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Stability of switched stochastic nonlinear systems

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1. Introduction

Hybrid systems are digital real-time systems which are embedded in analog environment. Analog part of the hybrid system is described with differential equations and discrete part of the hybrid systems is an event driven dynamics which can be described using concept from discrete event systems (Cassandras & Lafortune, 2008) and (Tabuada, 2009). In this paper we will consider the switched systems which can be viewed as higher-level abstraction of hybrid systems (Liberzon, 2003) and (Sun & Ge, 2005). We model each subsystem of a switched system by differential equation.

There are two ways for analysis of stability of switched deterministic systems. The first one is a construction of common Lyapunov function. Find the common Lyapunov functions is a difficult task (Narendra & Balakrishnan, 1994). The second one utilizes multiple Lyapunov functions for analysis of switched systems (Branicky, 1998). In this paper we will consider a stability of switched stochastic systems. We assume that (i) there is no jump in the state at switching instants and (ii) there is no Zeno behaviour, i.e. there is finite number of switches on every bounded interval of time. The situation with jump in the state of x at the switching instants is considered in (Guan et al., 2005) and (Li et al., 2005).

In recent years the stochastic hybrid systems become hot research topic. There are a few approaches to the problem. In the stochastic setting we have jump diffusion as the solution of stochastic differential equation driven by Levy process which is a linear combination of time, Brownian motion and pure jump process (Oksendal & Sulem, 2005). Close to deterministic hybrid systems is the concept of Piecewise deterministic Markov processes (Davis, 1993) and Stochastic hybrid systems (Hu et al., 2000). The most important difference among the models lies in where the randomness is introduced (Pola et al., 2001). Recently a few monographs are appeared which are devoted to Markov jump systems (Costa et al., 2005) and (Boukas, 2006). The monographs describe the processes that are subject to uncertain changes in their dynamics. Such kinds of systems can be described with Markov jump processes.

In this paper we will deal with stochastic stability of switched systems. Such problem for the systems in usual sense is covered in (Kozin, 1969), (Kushner, 1967) and (Hasminskii, 1980).

In the area of stochastic switched systems the important result is presented in (Chatterjee & Liberzon, 2004). In this paper is considered switched systems perturbed by a Wiener

process. Using multiple Lyapunov like functions the global asymptotic stability in probability is proved. In (Battilotti & De Santis, 2005) the novel notion of stochastic stability is introduced which guarantees a given probability that the trajectories of the system hit some target set in finite time and remain thereafter.

In this paper we find a set of conditions under which the stochastic switching system is exponentially m -stable. We use multiple Lyapunov function approach. The finite set of models is nonlinear stochastic systems. It is important to mention that the exponentially stable equilibrium is relevant for practice. Namely, such systems are robust to perturbations. After the main result, using Holder and generalized Chebyshev inequalities, it is proved, as a consequences of our result, that stochastic switched system is exponentially m_1 -stable for $\forall m_1 \in (0, m)$ and, also, is stable in probability.

2. Practical stochastic hybrid systems

The switching stochastic hybrid control is important tool for large class of real problems. We will briefly describe a few of such problems.

In (Hespanaha, 2005) is proposed the model for stochastic systems where transition between discrete models are triggered by stochastic events like transitions between states a continuous-time Markov chains. The rate at which transitions occur is allowed to depend bouth on the continuous and the discrete states of stochastic hybrid systems. Theory is applied for construction of stochastic models for on-off transmission control protocol (TCP) flows that considers both the congestion avoidance and slow-start modes.

In (Oh & Sastry, 2007) the algorithm for estimating states of a distributed networked system (DNCS) is described. A DCNS is extension of networking control systems (NCS) to model a distributed multi-agent system such as the Vicsek model where multiple agents communicate over a lassy communication channel. The best examples of such system include ad hoc wireless sensor networks and the network of mobile agents. The discrete time linear dynamic model of the DNCS with lossy links is the stochastic hybrid model.

Reference (Glower & Ligeros, 2004) describes the model for multiple flights from the point of view of an air traffic controller. The proposed model is multi-agent, hybrid and stochastic. It consists of many instances of flights, each with different aircraft dynamics, flight plan and flight management system. The different flights are canpled trough the effect of the mind which is modeled as a random field.

