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Pursuit-Evasion Games in Presence of Obstacles in Unknown Environments: towards an optimal pursuit strategy [*]

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Abstract

In this paper, we will incrementally build a complete pursuit algorithm to deal with a 2-players PEG in presence of a single unknown convex obstacle. We will first provide a sufficient condition to achieve capture without disappearance. Then, we will solve the *circular obstacle problem*, a particular problem highlighting a necessary trade-off between surveillance and capture. Next, the *pole problem*, as a generalization, of the convex obstacle problem will be tackled. The solution and the corresponding strategies will be detailed. A quasi-optimal pursuit strategy as regards the time to capture will be provided for the pole problem, and then transposed for the more general *convex obstacle problem*. For the cases leading to the evader victory in the pole problem, a last strategy allowing a maximal deviation of the line of disappearance will be added to complete to our pursuit algorithm. Finally, our complete pursuit algorithm will be adapted to use a heuristic minimization method instead of the strategy suggested by the resolution of the pole problem for the cases leading to the pursuer victory. Different heuristics, one being an approximation of the solution of the pole problem, will be compared with respect to the size of the capture basin and will highlight thez interest of our pursuit algorithm.

1. Introduction

Continuous differential games have been widely studied since the pioneering work of Issacs Isaacs (1965). In particular, pursuit-evasion games (PEGs) have received a great deal of attention, particularly in free spaces for problems such as the missile guidance Basar and Olsder. (1982); Espiau et al. (1992); Hájek (1975); Hutchinson et al. (1996); Isaacs (1965); Song and Um; (1996). Interesting recent works include the notion of *forward reachable sets* (related to maneuverability of the pursuer) for a team of pursuers against a fast moving evader Chung and Furukawa (2006); Chung et al. (2006). In contrast, PEGs in cluttered unknown environments, where obstacles imply specific movement constraints of the pursuer for maintaining visibility and the possibility for the evader to hide, represent a more recent problem, for which a definitive solution has not yet been found. The problem has been split into several classes. A first class of problems is addressed when the evader is not yet visible. Two major issues can be

discussed: the first consists in developing algorithms in order to find one or several static or dynamic evaders, in an environment either known or unknown, with either a single pursuer or a team of pursuers Chen et al. (2005); Gerkey et al. (2006); LaValle et al. (1997b); Park et al. (2001); Sachs et al. (2004); Suzuki and Yamashita (1992). These approaches suggest that before tracking an evader, efficient solutions to find it should be proposed. The second question, often referred as the Art Gallery Problem, consist in the efficient control of a team of robots so that every part of the environment could be visible by at least one pursuer, thus avoiding the intrusion of a robber in the art gallery Chvatal (1975); Gonzalez-Banos and Latombe (2001); O'Rourke (1983; 1987; 1998); Shermer (1992). Similar works focus on the problem of the positioning a minimum number of captors (movement captors or simply cameras) in the art gallery in order to remove invisible part of the environment. This first class of problems addressed the seeking the evader: what should be done when the evader is not yet visible.

Another major category of problems, that particularly interests us, arises as soon as the target(s) is/are visible. Approaches depend on the relative capabilities of the players, their relative knowledge, their objectives, and the number of pursuers and evaders the mission scenarios consider. A first historical question was raised by David Gal known as the Lion and Man Problem: a man (evader) and a lion (pursuer) are moving with the same speed within the non negative quadrant of the plane. In Sgall (2001), a solution that the author claims to be nearly optimal is proposed, consisting for the lion (whom coordinates are initially greater than the man's coordinates) in aligning himself between the man's position and a particular reference point (the center of the smallest circle, touching its initial position and both axis of the Euclidian space). More generally, interesting solutions for 2-players PEGs have been proposed when the map of the environment is known by the pursuer. The evader may be predictable LaValle et al. (1997a) or more interestingly unpredictable Isler et al. (2004); Murrieta-Cid et al. (2002; 2004; 2003). Thanks to the knowledge of the environment, scenarios mostly focus on maintaining the visibility of the target. Recent solutions rely on the use of a graph of mutual visibility: the environment is first subdivided into regions, and a graph that describes the visibility of each region by the others Murrieta-Cid et al. (2008). A NP-hard method based on this mutual visibility graph is proposed to provide a sufficient condition for maintaining the visibility. Another interesting study highlights that situations in which the target can never be captured Cheung (2005) may exist, even when the evader speed is smaller than the pursuer speed. Scenarios in which the pursuer has to stay at a fixed distance from the evader has also been tackled Muppirala et al. (2005).

In this paper, we consider a minimalist 2-players PEG in presence of obstacles, by assuming that: a) the map of the environment is also unknown; b) the exact positions of the pursuer and the evader are unknown. The mapping of the obstacles is not aimed since: 1) mapping is a classical and well documented problem, 2) PEGs in known environments is also well documented (but not definitively solved), 3) we assume that none of the opponents had time to do the mapping before the conflict, 4) we assume that the game ends upon the capture or the disappearance of the evader, 5) we obviously agree that the use of the map can be helpful for further pursuits in the same environment, 6) we hope in the future to deal with dynamic obstacles (moving obstacles, obstacles shape changes due to a non planar ground ...).

The problem of PEGs in unknown cluttered environment has not been extensively studied: provided solutions mostly aim at maintaining visibility in a classical indoor environment. In Gonzalez-Banos et al. (2002); Lee et al. (2002), the method is based on the minimization of a heuristic called escape risk. A more recent work proposed a better heuristic based on an approximated computation of what is called the vantage time Bandyopadhyay et al. (2006).

Interestingly, the authors point out that trying to maximize the instantaneous visibility, as done in Gonzalez-Banos et al. (2002); Lee et al. (2002), actually increases the latter probability of the evader disappearance, as opposed to a better balancing between closing the distance to the obstacle and maintaining visibility, which seems to offer a better global behavior of the algorithm.

Note that the capture as a termination mode in these previous studies was not considered; the sole objective was to maintain the visibility of the target as long as possible. Moreover, theses studies did not consider the evader as smart, resulting in a problem description without antagonist goals for the pursuer and the evader.

Hence, an interesting facet of this paper is to consider 2-player PEGs in presence of obstacles as an antagonist game, in which the evader is at least as smart as the pursuer and both of them have antagonist objectives. As Isaacs early said: *"... Difficulty of the problems when - and such is the essence of game theory - there are two opponents with conflicting aims and each is to make the best possible decisions understanding and taking into account that his antagonist is doing the same... If we seek conflicting objectives - and only such cases are of interest - the situation assumes something on the nature of the game."*

A last approach that should be cited before entering the heart of the paper is the use of genetic algorithm, inspired by evolutionist neuro-ethological data about the development of pursuit and evasion capabilities among the animal species Cliff and Miller (1996); Miller and Cliff (1994), in order to incrementally generate populations of pursuers and evaders that progress in parallel (ie: that co-evaluate) Choi et al. (2004); Eaton et al. (2002); Nitschke (2003).

In the following, we will incrementally build a complete pursuit algorithm to deal with a 2-players PEG in presence of a single unknown convex obstacle. We will first provide a sufficient condition to achieve capture without disappearance, based on the properties of the famous parallel pursuit. Then, we will solve the *circular obstacle problem*, a particular problem in which the evader, initially located on a circular obstacle, tries to hide behind it. The pursuer, initially located on the tangent to the obstacle crossing the evader position, has to capture the evader in minimum time or at least has to maximally delay its disappearance. Next, the *pole problem*, as a generalization of the convex obstacle problem, will be tackled. In this problem, the pursuer tries to capture the evader and the evader tries to rotate the line of sight in order to create a contact with a pole (corresponding to a point). The solution and the corresponding strategies will be detailed. A quasi-optimal pursuit strategy as regards the time to capture will be provided for the pole problem, and then transposed for the more general *convex obstacle problem*. For the cases leading to the evader victory in the pole problem, a last strategy allowing to maximally rotating the line of disappearance will be added to complete to our pursuit algorithm. Finally, our complete pursuit algorithm will be adapted to use a heuristic minimization method instead of the strategy suggested by the resolution of the pole problem for the cases leading to the pursuer victory. Different heuristics (inspired from the literature, or proposed here, one being an approximation of the solution of the pole problem) will be compared with respect to the size of the capture basin (initial conditions of the pursuer leading to capture without disappearance), highlighting the performance enhancement allowed by our solutions.

2. Interest of the convex obstacle problem

In this paper, a particular subset of the possible games, the *convex obstacle problem*, will be investigated. The rules are the following:

Rule 1: The map of the environment is initially unknown.

Rule 2: The pursuer is faster than the evader.

Rule 3: Each player knows the maximal speed of the other player.

Rule 4: The environment contains a single convex obstacle

Rule 5: The pursuer wins if it captures the evader in finite time while avoiding its disappearance

Rule 6: The evader wins if it succeeds in hiding or if it infinitely delays the capture.

The rule 1 has already been justified previously. The rule 2 is classical in PEGs since if the evader is faster or as fast as the pursuer, as a general rule, it can evade easily [1]. The rule 3 is also used since it largely extends the methods that can be developed. Moreover, the speed of an antagonist can be continuously estimated.

However, at first sight, one can wonder why the disappearance as a termination mode in an environment containing a single convex obstacle is interesting (rules 4, 5, 6). Indeed, in such an environment, even if the evader disappears for a while, the pursuer will eventually see it again and capture it by simply executing the following procedure: first it reaches the disappearance point and then it turns around the obstacle along its boundary. If the evader also moves along the boundary of the obstacle, the pursuer will obviously capture it. Otherwise, the pursuer can move along the obstacle boundary until being on a line orthogonal to the boundary of the obstacle crossing the position of the evader. In such a situation, capture is guaranteed without future disappearance by many pursuit strategies since the capture region is likely to not be altered by the obstacle.

Even if the obstacle is not convex but is simply such that each point of its boundary can be seen from at least one point outside the convex hull of the obstacle (let us call this kind obstacle a *nookless* obstacle), we could prove that capture is guaranteed. Indeed, after disappearance, if the pursuer simply follow the convex hull of the obstacle (which is the shortest path that allow to see all the points on the boundary of a nookless obstacle), either the evader has not entered the convex hull so the pursuer can always reach a position such that it is on the line orthogonal to the convex hull crossing the evader position (similar to the previous problem that consider a convex obstacle), or the evader has entered the convex hull. In this later case, the pursuer can always recover the sight of the evader by simply moving along the convex hull. Once the sight is recovered, the pursuer may use some strategies to prevent the evader to exit the convex hull (the problem becomes closer to a Lion and Man Problem for which solutions exist).

So, why should the disappearance be considered as a termination mode in the case of a single convex or even a nooklees obstacle. In presence of a single non-nookless obstacle or several obstacles, once the evader has disappeared, there is no deterministic guaranty to recover its sight. Indeed, when the pursuer sees a nook in the obstacle, either it enters the nook but the evader may simply have followed the convex hull, or the pursuer follows the convex hull but the evader may simply have entered the nook and may hide in a region that is not seeable from the convex hull. The same dilemma occurs with several obstacles: the pursuer can never know if the evader has turned around the obstacle behind which it has disappeared or if it is hidden behind another obstacle. To solve such situations, several pursuers seem to be required.

That is why it is very important to not lose the sight of the evader, and this is why the disappearance as a termination mode is very important, even if there is only a single convex

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[1] Actually, in *Lion and Man problems* Sgall (2001), the evader can be captured even if its speed is the same as the pursuer speed thanks to a line of sight pursuit for which the reference point is well chosen.

obstacle. An efficient pursuer in such a game will largely reduce the probability to face non-deterministic situations as described above in a more general case. Hence, the case of a PEG in presence of a single obstacle can be reduced to a PEG with a single convex obstacle in order to gain insight about the general problem. Moreover, although the convex obstacle problem is the simplest 2-players PEGs in presence of unknown obstacles, an optimal solution has not yet been found.

3. Sufficient capture condition under visibility constraint

In this section, a general sufficient condition that guaranty capture thanks to the properties of the famous parallel pursuit will be established. The region, where this condition holds, covers the major part of the environment.

Assume for a moment the absence of obstacles. The BSR (Boundary of Safe Region) is defined as the frontier of the region in which the evader E is able to go without being captured, what ever the pursuer P does. If the pursuer is faster than the evader, the classical BSR of the evader involved in a PEG in a free 2D space (no obstacles) is defined by an Apollonius circle Isaacs (1965); Nahin (2007); Petrosjan (1993). This definition is evader-centered. We define here the *pursuit region* related to a particular strategy as the set of positions that can be reached by the pursuer during the game when using a particular strategy. We define also the *capture region* related to a particular strategy as the set of positions where the capture can occur. Obviously, the capture region is included in the pursuit region. Finally, we introduce a short terminology about specific geometric objects such as disappearance vertex, line of disappearance and line of sight (see fig. 1.a)
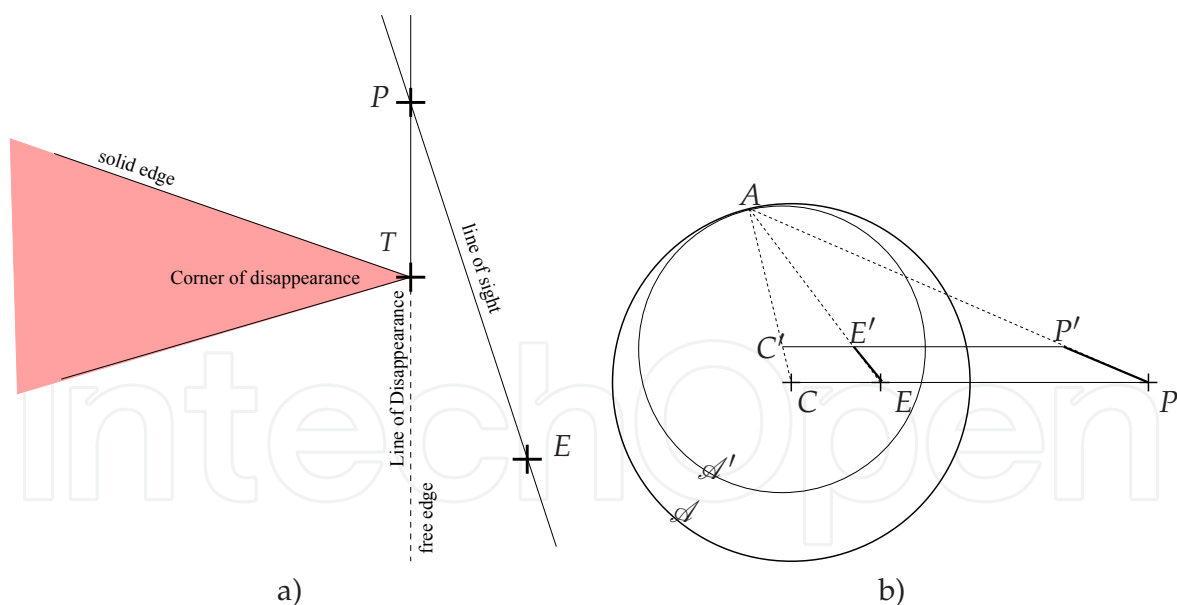

Fig. 1. a) Terminology: the line (PT) is the line of disappearance (i.e.: the tangent to the obstacle crossing the pursuer position), the line (PE) is the line of sight, and T is the disappearance vertex. b) Illustration of the Apollonius circle $\mathscr{A}$ for $\gamma = 4$ (the pursuer is twice faster than the evader), $E : (0,0)$ and $P : (6,0)$. $R = 4$ and $C : (-2,0)$. The Apollonius pursuit is equivalent to a parallel pursuit (all the lines of sight are parallel). Note that $\mathscr{A}'$ is included in $\mathscr{A}'$ and ththe two circle intersects in A.

