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Robust Model Predictive Control Algorithms for Nonlinear Systems: an Input-to-State Stability Approach

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This paper presents and compares two robust *MPC* controllers for constrained nonlinear systems based on the minimization of a nominal performance index. Under suitable modifications of the constraints of the Finite Horizon Optimization Control Problems (*FHOCP*), the derived controllers ensure that the closed loop system is Input-to-State Stable (*ISS*) with a robust invariant region, with relation to additive uncertainty/disturbance. Assuming smoothness of the model function and of the ingredients of the *FHOCP*, the effect of each admissible disturbance in the predictions is considered and taken into account by the inclusion in the problem formulation of tighter state and terminal constraints. A simulation example shows the potentiality of both the algorithms and highlights their complementary aspects.

Keywords: Robust *MPC*, Input to State Stability, Constraints, Robust design.

1. Introduction

Model predictive control (*MPC*) is an optimal control technique which deals with constraints on the states and the inputs. This strategy is based on the solution of a finite horizon optimization problem (*FHOCP*), which can be posed as a mathematical programming problem. The control law is obtained by means of the receding horizon strategy that requires the solution of the optimization problem at each sample time Camacho & Bordons (2004); Magni et al. (2009); Rawlings & Mayne (2009).

It is well known that considering a terminal cost and a terminal constraint in the optimization problem, the *MPC* stabilizes asymptotically a constrained system in absence of disturbances or uncertainties. If there exist uncertainties in the process model, then the stabilizing properties may be lost Magni & Scattolini (2007); Mayne et al. (2000) and these must be taken into account in the controller design. Recent results have revealed that nominal *MPC* may have

zero robustness, i.e. stability or feasibility may be lost if there exist model mismatches Grimm et al. (2004). Therefore it is quite important to analyze when this situation occurs and to find design procedures to guarantee certain degree of robustness. In Limon et al. (2002b); Scokaert et al. (1997) it has been proved that under some regularity condition on the optimal cost, the *MPC* is able to stabilize the uncertain system; however, this regularity condition may be not ensured due to constraints, for instance.

The synthesis of *NMPC* algorithms with robustness properties for uncertain systems has been developed by minimizing a nominal performance index while imposing the fulfillment of constraints for each admissible disturbance, see e.g. Limon et al. (2002a) or by solving a min-max optimization problem, see e.g. Chen et al. (1997); Fontes & Magni (2003); Magni et al. (2003); Magni, Nijmeijer & van der Schaft (2001); Magni & Scattolini (2005). The first solution calls for the inclusion in the problem formulation of tighter state, control and terminal constraints. The main advantage is that the on-line computational burden is substantially equal to the computational burden of the nominal *NMPC*. In fact, nominal prediction based robust predictive controllers can be thought as a nominal MPC designed in such a way that a certain degree of robustness is achieved. The main limitation is that it can lead to very conservative solutions. With a significant increase of the computational burden, less conservative results can be achieved by solving a min-max optimization problem.

Input-to-State Stability (*ISS*) is one of the most important tools to study the dependence of state trajectories of nonlinear continuous and discrete time systems on the magnitude of inputs, which can represent control variables or disturbances. The concept of *ISS* was first introduced in Sontag (1989) and then further exploited by many authors in view of its equivalent characterization in terms of robust stability, dissipativity and input-output stability, see e.g. Jiang & Wang (2001), Huang et al. (2005), Angeli et al. (2000), Jiang et al. (1994), Nešić & Laila (2002). Now, several variants of *ISS* equivalent to the original one have been developed and applied in different contexts (see e.g. Sontag & Wang (1996), Gao & Lin (2000), Sontag & Wang (1995), Huang et al. (2005)). The *ISS* property has been recently introduced also in the study of nonlinear perturbed discrete-time systems controlled with Model Predictive Control (*MPC*), see e.g. Limon et al. (2009), Raimondo et al. (2009), Limon et al. (2002a), Magni & Scattolini (2007), Limon et al. (2006), Franco et al. (2008), Magni et al. (2006). In fact, the development of *MPC* synthesis methods with enhanced robustness characteristics is motivated by the widespread success of MPC and by the availability of many MPC algorithms for nonlinear systems guaranteeing stability in nominal conditions and under state and control constraints.

In this paper two algorithms based on the solution of a minimization problem with respect to a nominal performance index are proposed. The first one, following the algorithm presented in Limon et al. (2002a), proves that if the terminal cost is a Lyapunov function which ensures a nominal convergence rate (and hence some degree of robustness), then the derived nominal *MPC* is an Input-to-State stabilizing controller. The size of allowable disturbances depends on the one step decreasing rate of the terminal cost.

The second algorithm, first proposed in a preliminary version in Raimondo & Magni (2006), shares with de Oliveira Kothare & Morari (2000) the idea to update the state of the nominal system with the value of the real one only each M step to check the terminal constraint. The use of a prediction horizon larger than a time varying control horizon is aimed to provide more robust results by means of considering the decreasing rate in a number of steps.

Both controllers are based on the Lipschitz continuity of the prediction model and of some of the ingredients of the *MPC* functional such as stage cost function and the terminal cost

function. Under the same assumptions they ensure that the closed loop system is Input-to-State-Stable (*ISS*) with relation to the additive uncertainty.

A simulation example shows the potentiality of both the algorithms and highlights their complementary aspects.

The paper is organized as follows: first some notations and definitions are presented. In Section 3 the problem is stated. In Section 4 the Regional Input-to-State Stability is introduced. In Section 5 the proposed *MPC* controllers are presented. In Section 6 the benefits of the proposed controllers are illustrated with several examples. Section 7 contains the conclusions. All the proofs are gathered in an Appendix in order to improve the readability.

2. Notations and basic definitions

Let $\mathcal{R}$, $\mathcal{R}_{\geq 0}$, $\mathcal{Z}$ and $\mathcal{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer numbers, respectively. For a given $M \in \mathcal{Z}_{\geq 0}$, the following set is defined $\mathcal{T}_M \triangleq \{kM, k \in \mathcal{Z}_{\geq 0}\}$. Euclidean norm is denoted as $|\cdot|$. Given a signal w, the signal's sequence is denoted by $\mathbf{w} \triangleq \{w(0), w(1), \cdots\}$ where the cardinality of the sequence is inferred from the context. The set of sequences of w, whose values belong to a compact set $\mathcal{W} \subseteq \mathcal{R}^m$ is denoted by $\mathcal{M}_{\mathcal{W}}$, while $\mathcal{W}^{sup} \triangleq \sup_{w \in \mathcal{W}}\{|w|\}$, $\mathcal{W}^{inf} \triangleq \inf_{w \in \mathcal{W}}\{|w|\}$. Moreover $\|\mathbf{w}\| \triangleq \sup_{k \geq 0}\{|w(k)|\}$ and $\|\mathbf{w}_{[\tau]}\| \triangleq \sup_{0 \leq k \leq \tau}\{|w(k)|\}$. The symbol *id* represents the identity function from $\mathcal{R}$ to $\mathcal{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions γ_1 and γ_2 from $\mathcal{R}$ to $\mathcal{R}$. Given a set $A \subseteq \mathcal{R}^n$, $|\zeta|_{\mathcal{A}} \triangleq \inf\{|\eta - \zeta|, \eta \in A\}$ is the point-to-set distance from $\zeta \in \mathcal{R}^n$ to A. The difference between two given sets $A \subseteq \mathcal{R}^n$ and $B \subseteq \mathcal{R}^n$ with $B \subseteq A$, is denoted by $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathcal{R}^n$ and $B \subseteq \mathcal{R}^n$, then the Pontryagin difference set C is defined as $C = A \sim B \triangleq \{x \in \mathcal{R}^n : x + \xi \in A, \forall \xi \in B\}$. Given a closed set $A \subseteq \mathcal{R}^n$, ∂A denotes the border of A. A function $\gamma : \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is of class $\mathcal{K}$ (or a "$\mathcal{K}$-function") if it is continuous, positive definite and strictly increasing. A function $\gamma : \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathcal{R}_{\geq 0} \times \mathcal{Z}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

3. Problem statement

In this paper it is assumed that the plant to be controlled is described by discrete-time nonlinear model:

$$x(k+1) = f(x(k), u(k)) + w(k), \ k \geq t, \ x(t) = \bar{x} \tag{1}$$

where $x(k) \in \mathcal{R}^n$ is the state of the system, $u(k) \in \mathcal{R}^m$ is the control variable, and $w(k) \in \mathcal{R}^n$ is the additive uncertainty. Notice that the additive uncertainty can model perturbed systems and a wide class of model mismatches. Take into account that these ones might depend on the state and on the input of the system, consider a real plant $x_{k+1} = \tilde{f}(x(k), u(k))$. Then the additive uncertainty can be taken as $w(k) = [\tilde{f}(x(k), u(k)) - f(x(k), u(k))]$. Note that if, as it will be assumed, x and u are bounded and f is Lipschitz, then w can be modeled as a bounded uncertainty. This kind of model uncertainty has been used in previous papers about robustness in *MPC*, as in Michalska & Mayne (1993) and Mayne (2000).

In the following assumption, the considered structure of such a model is formally presented.

Assumption 1.

1. *The uncertainty belongs to a compact set $\mathcal{W} \subset \mathcal{R}^n$ containing the origin, defined as*

$$\mathcal{W} \triangleq \{w \in \mathcal{R}^n : |w| \leq \gamma\} \tag{2}$$

   where $\gamma \in \mathcal{R}_{\geq 0}$.

2. *The system has an equilibrium point at the origin, that is $f(0,0) = 0$.*

3. *The control and state of the plant must fulfill the following constraints on the state and the input:*

$$x(k) \in X \tag{3}$$

$$u(k) \in U \tag{4}$$

   where X is and U are compact sets, both of them containing the origin.