Hybrid control, also, has the application in industrial processes. Namely, in the design of PID control systems there is often a choice between fast controllers giving a large overshoot. With hybrid controller very fast step response could be combined with good steady state regulation. The controller consists of a PID controller, a time-optimal controller and a selector. The stochastic hybrid control is need for basic weight regulation in pulp and paper processes (Astrom, 2006)

The solar energy plant (Costa et al., 2005) is another example of stochastic hybrid systems. It consists of a set of adjustable mirrors, capable of focusing sunlight on a tower that contains a boiler, trough which flows water. The power transferred to the boiler depends on the atmospheric conditions. Namely, whether it is sunny or cloudy day. With clear skies the boiler receives more solar energy and the water flow is greater than on cloudy conditions. It means that process dynamics is different for each of these conditions.

In (Filipovic, 2007) the problem of robust control of constrained linear dynamic system in the presence of a communication network with queues is considered. The communication network is between the process output and controller. It is assumed that the queue is at the sensor. The closed-loop system may face the problem of induced random delays caused by the communication network and that delay would deteriorate the system performance as well as stability. The described system is modeled as discrete – time jump linear systems with transition jumps being modeled as finite state Markov chains. Reference (Filipovic, 2008) describes the robustness of piecewise linear LQ control with prescribed degree of stability by using switching, low-and-high gain and over-saturation. It is shown that a robust controller with allowed over-saturation can exponentially stabilize linear uncertain system with prescribed exponential rate.

3. Models for hybrid systems and their importance

In this part of the chapter we will review some fundamental definitions for hybrid systems. First we will define deterministic hybrid systems (Abate, 2007).

Definition 1. A deterministic hybrid system is a collection

$$N^* = (Q, E, D, \Gamma, A, R)$$

where

- (i) $Q = \{q_1, q_2, \dots, q_m\}$ is a finite set of discrete modes.
- (ii) $E = \{e_{i,j}, (i, j) \in Q\} \subseteq Q \times Q$ is a set of edges, each of which is indexed by a pair of modes. In the edge $e_{i,j}$, $i = s(e)$ is its source and $j = t(e)$ its target.
- (iii) $D = \{D_1, D_2, \dots, D_m\}$ is a set of domains each of which is associated with a mode. Suppose that $D_q \subseteq \mathbb{R}^n$, $n, < \infty$, $\forall q \in Q$. The hybrid state space is
- (iv) $S = \bigcup_{q \in Q} q \times D_q$
- (v) $A = \{a_q, q \in Q\}$, $a_q : Q \times D \rightarrow D$ is the set of the vector fields which are assumed to be Lipschitz. Each vector field characterizes the continuous dynamics in the corresponding domain.
- (vi) $\Gamma = \{\gamma_{i,j} \subset D_i\} \subset S$, $j \neq i \in Q$ is the guards set. They represent boundary conditions and are associated with an edge
- (vii) $\forall i, j \in Q : \gamma_{i,j} \in \Gamma, \exists e_{i,j} \in E$

- (viii) $R : Q \times Q \times D \rightarrow D$ is a reset function associated with each element in Γ . With the point $s = (i, x) \in \gamma_{i,j}$ is associated a reset function $R(j, (i, x))$

The initial conditions will be taken from the set of hybrid values $Init \subseteq S$ ■

Hybrid systems have a problem known as the Zeno dynamics. Such behaviours happen when, in finite time interval, the hybrid trajectory jumps between specific domains infinitely many times. A hybrid system N^* is Zeno if for some execution

$$s(t), t \in \tau : \tau \rightarrow S \quad (1)$$

of N^* there exists a finite constant t_∞ such that

$$\lim_{i \rightarrow \infty} t_i = \sum_{i=0}^{\infty} (t_{i+1} - t_i) = t_\infty \quad (2)$$

We can distinguish two qualitatively different types of Zeno behaviour. For an execution $s(t), t \in \tau$ that is Zeno, $s(t), t \in \tau$ is

- (i) chattering Zeno (if there exists a finite constant C such that $t_{i+1} - t_i = 0$ for $\forall i \geq C$)
- (ii) genuinely Zeno (if $\forall i \in N, \exists k > 0 : t_{i+k+1} - t_{i+k} > 0$)

It is very important to make difference between the hybrid systems and switching systems (this kind of systems will be considered in this paper). The hybrid systems specify possible event conditions in terms of the variable of the model by introducing a guard set. The event times are then specified on the single trajectory and the sequence of these times varies depending on the single initial condition. The switching systems are characterized by event conditions that are a priori defined through a sequence of jumping times $\{t_k\}_{k \in N}$.

