  $m$  depends on the shape of left and right spread functions  $L$  and  $R$ , and the derivation order of them.

## 8. Numerical examples

In this Section we present some numerical examples which have been solved on both  $m$ -degree polynomial-form fuzzy number coefficients and general  $LR$  fuzzy number coefficients.

**Example 8.1** Consider the following 2 by 2 system of equations with  $m = 1$ :

$$\begin{cases} (-1 + 2r, 4 - 2r)x_1 + (-2 + 3r, 3 - 2r)x_2 = (-8 + 13r, 17 - 10r), \\ (1 + r, 4 - r)x_1 + (2r, 5 - 2r)x_2 = (2 + 8r, 23 - 8r), \end{cases}$$

For this system of equations by solving the least squares system we have  $x_1 = 2$  and  $x_2 = 3$ .

**Example 8.2** Consider the following 2 by 2 system of equations with  $m = 1$ :

$$\begin{cases} (-1 + r, 3 - r)x_1 + (1 + 2r, 4 - r)x_2 = (-12 + 11r, 17 - 8r), \\ (-1 + 2r, 3 - 2r)x_1 + (3r, 6 - 2r)x_2 = (-15 + 19r, 23 - 16r), \end{cases}$$

For this system of equations by solving the least squares system we have  $x_1 = -5$  and  $x_2 = 3$ .

**Example 8.3** Consider the following 2 by 2 system of equations with  $m = 2$ :

$$\begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 = \tilde{b}_1, \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 = \tilde{b}_2, \end{cases}$$

where

$$\begin{aligned} \tilde{a}_{11} &= (3r + r^2, 7 - 3r + 2r^2) \\ \tilde{a}_{12} &= (2r + r^2, 4 - 2r + 2r^2) \\ \tilde{a}_{21} &= (1 + 2r + r^2, 8 - 3r + r^2) \\ \tilde{a}_{22} &= (1 + 2r + r^2, 6 - 3r + 2r^2) \\ \tilde{b}_1 &= (48.45r + 17.1r^2, 111.15 - 48.45r + 34.2r^2) \\ \tilde{b}_2 &= (17.1 + 34.2r + 17.1r^2, 131.1 - 51.3r + 19.95r^2), \end{aligned}$$

For this system of equations by solving the least squares system we have  $x_1 = 14.85$  and  $x_2 = 2.25$ .

**Example 8.4** Consider the following 2 by 2 system of equations

$$\begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 = \tilde{b}_1, \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 = \tilde{b}_2, \end{cases}$$

where

$$\begin{cases} \tilde{a}_{11} = \tilde{a}_{22} = (2 + \ln\{(e - 1)r + 1\}, 4 - \ln\{(e - 1)r + 1\}), \\ \tilde{a}_{12} = \tilde{a}_{21} = (3 + \ln(2r + 1), 3 + 2\ln 3 - \ln(2r + 1)), \\ \tilde{b}_1 = (13 + \ln\{((e - 1)r + 1)^2(2r + 1)^3\}, 17 + 6\ln 3 - \ln\{((e - 1)r + 1)^2(2r + 1)^3\}), \\ \tilde{b}_2 = (12 + \ln\{((e - 1)r + 1)^3(2r + 1)^2\}, 18 + 4\ln 3 - \ln\{((e - 1)r + 1)^3(2r + 1)^2\}). \end{cases}$$

By choosing  $m = 3$ , we have

$$\begin{cases} \frac{a_{11}}{\tilde{a}_{11}} = \frac{a_{22}}{\tilde{a}_{22}} = 2.01846 + 1.48660r - 0.64225r^2 + 0.137185r^3, \\ \frac{a_{12}}{\tilde{a}_{12}} = \frac{a_{21}}{\tilde{a}_{21}} = 3.98154 - 1.48660r + 0.64225r^2 - 0.13718r^3, \\ \frac{b_1}{\tilde{b}_1} = \frac{b_2}{\tilde{b}_2} = 3.02605 + 1.66763r - 0.75820r^2 + 0.16313r^3, \\ \frac{b_1}{\tilde{b}_1} = \frac{b_2}{\tilde{b}_2} = 5.17117 - 1.66763r + 0.75820r^2 - 0.16313r^3, \\ \frac{b_1}{\tilde{b}_1} = 13.11509 + 7.97609r - 3.55909r^2 + 0.76375r^3, \\ \frac{b_1}{\tilde{b}_1} = 23.47658 - 7.97609r + 3.55909r^2 - 0.76375r^3, \\ \frac{b_2}{\tilde{b}_2} = 12.107499 + 7.79506r - 3.44314r^2 + 0.73781r^3, \\ \frac{b_2}{\tilde{b}_2} = 22.28695 - 7.79506r + 3.44314r^2 - 0.73781r^3, \end{cases}$$

and the 3–DNLS solution of this system is  $x_1 = 2.000000$  and  $x_2 = 3.000000$ .