4. *The state of the plant $x(k)$ can be measured at each sample time.*   □

The control objective consists in designing a control law $u = \kappa(x)$ such that it steers the system to (a neighborhood of) the origin fulfilling the constraints on the input and the state along the system evolution for any possible uncertainty and yielding an optimal closed performance according to certain performance index.

4. Regional Input-to-State Stability

In this section the *ISS* framework for discrete-time autonomous nonlinear systems is presented and Lyapunov-like sufficient conditions are provided. This will be employed in the paper to study the behavior of perturbed nonlinear systems in closed-loop with *MPC* controllers. Consider a nonlinear discrete-time system described by

$$x(k+1) = F(k, x(k), w(k)), \ k \geq t, \ x(t) = \bar{x} \tag{5}$$

where $F : \mathcal{Z}_{\geq 0} \times \mathcal{R}^n \times \mathcal{R}^r \to \mathcal{R}^n$ is locally Lipschitz continuous, $F(k, 0, 0) = 0$, $x(k) \in \mathcal{R}^n$ is the state, $w(k) \in \mathcal{R}^p$ is the input (disturbance), limited in a compact set $\mathcal{W}$ containing the origin $w(k) \in \mathcal{W}$. The solution to the difference equation (5) at time k, starting from state $x(0) = \bar{x}$ and for inputs $\mathbf{w}$ is denoted by $x(k, \bar{x}, \mathbf{w})$. Consider the following definitions.

Definition 1 (Robust positively invariant set). *A set $\Xi(k) \subseteq \mathcal{R}^n$ is a robust positively invariant set for the system (5), if $x(k, \bar{x}, \mathbf{w}) \in \Xi(k)$, $\forall k \geq t$, $\forall \bar{x} \in \Xi(t)$ and $\forall \mathbf{w} \in \mathcal{M}_{\mathcal{W}}$.*   □

Definition 2 (Magni et al. (2006) Regional *ISS* in $\Xi(k)$). *Given a compact set $\Xi(k) \subset \mathcal{R}^n$ containing the origin as an interior point, the system (5) with $\mathbf{w} \in \mathcal{M}_{\mathcal{W}}$, is said to be ISS (Input-to-State Stable) in $\Xi(k)$, if $\Xi(k)$ is robust positively invariant for (5) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that*

$$|x(k, \bar{x}, \mathbf{w})| \leq \beta(|\bar{x}|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|), \ \forall k \geq t, \ \forall \bar{x} \in \Xi(t). \tag{6}$$

                                                                                                              □

Definition 3 (Magni et al. (2006) ISS-Lyapunov function in Ξ). *A function $V: \mathcal{R}^n \to \mathcal{R}_{\geq}0$ is called an ISS-Lyapunov function in $\Xi(k) \subset \mathcal{R}^n$ for system (5) with respect to w, if:*

   1) *$\Xi(k)$ is a closed robust positively invariant set containing the origin as an interior point.*

2) *there exist a compact set $\Omega \subseteq \Xi(k)$, $\forall k \geq t$ (containing the origin as an interior point), a pair of suitable $\mathcal{K}_\infty$-functions α_1, α_2 such that:*

$$V(x) \geq \alpha_1(|x|), \forall x \in \Xi(k), \forall k \geq t \tag{7}$$

$$V(x) \leq \alpha_2(|x|), \forall x \in \Omega \tag{8}$$

3) *there exist a suitable $\mathcal{K}_\infty$-function α_3, a $\mathcal{K}$-function σ such that:*

$$\begin{aligned} \Delta V(x) &\triangleq & V(F(k,x,w)) - V(x) \\ &\leq & -\alpha_3(|x|) + \sigma(|w|), \forall x \in \Xi(k), \forall k \geq t, \ \forall w \in \mathcal{W} \end{aligned} \tag{9}$$

4) *there exist a suitable $\mathcal{K}_\infty$-functions ρ (with ρ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function) and a suitable constant $c_\theta > 0$, such that there exists a nonempty compact set $\Theta \subset \{x : x \in \Omega, d(x, \delta\Omega) > c_\theta\}$ (containing the origin as an interior point) defined as follows:*

$$\Theta \quad \triangleq \quad \{x : V(x) \leq b(\mathcal{W}^{sup})\} \tag{10}$$

where $b \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma$, with $\alpha_4 \triangleq \alpha_3 \circ \alpha_2^{-1}$.

□

The following sufficient condition for regional *ISS* of system (5) can be stated.

Theorem 1. *If system (5) admits an ISS-Lyapunov function in $\Xi(k)$ with respect to w, then it is ISS in $\Xi(k)$ with respect to w and $\lim_{k\to\infty} |x(k, \bar{x}, \mathbf{w})|_\Theta = 0$.*

Remark 1. *In order to analyse the control algorithm reported in Section 5.2, a time-varying system has been considered. However, because all the bounds introduced in the ISS Lyapunov function are time-invariant, Theorem 1 can be easily derived by the theorem reported in Magni et al. (2006) for time-invariant systems.* □

5. Nonlinear Model Predictive Control

In this section, the results derived in Theorem 1, are used to analyze the *ISS* property of two open-loop formulations of stabilizing *MPC* algorithms for nonlinear systems. The idea on the base of the two algorithms is the same one. However, there are important differences that, based on the dynamic system under consideration, give advantages to an algorithm rather than to the other in terms of domain of attraction and robustness. Notably, in the following it is not necessary to assume the regularity of the value function and of the resulting control law.

5.1 *MPC* **with constant optimization horizon**
The system (1) with $w(k) = 0$, $k \geq t$, is called *nominal model*. Let denote $u_{t_1,t_2} \triangleq \{u(t_1), u(t_1 + 1), \ldots, u(t_2)\}$, $t_2 \geq t_1$, a sequence of vectors and $u_{t_1,t_2}(t_3)$ the vector u_{t_1,t_2} at time t_3. If it is clear on the context the subscript will be omitted. The vector $\hat{x}(k|t)$ is the predicted state of the system at time k ($k \geq t$) obtained applying the sequence of inputs $u_{t,k-1}$ to the nominal model, starting from the real state $x(t)$ at time t, i.e. $\hat{x}(k|t) = f(\hat{x}(k-1|t), u(k-1))$, $k > t$, $\hat{x}(t|t) = x(t)$.

Assumption 2. *The function $f(\cdot, \cdot)$ is Lispchitz with respect to x and u in $X \times U$, with Lipschitz constants L_f and L_{fu} respectively.*

Remark 2. *Note that the following results could be easily extended to the more general case of $f(\cdot, \cdot)$ uniformly continuous with respect to x and u in $X \times U$. Moreover, note that in virtue of the Heine-Cantor, if X and U are compact, as assumed, then continuity is sufficient to guarantee uniform continuity Limon (2002); Limon et al. (2009).*

Definition 4 (Robust invariant region). *Given a control law $u = \kappa(x)$, $\bar{X} \subseteq X$ is a robust invariant region for the closed-loop system (1) with $u(k) = \kappa(x(k))$, if $\bar{x} \in \bar{X}$ implies $x(k) \in \bar{X}$ and $\kappa(x(k)) \in U$, $\forall w(k) \in \mathcal{W}$, $k \geq t$.* $\qquad\square$

Since there are mismatches between real system and nominal model, the predicted evolution using nominal model might differ from the real evolution of the system. In order to consider this effect in the controller synthesis, a bound on the difference between the predicted and the real evolution is given in the following lemma:

Lemma 1. *Limon et al. (2002a) Consider the system (1) satisfying Assumption 2. Then, for a given sequence of inputs, the difference between the nominal prediction of the state $\hat{x}(k|t)$ and the real state of the system $x(k)$ is bounded by*

$$|\hat{x}(k|t) - x(k)| \leq \frac{L_f^{k-t} - 1}{L_f - 1} \gamma, \quad k \geq t.$$

$\qquad\square$

To define the *NMPC* algorithms first let

$$
\begin{aligned}
B_\gamma^{k-t} &\triangleq \{z \in \mathcal{R}^n : |z| \leq \tfrac{L_f^{k-t}-1}{L_f-1}\gamma\} \\
X_{k-t} &\triangleq X \sim B_\gamma^{k-t} \\
&= \{x \in \mathcal{R}^n : x + y \in X, \forall y \in B_\gamma^{k-t}\}
\end{aligned}
$$

then define the following Finite Horizon Optimal Control Problem.

Definition 5 ($FHOCP^1$). *Given the positive integer N, the stage cost l, the terminal penalty V_f and the terminal set X_f, the Finite Horizon Optimal Control Problem ($FHOCP^1$) consists in minimizing, with respect to $u_{t,t+N-1}$, the performance index*

$$J(\bar{x}, u_{t,t+N-1}, N) \triangleq \sum_{k=t}^{t+N-1} l(\hat{x}(k|t), u(k)) + V_f(\hat{x}(t+N|t))$$

subject to

   (i) the nominal state dynamics (1) with $w(k) = 0$ and $x(t) = \bar{x}$;

  (ii) the state constraints $\hat{x}(k|t) \in X_{k-t}$, $k \in [t, t+N-1]$;

 (iii) the control constraints (4), $k \in [t, t+N-1]$;

 (iv) the terminal state constraint $\hat{x}(t+N|t) \in X_f$. $\qquad\square$

It is now possible to define a "prototype" of the first one of two nonlinear *MPC* algorithms: at every time instant t, define $\bar{x} = x(t)$ and find the optimal control sequence $u_{t,t+N-1}^o$ by solving the $FHOCP^1$. Then, according to the Receding Horizon (*RH*) strategy, define $\kappa^{MPC}(\bar{x}) = u_{t,t}^o(\bar{x})$ where $u_{t,t}^o(\bar{x})$ is the first column of $u_{t,t+N-1}^o$, and apply the control law

$$u = \kappa^{MPC}(x). \tag{11}$$

Although the $FHOCP^1$ has been stated for nominal conditions, under suitable assumptions and by choosing appropriately the terminal cost function V_f and the terminal constraint X_f, it is possible to guarantee the *ISS* property of the closed-loop system formed by (1) and (11), subject to constraints (2)-(4).