The hybrid models are more complicated than switched models.

To prove properties of a hybrid system which is a simulation of it and contains all of its behaviors. Properties of hybrid systems are then proved on the simulation and translated back to the original hybrid model.

Our final target is introduction of stochastic hybrid systems (SHS). In deterministic model N^* we will introduce probabilistic terms. An important work which has influenced the theory development for SHS is (Davis, 1993). In that reference the piecewise deterministic Markov processes are introduced.

The work (Gosh et al., 1997) has considered the optimal control for switching diffusions. This model describes the evolution of a process depending on a set of stochastic differential equations among which the process jumps according to state - dependent transitions intensities. The detailed treatment of hybrid switching diffusions is published recently (Yin & Zhu, 2010). Control of linear discrete-time stochastic systems subject both to multiplicative white noise and to Markovian jumping is considered in (Dragan et al., 2010).

Engineering applications include communication, fault detection and isolation, stochastic filtering, finance and so on.

Now we will introduce the general stochastic hybrid model according with reference (Bujorianu & Lygeros, 2006)

Definition 2: A general stochastic hybrid model is a collection

$$S_{SH} = (Q, n, A, B, W, \Lambda, \Gamma, R^A, R^\Gamma)$$

where

- (i) $Q = \{q_1, q_2, \dots, q_m\}$, $m \in \mathbb{N}$ is a set of discrete modes.
- (ii) $n : Q \rightarrow \mathbb{N}$ is the dimension of the domain associated with each mode. For $q \in Q$ the domain D_q is the Euclidean space $\mathbb{R}^{n(q)}$. The hybrid state space is

$$S = \bigcup_{q \in Q} \{q\} \times D_q$$

- (iii) The drift term in the continuous dynamics is

$$A = \{a_q, q \in Q\}, \quad a_q : D_q \rightarrow D_q$$

- (iv) The $n(q)$ - dimensional diffusion term in the continuous dynamics is

$$B = \{b_q, q \in Q\}, \quad b_q : D_q \rightarrow D_q \times D_q$$

- (v) The $n(q)$ - dimensional standard Wiener process is

$$W = \{w_q, q \in Q\}$$

- (vi) The transition intensity function is

$$\Lambda : S \times Q \rightarrow \mathbb{R}^+$$

$$j \neq i \in Q, \quad \lambda(s = (i, x), j) = \lambda_{i,j}(x)$$

- (vii) The reset stochastic kernel is

$$R^\Lambda : B(\mathbb{R}^{n(\cdot)}) \times Q \times S \rightarrow [0, 1]$$

- (viii) The closed guard set of the each of the domain is

$$\Gamma = \left\{ \bigcup_{\substack{j \neq i \\ (i,j) \in Q}} \gamma_{ij} \subset D_i \right\} \subset S$$

where γ_{ij} describes the jump events.

(ix) The reset stochastic kernel associated with point $s = (i, x) \in \gamma_{i,j}$ is

$$R^\Gamma : B(\mathbb{R}^{n(\cdot)}) \times Q \times S \rightarrow [0,1]$$

The γ_{ij} describes the reset probabilities associated with the elements in Γ .

The initial condition for the stochastic solution of the general stochastic hybrid model can be given from an initial probability distribution

$$\tilde{u} : B(S) \rightarrow [0,1] \quad \blacksquare$$

The important result (Brockett, 1983) shows that, even local asymptotic stabilization by continuous feedback, is impossible for some systems.

Result 1 (Brockett, 1983). Consider the following continuous system

$$\dot{x} = f(x, u) \quad , \quad x \in \mathbb{R}^n \quad , \quad u \in \mathbb{R}^m$$

and supposed that there exists a continuous feedback law

$$u = k(x) \quad , \quad k(0) = 0$$

which makes the origin a locally asymptotically stable equilibrium of the closed-loop system

$$\dot{x} = f(x, k(x))$$

Then the image of every neighborhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ under the map

$$(x, u) \rightarrow f(x, u)$$

contains some neighborhood of zero in \mathbb{R}^n \blacksquare

Above result provides a necessary condition for asymptotic stabilizability by continuous feedback. It means that, starting near zero and applying small controls, we must be able to move in all directions.