3.1 Apollonius pursuit properties

Let consider a PEG in the 2D plan with no obstacles, involving a single purser faster than a single evader. The following convention will be used:

- Points in the space are noted with capital letters (such as the point A).
- The coordinates of a point A are noted (x_a, y_a) and (r_a, θ_a) in a polar coordinates system.
- A vector between the origin of the coordinates system and a point A is noted $\boldsymbol{a}$.
- A vector between two points A and B will be noted $\overrightarrow{AB}$ but also $\boldsymbol{b} - \boldsymbol{a}$.
- The angle of a vector $\overrightarrow{AB}$ is noted θ_{AB}.
- $\|.\|$ is the Euclidian 2d-norm.
- The distance between two points A and B can be noted AB but also $\|\boldsymbol{b} - \boldsymbol{a}\|$.
- Geometrical objects are noted with calligraphically written letters (such as the circle $\mathscr{C}$).

The following notations and relations will be used:

- $\boldsymbol{p}$ is the position of the pursuer.
- $\boldsymbol{e}$ is the position of the evader.
- v_e is the maximal speed of the pursuer.
- v_p is the maximal speed of the evader.
- $v_e < v_p$, meaning that the pursuer is faster than the evader.
- $\gamma = k^2 = (\frac{v_p}{v_e})^2$ the square of the ratio k of the pursuer speed above the evader speed.
- $\gamma > 1$ since the pursuer is faster than the evader.

Let us remind some basics results about 2-players PEGs assuming straight line motion of the evader. To capture in minimum time, the optimal pursuit strategy is obviously a straight line motion towards the closest point of capture (a point of capture is such that the time to arrive to this point is the same for both the antagonists). If the evader adopts a straight line motion, the locus of interception $\mathscr{A}$ is the set of points $X : (x, y)$ such that $\frac{\|\boldsymbol{e}-\boldsymbol{x}\|}{v_e} = \frac{\|\boldsymbol{p}-\boldsymbol{x}\|}{v_p}$, more recognizable as:

$$k.\|\boldsymbol{e} - \boldsymbol{x}\| = \|\boldsymbol{p} - \boldsymbol{x}\| \tag{1}$$

$\mathscr{A}$ is an Apollonius circle with E and P as references points and k as parameter (eq. 1 is precisely the definition of an Apollonius circle). Such a circle can be noted $\mathscr{C}(E, P, k)$ The following expression are implied by the equation 1:

$$\begin{aligned} \left(\frac{\|\boldsymbol{e}-\boldsymbol{x}\|}{v_e}\right)^2 &= \left(\frac{\|\boldsymbol{p}-\boldsymbol{x}\|}{v_p}\right)^2 \\ \gamma.\|\boldsymbol{e} - \boldsymbol{x}\|^2 - \|\boldsymbol{p} - \boldsymbol{x}\|^2 &= 0 \end{aligned} \tag{2}$$

With a few substitutions and arrangements, it follows the equation of the Apollonius circle centered on C with the radius R:

$$\begin{cases} \boldsymbol{c} = \dfrac{\gamma.}{\gamma - 1}.(\boldsymbol{e} - \boldsymbol{p}) \\[2mm] R^2 = \dfrac{\gamma}{(\gamma - 1)^2}.\|\boldsymbol{e} - \boldsymbol{p}\|^2 \end{cases} \tag{3}$$

C is obviously aligned with E and P, and its the radius R only depends on the distance $\|e - p\|$ between the evader and the pursuer. Thus, C, located on the extension of the segment $[PE]$, can be expressed as $c = e - \frac{1}{\gamma - 1}(p - e)$. We finally note that the distance between C and P only depends on the distance between E and P as follow (this result will be used later):

$$\|c - p\| = \sqrt{\gamma}.R = \frac{\gamma}{\gamma - 1}\|e - p\| \tag{4}$$

The fig. 1.b illustrates the circle $\mathscr{A}$ for a given γ and for the given initial positions of the pursuer and the evader.

If the evader trajectory is a straight line toward a point A of the circle $\mathscr{A}$, there is no better strategy for the pursuer than going also to the point A, since it will go to A in straight line at its maximal speed. This strategy, often called Apollonius pursuit, is time-optimal for straight line motion. Any other pursuer movement will allow the evader to travel a distance greater than $\|e - a\|$.

A well known properties of the Apollonius pursuit is that any line (EP) during the game is parallel to the initial one. Indeed, as highlighted by the fig. 1.b, assume the evader has moved from E to E'. Let ρ be the ratio of the segment $[EA]$ that has been traveled by going from E to E' ($\frac{\|e' - e\|}{\|a - e\|} = \rho$). During the same time, the pursuer has moved to P', and obviously: $\frac{\|e' - e\|}{v_e} = \frac{\|p' - p\|}{v_p}$. The point A being of the Apollonius circle, it follows that: $\frac{\|a - e\|}{v_e} = \frac{\|a - p\|}{v_p}$. By dividing the two previous equality and with a few arrangement, it follows that:

$$\frac{\|e' - a\|}{\|e - a\|} = \frac{\|p' - a\|}{\|p - a\|} = 1 - \rho$$

The intercept theorem (or Thales theorem) implies that the line (EP) and $(E'P')$ are parallel. Hence, the Apollonius pursuit is more generally called the parallel pursuit, for antagonists that do not move in straight line during the game.

We introduced here the name Π-strategy to refer to the optimal parallel pursuit, the one continuously minimizing the distance $\|e - p\|$ (the notation Π-strategy is used in Petrosjan (1993)). The Π-strategy ensures that the pursuer will capture the evader inside the circle $\mathscr{A}$, whatever the evader does. This point and other properties of the Π-strategy is reminded in the followings.

3.2 Properties of the Π-strategy

If the evader does not move in straight line, the application Π depends on the current evader velocity:

$$\Pi : \quad \begin{array}{ccc} \mathbb{R}^6 & \to & \mathbb{R}^2 \\ (E, \overrightarrow{v_e}, P) & \mapsto & \overrightarrow{v_p} \end{array}$$

It is well known that, in free space (absence of obstacle), if the pursuer is faster than the evader, then the Π-strategy guaranties the capture of the evader inside the initial Apollonius circle $\mathscr{A}$ in finite time without disappearance. Moreover, the Apollonius circle $\mathscr{A}$ is the BSR of the evader (i.e.: the intersection of the capture regions of all the pursuit strategies).

Let us prove the first point. The Π-strategy will first be proved to allow for the evader capture inside the initial Apollonius circle $\mathscr{A}$ (ie: this will prove that the Apollonius circle $\mathscr{A}$ is the capture region related to the Π-strategy whatever the evader does). To prove this point, note that adopting the Π-strategy implies that the new Apollonius circle after an infinitesimal move of the evader and the pursuer is fully included in the initial Apollonius circle $\mathscr{A}$. Then, an upper

bound of the time to achieve the capture can be computed, by noting that the Π-strategy is at the equilibrium as regard a min-max approach. Finally, it will be reminded (thought it is trivial) that if the pursuer does not adopt the Π-strategy, the evader may be captured outside the circle, implying that the Apollonius circle is the BSR of the evader.

Let E' and P' be the point reached by the pursuer and the evader after an infinitesimal duration:

$$e' = e + v_e.dt$$
$$p' = p + v_p.dt$$

Let us call $\mathscr{A}'$ the new Apollonius circle centered on C' with radius R' related to the new positions P' and E' of the antagonists. As previously, $\rho = \frac{\|e'-e\|}{\|a-e\|} = \frac{\|p'-p\|}{\|a-p\|}$ is the ratio of the segment $[EA]$ and $[PA]$ respectively traveled by the evader and the pursuer by going respectively from E to E' and from P to P'. The coordinates E' and P' can be expressed as:

$$
\begin{aligned}
e' &= e + \rho.(a - e) & (5) \\
p' &= p + \rho.(a - p) & (6)
\end{aligned}
$$

By inserting these expression in the definition of the center and the radius of the Apollonius circle, and with a few arrangements, it follows the equation of the circle $\mathscr{A}'$:

$$
\begin{cases}
c' = c + \rho.(a - c) \\
R' = (1 - \rho).R
\end{cases}
\tag{7}
$$

We have shown here that the center C' of the circle $\mathscr{A}'$ belongs to the segment $[CA]$, which is a radius of $\mathscr{A}$. Obsviously, the point A belongs to the new circle $\mathscr{A}'$ since it is still located at the same time of travel from the antagonists. Hence $R' = \|a - c'\|$.

We now have to prove that $\mathscr{A}'$ is fully included in $\mathscr{A}$. Actually, we have to show that the two circles have at most a single intersection point which is precisely A. Let us provide a geometrical proof (see fig. 1.b for the illustration): consider two circles $\mathscr{A}$ and $\mathscr{A}'$ centered respectively on C and C'. The center $C' \neq C$ is located on a radius $[CA]$ with A a point of $\mathscr{A}$. The two circles intersect at least in A. Let $A' \neq A$ be a point of $\mathscr{A}$. To prove the full inclusion of $\mathscr{A}'$ in $\mathscr{A}$, we have to prove that $C'A' > C'A$.

If $CC'A'$ is a triangle then $CC' + C'A' > CA'$. Since $CA' = CA = CC' + C'A = CA$, then $CC' + C'A' > CC' + C'A$. Hence, we have $C'A' > C'A$. If $CC'A'$ is not a triangle, as $A' \neq A$, $[AA']$ is a diameter of $\mathscr{A}$ implying that $C'A' = CC' + CA'$. As $CC' \neq 0$, it is trivial that $C'A' > CA'$ which conclude the proof.

Of course, if the evader does not travel at maximal speed, the new positions will be such that the new maximal Apollonius circle (taking the maximal speed into account) is also included in the initial one.

Indeed, on the fig. 1.b, if the evader would not have moved at its maximal speed, the new pursuer position would be closer to the new evader position. The pursuer would actually aims a point $\widetilde{A}$ located on the segment $[EA]$. Indeed, for two Apollonius circles $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ sharing the same reference points E and P but with two different speed ratios, respectively k and $\widetilde{k}$ such that $\widetilde{k} > k > 1$ ($\widetilde{k}$ corresponds precisely to the Apollonius circle for an evader moving slower than v_e), all the points on the Apollonius circle with the parameter $\widetilde{k}$ (the higher) are

inside the other Apollonius circle. Note first that if one point of $\widetilde{\mathscr{A}}$ is inside $\mathscr{A}$, all the points of $\widetilde{\mathscr{A}}$ are inside $\mathscr{A}$, since the two circles cannot intersect (an intersection means that a single point is at two different distance ratios from E and P, which is impossible). Second, let O and $\widetilde{O}$ be the intersection of the segment $[EP]$ with the circles $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ respectively. It is clear that $\widetilde{O}$ belongs to the segment $[EO]$ since $\|e - \widetilde{o}\| = \frac{1}{k+1}\|e - p\| < \frac{1}{k+1}\|e - p\| = \|e - o\|$. As $\widetilde{O}$, a point of $\widetilde{\mathscr{A}}$ is inside $\mathscr{A}$, $\widetilde{\mathscr{A}}$ is inside $\mathscr{A}$. Thus, if the evader does not move at its maximal speed, the pursuer aims a point $\widetilde{A}$ located on the segment $[EA]$. The new pursuer position noted $\widetilde{P}'$ thus belongs to the segment $[E'P']$.

The new maximal Apollonius circle is obviously included in the initial one. Indeed, for two Apollonius circles $\mathscr{A}'$ and $\widetilde{\mathscr{A}}'$ sharing the inner reference point E' and the same speed ratio $k > 1$, but such that the outer reference points, respectively P' and $\widetilde{P}'$ are different: $\widetilde{P}'$ belongs to $[E'P']$ (the two circles $\mathscr{A}'$ and $\widetilde{\mathscr{A}}$ correspond precisely to the maximal Apollonius circles after an infinitesimal movement of the evader respectively at maximal speed and at a slower speed), $\widetilde{\mathscr{A}}'$ is inside $\mathscr{A}'$. Note first that if one point of $\widetilde{\mathscr{A}}'$ is inside $\mathscr{A}'$, all the points of $\widetilde{\mathscr{A}}'$ are inside $\mathscr{A}'$, since the two circles can not intersect (an intersection means that a single point is a the same distance ratio from E' and two different points P' and $\widetilde{P}'$ belonging to $[E'P']$, which is impossible). Second, let O' and $\widetilde{O}'$ be the intersection of the segment $[E'P']$ with the circles $\mathscr{A}'$ and $\widetilde{\mathscr{A}}'$ respectively. It is clear that $\widetilde{O}'$ belong to the segment $[E'O']$ since $\|e' - o'\| = \frac{1}{k+1}\|e' - p'\| > \frac{1}{k+1}\|e' - \widetilde{p}'\| = \|e - \widetilde{o}'\|$. As $\widetilde{O}'$, a point of $\widetilde{\mathscr{A}}'$ is inside $\mathscr{A}'$, $\widetilde{\mathscr{A}}'$ is inside $\mathscr{A}'$. Since $\widetilde{\mathscr{A}}' \subset \mathscr{A}' \subset \mathscr{A}$, the new Apollonius circle after an infinitesimal movement in inside the initial one whatever the evader does.