**Example 8.5** Consider the following 2 by 2 system of equations

$$\begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 = \tilde{b}_1, \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 = \tilde{b}_2, \end{cases}$$

where

$$\begin{cases} \tilde{a}_{11} = \tilde{a}_{21} = (\ln(r + 1), \ln(4 - 2r)), \\ \tilde{a}_{12} = (\ln(2r + 1), \ln(4 - r)), \\ \tilde{a}_{22} = (\ln(r + 1), \ln(5 - 3r)), \\ \tilde{b}_1 = (\ln\{(r + 1)^2(2r + 1)\}, \ln\{(4 - 2r)^2(4 - r)\}), \\ \tilde{b}_2 = (3\ln(r + 1), \ln\{(4 - 2r)^2(5 - 3r)\}). \end{cases}$$

the 1–DNLS solution of this system is  $x_1 = 2.022850$  and  $x_2 = 0.06491$ , meanwhile the 2–DNLS solution of this system is  $x_1 = 2 + 0.0 \times 10^{-6}$  and  $x_2 = 1 + 0.0 \times 10^{-6}$ , also the 3–DNLS solution of this system is  $x_1 = 2 + 0.0 \times 10^{-13}$  and  $x_2 = 1 + 0.0 \times 10^{-13}$ .

### 9. Conclusion

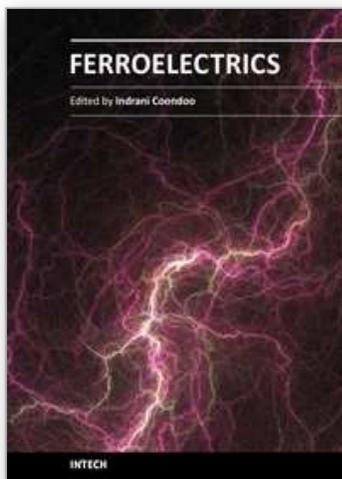
In this chapter a new method was proposed to solve a system of linear equations. If all coefficients are polynomial form fuzzy numbers then the system may have an exact solution but otherwise we can approximate fuzzy coefficients with  $m$ -degree polynomial-form fuzzy numbers and find an approximated solution of the system. Choosing  $m$  depends on the shape of left and right spread functions  $L$  and  $R$ , and the derivation order of them. The presented method can be applied on any system of equations with  $LR$  fuzzy number coefficients.

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## **Ferroelectrics**

Edited by Dr Indrani Coondoo

ISBN 978-953-307-439-9

Hard cover, 450 pages

**Publisher** InTech

**Published online** 14, December, 2010

**Published in print edition** December, 2010

Ferroelectric materials exhibit a wide spectrum of functional properties, including switchable polarization, piezoelectricity, high non-linear optical activity, pyroelectricity, and non-linear dielectric behaviour. These properties are crucial for application in electronic devices such as sensors, microactuators, infrared detectors, microwave phase filters and, non-volatile memories. This unique combination of properties of ferroelectric materials has attracted researchers and engineers for a long time. This book reviews a wide range of diverse topics related to the phenomenon of ferroelectricity (in the bulk as well as thin film form) and provides a forum for scientists, engineers, and students working in this field. The present book containing 24 chapters is a result of contributions of experts from international scientific community working in different aspects of ferroelectricity related to experimental and theoretical work aimed at the understanding of ferroelectricity and their utilization in devices. It provides an up-to-date insightful coverage to the recent advances in the synthesis, characterization, functional properties and potential device applications in specialized areas.

### **How to reference**

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Majid Amirfakhrian (2010). Numerical Solution of a System of Polynomial Parametric Form Fuzzy Linear Equations, *Ferroelectrics*, Dr Indrani Coondoo (Ed.), ISBN: 978-953-307-439-9, InTech, Available from: <http://www.intechopen.com/books/ferroelectrics/numerical-solution-of-a-system-of-polynomial-parametric-form-fuzzy-linear-equations>

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