Assumption 3. *The function $l(x,u)$ is such that $l(0,0) = 0$, $l(x,u) \geq \alpha_l(|x|)$ where α_l is a $\mathcal{K}_\infty$-function. Moreover, $l(x,u)$ is Lipschitz with respect to x and u, in $X \times U$, with constant L_l and L_{lu} respectively.*

Remark 3. *Notice that if the stage cost $l(x,u)$ is a piece-wise differentiable function in X and U (as for instance the standard quadratic cost $l(x,u) = x'Qx + u'Ru$) and X and U are bounded sets, then the previous assumption is satisfied.*

Assumption 4. *The design parameter V_f and the set $\Phi \triangleq \{x : V_f(x) \leq \alpha\}$, $\alpha > 0$, are such that, given an auxiliary control law κ_f,*

1. $\Phi \subseteq X_{N-1}$;
2. $\kappa_f(x) \in U, \forall x \in \Phi$;
3. $f(x, \kappa_f(x)) \in \Phi, \forall x \in \Phi$;
4. $\alpha_{V_f}(|x|) \leq V_f(x) < \beta_{V_f}(|x|), \forall x \in \Phi$, where α_{V_f} and β_{V_f} are $\mathcal{K}_\infty$-functions;
5. $V_f(f(x, \kappa_f(x))) - V_f(x) \leq -l(x, \kappa_f(x)), \forall x \in \Phi$;
6. V_f is Lipschitz in Φ with a Lipschitz constant L_v.

Remark 4. *The assumption above can appear quite difficult to be satisfied, but it is standard in the development of nonlinear stabilizing MPC algorithms. Moreover, many methods have been proposed in the literature to compute V_f, Φ satisfying the Assumption 4 (see for example Chen & Allgöwer (1998); De Nicolao et al. (1998); Keerthi & Gilbert (1988); Magni, De Nicolao, Magnani & Scattolini (2001); Mayne & Michalska (1990)).*

Assumption 5. *The design parameter $X_f \triangleq \{x \in \mathcal{R}^n : V_f(x) \leq \alpha_v\}$, $\alpha_v > 0$, is such that for all $x \in \Phi$, $f(x, k_f(x)) \in X_f$.*

Remark 5. *If Assumption 4 is satisfied, then, a value of α_v satisfying Assumption 5 is the following*

$$\alpha_v = (id + \alpha_l \circ \beta_{V_f}^{-1})^{-1}(\alpha).$$

For each $x(k) \in \Phi$ there could be two cases. If $V_f(x(k)) \leq \alpha_v$, then, by Assumption 4, $V_f(x(k+1)) \leq \alpha_v$. If $V(x(k)) > \alpha_v$, then, by point 4 of Assumption 4, $\beta_{V_f}(|x(k)|) \geq V_f(x(k)) > \alpha_v$, that means $|x(k)| > \beta_{V_f}^{-1}(\alpha_v)$. Therefore, by Assumption 3 and point 4 of Assumption 4, one has

$$
\begin{aligned}
V_f(x(k+1)) &\leq V_f(x(k)) - l(x(k), \kappa_f(x(k))) \leq V_f(x(k)) - \alpha_l(|x(k)|) \\
&\leq \alpha - \alpha_l \circ \beta_{V_f}^{-1}(\alpha_v)
\end{aligned}
$$

for all $V_f(x(k+1)) \leq \alpha_v$. Then, $\alpha_v = \alpha - \alpha_l \circ \beta_{V_f}^{-1}(\alpha_v)$ satisfy the previous equation. After some manipulations one has $\alpha_v = (id + \alpha_l \circ \beta_{V_f}^{-1})^{-1}(\alpha)$.   □

Let $X^{MPC}(N)$ be the set of states of the system where an admissible solution of the $FHOCP^1$ optimization problem exists.

Definition 6. *Let $\alpha_1 = \alpha_3 = \alpha_l$, $\alpha_2 = \beta_{V_f}$, $\Xi = X^{MPC}(N)$, $\Omega = \Phi$, $\sigma = L_J$, where $L_J \triangleq L_{V_f} L_f^{N-1} + L_l \frac{L_f^{N-1}-1}{L_f-1}$.*

Assumption 6. *The values w are such that point 4 of Definition 2 is satisfied with $V(x) \triangleq J(x, u_{t,t+N-1}^o, N)$.*

Remark 6. *From this assumption it is inferred that the allowable size of disturbances is related with the size of the local region Ω where the upper bound of the terminal cost is found. This region can be enlarged following the way suggested in Limon et al. (2006). However, this might not produce an enlargement of the allowable size since the new obtained bound is more conservative.*   □

The main peculiarities of this $NMPC$ algorithm are the use in the $FHOCP^1$ of: (i) tightened state constraints along the optimization horizon; (ii) terminal set that is only a subset of the region where the auxiliary control law satisfies Assumption 4 in order to guarantee robustness (see Assumptions 4 and 5).

Let introduce now following theorem.

Theorem 2. *Let a system be described by a model given by (1). Assume that Assumptions 1-6 are satisfied. Then the closed loop system (1), (11) is ISS with robust invariant region $X^{MPC}(N)$ if the uncertainty is such that*

$$\gamma \leq \frac{\alpha - \alpha_v}{L_v L_f^{N-1}} \tag{12}$$

5.2 *MPC* **with time-varying control horizon**

In this sub-section the second algorithm will be shown. It is based on the same ideas of the first one and it is motivated by the attempt to reduce its intrinsic conservativity.

The second Finite Horizon Optimal Control Problem ($FHOCP^2$) to be introduced is characterized by using a time varying control horizon $N_c(t)$ and a (time invariant) prediction horizon N_p. The control horizon is given by

$$N_c(t) \triangleq \left(\left\lfloor \frac{t}{M} \right\rfloor + 1 \right) M - t$$

where $\lfloor \cdot \rfloor$ indicates the integer part operator and M is a parameter which determines its maximum value, i.e. $N_c(t) \in [1, M]$.

Definition 7 ($FHOCP^2$)**.** *Given a stabilizing control law κ_f the maximum control horizon M, the prediction horizon N_p, the stage cost l, and the terminal penalty V_f, the Finite Horizon Optimal Control Problem ($FHOCP^2$) consists in minimizing, with respect to $u_{t,t+N_c(t)-1}$, the performance index*

$$J(\bar{x}, u_{t,t+N_c(t)-1}, N_c(t), N_p) \triangleq \sum_{k=t}^{t+N_p-1} l(\hat{x}(k|t), u(k)) + V_f(\hat{x}(t+N_p|t))$$

subject to

- (i) the nominal state dynamics (1) with $w(k) = 0$ and $\bar{x} = x(t)$;

- (ii) the state constraints $\hat{x}(k|t) \in X_{k-t}$, $k \in [t, ..., t + N_c(t) - 1]$;

- (iii) the control constraints (4), $k \in [t, ..., t + N_c(t) - 1]$;

- (iv) the terminal state constraint $\tilde{x}(t + N_c(t)|t + N_c(t) - M) \in X_f$ where $\tilde{x}$ denotes the nominal prediction of the system considering as initial condition $x(t + N_c(t) - M)$ and applying the sequence of control inputs $\tilde{u}_{t+N_c(t)-M,t+N_c(t)-1}$ defined as

$$\tilde{u}_{t+N_c(t)-M,t+N_c(t)-1}(k) = \begin{cases} u^o_{k,k} & \text{if } k < t \\ u_{t,t+N_c(t)-1}(k) & \text{if } k \geq t \end{cases}$$

- (v) the control signal

$$u(k) = \begin{cases} u_{t,t+N_c(t)-1}(k), \ k \in [t, t + N_c(t) - 1] \\ \kappa_f(\hat{x}(k|t)), \ k \in [t + N_c(t), t + N_p - 1] \end{cases} \tag{13}$$

$\square$

It is now possible to introduce the second $NMPC$ algorithm in the following way: at every time instant t, define $\bar{x} = x(t)$ and find the optimal control sequence $u^o_{t,t+N_c(t)-1}$ by solving the $FHOCP^2$. Then, according to the RH strategy, define $\kappa^{MPC}(t, \bar{x}, \tilde{x}(t|t + N_c(t) - M)) = u^o_{t,t}(\bar{x}, \tilde{x}(t|t + N_c(t) - M))$ where $u^o_{t,t}(\bar{x}, \tilde{x}(t|t + N_c(t) - M))$ is the first column of $u^o_{t,t+N_c(t)-1}$, and apply the control law

$$u(t) = \kappa^{MPC}(t, x(t), \tilde{x}(t|t + N_c(t) - M)). \tag{14}$$

Note that the control law is time variant (periodic) due to the time variance of the control horizon $N_c(t)$ and depends also on $\tilde{x}(t|t + N_c(t) - M)$.
Therefore, defining

$$\xi(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t|t + N_c(t) - M)) \end{bmatrix} = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \in \mathcal{R}^{2n},$$

the closed-loop system formed by (1) and (14) is given by

$$\xi(k+1) = \tilde{F}(k, \xi(k), w(k)), \ k \geq t, \xi(t) = \bar{\xi} \tag{15}$$

where

$$\tilde{F}(k, \xi(k), w(k)) = \begin{bmatrix} f(\xi_1(k), \kappa^{MPC}(k, \xi_1(k), \xi_2(k))) + w(k) \\ \begin{cases} f(\xi_2(k), \kappa^{MPC}(k, \xi_1(k), \xi_2(k))), \ \forall (k+1) \notin \mathcal{T}_M \\ f(\xi_1(k), \kappa^{MPC}(k, \xi_1(k), \xi_2(k))) + w(k), \ \forall (k+1) \in \mathcal{T}_M \end{cases} \end{bmatrix}$$