Moreover, as the only intersection of the circle $\mathscr{A}$ and $\mathscr{A}'$ is precisely the point A aimed by the evader, it is obvious that as soon as the evader does not travel in straight line at its maximal speed, it will allow the pursuer to capture it closer to its initial position. Indeed, if the evader change its direction of motion at time $t > 0$ even at maximal speed, the new Apollonius circle will no longer have any contact point with the initial circle $\mathscr{A}$. Hence, for any point E_1 inside $\mathscr{A}$ reached by the evader while the pursuer has reached P_1, the greatest distance between the evader and the pursuer (the distance $\|e_1 - p_1\|$) is obtained for a straight line motion of the evader at maximal speed.

The capture occurs in finite time, since a bound to the time to capture exists. As the time to capture is linear with the traveled distance, resulting form the integration of the infinitesimal movements of the pursuer and the evader, let us first compute the movement of the evader that maximizes $\|e' - p'\| = (1 - \rho).\|e - p\|$ after an infinitesimal movement. Note that this direction minimizes ρ. We also have that $\rho = \frac{dt.v_e}{\|e-a\|}$. The point A that minimizes ρ also maximizes $\|e - a\|$. Let us express A in a different manner as before:

$$\begin{cases} x_k^a = x_k^c + R_k^c.cos(\alpha) \\ y_k^a = y_k^c + R_k^c.sin(\alpha) \end{cases} \tag{8}$$

We now look for the $\alpha^* \in [0, 2\pi[$ that maximizes the distance $\|e - a\|$:

$$\alpha^* = \underset{\alpha \in [0,2\pi[}{\arg\max} \left(e - a\right) = \underset{\alpha \in [0,2\pi[}{\arg\max} \left(e - a\right)^2$$

With a few arrangements, the problem becomes:

$$\alpha^* \quad = \quad \underset{\alpha \in [0, 2\pi[}{\arg\max} \left((x_c - x_e) cos(\alpha) + \right. \tag{9}$$

$$\left. (y_c - y_e) sin(\alpha) \right) \tag{10}$$

By studying the variation of this function with respect to α, we have that $\alpha^* = \theta_{EC}$ where θ_{EC} is the direction of the vector $\overrightarrow{EC}$. Hence, the strategy of the evader in order to maximizes the future distance to the pursuer after infinitesimal movement is simply to run away (θ_{EC} being precisely the opposite direction of the pursuer along the line of sight). In parallel, the worst evader strategy is to go toward the pursuer, since the direction of the pursuer $-\theta_{EC}$ also minimizes the future distance between the antagonists. In both case, the Π-strategy leads the pursuer to simply aim an optimal evader like in a pure pursuit strategy, known as the optimal pursuit (against any motion of the evader). The Π-strategy is time optimal for any straight line motion of the evader but also against the optimal evasion strategy (which is a straight line motion). The Π-strategy respects the equilibrium of the min-max approach.

The maximal time to capture t^* corresponds to the optimal value of a PEG involving 2 players with simple motion in free space:

$$t^* = \frac{\|\boldsymbol{p} - \boldsymbol{e}\|}{v_p - v_e}$$

The Π-strategy allows for the evader capture in finite time.

To finally prove that the capture region of the Π-strategy is the BSR of the evader, it is sufficient to notice two facts: first, there exists a strategy, the Π-strategy, that allows for the capture inside the initial Apollonius circle $\mathscr{A}$. Second, if the evader travels in straight line, any other strategy different from the Π-strategy will allow the evader to go outside $\mathscr{A}$. The Apollonius circle is the BSR of the evader.

Finally, let S_r and S_l be the points such that the lines (PS_r) and (PS_l) are the right and left tangent lines to the circle $\mathscr{A}$ starting from P. The union of the triangle $PS_r S_l$ and the circle $\mathscr{A}$ represents the pursuit region (the set of all the pursuer-evader positions during the game) related to the Π-strategy.

3.3 A sufficient condition to guaranty capture without disappearance

Our goal is to provide here a general sufficient condition to guaranty capture under visibility constraint. For convenience, we adopt the same terminology as used in Bandyopadhyay et al. (2006); Gonzalez-Banos et al. (2002); Lee et al. (2002): the set of points that are visible from the pursuer at time t defines a region called the *visibility region* The visibility region is composed by both *solid edges* and *free edges*. A solid edge represents an observed part of the physical obstacles of the environment as opposed to a free edge, which is caused by an occlusion (see fig. 1.a) and is aligned with the pursuer position. In order to hide, the evader must cross a free edge. Any point of a free edge is called an escape point. All the points belonging to the free edges are potential escape points. The disappearance corresponds to the intersection of the light of sight with an obstacle.

An obvious capture condition under visibility constraint is the following:

Condition 3.1. *If the Apollonius circle $\mathscr{A}$ does not intersect neither any free edge, nor any obstacle, then the capture is guaranteed without disappearance by adopting the Π-strategy.*
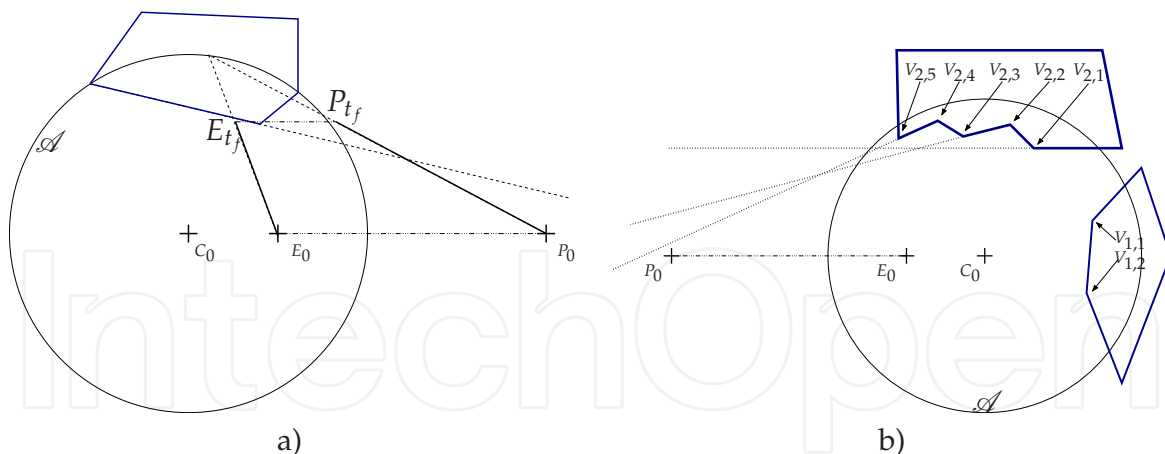
Fig. 2. a) Illustration of the creation of a free edge during the game that did not exist at the beginning of the game: at the beginning, the pursuer sees all the obstacle boundaries (blue polygon) that belong to the pursuit region. Here, $\gamma = 4$: the pursuer is twice faster than the evader. There clearly exists an evasion strategy that will break the line of sight with the obstacle at time t_f if the pursuer adopts the Π-strategy. b) Identification of the vertices that can break the line of sight if the pursuer adopts the Π-strategy. For each vertex inside the Apollonius circle, consider a point E' at a very small distance on the right of the vertex. If the evader can reach E' (i.e. the segment $[EE']$ does not intersect any obstacle edge), the Π-strategy fails. In this example, only the vertices $V_{2,3}$ and $V_{2,5}$ prevents the Π-strategy to capture the evader without disappearance. For all the other vertices, it is clear that E' is inside an obstacle, implying that the evader is not able to reach these escape points.

Indeed, the absence of free edge in the Apollonius circle implies the absence of obstacle in the part of the pursuit region which is outside the Apollonius circle. Since the Apollonius circle does not intersect any obstacle, the pursuit region of the Π-strategy is empty. Everything is thus as a PEG in free space if the pursuer adopts the Π-strategy since none of the possible segment $[E'P']$ can intersect an obstacle.

At first sight, one could think that if the initial Apollonius circle does not intersect any free edge, then capture is guaranteed. The fig. 2.a illustrates an example without any free edge intersecting the initial Apollonius circle, illustrating anyway a movement of the evader that will lead to break the line of sight if the pursuer adopts the Π-strategy. Nevertheless, such situations only happen for particular obstacle shapes that intersect the Apollonius circle. Hence, it is possible to refine the capture condition by refining which kind of obstacles is allowed in the capture region.

Our general sufficient condition to guaranty capture without disappearance is the following:

Condition 3.2. *If the Apollonius circle does not intersect any free edge and if the shape of the obstacles inside the Apollonius circle can not lead to break the line of sight if the pursuer adopts the Π-strategy, then capture is guaranteed without disappearance by adopting the Π-strategy.*

We propose here a simple method to verify if the evader is able to hide from a pursuer using the Π-strategy or not. To simplify, rotate and translate the initial coordinate system such that the new purser position is the origin of the new coordinates system, the line of sight becomes the abscise, and the abscise of E is positive (translation of a vector $-\boldsymbol{p}$ and rotation of an angle

$-\theta_{PE}$ with θ_{PE} the orientation of the vector $\overrightarrow{PE}$ in the initial coordinates system). The figure 2.b is drawn after this transformation. Then, for each vertex $V : (x_v, y_v)$ of the obstacle inside the Apollonius circle, let E' be the point at an arbitrarily small distance ϵ on the right of the vertex V: $x_{e'} = x_v + \epsilon$ and $y_{e'} = y_v$. If E' is inside the obstacle, the evader cannot use the vertex V to hide from a pursuer using the Π-strategy since it would need to cross an obstacle edge. Hence, capture without disappearance is guaranteed by adopting the Π-strategy, if for all obstacle vertices V_k inside the Apollonius circle and all the related E'_k, none of the segments $[EE'_k]$ intersects any obstacle edge. For example, in the figure 2.b, the vertices $\{V_{2,3}, V_{2,5}\}$ prevent to verify this condition, thus prevent to guaranty capture.

In the following, as soon as the condition 3.2 holds, the pursuer will adopt the Π-strategy to terminate the game.

3.4 Region of adoption of the Π-strategy

Given a convex obstacle and a position of the pursuer, let us compute the set of initial evader positions such that the Π-strategy guaranties capture, thanks to the condition thanks 3.2 (refer to fig 4.a).

First, note that, for a convex obstacle, the two contact points of the left and right tangents to the obstacle starting from the pursuer position are the only disappearance vertices. Moreover all the points between the left and right lines of disappearance are visible from P. If the evader is between the two lines of disappearance, and if its time to go to a given disappearance vertex is greater than the time for the pursuer to go to the same vertex, then the Π-strategy guaranties capture.

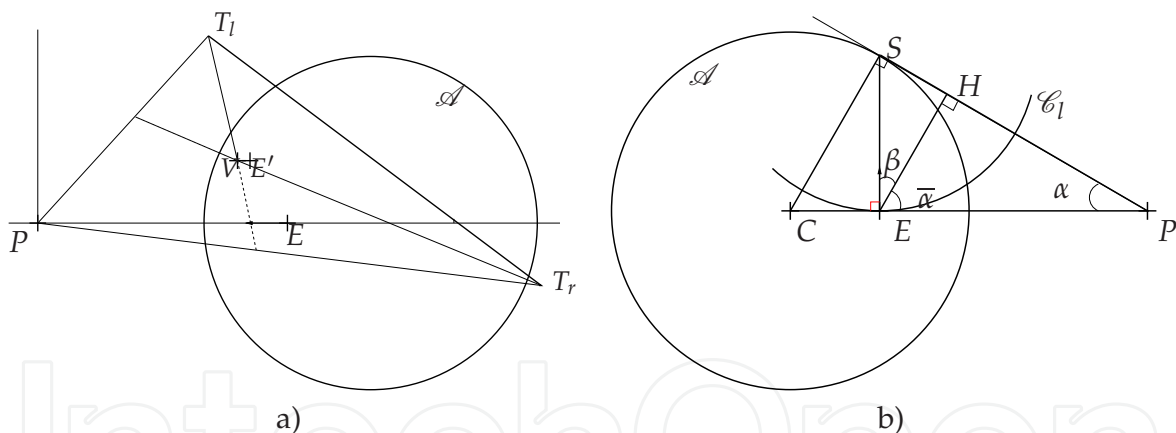a)   b)

Fig. 3. a) If the evader is between the left and right line of disappearance and if the disappearance vertices do not belongs to the Apollonius circle, then, any disappearance point E' is included in the triangle $T_l V T_r$ (V being the obstacle point creating the occlusion with the pursuer), hence is inside the obstacle, which is impossible. The Π-strategy allows for the capture. b) Computation of the minimal distance between the evader and the line of disappearance, in order to guaranty that the Apollonius circle does not contain any free edge. Here, the line (PS) is the line of disappearance and the disappearance vertex T is assumed to belong to the segment $[PS]$. We demonstrate is the text that the distance EH is linear with respect to the distance PH. The line (ES) is perpendicular to the line (EP) ($\beta = \alpha$). This information helps to determine the set of the position of the evader such that no free edge intersects the Apollonius circle for a given obstacle and a given pursuer's position, when the evader is not between the two lines of disappearance (see fig. 4.a.a).