Let us consider the class of nonholonomic systems

$$\dot{x} = \sum_{i=1}^m g_i(x)u_i = G(x)u \quad (3)$$

where

$$x \in R^n, \quad u \in R^m \quad \text{and} \quad G \in R^{n \times m} \quad (4)$$

Nonholonomy means that the system is subject to constraints involving both the state x (position) and its derivative \dot{x} (velocity). Under the assumption that

$$\text{rank } G(0) = m < n \quad (5)$$

the above nonholonomic system violates Brockett condition. For that case we have following result (Liberzon, 2003).

Result 2. The nonholonomic system (3), for which is satisfied condition (5), cannot be asymptotically stabilized by a continuous feedback law ■

In the singular case, when $G(x)$ drops rank at 0, the Result 2 does not hold. The full-rank assumption imposed on G is essential. Nonholonomic system satisfying this assumption are nonsingular systems.

The good example of superiority of hybrid system is the problem of parameter estimation of option pricing in quantitative finance. The Black-Scholes model

$$\frac{\partial c(t)}{\partial t} + rS(t)\frac{\partial c(t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c(t)}{\partial S^2(t)} = rc(t) \quad (6)$$

where $c(t)$ is call option, σ is volatility, r is interest rate and $S(t)$ is price of the stock at time t . Traditionally, a geometric Brownian motion (GBM) is used to capture the dynamics of the stock market by using a stochastic differential equation with deterministic expected returns and nonstochastic volatilities. It does give a reasonable good description of the market in the short time period. But, in the long run, it fails to describe the behaviors of stock price owing the nonsensitivity to random parameters changes. Because, the modification of the model is need. According with those observations the price of the stock will be described by the following stochastic hybrid model.

$$dS(t) = \mu S(t)dt + \sigma S(t)dw(t) \quad (7)$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ represent the expected rate of return and volatility of the stock price and $w(\cdot)$ is a standard one-dimensional Brownian motion. The $\alpha(\cdot) = \{\alpha(t) : t \geq 0\}$ is a finite-state Markov chain which is independent of the Brownian motion.

We will find the optimal value of σ by using stochastic optimization. For that we shall use stochastic approximation when the noise is non Gaussian (Filipovic, 2009)

$$\sigma_{n+1} = \Pi[\sigma_n - \gamma_n c_\sigma(\sigma_n) \nu(c(\sigma_n) - c_n)] \quad (8)$$

where $\psi(\cdot)$ is Huber function

$$\psi(x) = \min(|x|, k) \operatorname{sgn} x, \quad 1 < k < 3 \quad (9)$$

and given by minimization of the following function

$$H(x) = -\log p^*(x) \quad (10)$$

where $p^*(\cdot)$ is the least favorable density for a priori known class of distribution to which stochastic noise belongs.

In relation (12) $c_\sigma(\cdot)$ denotes the derivative of $c(\cdot)$ and $\Pi = \Pi_{[0, M]}$, $M > 0$ is a projection operator given by

$$\Pi(\sigma) = \begin{cases} 0, & \text{if } \sigma < 0 \\ M, & \text{if } \sigma > M \\ \sigma, & \text{if otherwise} \end{cases} \quad (11)$$

The $\{\gamma_n\}$ is a sequence which satisfies following conditions

$$\lim_{n \rightarrow \infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty, \quad \sum_{n=1}^{\infty} \gamma_n^2 < \infty \quad (12)$$

The above scheme (hybrid stochastic model + stochastic approximation) predicts more accurate option price than traditional Black-Scholes model does.

4. Application of switched stochastic nonlinear systems in air traffic management

Switched systems have been studied dominantly in the deterministic frame. On the other hand the stochastic switched systems are rather young. We have many possibilities to introduce randomness into the traditional switching systems framework. The one way is to assume that the dynamics is governed by stochastic differential equations. Another one is to make the discrete jumps random according to a Markov transitions matrix whereby the continuous dynamics is deterministic. If the transition matrix is independent of the state we have setting similar to that of Markov jump linear systems.

Here we will consider situation when the continuous part of the system is described with stochastic differential equations. Such model describes much real situations: communication networks, distributed network systems, solar energy plant, cardiac stimulators, encephalogram analyzers and air traffic control system. In the sequel we will shortly describe last kind of systems.