Definition 8. *Let* $X^{MPC}(t, N_p) \in \mathcal{R}^{2n}$ *be the set of states* $\xi(t)$ *where an admissible solution of the* $FHOCP^2$ *exists.* $\square$

Noting that $x(t) = \tilde{x}(t|t + N_c(t) - M))$, $\forall t \in \mathcal{T}_M$ since $N_c(t) = M$, the closed-loop system (1), (14) for $k \in \mathcal{T}_M$ is time invariant since the control law is time invariant and

$$x(k + M) = \bar{F}(x(k), w_{k,k+M-1}), \ \forall k \in \mathcal{T}_M, \ k \geq t, \ x(t) = \bar{x}. \tag{16}$$

Definition 9. *Let $X_M^{MPC}(N_p) \in \mathcal{R}^n$ be the set x of states of the system (1) where an admissible solution of the $FHOCP^2$ exists $\forall t \in \mathcal{T}_M$.* $\square$

As in the previous algorithm, although the $FHOCP^2$ has been stated for nominal conditions, under suitable assumptions and by choosing accurately the terminal cost function V_f and the terminal constraint X_f, it is possible to guarantee the *ISS* property of the closed-loop system formed by (1) and (14), subject to constraints (2)-(4).

Assumption 7. *The auxiliary control law κ_f is Lipschitz in Φ with a Lipschitz constant L_κ where $\Phi \triangleq \{x \in X_{M-1} : V_f(x) \leq \alpha\}$, $\alpha > 0$.*

Remark 7. *Note that, an easy way to satisfy Assumption 7 is to choose κ_f linear, e.g. the solution of the infinite horizon optimal control problem for the unconstrained linear system.*

Assumption 8. *The design parameter $X_f \triangleq \{x \in \mathcal{R}^n : V_f(x) \leq \alpha_v\}$ is such that, considering the system (1), with $u = \kappa_f(x)$ and $w(k) = 0$, for all $x(t) \in \Phi$ results $\hat{x}(t + M|t) \in X_f$ and $\hat{x}(k|t) \in X_{k-t}$, $k \in [t, t + M - 1]$.*

Definition 10. *Let $\alpha_1 = \alpha_3 = \alpha_l$, $\alpha_2 = \beta_{V_f}$, $\Xi = X_M^{MPC}(N_p)$, $\Omega = \Phi$, $\sigma = L_J^M$, where*

$$L_J^M \triangleq \sum_{k=t}^{t+M-1} \left\{ L_l \frac{L_f^{N_c(k)-1} - 1}{L_f - 1} + L_{lx} L_f^{N_c(k)-1} \frac{L_x^{N_p - N_c(k)+1} - 1}{L_x - 1} + L_v L_f^{N_c(k)-1} L_x^{N_p - N_c(k)+1} \right\}$$

with $L_x \triangleq (L_f + L_{fu} L_\kappa)$ and $L_{lx} \triangleq (L_l + L_{lu} L_\kappa)$.

Assumption 9. *The values w are such that point 4 of Definition 2 is satisfied with $V(x) \triangleq J(x, u_{t,t+M-1}^o, M, N_p)$.*

The main peculiarities of this *NMPC* algorithm, with respect to the one previously presented, are the use in the $FHOCP^2$ of: (i) a time varying control horizon; (ii) a control horizon that is different from prediction horizon; (iii) the fact that the real value of the state is updated only each M step to check the terminal constraint while it is updated at each step for the computation of cost. These modifications allows to relax Assumption 5 with Assumption 8. In this way it could be possible to enhance the robustness. The idea to use the measure of the state only each M step has been already used in an other context in contractive *MPC* de Oliveira Kothare & Morari (2000).

Theorem 3. *Let a system be described by a model given by (1). Assume that Assumptions 1-4, 7-9 are satisfied. Then the closed loop system (15) is ISS with robust invariant region $X^{MPC}(t, N_p)$ if the uncertainty is such that*

$$\gamma \leq \frac{\alpha - \alpha_v}{L_v \frac{L_f^M - 1}{L_f - 1}} \tag{17}$$

Different from Magni, De Nicolao, Magnani & Scattolini (2001) the use of a prediction horizon longer than the control horizon does not affect the size of the robust invariant region because the terminal inequality constraint has been imposed at the end of the control horizon. However the following theorem proves that this choice has positive effect on the performance.

Theorem 4. *Magni, De Nicolao, Magnani & Scattolini (2001) Letting $l(x,u) = x'Qx + u'Ru$, $Q > 0, R > 0$, $u = -K^{LQ}x$ the solution of the infinite horizon optimal control problem for the unconstrained linear system*

$$x(k+1) = Ax(k) + Bu(k)$$

with $A = \partial f(x,u)/\partial x|_{x=0,u=0}$, $B = \partial f(x,u)/\partial u|_{x=0,u=0}$, for each given N_c, if $\kappa_f(x) = -K^{LQ}x$, then $\lim_{N_p \to \infty} \partial \kappa^{MPC}(x)/\partial x|_{x=0} = K^{LQ}$.

In conclusion, Theorems 2 and 3 proven that both the algorithm guarantee the *ISS* of the closed-loop system. However a priori it is not possible to establish which of the two algorithms give more robustness. This because of the dependance from the values of L_f, M, N_p of the bounded on the maximum disturbance allowed. Therefore, based on the dynamic system in object, it will be used an algorithm rather than the other.

6. Examples

The objective of the examples is to show that, based on the values of certain parameters, one algorithm can be better than the other. In particular two examples are shown: in the first one the algorithm based on $FHOCP^1$ is better than the one based on $FHOCP^2$ in terms of robustness; in the second one the contrary happens.

6.1 Example 1

Consider the uncertain nonlinear system given by

$$
\begin{aligned}
x_1(k+1) &= 0.55x_1(k) + 0.12x_2(k) + (0.01 - 0.6x_1(k) + x_2(k) + \Lambda_1)u(k) \\
x_2(k+1) &= 0.67x_2(k) + (0.15 + x_1(k) - 0.8x_2(k) + \Lambda_2)u(k)
\end{aligned}
$$

where Λ_1 and Λ_2 are the parameters of the system model uncertainty. The control is constrained to be $|u| \leq u_{\max} = 0.2$. Defining $w = [\Lambda_1 u^T \ \Lambda_2 u^T]^T$ the disturbance is in the form (1) and the nominal system is in the form $x(k+1) = Ax + Bu + Cxu$. Considering the ∞-norm, the Lipschitz constant of the system is

$$L_f = \max_u(|A + Cu|_\infty) = \max\{|A + 3C|_\infty, |A - 3C|_\infty\} = 1.03.$$

In the formulation of the $FHOCP^1$ and $FHOCP^2$ the stage is $l(x,u) = x'Qx + u'Ru$ with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$ and the auxiliary control law $u = -K^{LQ}x$ is derived by solving an Infinite Horizon optimal control problem for the linearized system around the origin

$$
\begin{aligned}
x_1(k+1) &= 0.55x_1(k) + 0.12x_2(k) + 0.01u(k) \\
x_2(k+1) &= 0.67x_2(k) + 0.15u(k)
\end{aligned}
$$

with the same stage cost. The solution of the associated Riccati Equation is $P = \begin{bmatrix} 1.4332 & 0.1441 \\ 0.1441 & 1.8316 \end{bmatrix}$ so that the value of K^{LQ} is $K^{LQ} = \begin{bmatrix} -0.0190 & -0.1818 \end{bmatrix}$. The value of the

Lipschitz constant L_κ of the auxiliary control law is $L_\kappa = |K^{LQ}|_\infty = 0.1818$. The terminal penalty $V_f(x) = \beta x'Px$, where $\beta = 1.2$ satisfies

$$\lambda_{\max}(Q + K^{LQ'}RK^{LQ}) < \beta\lambda_{\min}(Q + K^{LQ'}RK^{LQ})$$

in order to verify Assumption 7. Therefore, considering the presence of the constraint on the control, the linear controller $u = -K^{LQ}x$ stabilizes the system only in the invariant set Φ, $\Phi = \{x : 1.2x'Px \le \alpha = 0.2\}$ The value of the Lipschitz constant L_v is $L_v = \max_{x \in \Phi} |2\beta Px|_\infty = 2.4|Px|_\infty = 1.3222$. For the algorithm based on $FHOCP^2$ the final constraint X_f depends on the value M while for the algorithm based on $FHOCP^1$ it results $X_f = \{x : 3x'Px \le 0.0966\}$. In Figure 1.a the maximum value of γ that satisfies (12) (solid line) and the one that satisfies the (17) (dotted line) for different values of M, are reported. In this example the algorithm based on the $FHOCP^1$ guarantees major robustness than the one based on $FHOCP^2$.
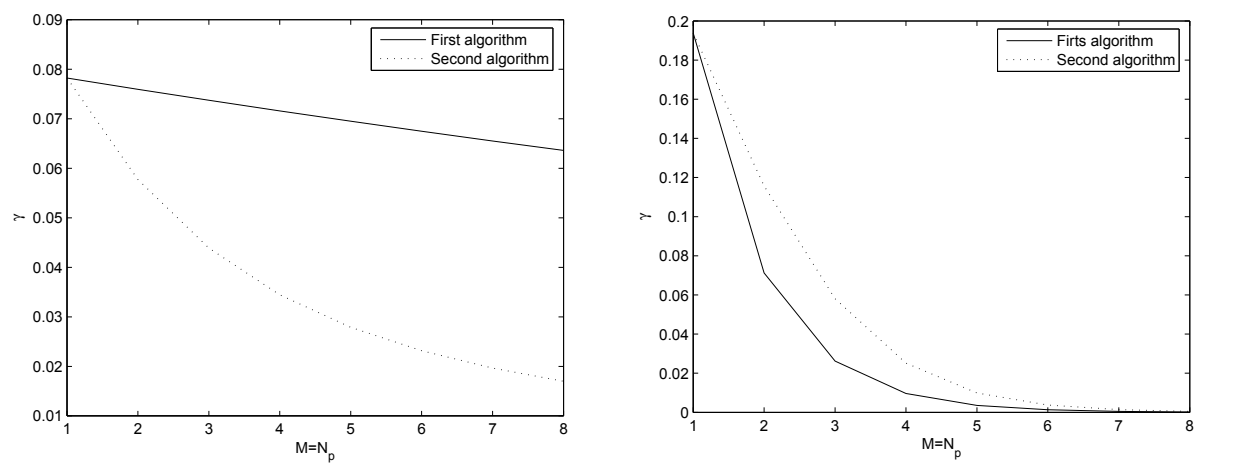
(a) Example 1: comparison of γ between the two algorithms.