Indeed, the first point of the condition 3.2 is verifyied because there is no free edge inside the initial Apolonius circle. Moreover, the obstacle being convex, the vertices belonging to the Apollonius circle cannot break the line of sight if the pursuer adopts the Π-strategy (second point of the condition 3.2). This can be demonstrated by noting that for any future possible position of the evader E' inside the Apollonius circle and the corresponding position P' of the pursuer, an occlusion implies the presence of a point V of the obstacle between E' and P', which is impossible due to the convexity of the obstacle. Indeed, perform first the translation of a vector $-\boldsymbol{p}$ and the rotation of an angle $-\theta_{PE}$ (see fig. 3.a) in order to simplify. E being between the two tangents (PT_l) and (PT_r), it follows that $0 \leq \theta_{PT_l} \leq \pi$ and $-\pi \leq \theta_{PT_r} \leq 0$. The obstacle being convex, all the visible points of the obstacle between the two tangents belong to the triangle PT_rT_l. Consider a possible disappearance point E' in the Apollonius circle. If E' (with a positive abscise) is not in the triangle PT_rT_l, E' is not a disappearance point since it is clear that there is not any point of the obstacle (all belonging to the triangle PT_rT_l) between E' and all the possible P' on the left of E'. Hence, E' being a disappearance point, there exists a point V of the obstacle between E' and P': $y_v = y_{e'}$ and $x_v = x_{e'} - \epsilon$ with $\epsilon > 0$. V inside the triangle PT_rT_l implies $\theta_{PT_r} \leq \theta_{PV} \leq \theta_{PT_l}$. It is obvious by construction that the point E' is inside the triangle T_lVT_r since $\theta_{PT_r} < \theta_{VT_r} < (\theta_{VE'} = 0) < \theta_{VT_l} < \theta_{PT_l}$ and E' located in the same half-plan as P (the left one) relatively to the line (T_lT_r). The obstacle being convex, the segments $[T_lV]$ and $[VT_r]$ belong to the obstacle: hence the triangle T_lVT_r to the obstacle, which is impossible since E', a point of this triangle, is, by essence of a disappearance point, outside the obstacle.

To sum-up, if the evader is between the two line of disappearance and if the two vertices of disappearance are outside the Apollonius circle, the Π-strategy guaranties capture without disappearance. In practice, the verification of the second point of the condition 3.2 requires to be checked only if the evader is not between the the two lines of disappearance.

Second, if the evader is not between the two lines of disappearance, what are the positions the set of evader positions such that the Apollonius circle is tangent to a free edge? Note first that if the evader can arrive to a disappearance vertex T before the pursuer ($k.ET < PT$), this vertex belongs to the Apollonius pursuit and the capture cannot be guaranteed[2]. Otherwise, the fig 3.b helps us to compute the minimal distance between the evader and a free edge to guaranty capture under visibility constraint by adopting the Π-strategy. In fig 3.b, (PS) is the line of disappearance, H is the projection of E on the line (PS) and EH is then the distance between the line of disappearance and the evader. The disappearance vertex T belongs to the segment $[PS]$, otherwise the circle would be tangent to the line of disappearance but not tangent with the corresponding free edge (each vertex inside the Apollonius circle should be verified to lead or not to a future line of sight occlusion). The center of the Apollonius circle is noted C and, of course, the line (CS) and (PS) are perpendicular.

We are looking for an expression of the distance EH with respect to the distance HP. The Pythagor theorem also implies that $EH^2 = ES^2 - HS^2$. As the line (EH) and (CS) are parallel and due to the Thales theorem, note that:

$$\frac{CP}{EP} = \frac{PS}{HP} = \frac{CS}{EH} = \frac{\gamma}{\gamma - 1}$$

S being on the Apollonius circle, it follows that:

$$ES^2 = \frac{PS^2}{\gamma} = \frac{\gamma}{(\gamma - 1)^2}.HP^2$$

---

[2] Until the end of the section, the distance between two points A and B will simply be noted AB.

H belonging to $[PS]$ we have:

$$HS^2 = (PS - HP)^2 = \frac{1}{(\gamma - 1)^2}.HP^2$$

Hence:

$$EH = HP.\frac{1}{\sqrt{\gamma - 1}}$$

The frontier between the evader positions such that the Apollonius circle intersects the line of disappearance or does not is a line (d_l) starting from P. The angle α between this line and the line of disappearance is constant:

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{\gamma - 1}}\right)$$

Note also that the line (ES) and (EP) are perpendicular. Indeed, in the rectangular triangle PHE, the angle $\widehat{PEH} = \overline{\alpha} = \frac{\pi}{2} - \alpha$. In the rectangular triangle SEH, the angle $\beta = \widehat{SEH}$ is such that

$$\beta = \tan^{-1}\left(\frac{SH}{EH}\right)$$

Since $SH = \frac{1}{\gamma - 1}.HP$ and $EH = \frac{1}{\sqrt{\gamma - 1}}.HP$, it follows:

$$\beta = \tan^{-1}(\frac{1}{\sqrt{\gamma - 1}}) = \alpha$$

Thus, (ES) and (EP) are perpendicular since $\widehat{PES} = \alpha + \overline{\alpha} = \frac{\pi}{2}$.

Building of the set of the evader positions such that the condition 3.2 holds is now trivial. Indeed, in the fig 3.b, assume that $S = T$ (S is the disappearance vertex T). The circle centered on $S = T$ and crossing E is noted $\mathscr{C}_l$ in fig.3.b and is such that $PS = k.ES$ (or $PT = k.ET$). As (EP) is perpendicular with (ES) (hence with (ET)), the angle α is such that the line (d_l) is tangent to the circle $\mathscr{C}_l$. The contact point between the circle $\mathscr{C}_l$ and its tangent starting from P is the starting point of the frontier between the evader positions such that the Apollonius circle is tangent to the related free edge.

The evader positions such that the condition 3.2 holds are drawn as the white region in fig 4.a. For the proposed obstacle, the Π-strategy allows for the capture in the whole region where the Apollonius circle includes a part of the obstacle but no free edges. In the following, we will focus on the strategy to adopt when the evader position does not belongs to the region where the Π-strategy guaranties capture.

4. The circular obstacle problem

In order to gain insight about what should be done if the condition 3.2 does not hold, the circular obstacle problem defined in fig. 4.b will be investigated. The solving of this game will highlight the existence of a necessary trade-off between maximizing visibility and minimizing the time to capture.

In this game, the evader moves along the boundary of a circular obstacle $\mathscr{C}_e$ (the radius of $\mathscr{C}_e$ is R_e and C is the center). The pursuer is initially located on the tangent to $\mathscr{C}_e$ crossing the evader position. The pursuer tries to capture the evader as fast as possible while maintaining its visibility, or at least it tries to delay the evader disappearance as long as possible. The evader
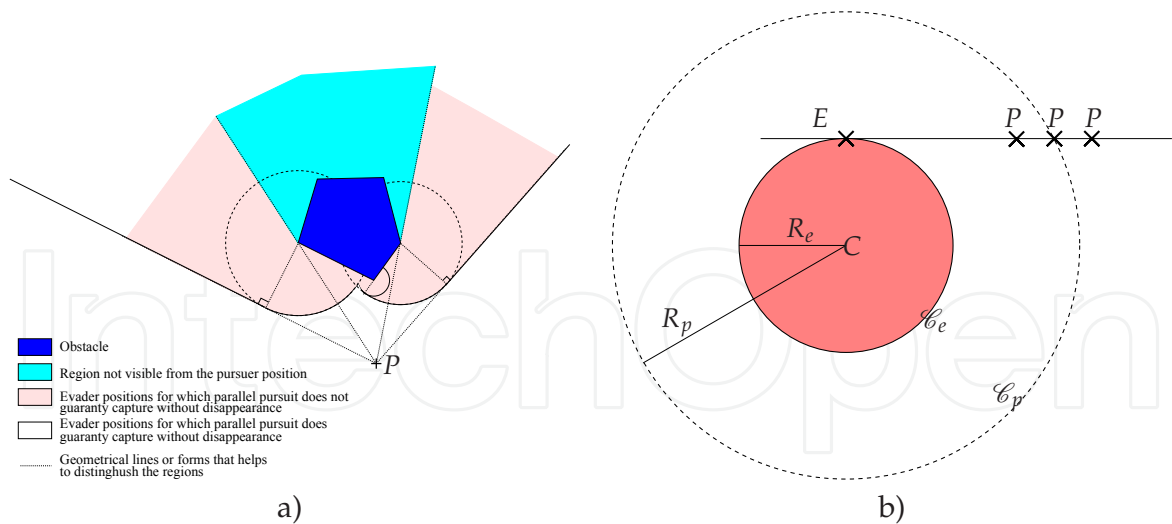
a)   b)

Fig. 4. a) For a given obstacle (convex) and a given position of the pursuer, the figure distinguishes the set of the evader positions such that the Π-strategy guaranties capture without disappearance (region not colored) from the positions for which a free edge intersect the Apollonius circle (region colored in light pink). The two dotted circles are the evader positions such that the time to go to the disappearance vertex is equal for both the antagonists. b) The circular obstacle problem: The evader is moving along the boundary of a circular obstacle $\mathscr{C}_e$, centered on O with the radius R_e. $\mathscr{C}_p$ is another circle centered on O with a radius R_p defined such that $\frac{R_p}{R_e} = \frac{v_p}{v_e} = k$ (the speeds are constant). The pursuer is initially located on the tangent of $\mathscr{C}_e$ touching the evader position. It tries to capture the evader in minimum time while maintaining its visibility (or at least it tries to delay the time to disappearance as long as possible). This corresponds to stay on the tangent, as it will be shown. What is its trajectory, if it starts at a distance $r(0)$? Or at least, what are the kinematics equations of its trajectory.

is initially on the boundary of the obstacle. Hence, moving along the boundary is obviously optimal in order to disappear since this movement maximally deviate the half-plane from which the evader is visible. Assume that $\mathscr{C}_p$ is another circle centered on C with a radius R_p defined such that $\frac{R_p}{R_e} = \frac{v_p}{v_e} = k$ (the speeds are constant, as usual). Hence, $\omega_e = \frac{v_e}{R_e} = \frac{v_p}{R_p}$ is then the angular speed of the evader. The pursuer's speed is $v_p = \omega_e.R_p$.

From the pursuer point of view, minimizing the time to capture (or maximizing the time of visibility maintenance if the capture is impossible) while maintaining the visibility is equivalent to stay on the tangent. Indeed, the fig 5.a illustrate how the pursuer can consume its velocity $v_p.dt$ depending on its distance r to the center C (to simplify the notation, the coordinate of the pursuer P in the polar coordinates system centered on C are r and θ, respectively the radius and the angle of the point P). The evader performs an infinitesimal angular movement $d\phi_e = \omega_e.dt$ between t and $t + dt$. Let $\mathscr{C}_v$ be the circle corresponding to the locus of the possible pursuer positions after an infinitesimal movement. With respect to r, the circles centered on P on the fig 5.a represent the possible positions $\mathscr{C}_v$ the pursuer can reach by consuming its velocity $v_p.dt$.

First, it is clear that the only solution for the pursuer to maintain the evader visibility is to aim a point on the circle $\mathscr{C}_v$ that will be in the half-plane from which the evader is visible at $t + dt$. Second, among all the choices, the best local choice to either capture as fast as possible or at
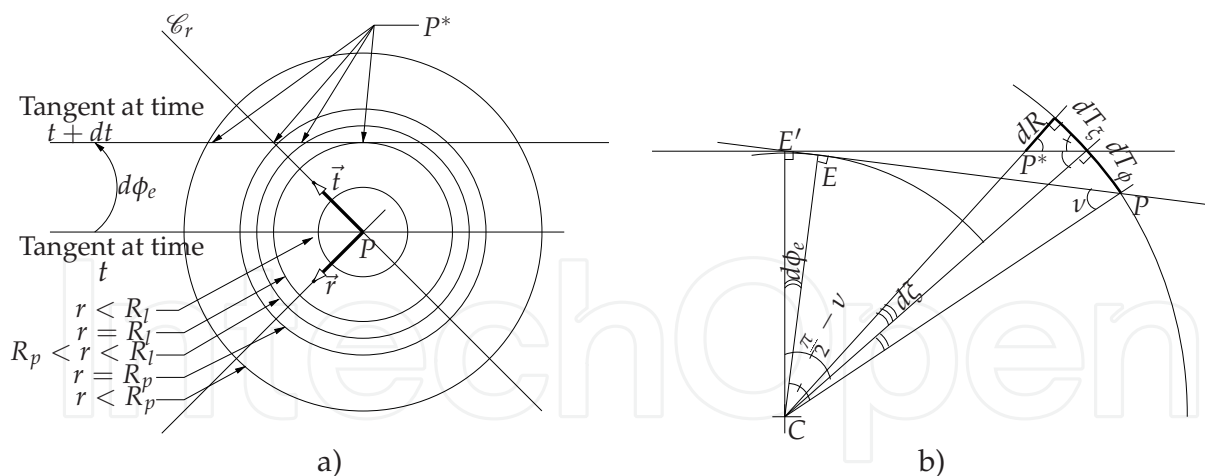
Fig. 5. a) The circular obstacle problem: For an infinitesimal rotation of the evader ($d\phi_e$), how can the pursuer consume its velocity $v_p.dt$ when located at a distance r from the rotation center C. $\vec{r}$ and $\vec{t}$ are the radial and tangential unitary vectors of the polar coordinate system centered on C. The five circles ($\mathscr{C}_v$) centered on P represents the possible positions the pursuer can reach with respect to r, by consuming its velocity $v_p.dt$. From inside to outside: $r > R_l$ implies that the circle $\mathscr{C}_v$ does not intersect the new tangent (instantaneous disappearance as soon as the evader moves); $r = R_l$ is the limit case after which disappearance is instantaneous; $R_p < r < R_l$ implies that the pursuer can maintain the visibility until $r = R_l$ (where the disappearance occurs); $r = R_p$ implies that the pursuer can infinitely maintain the visibility but can not close the distance to the evader (infinite game duration); and $r < R_p$ implies that the pursuer is able to decrease the distance to the evader while maintaining visibility (capture is guaranteed in finite time). b) Decomposition of the pursuer velocity for an infinitesimal movement of the evader.

least maintain the visibility as long as possible is to aim the point which is the closer to the future evader position: this point is actually the intersection of the circle $\mathscr{C}_v$ with the future tangent that minimizes r at time $t + dt$ (hence minimizing its distance L to the evader since $L^2 = r^2 - R_e^2$ as long as the pursuer is on the tangent line to the obstacle). Three cases are possible, with respect to r and to the number of intersections of the circle $\mathscr{C}_v$ with the future tangent:

- $r > R_l$: no intersection point exists: the disappearance is instantaneous as soon as the evader moves.