In the current organization of Air Traffic Management the centralized Air Traffic Control (ATC) is a complete control of the air traffic and responsible for safety (Prandini et al., 2000). The main objective of ATC is to maintain safe separation whereby minimum safe separation can vary with the density of the traffic and the region of airspace. To improve performance of ATC owing the increasing levels of traffic, research has been devoted to create tools for Conflict Detection and Conflict Resolution. In Conflict Detection one has to evaluate the possibility in the future position of aircraft while they follow their flight plans. As the model for prediction of the future position of aircraft can be used stochastic differential equations (Glower & Lygeros, 2004). On the basis of the prediction one can evaluate matrices related to safety (for example, conflict probability over a certain time horizon). For Conflict Resolution it is need to calculate suitable maneuvers to avoid a predicted conflict. A framework for such problem can be Monte Carlo Markov Chains which is based on Bayesian statistics. Here we will consider the stochastic model in the form of family of stochastic differential equations (stochastic switched systems). The switched systems are nonlinear. It is assumed that there is no jump in the state of switching instants and there is no Zero behavior, i.e. there is finite number of switches on energy bounded interval. For such system we will find a set of conditions under which the stochastic switching system is exponentially m-stable. The exponentially stable equilibrium is relevant for practice because such systems are robust in perturbation.

5. Formulation of the main problem

Let us suppose that S_1 and S_2 are subset of Euclidean space, $C[S_1, S_2]$ denotes the space of all continuous functions $f : S_1 \rightarrow S_2$ and $C^2[S_1, S_2]$ is a space functions which are twice continuously differentiable. We now introduce some functions according with (Isidori, 1999). A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\alpha(r) \rightarrow \infty$ for $r \rightarrow \infty$ the function is said to belongs to class K_∞ . A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ belongs to class KL if, for each s the mapping $\beta(r, s)$ belongs to class K with respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $t \rightarrow \infty$.

Let (Ω, F, P) is a complete probability space. We define a family of stochastic differential equations as (Oksendal, 2000)

$$dx(t) = f_p(x(t))dt + G_p(x(t))dw(t) \quad (13)$$

where $x \in R^n$ is a state of system, w is an s -dimensional normalized Wiener process defined on probability space Ω , dx is a stochastic differential of x , P is an index set, $f_p : R^n \rightarrow R^n$ and $G_p : R^n \rightarrow R^{n \times s}$ are corresponding functions. The quantities in the relation (1) ensure existence and uniqueness of stochastic differential equations (Oksendal, 2000).

For definition of switched system generated by the family (1) we will introduce a switching signal. This is a piecewise constant function.

$$\sigma : [0, \infty) \rightarrow P^S \quad (14)$$

Such a function has a finite number of discontinuities on every bounded time interval and takes constant values on every interval between two consecutive switching times. The σ is a continuous from the right everywhere

$$\sigma(t) = \lim_{\tau \rightarrow t^+} \sigma(\tau) , \quad \forall \tau > 0 \quad (15)$$

The switched system for the family (1) generated by σ is

$$dx(t) = f_{\sigma(t)}(x(t))dt + G_{\sigma(t)}(x(t))dw(t) \quad (16)$$

For the system (16) we denote the switching instants with t_i , $i = 1, 2, \dots$, $t_0 = 0$ and the sequence $\{t_i\}_{i \geq 0}$ is strictly increasing

The infinitesimal generator for every system from family (13) is (Oksendal, 2000)

$$L_p = f'_p(x) \frac{\partial}{\partial x} + \frac{1}{2} T_r G_p(x) G_p^T(x) \frac{\partial^2}{\partial x^2} \quad (17)$$

where T_r is trace of square matrix.

We now will introduce the concept of stochastic m -stability (Afanasev et al., 1989).

Definition 1. Trivial solution of equation (1) is exponentially m -stable if for some constants $(k_1, k_2) > 0$ is valid.

$$E[\|x(t)\|^m] \leq k_1 \|x(t_0)\|^m \exp\{-k_2(t-t_0)\} , \quad \exists m > 0$$

where $E[\cdot]$ is mathematical expectation.