(b) Example 2: comparison of γ between the two algorithms.
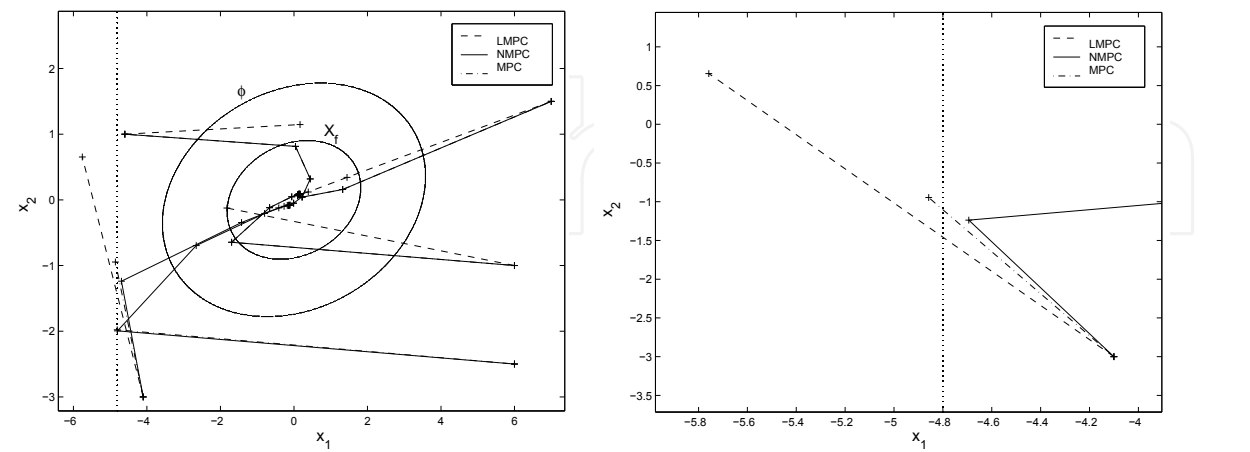
(c) Example 2: closed loop state evolution.

(d) Example 2: detail of the closed-loop state evolution with initial state (-4.1;-3).

6.2 Example 2

This example shows a case in which the algorithm based on $FHOCP^2$ gives a better solution. Consider the uncertain nonlinear system

$$\begin{aligned}
x_1(k+1) &= x_2(k) + (0.3x_2(k) + \Lambda_1)u \\
x_2(k+1) &= -0.32x_1(k) + 1.8x_2(k) + (1 - 0.2x_2(k) + \Lambda_2)u
\end{aligned}$$

where Λ_1 and Λ_2 are the parameters of the system model uncertainty. The control is constrained to be $|u| \leq u_{\max} = 3$ and the state x_1 is constrained to be $x_1 \geq -4.8$. Considering the ∞-norm, the Lipschitz constant of the system is

$$L_f = \max_u(|A + Cu|_\infty) = \max\{|A + 3C|_\infty, |A - 3C|_\infty\} = 2.72.$$

In the formulation of the $FHOCP^1$ and $FHOCP^2$ the stage is $l(x,u) = x'Qx + u'Ru$ with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$ and the auxiliary control law $u = -K^{LQ}x$ is derived by solving an Infinite Horizon optimal control problem for the linearized system around the origin

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= -0.32x_1(k) + 1.8x_2(k) + u
\end{aligned}$$

with the same stage cost. The solution of the associated Riccati Equation is $P = \begin{bmatrix} 1.0834 & -0.4428 \\ -0.4428 & 4.3902 \end{bmatrix}$ so that the value of K^{LQ} is $K^{LQ} = \begin{bmatrix} -0.2606 & 1.3839 \end{bmatrix}$. The value of the Lipschitz constant L_κ of the auxiliary control law is $L_\kappa = |K^{LQ}|_\infty = 1.3839$. The terminal penalty $V_f(x) = \beta x'Px$, where $\beta = 3$, satisfies

$$\lambda_{\max}(Q + K^{LQ\prime}RK^{LQ}) < \beta\lambda_{\min}(Q + K^{LQ\prime}RK^{LQ})$$

in order to verify Assumption 7. Therefore, considering the presence of the constraint on the control, the linear controller $u = -K^{LQ}x$ stabilizes the system only in the invariant set Φ, $\Phi = \{x : 3x'Px \leq \alpha = 40.18\}$. The value of the Lipschitz constant L_v is $L_v = \max_{x \in \Phi}|2\beta Px|_\infty = 6|Px|_\infty = 45.9926$. For the algorithm based on $FHOCP^2$ the final constraint X_f depends on the value M while for the algorithm based on $FHOCP^1$ it results $X_f = \{x : 3x'Px \leq 31.2683\}$. In Figure 1.b the maximum value of γ that satisfies (12) (solid line) and the one that satisfies the (17) (dotted line) for different values of M, are reported. In this example, the advantage of the algorithm based on the $FHOCP^2$ with respect to first one is due to the fact that the auxiliary control law can lead the state of the nominal system from Φ to X_f in M steps rather than in only one. Hence, since the difference between Φ and X_f is bigger, then a bigger perturbation can be tolerated. In Figure 1.c the state evolutions of the nonlinear system obtained with different control strategies with initial condition

x_{01}	6	−4.1	7	6	−4.6
x_{02}	−2.5	−3	1.5	−1	1

and $\gamma = 0.0581$ are reported: in solid line, using the new algorithm ($NMPC$), with $N_p = 10$ and $M = 3$, in dashed line, using the new algorithm but with the linearized system in the solution of the $FHOCP$ ($LMPC$) and in dash-dot line the results of a nominal MPC (MPC) with $N_p = 10$ and $N_c = 3$. It is clear that, since the model used for the $FHOCP$ differs from the nonlinear model, using $LMPC$ feasibility is not guaranteed along the trajectory as shown with

initial states $[-4.6; 1]$, $[-4.1; -3]$, $[6; -1]$. Also with the nominal *MPC*, as shown with initial states $[-4.1; -3]$, $[6; -2.5]$, since uncertainty is not considered, feasibility is not guaranteed. Figure 1.d shows a detail of the unfeasibility phenomenon from the first to the second time instant with initial state $[-4.1; -3]$. The state constraint infact is robustly fulfilled only with the *NMPC* algorithm. For the other initial states, the evolutions of the three strategies are close.

7. Conclusions

In this paper two design procedures of nominal *MPC* controllers are presented. The objective of these algorithms is to provide some degree of robustness when model mismatches are present. Regional Input-to-State Stability (*ISS*) has been used as theoretical framework of the closed loop analysis. Both controllers assume the Lipschitz continuity of the model and of the stage cost and terminal cost functions. Robust constraint satisfaction is ensured by introducing restricted constraints in the optimization problem based on the estimation of the maximum effect of the uncertainty. The main differences between the proposed algorithms are that the second one uses a time varying control horizon and, in order to check the terminal constraints, it updates the state with the real one just only each M steps. Theorem 2 and Theorem 3 give sufficient condition on the maximum uncertainty in order to guarantee regional *ISS*. The bounds depend on both system parameters and control algorithm parameters. These conditions, even if only sufficient, give an idea on the algorithm that it is better to use for a particular system.