- $r = R_l$: there exists a single intersection: this corresponds to the limit case, after which the disappearance is instantaneous. Indeed, in this case, as the radius r decreases, the next situation will correspond to the first case.

- $r < R_l$: two intersection points exist: in this case, the pursuer must aim the point P^* that minimizes r at time $t + dt$ (the left one in the figure 5.a). Let us call r^* the radius of P^*. Three sub-cases are possible:

    - $r^* < r$: the pursuer close the distance to the evader ($\dot{r} < 0$) and the capture will eventually occurs in finite time.

    - $r^* = r$: the pursuer stay on the circle with the radius r ($\dot{r} = 0$), and the game duration will be infinite.

– $r^* > r$: the pursuer gets away from the evader ($\dot{r} > 0$): it can only maintain visibility by increasing the distance to the evader, which will eventually disappear.

Let us express the radial and tangential components of the pursuer speed in order to locally minimize the distance to the evader under visibility constraint. The fig 5.b illustrates the different variables to solve the problem, considering an infinitesimal angular movement $d\phi_e$ of the evader.

The pursuer movement can be decomposed into one radial and two tangential components (P^* is the aimed position at $t + dt$):

- $dT_\phi = r.d\phi_e$: the tangential component in order to maintain visibility while remaining at the same distance from the evader.

- $dT_\zeta = r.d\zeta$: the tangential component in order to reach the line (CP^*) after having performed dT_ϕ.

- dR: the radial component in order to reach the point P^* after having perform dT_ϕ and dT_ζ.

The infinitesimal velocity vector is expressed as follows:

$$\overrightarrow{v_p}.dt = -dR.\overrightarrow{r} + (dT_\phi + dT_\zeta)\overrightarrow{t} \tag{11}$$

with $\overrightarrow{r}$ and $\overrightarrow{t}$ the radial and tangential unitary vectors of the polar coordinate system centered on C.

Let us compute dT_ζ as a function of dR. The angle $\nu = (\overrightarrow{PE}, \overrightarrow{PC})$ is really helpful, since $\nu = \sin^{-1}(\frac{R_e}{r})$, and $\tan(\nu) = \frac{dT_\zeta}{dR}$ by construction (see fig 5.b). We deduce that:

$$
\begin{aligned}
dT_\zeta &= dR.\tan(\nu) \\
&= \tan(\sin^{-1}(\frac{R_e}{r})) \\
&= \frac{R_e}{\sqrt{r^2 - R_e^2}}.dR
\end{aligned}
\tag{12}
$$

The pursuer velocity can now be expressed as follows:

$$\overrightarrow{v_p} = \dot{r}.\overrightarrow{r} + (\omega_e.r - \dot{r}.\frac{R_e}{\sqrt{r^2 - R_e^2}})\overrightarrow{t} \tag{13}$$

This expression of the pursuer velocity allows to remain on the tangent (valid for P^* and also for the second intersection point of the future tangent with the circle $\mathscr{C}_v$) and is valid in any case if of course $r \leq R_l$ (remember that R_l is the limit case).

The pursuer speed being constant, we obtain the following differential equation (norm of the pursuer velocity):

$$\dot{r}^2\frac{r^2}{r^2 - R_e^2} + \dot{r}.\frac{2.r\omega_e.R_e}{\sqrt{r^2 - R_e^2}} + \omega_e^2.r^2 - v_p^2 = 0 \tag{14}$$

This equation is quadratic and admits two expressions of $\dot{r}$ (noted $\dot{r}_-$ and $\dot{r}_+$ such that $\dot{r}_- < \dot{r}_+$), corresponding to the two intersections of the future tangent with the locus $\mathscr{C}_v$ of the future pursuer position:

$$\dot{r}_- = \frac{\omega_e.\sqrt{r^2 - R_e{}^2}}{r}.(R_e - \sqrt{R_p^2 + R_e^2 - r^2})$$

$$\dot{r}_+ = \frac{\omega_e.\sqrt{r^2 - R_e{}^2}}{r}.(R_e + \sqrt{R_p^2 + R_e^2 - r^2}) \qquad (15)$$

Reaching P^* obviously corresponds to use the smallest expression $\dot{r}_-$. Moreover, the radius R_l (the limit case for which the circle $\mathscr{C}_v$ has a single intersection with the future tangent) can be computed easily since this is the one for which $\dot{r}_- = \dot{r}_+$:

$$\dot{r}_- = \dot{r}_+$$

$$\leftrightarrow \quad \sqrt{R_p^2 + R_e^2 - R_l^2} = -\sqrt{R_p^2 + R_e^2 - R_l^2}$$

$$\leftrightarrow \quad R_l = \sqrt{R_e^2 + R_p^2} \qquad (16)$$

The kinematics equation of the pursuer trajectory, consisting in locally minimizing the distance to the evader under the visibility constraint for the circular obstacle problem is:

$$\begin{cases} \dot{r} = \dfrac{\omega_e.\sqrt{r^2 - R_e{}^2}}{r}.(R_e - \sqrt{R_p^2 + R_e^2 - r^2}) \\[2mm] \dot{\theta} = \omega_e - \dfrac{\dot{r}.R_e}{r.\sqrt{r^2 - R_e^2}} \end{cases} \qquad (17)$$

Unfortunately, the pursuer trajectory cannot be expressed thanks to classical known functions. The fig. 6.a and 6.b shows the course of the game for an initial position of the pursuer close to but inside the limit circle $\mathscr{C}_p$ ($r(0) < R_p$), on the limit circle ($r(0) = R_p$), and close to but outside the limit circle ($r(0) > R_p$). The trajectories are generated with a numerical solver of differential equation (ode45) provided by Matlab®. Each time step can be seen as a new initial condition, so these trajectories contains almost all the trajectories for initial conditions such that $r(0) \leq R_l$.

The resolution of this game is interesting for at least three reasons: the first one is that most of the methods for visibility maintenance in known environment provided until now assumed a polygonal environment in order to decompose it into a finite number of sub-regions. In the case of a circular obstacle, the number of regions would be infinite and the known methods cannot be applied. The second reason is that this game clearly illustrates the trade-off between fast capture and visibility maintenance in PEGs in presence of obstacles, if the visibility maintenance is a hard constraint of the game. The last one is that this resolution gives insight on what should be done in unknown environment: it seems that doing the minimal but necessary effort to maintain visibility and consuming the spare power in reducing the distance to the future evader position is a relevant strategy, actually locally optimal. The only constraint is to estimate what would be this minimal necessary effort for visibility maintenance.
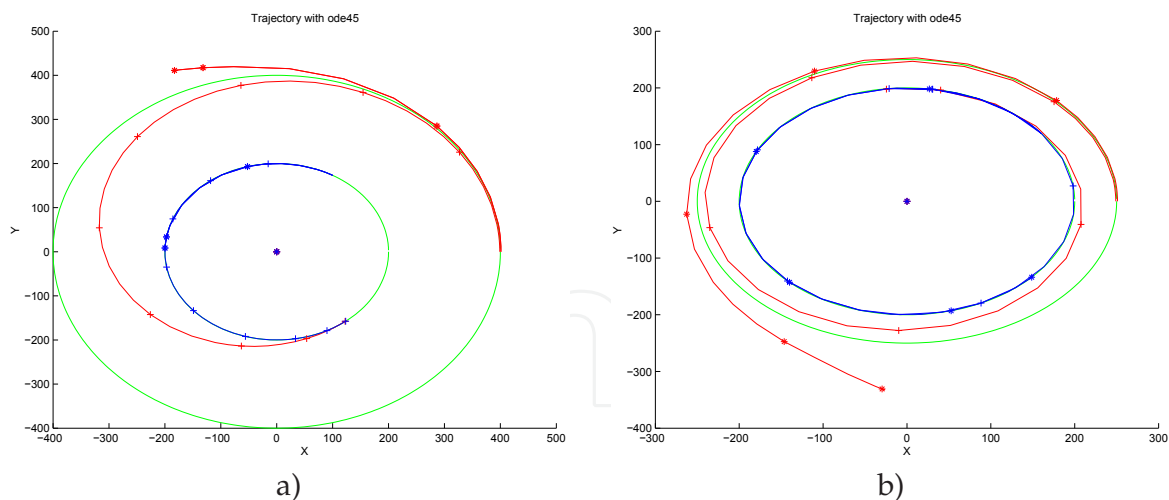
a)   b)

Fig. 6. The circular obstacle problem. a) Here, $R_p = 400$, $v_p = 4$, $R_e = 200$, $v_e = 2$. $r(0) = 399$ for the inner trajectory and $r(0) = 401$ for the outter one. The inner green circle is the obstacle ($\mathscr{C}_e$), and the outer green circle is the limit circle $\mathscr{C}_p$ (These circle correspond to the infinite trajectories of the pursuer and the evader when $r(0) = R_p$). The red and the blue trajectories are respectively the trajectory of the pursuer and the evader. The crosses and the star on the trajectories, plotted at the same time step, help to verify that the pursuer is always on the tangent to the circle, touching the evader position. b) Here, $R_p = 250$, $v_p = 2.5$, $R_e = 200$, $v_e = 2$. $r(0) = 249$ for the inner trajectory and $r(0) = 251$ for the outer one.

5. A 2-person PEG biased by a single unknown convex obstacle: Construction of an algorithm

5.1 The pole problem

Let us consider some given initial condition for the convex obstacle problem. As illustrated by the fig. 1.a from the pursuer point of view, the evader will try to hide by crossing the line of disappearance forward the segment $[PT]$ (by crossing the free edge). This line of disappearance can be seen as a stick, anchored on the fixed disappearance vertex T, and such that the pursuer controls its orientation.

As the pursuer does not know the shape of the obstacle outside its visibility region, the worst case would be an extremely sharp obstacle. Hence, a simplification of the convex obstacle problem is to consider the disappearance vertex T as a simple pole or a punctual obstacle. T is now taken as the center of a polar coordinate system as illustrated in the fig. 7. The position of the evader and the pursuer are now respectively noted (r_e, θ_e) and (r_p, θ_p).

The pole problem is defined as follows:

- The evader wins if it can change the sign of the angle α or if it can arrive to the pole before the pursuer ($r_e = 0$) where a final infinitesimal move terminate the game.

- The pursuer wins if it can arrive to the pole before the evader ($r_p = 0$ and $r_p < r_e$) while maintaining the sign of α.

Obviously, if $\frac{r_p}{k} \geq r_e$, then the evader wins whatever the pursuer does by simply going toward the pole ($\dot{r}_e = -v_e$ and $\dot{\theta}_e = 0$). On the fig. 7, for the drawn position of the pursuer and $k = 2$, the initial positions of the evader such that $\frac{r_p}{k} \geq r_e$ is the colored semi-circle (the problem is symmetrical for $\alpha < 0$).
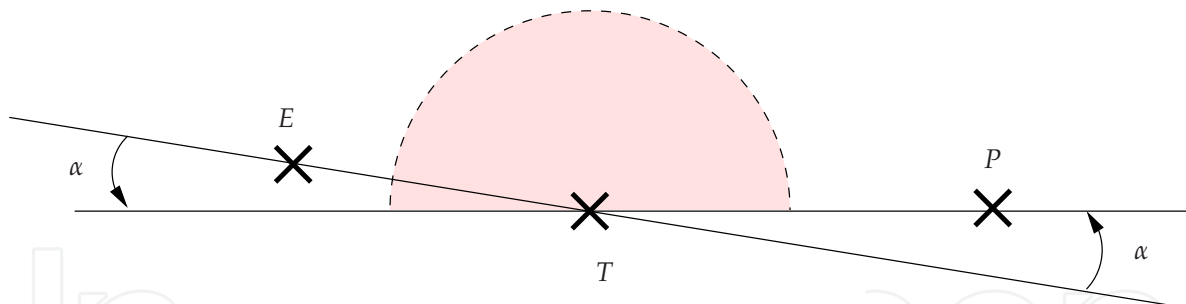
Fig. 7. The pole problem: an approximation of the convex obstacle problem. T is the disappearance vertex, which can be seen as a simple pole by the pursuer in a worst case scenario. Let α be the angle between the lines (ET) and (EP). The evader wins the game if it can change the sign of the angle α or if it can arrive to the pole before the pursuer. The pursuer wins if it can avoid the evader to win, and if it can arrive to the pole before the evader. The colored semi-circle represents the positions such that the evader can arrive to the pole before the pursuer by simply moving in straight line (the radius is $\frac{r_p}{k}$ and with $k = 2$). Inside the semi-circle, the evader wins, and otherwise the pursuer wins as shown in the text.

If $\frac{r_p}{k} < r_e$, whatever the evader does, it will be shown there exists a pursuit strategy that guaranties the pursuer victory. Suppose that initially $\alpha > 0$ as in the fig. 7. In order to avoid the evader disappearance, the pursuer must maintain $\alpha > 0$ and it is sufficient to arrive at the pole before the evader to ensure the victory. It is only sufficient because at a given moment of the game, the capture may be guaranteed by adopting the Π-strategy (condition 3.2). To preserve the sign of α, a simple strategy is to rotate at the same angular speed as the evader and to spend the spare power of the velocity vector in decreasing the distance to the pole (let us call this strategy the α-invariant strategy). The kinematics equation of the pursuer adopting the α-invariant strategy is:

$$\begin{cases} \dot{\theta}_p = \dot{\theta}_e \\ \dot{r}_p = -\sqrt{v_p{}^2 - r_p{}^2.\dot{\theta}_p{}^2} \end{cases} \tag{18}$$

Let us show that the α-invariant enables the pursuer to arrive to the pole in finite time. We have the following relations (with $\dot{r}_p < 0$, and $\dot{\theta}_e > 0$ as in the figure 7):

$$\begin{aligned} v_p^2 &= \dot{r}_p{}^2 + r_p{}^2.\dot{\theta}_e{}^2 \\ v_e^2 &= \frac{v_p^2}{k^2} = \dot{r}_e{}^2 + r_e{}^2.\dot{\theta}_e{}^2 \end{aligned}$$

Assume that $\frac{r_p}{k} < r_e$ (it is at least true for $t = 0$), we deduce that:

$$r_e{}^2 > \frac{r_p{}^2}{k^2}$$

$$r_e{}^2 \cdot \dot{\theta}_e{}^2 > \frac{r_p{}^2 \cdot \dot{\theta}_e{}^2}{k^2}$$

$$\frac{v_p{}^2}{k^2} - \dot{r}_e{}^2 > \frac{v_p{}^2}{k^2} - \frac{\dot{r}_p{}^2}{k^2}$$

$$\dot{r}_e{}^2 < \frac{\dot{r}_p{}^2}{k^2} \qquad (19)$$

Whatever the sign of $\dot{r}_e$, since $\dot{r}_p < 0$, we have the following relation: $\frac{\dot{r}_p}{k} < \dot{r}_e$.
Let us express the derivative at time t as a limit:

$$\dot{r}_e(t) = \lim_{dt \to 0} \frac{r_e(t + dt) - r_e(t)}{dt}$$

$$\dot{r}_p(t) = \lim_{dt \to 0} \frac{r_p(t + dt) - r_p(t)}{dt}$$

Assume $\frac{\dot{r}_p(t)}{k} < \dot{r}_e(t)$ and $\frac{r_p(t)}{k} < r_e(t)$ (at least true for $t = 0$), it follows that

$$\lim_{dt \to 0} \frac{r_e(t + dt) - r_e(t)}{dt} > \lim_{dt \to 0} \frac{r_p(t + dt) - r_p(t)}{k.dt}$$

$$\lim_{dt \to 0} \frac{r_e(t + dt)}{dt} - \frac{r_p(t)}{k.dt} > \lim_{dt \to 0} \frac{r_e(t + dt) - r_e(t)}{dt}$$

Hence,

$$r_e(t + dt) > \frac{r_p(t + dt)}{k}$$

We have shown that, if $r_e > k.r_p$, the α-invariant strategy implies that $\frac{\dot{r}_p}{k} < \dot{r}_e$. We have shown that if $r_e(t) > k.r_p(t)$ and $\frac{\dot{r}_p(t)}{k} < \dot{r}_e(t)$, then $r_e(t + dt) > k.r_p(t + dt)$. Hence, if the pursuer uses the α-invariant strategy and if initially $r_e(0) > k.r_p(0)$, then $r_e(t) > k.r_p(t)$ for all $t > 0$.
Let t^* be the instant such that $r_p(t^*) = 0$ (the pursuer arrives at the pole). It is clear that the evader is not yet arrived at the pole since $r_e(t^*) > k.r_p(t^*) = 0$.
It finally has to be proved that the pursuer is able to arrive at the pole in finite time. Consider the following relation:

$$r_p = m.k.r_e$$

with $m > 0$ a temporal function. If we can show that the $\dot{r}_p$ admit a negative upper bound, the pursuer is obviously able to arrive to the pole. Let us compute the upper limit of $\dot{r}_p$:

$$\max \dot{r}_p = -\sqrt{v_p{}^2 - r_p{}^2 . \max(\dot{\theta}_e)^2}$$

$$= -\sqrt{k^2.v_e{}^2 - r_p{}^2 \frac{v_e{}^2}{r_e{}^2}}$$

$$= -k.v_e.\sqrt{(1 - m^2)} \qquad (20)$$

Now, if we can prove that $\dot{m} < 0$, then we would have proved that whatever the evader does, the α-strategy enables the pursuer to decrease faster and faster the distance to the pole, guarantying to arrive to the pole in finite time.

$$\dot{m} \quad = \quad \frac{1}{k} \cdot \frac{\dot{r}_p r_e - \dot{r}_e r_p}{r_e{}^2} \tag{21}$$

The sign of $\dot{m}$ depends on $\dot{r}_p r_e - \dot{r}_e r_p$. If $\dot{r}_e > 0$, it is clear that $\dot{m} < 0$ since it is known that $\dot{r}_p < 0$. If $\dot{r}_e < 0$, it follows $\dot{r}_p < k\dot{r}_e$ and that $k.r_e - r_p > 0$, we have:

$$\begin{aligned} \dot{r}_p r_e - \dot{r}_e r_p \quad &< \quad \dot{r}_e (k.r_e - r_p) \\ &< \quad 0 \end{aligned} \tag{22}$$

In any case, $\dot{m} < 0$, which means that the pursuer is able, even in worst case, to converge faster and faster to the pole.

A last remarks is that α can be arbitrarily small without changing the solution of the problem. Hence, a faster pursuit strategy than the α-invariant strategy is to aim the pole as long as $|\alpha| > \epsilon$, with ϵ a security margin arbitrarily chosen, and to use the α-invariant strategy when $|\alpha| = \epsilon$. We call this strategy the α-minimal strategy which is defined as follow:

$$if \; |\alpha| > \epsilon \quad \begin{cases} \dot{\theta}_p = & 0 \\ \dot{r}_p = & v_p \end{cases}$$

$$\tag{23}$$

$$and \; otherwise \quad \begin{cases} \dot{\theta}_p = & \theta_e \\ \dot{r}_p = & -\sqrt{v_p{}^2 - r_p{}^2 . \dot{\theta}_e{}^2} \end{cases}$$

This strategy is locally quasi-optimal since it does the minimal necessary effort to maintain visibility and maximally reduce the distance to the victory position. Interestingly, going toward the pole is equivalent to aim a future position of an evader trying to hide by crossing the free edge. It corresponds to an adaptive proportional pursuit such that the reference point is not a future position of the evader but rather a potential escape point, being precisely on the line of disappearance. The solution of the pole problem is the following:

- If $r_p < k.r_e$ and if the pursuer adopts the α-minimal strategy (or even the α-invariant strategy), the capture without disappearance is guaranteed in finite time.

- If $r_p \geq k.r_e$ and if the evader goes directly towards the pole, disappearance is guaranteed.

5.2 From the pole problem to the convex obstacle problem

The difference between the pole problem and the convex obstacle problem is that the antagonists can not rotate indefinitely around the pole. Indeed, the point T belongs to two obstacle edges. As a consequence, on one hand the α-minimal strategy no longer guaranties capture if $r_e > \frac{r_p}{k}$, but only guaranties to see the next obstacle edge without disappearance. On the other hand $r_e \leq \frac{r_p}{k}$ does no guaranty the evader disappearance. Indeed, if $r_e \leq \frac{r_p}{k}$, the pursuer can rotate the line of disappearance by performing a tangential movement in order to hope to see

the hidden part of the obstacle before the evader disappearance. Two cases must be considered according the position of the orthogonal projection H of the evader position on the line of disappearance (PT).

If the projection H of the position of the evader on the line of disappearance is not forward $[PT]$ (see fig. 8.a), then the evader may disappear by simply reaching the point T. The best the pursuer can do is to spent the time required for the evader to go to T in maximally rotating the line of disappearance. Consider a circle centered on P, with the radius $k.r_e$ (the distance the pursuer can travel while the evader tries to reach the point T). To maximally deviate the line of disappearance, the pursuer must aim the tangent line to this circle crossing the point T, as illustrated on the fig 8.a. For a given position r_e of the evader, the maximal deviation of the line of disappearance is the angle $\delta^* = \sin^{-1}(\frac{k.r_e}{r_p})$ and the distance between the pursuer and the point T will be $r_p^+ = \sqrt{r_p{}^2 - k^2.r_e{}^2}$ at the end of the complete movement. The velocity vector of the pursuer must form an angle $\overline{\delta^*} = \cos^{-1}(\frac{k.r_e}{r_p})$ with the line (PT). Of course, as soon as a new obstacle edge becomes visible, the pursuer has to wonder about its new strategy according to what it sees. Let us call ω the angle between the current line of disappearance and the next visible obstacle edge. If $\omega > \delta^*$, the evader is able disappear whatever the pursuer does. If $\omega \leq \delta^*$, the game will continue since the pursuer will see a new obstacle edge before the evader disappearance. In the following, this strategy will be called the $MD - LoD$ strategy (standing for Maximal Deviation of the Line of Disappearance).

If H is located on the extension of the segment $[PT]$ (see fig. 8.b), the problem is more complex. Indeed, there may exist a future free edge the evader can reach during the game. These future free edges lie outside the pursuer visibility region, and form an angle δ with the line (PT). A particular future line of disappearance is noted (d_δ).

Note that the possible angle δ such that the evader can reach the line d_δ are bounded: $\delta + \alpha \leq \frac{\pi}{2}$ ($\alpha = \widehat{ETP'}$, with P' the symmetrical point of P with respect to T), because reaching a line such that $\delta + \alpha > \frac{\pi}{2}$ is equivalent to reach the point T and is also equivalent to reach the line d_δ with $\delta = \frac{pi}{2} - \alpha$.

In order to hide by crossing a line (d_δ), the best evader motion is to aim its own projection on this line. Let D_δ be this projection. Let $\overline{D_\delta}$ be the projection of P on the line (d_δ). Going straight to $\overline{D_\delta}$ would then be the best solution for the pursuer to avoid the disappearance of the evader. Hence, the evader looks for a line such that $k.\|e - d_\delta\| - \|p - \overline{d_\delta}\| < 0$ (the time for the evader to go the D_δ is smaller than the time for the pursuer to go to $\overline{D_\delta}$). The evader looks for a δ above which $k.r_e.sin(\alpha + \delta) - r_p.sin(\delta) < 0$.

First, if $k.r_e > r_p$ (region A if the fig. 8.b), such a line does not exist. Indeed:

$$k.r_e \quad > \quad r_p \tag{24}$$
$$k.r_e.sin(\delta) \quad > \quad r_p.sin(\delta) \tag{25}$$
$$k.r_e.sin(\delta + \alpha) \quad > \quad r_p.sin(\delta) \tag{26}$$
$$k.r_e.sin(\delta + \alpha) - r_p.sin(\delta) \quad > \quad 0 \tag{27}$$
$$\tag{28}$$

Hence, if the evader is in the region A, the pursuer will use the α-minimal strategy as suggested by the resolution of the pole problem in order to maintain the evader visibility until seeing a new vertex that "deals new cards".
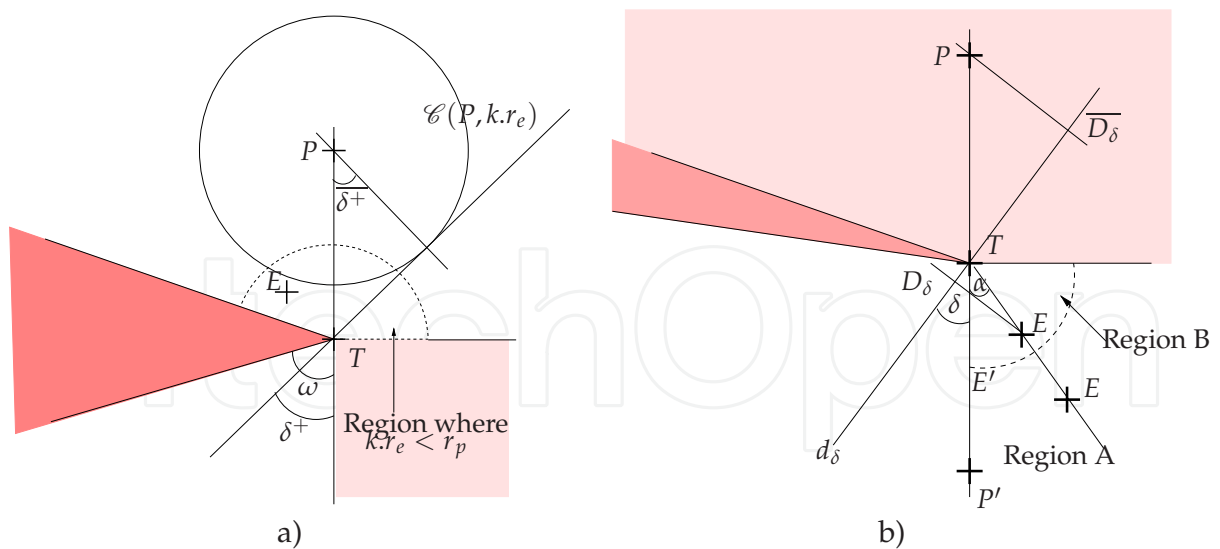
a)   b)

Fig. 8. a) How can the pursuer maximally rotate the line of disappearance if the $r_e < \frac{r_p}{k}$ and if the projection H of E on the line (PT) belongs to the segment $[PT]$ (the light pink region is not considered for the moment). The pursuer must draw the circle centered on its position with a radius $k.r_e$ (corresponding to the position it can reach in straight line while the evader is going to T) and must aim the contact point of the tangent to the circle crossing the point T (this tangent is the maximally rotated line of disappearance the pursuer can create while the evader goes to T). More simply, its velocity vector must from an angle $\overline{\delta^+} = cos^{-1}(\frac{k.r_e}{r_p})$ with the line (PT). In this example, whatever the pursuer does, the evader is able to disappear by reaching the point T because $\omega > \delta^+$. If $\omega \leq \delta^+$, the game would continue whatever the evader does. b) What happens if the projection H of E on the line (PT) is on the extension of the segment $[PT]$ (the light pink region has been treated previously). If the evader is in the region A ($k.r_e > r_p$), it cannot reach any line d_δ before the pursuer. The pursuer will adopt a α-minimal strategy until it sees a new disappearance vertex. If the evader is in the region B ($k.r_e \leq r_p$), there always exist a line d_δ the evader can reach before the pursuer, at least for $\delta = \frac{\pi}{2} - \alpha$ which corresponds for the evader to reach the point T. The pursuer can only deviate the line of disappearance to hope to see the hidden part of the obstacle before the evader reaches a line of disappearance.