8. Appendix

Lemma 2. *Let $x \in X_{k-t}$ and $y \in \mathcal{R}^n$ such that $|y - x| \leq L_f^{k-t-1}\gamma$. Then $y \in X_{k-t-1}$.*

Proof : Consider $e_{k-t-1} \in B_\gamma^{k-t-1}$, and let denote $z = y - x + e_{k-t-1}$. It is clear that

$$|z| \leq |y - x| + |e_{k-t-1}| \leq L_f^{k-t-1}\gamma + \frac{L_f^{k-t-1} - 1}{L_f - 1}\gamma = \frac{L_f^{k-t} - 1}{L_f - 1}\gamma$$

thus, $z \in B_\gamma^{k-t}$. Taking into account that $x \in X_{k-t}$, for all $e_{k-t-1} \in B_\gamma^{k-t-1}$, it results that $y + e_{k-t-1} = (x + z) \in X$. This yields that $y \in X_{k-t-1}$.   ■

Proof of Theorem 2: Firstly, it will be shown that region $X^{MPC}(N)$ is robust positively invariant for the closed loop system: if $x(t) \in X^{MPC}(N)$, then $x(t+1) = f(x(t), u^o(t)) + w(t) \in X^{MPC}(N)$ for all $w(t) \in W$. This is achieved by proving that for all $x(t) \in X^{MPC}(N)$, there exists an admissible solution of the optimization problem in $t + 1$, based on the optimal solution in t, i.e. $\bar{u}_{t+1,t+N} = [u^o_{t+1,t+N-1}, k_f(\hat{x}(t+N|t+1))]$. Let denote $\bar{x}(k|t+1)$ the state obtained applying the input sequence $\bar{u}_{t+1,k-1}$ to the nominal model with initial condition $x(t+1)$. In order to prove that the sequence $\bar{u}_{t+1,t+N}$ is admissible, it is necessary that

a) $\bar{u}(k) \in U$, $k \in [t+1, t+N]$: it follows from the feasibility of $u^o_{t,t+N-1}$ and the fact that $\kappa_f(x) \in U, \forall x \in X_f \subseteq \Phi$.

b) $\bar{x}(t+N+1|t+1) \in X_f$: first, it is going to be shown that $\bar{x}(t+N|t+1) \in \Phi$. Taking
into account that $|\bar{x}(t+N|t+1) - \hat{x}(t+N|t)| \leq L_f^{N-1}\gamma$ then

$$V_f(\bar{x}(t+N|t+1)) \leq V_f(\hat{x}(t+N|t)) + L_v L_f^{N-1}\gamma \leq \alpha_v + L_v L_f^{N-1}\gamma \leq \alpha.$$

Therefore $\bar{x}(t+N|t+1) \in \Phi$ and hence, applying the auxiliary control law, $\bar{x}(t+N+1|t+1) \in X_f$.

c) $\bar{x}(k|t+1) \in X_{k-t-1}$, $k \in [t+1, t+N]$: considering that $|x(t+1) - \hat{x}(t+1|t)| \leq \gamma$ by
recursion $|\bar{x}(k|t+1) - \hat{x}(k|t)| \leq L_f^{k-t-1}\gamma$ for $k \in [t+1, t+N]$. Since $\hat{x}(k|t) \in X_{k-t}$,
then, by Lemma 2, $\bar{x}(k|t+1) \in X_{k-t-1}$. Moreover, since $\bar{x}(t+N|t+1) \in \Phi \subseteq X_{N-1}$,
the proof is completed.

Now, in order to show that the closed loop system (1), (11) is *ISS* in $X^{MPC}(N)$, let verify
that $V(\bar{x}, N) \triangleq J(\bar{x}, u^o_{t,t+N-1}, N)$ is an *ISS*-Lyapunov function in $X^{MPC}(N)$. First note that by
Assumption 3

$$V(\bar{x}, N) \geq \alpha_l(|\bar{x}|), \quad \forall \bar{x} \in X^{MPC}(N). \tag{18}$$

Moreover, in view of Assumption 4, $\tilde{u}_{t,t+N} = [u^o_{t,t+N-1}, k_f(\hat{x}(t+N|t))]$ is an admissible, possible suboptimal, control sequence for the $FHOCP^1$ with horizon $N+1$ at time t with cost

$$\begin{aligned} J(\bar{x}, \tilde{u}_{t,t+N}, N+1) &= V(\bar{x}, N) - V_f(\hat{x}(t+N|t)) + V_f(\hat{x}(t+N+1|t)) \\ &\quad + l(\hat{x}(t+N|t), k_f(\hat{x}(t+N|t))). \end{aligned}$$

Since $\tilde{u}_{t,t+N}$ is a suboptimal sequence, $V(\bar{x}, N+1) \leq J(\bar{x}, \tilde{u}_{t,t+N}, N+1)$ and, using point 5 of
Assumption 4, it follows that $J(\bar{x}, \tilde{u}_{t,t+N}, N+1) \leq V(\bar{x}, N)$. Then

$$V(\bar{x}, N+1) \leq V(\bar{x}, N), \quad \forall \bar{x} \in X^{MPC}(N)$$

with $V(\bar{x}, 0) = V_f(\bar{x})$, $\forall \bar{x} \in \Phi$. Therefore

$$V(\bar{x}, N) \leq V(\bar{x}, N-1) \leq V_f(\bar{x}) < \beta_{V_f}(|\bar{x}|), \quad \forall \bar{x} \in \Phi. \tag{19}$$

Moreover, let define ΔJ as

$$\begin{aligned} \Delta J &\triangleq J(x(t+1), \bar{u}_{t+1,t+N}, N) - J(x(t), u^o_{t,t+N-1}, N) \\ &= -l(x(t), u^o(t)) + \sum_{k=t+1}^{k=t+N-1} \{l(\bar{x}(k|t+1), \bar{u}(k)) - l(\hat{x}(k|t), u^o(k))\} \\ &\quad + l(\bar{x}(t+N|t+1), \bar{u}(t+N)) + V_f(\bar{x}(t+N+1|t+1) - V_f(\hat{x}(t+N|t)). \end{aligned} \tag{20}$$

From the definition of $\bar{u}$, $\bar{u}(k) = u^o(k)$, for $k \in [t+1, t+N-1]$, and hence $l(\bar{x}(k|t+1), \bar{u}(k)) - l(\hat{x}(k|t), u^o(k)) \leq L_l L_f^{k-t-1}\gamma$ and analogously

$$V_f(\bar{x}(t+N|t+1) - V_f(\hat{x}(t+N|t)) \leq L_v L_f^{N-1}\gamma.$$

Substituting these expressions in (20) and considering that $\bar{x}(t+N|t+1) \in \Phi$, from Assumption 4, there is

$$\begin{aligned} \Delta J &\leq [l(\bar{x}(t+N|t+1), \bar{u}(t+N)) + V_f(\bar{x}(t+N+1|t+1) - V_f(\bar{x}(t+N|t+1)] \\ &\quad - l(x(t), u^o(t)) + L_J\gamma \leq -l(x(t), u^o(t)) + L_J\gamma \end{aligned}$$

where $L_J \triangleq L_v L_f^{N-1} + L_l \frac{L_f^{N-1}-1}{L_f-1}$. Considering that by Assumption 3, $l(x,u) \geq \alpha_l(|x|)$ and the optimality of the solution, then

$$V(x(t+1),N) - V(x(t),N) \leq \Delta J \leq -\alpha_l(|x(t)|) + L_J\gamma, \ \forall x \in X^{MPC}(N) \quad (21)$$

Therefore, by (18), (19) and (21), $V(\bar{x},N)$ is an *ISS*-Lyapunov function of the closed loop system (1), (11), and hence, the closed-loop system is *ISS* with robust invariant region $X^{MPC}(N)$. ∎

Proof of Theorem 3: Firstly, it will be shown that region $X^{MPC}(t,N_p)$ is robust positively invariant for the closed-loop system. This is achieved by proving that for all $\xi(t) \in X^{MPC}(t,N_p)$, there exists an admissible solution $\bar{u}_{t+1,t+1+N_c(t+1)-1}$ of the optimization problem in $t+1$, based on the optimal solution in t. This sequence is given by

$$\bar{u}_{t+1,t+1+N_c(t+1)-1}(k) = \begin{cases} u^o_{t,t+N_c(t)-1}(k) & \text{if } t+1 \notin \mathcal{T}_M \\ \kappa_f(\hat{x}(k|t+1)) & \text{if } t+1 \in \mathcal{T}_M \end{cases}$$

for $k \in [t+1,\cdots,t+1+N_c(t+1)-1]$. Notice that if $t+1 \notin \mathcal{T}_M$, $N_c(t+1) = N_c(t)-1$ and hence the sequence is well defined.

Moreover, since necessary for the *ISS* proof, it will be shown that, starting from the (nominal) state $\hat{x}(t+1|t)$, the sequence $\bar{u}'_{t+1,t+1+N_c(t+1)-1}$ is admissible. This is given by

$$\bar{u}'_{t+1,t+1+N_c(t+1)-1}(k) = \begin{cases} u^o_{t,t+N_c(t)-1}(k) & \text{if } t+1 \notin \mathcal{T}_M \\ \kappa_f(\hat{x}(k|t)) & \text{if } t+1 \in \mathcal{T}_M \end{cases}$$

for $k \in [t+1,\cdots,t+1+N_c(t+1)-1]$.

In order to prove that the two sequences are admissible, it is necessary that
1) $\tilde{x}(t+1+N_c(t+1)|t+1+N_c(t+1)-M) \in X_f$ with $\tilde{u}_{t+1+N_c(t+1)-M,t+1+N_c(t+1)-1}$ derived from both $\bar{u}$ and $\bar{u}'$;
2) $\hat{x}(k|t+1) \in X_{k-t-1}, k \in [t+1,t+1+N_c(t+1)-1]$ with input $\bar{u}$;
3) $\hat{x}(k|t) \in X_{k-t}, k \in [t+1,t+1+N_c(t+1)-1]$ with input $\bar{u}'$;
4) $\bar{u}(k) \in U, \bar{u}'(k) \in U, k \in [t+1,t+1+N_c(t+1)-1]$.