Second, if $\frac{r_p}{k} > r_e$ (region B of the fig 8.b), everything becomes drastically more complex. Hence, we decided to not tackle the case in this article. In the following, the pursuer will adopt the α-minimal strategy if the evader is in the region B of the fig 8.b, but we are aware that this case should be considered very thoroughly and carefully in order to determine an efficient strategy. We are also aware that the α-minimal strategy may lead to the evader disappearance in this region.

5.3 An incrementally built pursuit algorithm for the convex obstacle problem
Incrementally, a complete pursuit strategy, which combined three strategies and determine which one is the most relevant according to the current situation, has been built:

- Π-strategy: if the condition 3.2 holds, the pursuer will adopt the Π-strategy to conclude the game. All the games the pursuer can win will finish by the adoption of the Π-strategy.

- MD-LoD strategy: if the projection of the evader position H on the line (PT) is not on the extension of the segment [PT] and if $\frac{r_p}{k} > r_e$, the pursuer uses the MD-LoD strategy to maximally deviate the line of disappearance in order to hope to get the sight of the next hidden edge of the obstacle. If the two possible vertices of disappearance (the left one and the right one) verify this condition, the pursuer should deviate the line of disappearance for which $t_p - t_e$ is the higher, t_p and t_e being the time to reach a given disappearance vertex for respectively the pursuer and the evader.

- α-minimal strategy: If the projection H on the closest disappearance line if forward the segment $[PT]$, then the α-minimal strategy will be used as suggested by the resolution of the pole problem. Yet, if $r_e < \frac{r_p}{k}$, we noticed that a better strategy, which is not under the scope of this article, is likely to exist and should be built.

The figure 9 illustrates on a given example (a given position of the pursuer and a given convex) which strategy is used according to the evader positions. In the following, the concept underlying the α-minimal strategy will be compared with other heuristics that has been proposed in the literature or that appears relevant.

6. Heuristic comparison

In this section, our pursuit algorithm, especially the interest of the α-minimal strategy, will be evaluated. A measure to compare the efficiency of different algorithms is proposed: the
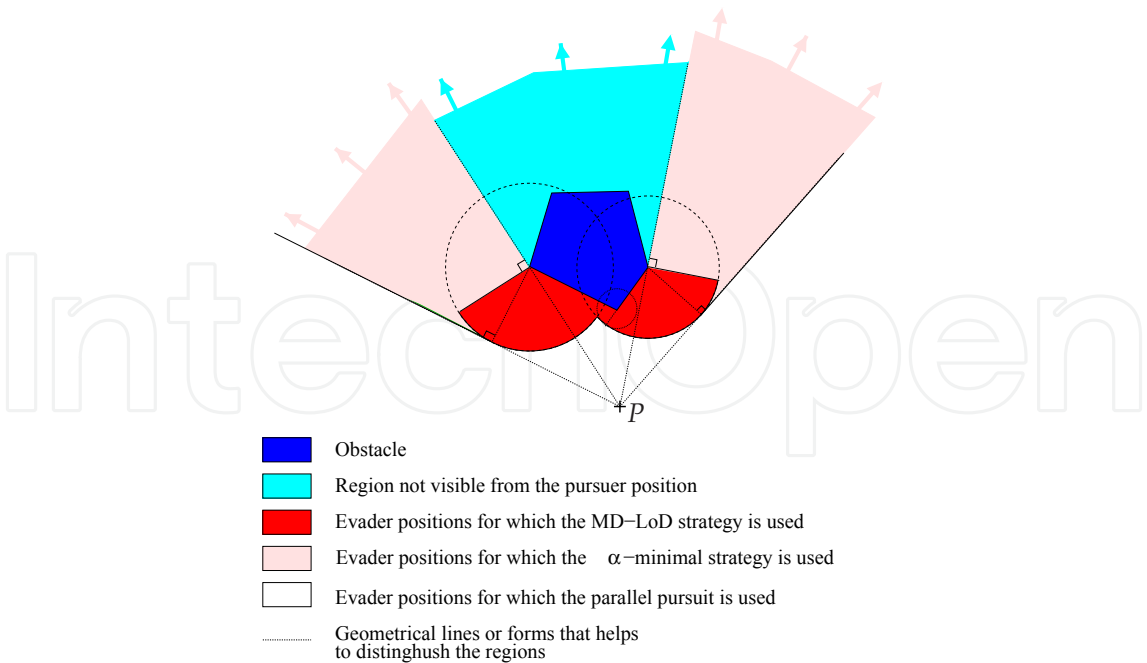

Fig. 9. Strategy used with respect to the evader position, for a fixe position of the pursuer and a given convex obstacle.

size of the capture basins. The proposed methods in the literature for the problem of visibility maintenance are based on heuristics. Our experiment will consist in building the capture basin of our pursuit algorithm if a particular heuristic is used instead of the α-minimal strategy. Heuristics inspired by Bandyopadhyay et al. (2006); Gonzalez-Banos et al. (2002), as well as other simple heuristics that appears relevant for the problem, one approximating the α-minimal strategy, will be compared. We choose to not directly implement the α-minimal strategy because it requires more information that the other heuristics (the evader angular speed related to the disappearance vertex). Anyway, it will be shown that the heuristic approximating the α-minimal strategy largely outperforms the other heuristics. A last important point concerns the evader strategy. Although this article does not deal with evasion, we need that the pursuer plays against a relatively smart evader. In this section, an evasion strategy (which we do not claim to be optimal but simply smart) will be proposed and uses against the pursuer in the experiments.

6.1 Capture and evasion basin

In order to compare pursuit heuristics, a measure is needed. The method proposed here is somehow inspired by the dynamical systems theory. Let the couple evader-pursuer be a coupled dynamical system. As each opponent state is completely determined by 3 coordinates (x, y, θ) (or 2 for simple motion without any constraints on the turning rate), the dynamical system is defined by 6 dimensions. The topology of the obstacles corresponds to a high dimensional set of parameters. An important criterion that can be taken into account to justify that a pursuit algorithm is better than another one is the volume of the capture basin (ie: the set of initial conditions such that the pursuer eventually wins the game): the wider the capture basin, the better the pursuit algorithm for this environment. An optimal algorithm should be such that all the capture basins related to other algorithms are included in the capture basin of this optimal algorithm for any convex obstacles.

As it is particularly difficult to represent such a basin for a 2-players PEG (at least 4 dimensions), and even more difficult to analyze it, the heuristic comparison will be reduced in the followings to the building of capture basins, assuming that a set of initial conditions is fixed (the initial state of the evader). Then, for a given paving of the environment, we build the capture basin of each heuristic in 2 dimensions. For example, in the circular obstacle problem, the capture basin related to the optimal pursuit-evasion strategy is the ring defined by the set of points (r, θ) such that $R_e \leq r < R_p$ and the evasion basin is obviously the set of points such that $r > R_p$.

6.2 List of the variables

Before describing the different heuristics that will be compared, let us first give the list of the variables on which they rely. Then we will define the evader strategy. The obstacle is a polygon (or at least a segment) G when it is seen by the pursuer and G^E when it is seen by the evader (see fig 10).

- T_r and T_l are the two vertices of the polygon G such that the lines (PT_r) and (PT_l) are the right and left tangents to the polygon G starting from P (the two lines of disappearance). The two free edges correspond to the extension of the segments $[PT_l]$ and $[PT_r]$.

- H_r and H_l are respectively the projections of the evader on the line of disappearance (PT_r) and (PT_l).

- r_r and r_l are the distances between the pursuer and the vertices T_r and T_l respectively: $r_r = \|\boldsymbol{p} - \boldsymbol{t_r}\|$ and $r_l = \|\boldsymbol{p} - \boldsymbol{t_l}\|$.

- r'_r and r'_r: if H_r (resp. H_l) is forward $[PT_r]$ (resp. forward $[PT_l]$), we introduce $r'_r = \|\boldsymbol{t_r} - \boldsymbol{h_r}\|$ (resp. $r'_l = \|\boldsymbol{t_l} - \boldsymbol{h_l}\|$) the distance between T_r and H_r (resp. between T_l and H_l).

- h_r and h_l are the distances the evader has to travel in order to hide by crossing (PT_r) and (PT_l) forward $[PT_r]$ and forward $[PT_l]$ respectively. If the path to hide is a broken line, the distance must be computed accordingly.

- l: distance between the line of sight and the disappearance vertex.

To summarize, with the subscript $x = \{r, l\}$ specifying that either the right or left disappearance vertex is considered, r_x is the distance between the pursuer P and the disappearance vertex T_x, h_x is the distance the evader must travel to reach the closest disappearance point on the line of disappearance (PT_x) (the point H_x if H_x is forward $[PT_x]$ or the point T_x otherwise) and r'_x is the distance between the disappearance vertex T_x and the projection H of the evader on the extension of the segment $[PT_x]$ (if the projection on the free edge does not exist because H_x is not forward $[PT_x]$, then $r'_x = 0$).
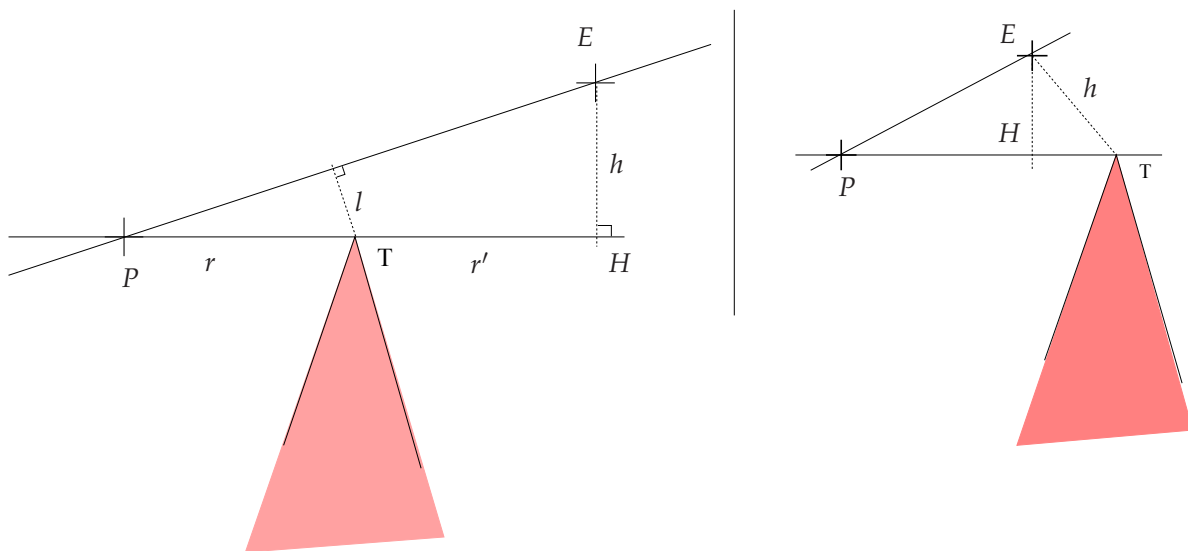

Fig. 10. Distances and points used for the computation of the heuristics by the pursuer. Here, we consider the left line of disappearance. $r = \|\boldsymbol{p} - \boldsymbol{t}\|$, wherever the evader is. As regard the evader E, if its projection H on the line (PT) is not forward the segment $[PT]$, $h = \|\boldsymbol{e} - \boldsymbol{t}\|$ and $r' = 0$. If H is forward $[PT]$, $h = \|\boldsymbol{e} - \boldsymbol{h}\|$, $r'_r = \|T - H\|$ and finally l is the distance between the line of disappearance vertex and the line of sight. These definitions hold for both the right and left line of disappearance. Hence each variable can be noted with a subscript $x = \{r, l\}$ specifying which line is considered: (r_x, r'_x, h_x).

6.3 Heuristics-based pursuit algorithm under visibility constraint

In order to equitably compare the different heuristic and to tackle the problem of a real game where the players have opposite objectives, we first need a smart evader. Evasion by hiding is not a trivial problem. An obvious local strategy for the evader is to locally aim the most secure disappearance point (H_x or T_x) or at least to run-away in order to delay an unpreventable capture. Evasion strategies not being under the scope of this article, the algorithm to choose the most secure disappearance point is not provided here.

6.3.1 Foreword

The problem of a 2-players PEG in an unknown cluttered environment has recently been tackled. The proposed solutions consist in locally minimizing either the *escape risk* Gonzalez-Banos et al. (2002); Lee et al. (2002) or the *vantage time* Bandyopadhyay et al. (2006). The sole problem of the surveillance was addressed in these works: the termination modes were either the duration of the game or on the disappearance of the evader. Interestingly, the vantage time minimization (let us call this approach VTM) seems to outperform the escape risk minimization (ERM). The authors have highlighted that the surveillance is enhanced by a good balancing between the radial movement (the movement towards the disappearance point) and the tangential movement (orthogonal to the line of disappearance). The ERM gives a too high influence to the tangential movement, and increase the latter probability for a smart evader to escape. On the contrary, the VTM give a higher influence to the radial movement. By an early decrease of the distance to the disappearance vertex, the influence of the future tangential movement is higher and allows for longer visibility maintenance. Here, we claim that the most interesting balancing between the radial and the tangential components of the velocity actually corresponds to a minimal necessary effort in visibility maintenance in order to maximally close the distance to the disappearance vertex (α-minimal strategy). As the evader must aim the disappearance line in order to disappear, closing the distance to the disappearance vertex somehow corresponds to for the pursuer to move towards a future position of an evader that would try to hide.

6.3.2 List of the heuristics

As previously said, the minimization of the different heuristics will be used instead of the α-minimal strategy in our global pursuit algorithm. One of these heuristics leads to a pursuit behavior that is very close to the α-minimal strategy.