1) First note that if $t+1 \notin \mathcal{T}_M$, then $\bar{u}(k) = \bar{u}'(k) = u^o(k), k \in [t+1,t+1+N_c(t+1)-1]$. This yields to $\tilde{x}(k|t+N_c(t)-M) = \tilde{x}(k|t+1+N_c(t+1)-M)$ for all $k \in [t+1+N_c(t+1)-M,t+1+N_c(t+1)]$ and hence

$$\tilde{x}(t+1+N_c(t+1)|t+1+N_c(t+1)-M) = \tilde{x}(t+N_c(t)|t+N_c(t)-M) \in X_f.$$

On the contrary, if $t+1 \in \mathcal{T}_M$ then $\bar{u}_{t+1,t+1+N_c(t+1)-1}(k) = \kappa_f(\hat{x}(k|t+1))$ and $\bar{u}'_{t+1,t+1+N_c(t+1)-1}(k) = \kappa_f(\hat{x}(k|t))$. We are going to prove that both sequence satisfies the terminal constraint:

- Consider the sequence $\bar{u}$ and let denote $\tilde{u}$ and $\tilde{x}$ the sequence and predictions derived from $\bar{u}$. In virtue of Lemma 1 and the fact that $N_c(t) = 1$, the following inequality holds

$$|x(t+1) - \tilde{x}(t+1|t+N_c(t)-M)| \leq \frac{L_f^M-1}{L_f-1}\gamma \quad (22)$$

and by point 5 of Assumption 4 it follows that

$$V_f(x(t+1)) - V_f(\tilde{x}(t+1|t+N_c(t)-M))$$

$$\leq L_v|x(t+1) - \tilde{x}(t+1|t+N_c(t)-M)| \leq L_v \frac{L_f^M - 1}{L_f - 1}\gamma$$

Hence, considering that $\tilde{x}(t+1|t+N_c(t)-M) \in X_f$ and the uncertainty satisfies (17), then

$$V_f(x(t+1)) \leq V_f(\tilde{x}(t+1|t+N_c(t)-M)) + L_v \frac{L_f^M - 1}{L_f - 1}\gamma \leq \alpha_v + L_v \frac{L_f^M - 1}{L_f - 1}\gamma \leq \alpha \qquad (23)$$

and therefore $x(t+1) \in \Phi$. Hence, from Assumption 8, $\kappa_f(\hat{x}(k|t+1))$ steers the nominal state in X_f in M steps. Then $\bar{u}_{t+1,t+N_c(t+1)-1}$ satisfies the constraint.

- Let consider now $\bar{u}'$ and let denote $\tilde{u}'$ and $\tilde{x}'$ the sequence and predictions derived from $\bar{u}'$. Since $\hat{x}(t+1|t) = f(x(t), u_{t,t}^o)$ we have that

$$
\begin{aligned}
&|\hat{x}(t+1|t) - \tilde{x}'(t+1|t+N_c(t)-M)| \\
=\;& |f(x(t), u^o(t)) - f(\tilde{x}'(t|t+N_c(t)-M), u^o(t))| \\
\leq\;& L_f|x(t) - \tilde{x}'(t|t+N_c(t)-M)|
\end{aligned}
$$

and from (22) $|\hat{x}(t+1|t) - \tilde{x}'(t+1|t+N_c(t)-M)| \leq L_f \frac{L_f^{M-1}-1}{L_f-1}\gamma$. Finally, following the same idea used to derive (23)

$$
\begin{aligned}
V_f(\hat{x}(t+1|t)) &\leq V_f(\tilde{x}'(t+1|t+N_c(t)-M)) + L_v L_f \frac{L_f^{M-1}-1}{L_f-1}\gamma \\
&< \alpha_v + L_v \frac{L_f^M - 1}{L_f - 1}\gamma \leq \alpha. \qquad (24)
\end{aligned}
$$

Therefore $V_f(\hat{x}(t+1|t)) < \alpha$ and consequently $\hat{x}(t+1|t) \in \Phi$. Hence $\kappa_f(\hat{x}(k|t))$ steers the nominal state in X_f in M steps. Then $\bar{u}'_{t+1,t+N_c(t+1)-1}$ satisfies the constraint.

2) Consider the sequence of inputs $\bar{u}$ and assume that $t+1 \notin \mathcal{T}_M$, then, since by optimality of solution at time t, $\hat{x}(k|t) \in X_{k-t}$ and

$$|\hat{x}(k|t+1) - \hat{x}(k|t)| \leq L_f^{k-t-1}\gamma, \; k \in [t+1, t+1+N_c(t+1)-1]$$

from Lemma 2, it follows that $\hat{x}(k|t+1) \in X_{k-t-1}$. If $t \in \mathcal{T}_M$ then $x(t+1) \in \Phi$ as shown in (23), and from Assumptions 4, 7, the constraints satisfaction is directly derived.

3) Consider that the sequence $\bar{u}'_{t+1,t+1+N_c(t+1)-1}$ is applied from the state $\hat{x}(t+1|t)$. If $t+1 \notin \mathcal{T}_M$ then the constraints are satisfied since $\hat{x}(k|t) \in X_{k-t}$. If $t+1 \in \mathcal{T}_M$, as shown in (24), $\hat{x}(t+1|t) \in \Phi$ and then, by Assumptions 4, 7, constraints satisfaction is directly derived.

4) From the admissibility of $u_{t,t+N_c(t)-1}^o$ and the fact that for all $x \in \Phi$, $\kappa_f(x) \in U$, it follows that $\bar{u}(k) \in U, \bar{u}'(k) \in U, k \in [t+1, t+1+N_c(t+1)-1]$.

Now, in order to show that the closed loop system (15) is *ISS* in $X^{MPC}(t, N_p)$, it is first proven that the closed-loop system (16), defined for each $t \in \mathcal{T}_M$, is *ISS* in $X_M^{MPC}(N_p)$.

In order to prove the first part let verify that $V(\bar{x}, M, N_p) \triangleq J(\bar{x}, u^o_{t,t+M-1}, M, N_p)$, is an *ISS*-Lyapunov function for the system (16).

Let denote $\bar{x}(k|t+1)$ and $\bar{x}'(k|t)$ the state evolution obtained with input $\bar{u}(k)$ and initial state $x(t+1)$ and with input $\bar{u}'(k)$ and initial state $x(t+1|t)$ respectively. Let call $J^*(t,x)$, $\bar{J}(x)$ and $\bar{J}'(x)$ the optimal cost and the cost relative to the admissible sequences $\bar{u}$ and $\bar{u}'$ respectively. First note that by Assumption 3

$$V(\bar{x}, M, N_p) \geq \alpha_l(|\bar{x}|), \ \forall \bar{x} \in X^{MPC}_M(N_p). \tag{25}$$

Moreover, $\tilde{u}_{t,t+M-1} = u^o_{t,t+M-1}$, where $u^o_{t,t+M-1}$ is the optimal control sequence for the $FHOCP^2$ with prediction horizon N_p, is an admissible, possible suboptimal, control sequence for the $FHOCP^2$ with control horizon M and prediction horizon $N_p + 1$ at time t with cost

$$
\begin{aligned}
J(\bar{x}, \tilde{u}_{t,t+M-1}, M, N_p+1) = \ & V(\bar{x}, M, N_p) - V_f(\hat{x}(t+N_p|t)) + V_f(\hat{x}(t+N_p+1|t)) \\
& + l(\hat{x}(t+N_p|t), k_f(\hat{x}(t+N_p|t))).
\end{aligned}
$$

Since $\tilde{u}_{t,M-1}$ is a suboptimal sequence $V(\bar{x}, M, N_p+1) \leq J(\bar{x}, \tilde{u}_{t,t+M-1}, M, N_p+1)$ and, using point 5 of Assumption 4, it follows that $J(\bar{x}, \tilde{u}_{t,t+M-1}, N_p+1) \leq V(\bar{x}, M, N_p)$. Then $V(\bar{x}, M, N_p+1) \leq V(\bar{x}, M, N_p)$, $\forall \bar{x} \in X^{MPC}_M(N_p)$, $N_p \geq M$. In particular, it is true that $V(\bar{x}, M, N_p) \leq V(\bar{x}, M, M)$, $\forall \bar{x} \in X^{MPC}_M(M)$. Now, in view of Assumption 4, $\tilde{u}_{t,t+M} = [u^o_{t,t+M-1}, k_f(\hat{x}(t+M|t))]$ is an admissible, possible suboptimal, control sequence for the $FHOCP^2$ with horizon $M+1$ with cost

$$
\begin{aligned}
J(\bar{x}, \tilde{u}_{t,t+M}, M+1, M+1) = \ & V(\bar{x}, M, M) - V_f(\hat{x}(t+M|t)) + V_f(\hat{x}(t+M+1|t)) \\
& + l(\hat{x}(t+M|t), k_f(\hat{x}(t+M|t))).
\end{aligned}
$$

Since $\tilde{u}_{t,t+M}$ is a suboptimal sequence $V(\bar{x}, M+1, M+1) \leq J(\bar{x}, \tilde{u}_{t,t+M}, M+1, M+1)$ and, using point 5 of Assumption 4, it follows that $J(\bar{x}, \tilde{u}_{t,t+M}, M+1) \leq V(\bar{x}, M, M)$. Then $V(\bar{x}, M+1, M+1) \leq V(\bar{x}, M, M)$, $\forall \bar{x} \in X^{MPC}_M(M)$ with $V(\bar{x}, 0, 0) = V_f(\bar{x})$, $\forall \bar{x} \in \Phi$. Therefore

$$V(\bar{x}, M, M) \leq V(\bar{x}, M-1, M-1) \leq V_f(\bar{x}) < \beta_{V_f}(|\bar{x}|), \ \forall \bar{x} \in \Phi. \tag{26}$$

Moreover, let calculate

$$
\begin{aligned}
& \bar{J}'(\hat{x}(t+1|t)) - J^*(t, x(t)) \\
= \ & \sum_{k=t+1}^{t+1+N_c(t+1)-1} l(\bar{x}'(k|t), \bar{u}'(k)) + \sum_{k=t+1+N_c(t+1)}^{t+N_p} l(\bar{x}'(k|t), \kappa_f(\bar{x}'(k|t))) \\
& - \sum_{k=t}^{t+N_c(t)-1} l(\hat{x}(k|t), u^o(k)) - \sum_{k=t+N_c(t)}^{t+N_p-1} l(\hat{x}(k|t), \kappa_f(\hat{x}(k|t))) \\
& + V_f(\bar{x}'(t+1+N_p|t)) - V_f(\hat{x}(t+N_p|t)).
\end{aligned}
$$

Since, both the state evolutions are obtained with initial condition $\hat{x}(t+1|t)$ and the same input sequence from time $t+1$ and until $t+N_p-1$, there is $\bar{x}'(k|t) = \hat{x}(k|t)$, $k \in [t+1, t+N_p]$ so that

$$
\begin{aligned}
\bar{J}'(\hat{x}(t+1|t)) - J^*(t, x(t)) = \ & l(\hat{x}(t+N_p|t), \kappa_f(\hat{x}(t+N_p|t)) - l(x(t), u^o(t)) \\
& + V_f(\hat{x}(t+1+N_p|t)) - V_f(\hat{x}(t+N_p|t)).
\end{aligned}
$$

Using point 5 of the Assumption 4

$$\bar{J}'(\hat{x}(t+1|t)) - J^*(t,x(t)) \leq -l(x(t),\kappa^{MPC}(t,x(t))).\tag{27}$$

Let consider now the difference

$$
\begin{aligned}
\bar{J}(x(t+1)) - \bar{J}'(\hat{x}(t+1|t)) &= \sum_{k=t+1}^{t+N_c(t)-1} \{l(\bar{x}(k|t+1),\bar{u}(k)) - l(\bar{x}'(k|t),\bar{u}'(k))\} \\
&+ \sum_{k=t+N_c(t)}^{t+N_p} \{l(\bar{x}(k|t+1),\kappa_f(\bar{x}(k|t+1))) \\
&\quad - l(\bar{x}'(k|t),\kappa_f(\bar{x}'(k|t)))\} \\
&+ V_f(\bar{x}(t+1+N_p|t+1)) - V_f(\bar{x}'(t+1+N_p|t)).
\end{aligned}
$$

Note that $\bar{u}(k) = \bar{u}'(k)$, $k \in [t+1, t+N_c(t)-1]$, while the signals are different for $k > t+N_c(t)-1$. Since $|\bar{x}(k|t+1) - \bar{x}'(k|t)| \leq L_f^{k-t-1}\gamma$ from Assumption 3 it is derived that

$$|l(\bar{x}(k|t+1),\bar{u}(k)) - l(\bar{x}'(k|t),\bar{u}'(k))| \leq L_l L_f^{k-t-1}\gamma, \; k \in [t+1,....,t+N_c(t)-1]$$

Therefore, an upper bound for the first part of the summation is given by

$$\sum_{k=t+1}^{t+N_c(t)-1} \{l(\bar{x}(k|t+1),\bar{u}(k)) - l(\bar{x}'(k|t),\bar{u}'(k))\} \leq L_l \frac{L_f^{N_c(t)-1}-1}{L_f-1}\gamma.\tag{28}$$

For $k > t+N_c(t)$, where $\bar{u}'$ and $\bar{u}$ are obtained applying the auxiliary control law to $\bar{x}(k|t+1)$ and $\bar{x}'(k|t)$ respectively, the upper bound is obtained using Assumptions 3 and 7, $l(\bar{x}(k|t+1),\bar{u}(k|t+1)) - l(\bar{x}'(k|t),\bar{u}'(k|t)) \leq (L_l + L_{lu}L_\kappa)|\bar{x}(k|t+1) - \bar{x}'(k|t)|$ and Assumption 2, $|\bar{x}(k+1|t+1) - \bar{x}'(k+1|t)| \leq (L_f + L_{fu}L_\kappa)|\bar{x}(k|t+1) - \bar{x}'(k|t)|$. Moreover $|\bar{x}(t+N_c(t)|t+1) - \bar{x}'(t+N_c(t)|t)| \leq L_f^{N_c(t)-1}\gamma$ and defining $L_x \triangleq (L_f + L_{fu}L_\kappa)$ and $L_{lx} \triangleq (L_l + L_{lu}L_\kappa)$, the following upper bound is obtained

$$
\begin{aligned}
\sum_{k=t+N_c(t)}^{t+N_p} \{l(\bar{x}(k|t+1),\bar{u}(k|t+1)) - l(\bar{x}'(k|t),\bar{u}'(k|t))\} &\leq L_{lx} \sum_{k=t+N_c(t)}^{t+N_p} |\bar{x}(k|t+1) - \bar{x}'(k|t)| \\
&\leq L_{lx} \sum_{k=t+N_c(t)}^{t+N_p} L_x^{k-t-N_c(t)}|\bar{x}(t+N_c(t)|t+1) - \bar{x}'(t+N_c(t)|t)| \\
&\leq L_{lx}L_f^{N_c(t)-1}\frac{L_x^{N_p-N_c(t)+1}-1}{L_x-1}\gamma.
\end{aligned}
$$

Finally in order to compute an upper bound for the difference of terminal penalties note that $|\bar{x}(t+N_p+1|t+1) - \bar{x}'(t+N_p+1|t)| \leq L_f^{N_c(t)-1}L_x^{N_p-N_c(t)+1}\gamma$ and using point 6 of Assumption 4, $V_f(\bar{x}(t+N_p+1|t+1)) - V_f(\bar{x}'(t+N_p+1|t)) \leq L_v L_f^{N_c(t)-1}L_x^{N_p-N_c(t)+1}\gamma$. Therefore the following bound is obtained

$$
\begin{aligned}
\bar{J}(x(t+1)) - \bar{J}'(\hat{x}(t+1|t)) &\leq L_l\frac{L_f^{N_c(t)-1}-1}{L_f-1}\gamma + L_{lx}L_f^{N_c(t)-1}\frac{L_x^{N_p-N_c(t)+1}-1}{L_x-1}\gamma \\
&+ L_v L_f^{N_c(t)-1}L_x^{N_p-N_c(t)+1}\gamma.
\end{aligned}
$$

Defining

$$L_J(t) \quad \triangleq \quad L_l \frac{L_f^{N_c(t)-1} - 1}{L_f - 1} + L_{lx} L_f^{N_c(t)-1} \frac{L_x^{N_p-N_c(t)+1} - 1}{L_x - 1} + L_v L_f^{N_c(t)-1} L_x^{N_p-N_c(t)+1}$$

it follows that $\bar{J}(x(t+1)) \leq \bar{J}'(\hat{x}(t+1|t)) + L_J(t)\gamma$. Considering that $J^*(t+1, x(t+1))$ is the optimal solution at time $t+1$, $J^*(t+1, x(t+1)) \leq \bar{J}(x(t+1)) \leq \bar{J}'(\hat{x}(t+1|t)) + L_J(t)\gamma$. From (27) it is possible to conclude $J^*(t+1, x(t+1)) - J^*(t, x(t)) \leq -l(x(t), \kappa^{MPC}(t, x(t))) + L_J(t)\gamma$. and by Assumption 3

$$J^*(t+1, x(t+1)) - J^*(t, x(t)) \leq -\alpha_l(|x(t)|) + L_J(t)\gamma. \tag{29}$$

Now, since $V(x(t), M, N_p) = J^*(t, x(t))$, $\forall t \in \mathcal{T}_M$, using (29), there is

$$
\begin{aligned}
V(x(t+M), M, N_p) - V(x(t), M, N_p) \quad &\leq \quad \sum_{k=t}^{t+M-1} -\alpha_l(|x(k)|) + L_J(k)\gamma \\
&\leq \quad -\alpha_l(|x(t)|) + \sum_{k=t}^{t+M-1} L_J(k)\gamma.
\end{aligned}
\tag{30}
$$

Therefore, by (25), (26) and (30), $V(x, M, N_p)$ is an ISS-Lyapunov function for the closed-loop system (16) and hence, the closed-loop system is ISS with robust invariant region $X_M^{MPC}(N_p)$. Now, to conclude the proof, it is necessary to demonstrate that, for $t \notin \mathcal{T}_M$, the system (15) is ISS in $X^{MPC}(t, N_p)$. Since the model predictive control law (14) is admissible for the $FHOCP^2$, the closed-loop system (15) is such that $\xi_1(t+nM) \in \Phi$, $\forall t \in \mathcal{T}_M$, $\forall n \in \mathcal{Z}_{>0}$. Hence, in order to prove that the system (15) is ISS in $X^{MPC}(t, N_p)$, it is sufficient to prove that the system (15) is ISS in Φ.
Noting that

$$
\begin{aligned}
\alpha_l(|x(t+i)|) \leq J^*(t+i, x(t+i)) \quad &\leq \quad V(x(t), M, N_p) - \alpha_l(|x(t)|) + \sum_{k=t}^{t+i-1} L_J(k)\gamma \\
&\leq \quad \beta_{V_f}(|x(t)|) + \sum_{k=t}^{t+i-1} L_J(k)\gamma, \; \forall x(t) \in \Phi
\end{aligned}
$$

considering that for any $\mathcal{K}_\infty$-function γ, $\gamma(r+s) \leq \gamma(2r) + \gamma(2s)$, there is

$$|x(t+Mn+i)| \leq \alpha_l^{-1}(2\beta_{V_f}(|x(t+Mn)|)) + \alpha_l^{-1}(2\sum_{k=t}^{t+i-1} L_J(k)\gamma), \; \forall x(t) \in \Phi \tag{31}$$

for all $n \in \mathcal{Z}_{>0}$ and $i \in [0, \cdots, M-1]$. Since the closed-loop system (16) is ISS with robust invariant region $X_M^{MPC}(N_p)$, there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$, and a $\mathcal{K}_\infty$-function λ such that $|x(t+Mn)| \leq \beta(|x(t)|, n) + \lambda(\gamma)$, $\forall n \in \mathcal{Z}_{\geq 0}, \forall x(t) \in X_M^{MPC}(N_p)$. Applying this to (31), there is $|x(t+Mn+i)| \leq \tilde{\beta}(|x(t)|, n) + \tilde{\lambda}(\gamma)$, $\forall x(t) \in \Phi$. Hence, in conclusion, the system (15) is ISS in $X^{MPC}(t, N_p)$. ∎

9. References

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