The first heuristic H_{ER} is inspired by the escape risk function proposed in Gonzalez-Banos et al. (2002); Lee et al. (2002):

$$H'_{ER} = \max_{x \in \{r,l\}} \left(\frac{v_e . r_x}{v_p . h_x} \right)$$

Step after step, the pursuer should choose to move in order to minimize H_{ER}. An average among all the free edges could have been used instead of the max operator as in Gonzalez-Banos et al. (2002); Lee et al. (2002) but the resulting behavior would leads the pursuer to equilibrated the escape risk among all the free edge influences, whereas the max operator leads to prior focus on the riskiest free edge. A preferable method is to estimate the most critical free edge $x^* = \{r, l\}$ (it is trivial in the region where the heuristic minimization is used as illustrated by the fig. 9). Hence, the following heuristic H_{ER} is equivalent to the heuristic H'_{ER}:

$$H_{ER} = \frac{r}{h}$$

with $r = r_{x^*}$ and $h = h_{x^*}$. In the following, we will use this more simple formalism ($r' = r'_{x^*}$). The constant $\frac{v_e}{v_p}$ is removed, since it has no influence on the local minimization.

The second heuristic inspired by Bandyopadhyay et al. (2006) aims at reducing the vantage time, which corresponds to the time required to push the evader in the area such that the distance to hide is greater than the distance to avoid hiding (assuming that the current evader velocity will not change). The authors proposed an approximated computation of this time. They first estimate the vertex T_x behind which the evader tries to hide (equivalent to find x^*). Here, the most critical escape path $x^* \in \{r, l\}$ is first computed. The velocity vector

$v_r \overrightarrow{r} + v_t \overrightarrow{t}$ ($\overrightarrow{r}$ and $\overrightarrow{t}$ are the unit vectors in the tangential and the radial direction) that minimizes the vantage time, also minimizes the risk defined as:

$$H_{VT} = \frac{r - h}{v_r + v_t(r'/r) - v_e}$$

The authors deduced that the correct velocity vector is: $(v_p/\sqrt{r^2 + r'^2})(r \overrightarrow{r} + r' \overrightarrow{t})$, by differentiating H_{VT} with respect to v_t and v_r.

A third heuristic we introduce here simply compares the distance the pursuer has to travel to avoid hiding (by reaching the vertex that may break the line of sight) with the distance for the evader to reach the related free edge. $x^* \in \{r, l\}$ is first computed. This heuristic (let it be called *spatial hidability*) is the following:

$$H_{SH} = r - h$$

Note that this heuristic is simpler but very close to H_{VT}, since the vantage time estimation results from the integration of the expression $r - h$.

We also proposed a forth heuristic that compares the time needed by the pursuer to avoid hiding with the time for the evader to reach the related free edge (let us call it *temporal hidability*), knowing $x^* \in \{r, l\}$:

$$H_{TH} = r - k.h$$

Finally, we propose a last heuristic which approximates the α-minimal strategy. $x^* \in \{r, l\}$ is first computed. As long as the distance l between the line of sight and the disappearance vertex T is greater than a given security distance l_0, the heuristic minimization should lead the pursuer to aim the disappearance vertex. When the distance l become smaller than l_0, the heuristic minimization should lead the pursuer to use part of its velocity to perform a tangential movement. The following heuristic called H_{AMA} (standing for Alpha-Minimal Approximation) provides such a behavior:

$$H_{AMA} = r - \left(\frac{l_0}{l}\right)^n.h$$

with $n > 1$ that can be adapted ($n = 2$ in the following to delay the beginning of the tangential movement). If l is greater than l_0, $\left(\frac{l_0}{l}\right)^n.h$ is negligible as compared with r and the pursuer will aim the disappearance vertex. If l becomes smaller than l_0, r becomes negligible as compared with $\left(\frac{l_0}{l}\right)^n.h$ leading the pursuer to perform a tangential movement.

For comparison, the direction of movement for each heuristic, computed by differentiating the heuristic, is given in the table 1. We note that the heuristics H_{VT} and H_{SH} have the same gradient direction.

In the following, the capture basin of our pursuit algorithm, embedding each one of the proposed heuristic instead of the α-minimal strategy will be compared for different obstacle shapes against the smart evasion strategy introduced previously.

6.4 Results in a virtual environment

In fig 11.a, 11.b, and 11.c, the capture basins of our pursuit algorithm, using the minimization each of the heuristics H_{ER}, H_{TM}, H_{SH}, H_{VT} and H_{AMA}, are displayed for different obstacles. The capture basin of the pure pursuit is also displayed. The 2 dimensions of the capture basin correspond to the initial positions (x, y) of the pursuer from which it achieves the capture of the evader always starting at the same position $(60, 75)$ (the length unit is the meter). The

Algo.	Expression	radial	component	tangential	component
ERM	$\frac{r}{h}$	$-r$	$\overrightarrow{r'}$	$\frac{r^2}{h}.r'$	$\overrightarrow{t}$
THM	$r - k.h$	$-r$	$\overrightarrow{r'}$	$k.r'$	$\overrightarrow{t}$
VTM	$\frac{r-h}{v_r+v_t(r'/r)-v_e}$	$-r$	$\overrightarrow{r'}$	r'	$\overrightarrow{t}$
SHM	$k - h$	$-r$	$\overrightarrow{r'}$	r'	$\overrightarrow{t}$
$AHAM_{n=2}$	$r - \left(\frac{l_0}{l}\right)^2.h$	$-r$	$\overrightarrow{r'}$	$\left(\frac{l_0}{l}\right)^2.r'$	$\overrightarrow{t}$

Table 1. Direction of the gradient of the proposed heuristics 11.

speed are $v_e = 2$ and $v_p = 4 \; m.s^{-1}$. As foreseen, the best algorithm is undoubtedly the AHA-minimization. The related capture basin includes almost all the other capture basins. Inspired from the solving of the pole problem, this strategy leads the pursuer to aim the disappearance vertex as fast as possible while minimally counter-balancing the movements of the evader when the line of sight and the line of disappearance are very close: the direction of the movement is such that the pursuer does not change the orientation of the disappearance line excepted in order to compensate the evader tangential move when the disappearance is imminent. The fast reaching of the disappearance vertex allows for easier visibility maintenance because the required leverage to compensate the evader tangential movement will be minimal when the disappearance becomes imminent.

Moreover, by aiming the disappearance vertex, the pursuer performs an adaptive proportional navigation since it aims a future position of the evader (obviously, the evader aims a point on the disappearance line). The resulting behavior is between a pure pursuit and the Π-strategy: the pursuer moves along the shortest path to the potential points of capture.

The fig 11.c provides an example of a game in our 2D virtual environment: The evader uses the strategy described previously and the pursuer uses our pursuit algorithm with the AMA heuristic minimization (the one approximating the α-minimal strategy). In this situation, only the AMA minimization allows for the capture.

7. Conclusion

In this article, the problem of pursuit under visibility constraint in an unknown cluttered environment has been tackled. First, a sufficient condition of capture in the presence of unknown obstacles has been established. The Π-strategy consisting in an optimal parallel pursuit guaranties the capture without disappearance if the capture region (the Apollonius circle) does not contains any free edge and if the obstacles included in the capture region can not break the future line of sight. We then wonder what should be done in other situations.

We first solved the circular obstacle problem, a particular game in which the evader moves along the boundary of a circular obstacle and the pursuer is initially located on tangent line to the obstacle touching the evader position. The resolution highlighted that, under visibility constraint, the pursuit algorithm that locally optimizes the time to capture leads in parallel to perform the minimal necessary effort in maintaining the visibility. The pole problem has then been investigated. This game is an approximation of an extremely sharp obstacle vertex. The solution showed that the pursuer wins if it can arrive to the pole before the evader, by simply compensating the rotation of the line of sight with a rotation of the line of disappearance. Otherwise, the evader wins by simply reaching the pole. This has led us to propose a pursuit strategy called the α-minimal strategy consisting in moving towards the disappearance vertex
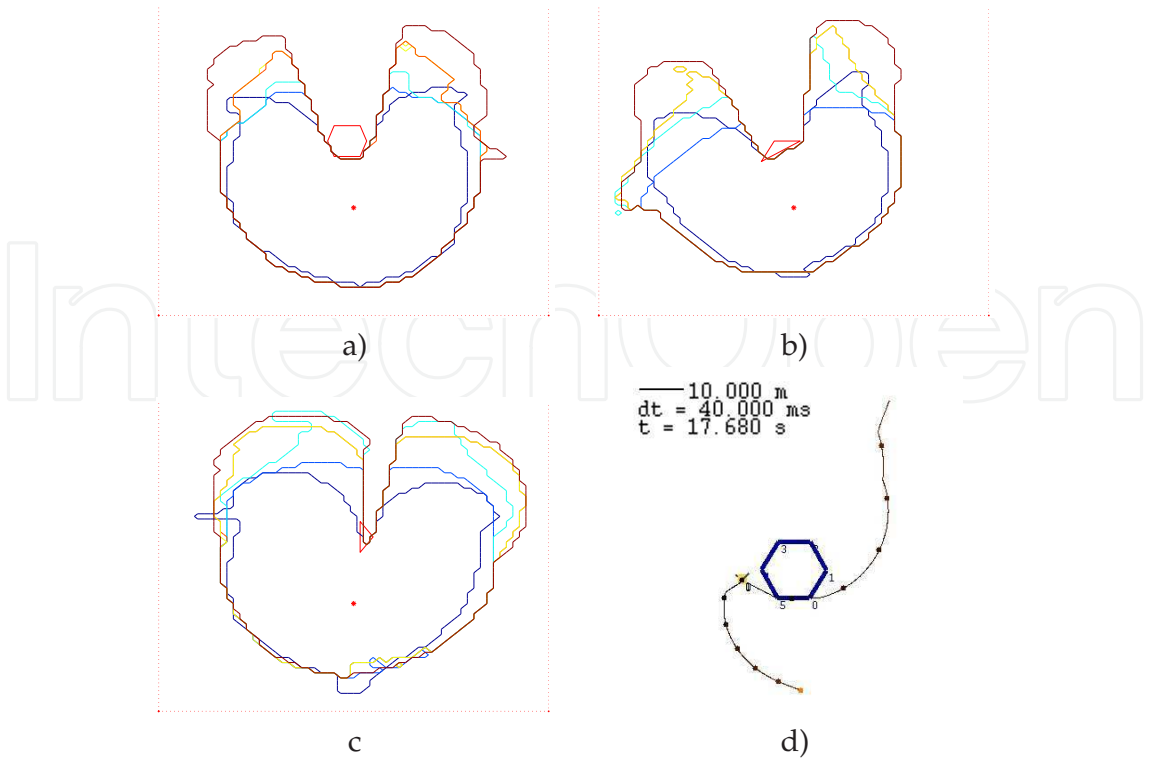
Fig. 11. a,b,c) Capture basin of each algorithm: The position of the evader is fixed (the red crosses). The red polygon is the obstacle. The dark blue basin correspond to a simple pure pursuit, the light blue one to the ERM, the cyan one to the THM, the orange and yellow one (almost superimposed) to the SHM and the VTM respectively, and the maroon one to the AMAM. Expected for particular cases due to the non-optimal behavior of the evader, the capture basin of the strategy approximating the α-minimal strategy includes all the other capture basins. d) Illustration of the AMA heuristic minimization that approximates the α-minimal strategy. The pursuer aims the disappearance vertex as long as possible and begins to perform a tangential movement when disappearance becomes imminent. The other heuristic do not allow for the evader visibility maintenance.

as fast as possible while preventing the imminent evader disappearance by a minimal compensation of the line of sight rotation. In the general case of a convex obstacle, this strategy guaranties the pursuer to see the next obstacle vertex without evader disappearance in the region where it wins the pole problem. In the region where the evader wins the pole problem, the generalization to the case of a convex obstacle is harder. If the projection of the evader on the line of disappearance is not on the related free edge, we established a pursuit behavior that aims at maximally rotating the line of disappearance before the evader disappearance, in order to hope to see the hidden part of the obstacle. This strategy called MD-LoD, standing for maximal Deviation of the Line of Disappearance, allows extending the capture basin in particular situations. If the projection of the evader position belongs to a free edge, the analysis becomes much more difficult and was not under the scope of this article.

Incrementally, a pursuit algorithm has been built. It combines the Π-strategy if it guaranties capture without disappearance, the $MD - LoD$ strategy when the evader is able to arrive to the disappearance vertex before the pursuer (and if it projection of the line of sight does not

belongs to a free edge) and the α-minimal strategy otherwise. Finally, we compared the capture basins of our pursuit algorithm modified such that the minimization of a given heuristic is used instead of the α-minimal strategy. Two of these heuristics were inspired by previous heuristics found of the literature (escape risk Gonzalez-Banos et al. (2002); Lee et al. (2002) and vantage time Bandyopadhyay et al. (2006)), two of them appeared relevant to the problem (spatial and temporal hidability) and the last one was built to approximate the α-minimal strategy. As foreseen, the strategy consisting in closing the distance to the disappearance vertex as fast as possible and doing the minimal necessary effort to maintain visibility extends the capture basin.

All along the article, even though the building of an evasion strategy was not addressed, the evader has always been considered as intelligent. For the simulation, we propose a geometrical method to locally aim the most secure instantaneous escape point or to run away if disappearance is impossible. In particular situation, it is clear that a better evasion strategy exists as highlighted by the pole problem.

In future work, it will be important to provide more global evasion strategies in order to evaluate how far the one we proposed is from an optimal and to imagine the possible evolutions of our algorithm. The concepts underlying the building of our pursuit algorithm, especially the α-minimal strategy and the sufficient capture condition we established based on the properties of the Π-strategy, should be also considered to tackle the problem of several unknown non-convex obstacles. Based on the insight provided by this study, it is also possible to investigate new pursuit concepts involving several pursuers in presence of multiple obstacles, not necessarily nookless.

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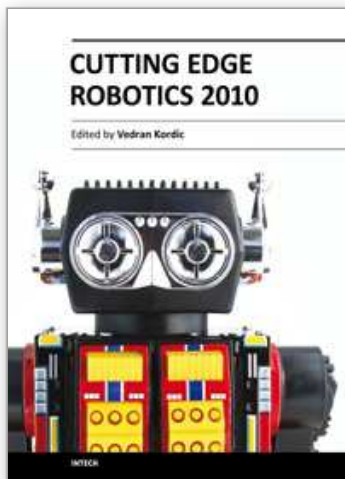
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Edited by Vedran Kordic

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