For the every $p \in P$ also is valid next result (Afanasev et al., 1989)

Lemma 1. Let us consider the system from family (1) with index p . Suppose that exists function $V_p \in C^2[R^n, R_{\geq 0}]$ and constants $a, b, c, m > 0$. For function V_p is a valid next assumption

$$1^\circ \quad a\|x\|^m \leq V_p(x) \leq b\|x\|^m$$

$$2^\circ \quad L_p V_p(x) \leq -c\|x\|^m$$

Then, for fixed p , the stochastic system (1) is exponentially m -stable ■

6. Exponential stability of switched systems

Now we will formulate the main result of this paper.

Theorem 1. Let us suppose that for system (4) is satisfied

1° $x(0)$ is deterministic quantity

2° index set P^S is finite, i.e.

$$P^S = \{1, 2, \dots, N\}$$

3° function $V_p \in C^2[R^n, R_{\geq 0}]$ for $\forall p \in P^S$

4° function $U \in KL$ where functions in KL has the form $kre^{-\gamma s}$, $k > 0$, $\gamma > 0$

5° for $\forall x \in R^n$, $\forall m \in (0, \infty)$, $\forall p \in P^S$ and $a, b > 0$

$$a\|x(t)\|^m \leq V_p(x(t)) \leq b\|x(t)\|^m$$

6° for $\forall x \in R^n$, $\forall m \in (0, \infty)$, $\forall p \in P^S$ and $c > 0$

$$L_p V_p(x(t)) \leq -c\|x(t)\|^m$$

7° for $\forall p \in P^S$ and pair of switching instants

$$\sigma(t_i) = \sigma(t_j) = p, \quad \forall (t_i, t_j), \quad i < j \text{ and}$$

$$\sigma(t_n) \neq p \text{ for } i < k < j$$

the inequality

$$E[V_p(x(t_j))] - E[V_p(x(t_i))] \leq -E[U(\|x(t_i)\|)]$$

is satisfied.

Then the system (4) is exponentially m -stable ■

Proof: Let us consider subsystem (13) for fixed p . From probabilistic interpretation of infinitesimal generator we have

$$E[V_p(x(t)) - V_p(x(t_0))] = E \int_{t_0}^t L_p V_p(x(\tau)) d\tau \quad (18)$$

By differentiation of both sides of relation (18) and using conditions 5° and 6° of theorem, we have

$$\frac{d}{dt} E[V_p(x(t))] = E[L_p V_p(x(t))] \leq -c \|x(t)\|^m \leq -\frac{c}{b} E[V_p(x(t))] \quad (19)$$

From last relation we have

$$E[V_p(x(t))] \leq E[V_p(x(t_0))] \exp\left\{-\frac{c}{b}(t-t_0)\right\} \quad (20)$$

Using conditions 1° and 5° of theorem follows

$$E[V_p(x(t))] \leq b \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t-t_0)\right\} \quad (21)$$

Let is, now, consider the interval $[t_0, t_1[$. From relation (21) and assumption 5° and 6° of theorem one can get

$$E[V_{\sigma(t_0)}(x(t_1))] \leq E[V_{\sigma(t_0)}(x(t_0))] \exp\left\{-\frac{c}{b}(t_1-t_0)\right\} \leq b \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_1-t_0)\right\} \quad (22)$$

But, over the same interval, for $p \neq \sigma(t_0)$ using assumption 5° of theorem, the estimate

$$E[V_p(x(t_1))] \leq E[b \|x(t_1)\|^m] = \frac{b}{a} E[a \|x(t_1)\|^m] \leq \frac{b}{a} E[V_p(x(t_1))] \quad (23)$$

holds true. Using last two equations follows

$$E[V_p(x(t_1))] \leq b \left(\frac{b}{a}\right) \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_1-t_0)\right\} \quad (24)$$

Now, consider the interval $[t_1, t_2[$. From (20) and (24) we have

$$\begin{aligned} E[V_{\sigma(t_1)}(x(t_2))] &\leq E[V_{\sigma(t_1)}(x(t_1))] \exp\left\{-\frac{c}{b}(t_2-t_1)\right\} \leq \\ &b \left(\frac{b}{a}\right) \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_1-t_0)\right\} \cdot \exp\left\{-\frac{c}{b}(t_2-t_1)\right\} = \\ &= b \left(\frac{b}{a}\right) \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_2-t_0)\right\} \end{aligned} \quad (25)$$

Over the same interval $p \neq \sigma(t_1)$ we have

$$E[V_p(x(t_2))] \leq E[b\|x(t_2)\|^m] = \frac{b}{a} E[a\|x(t_2)\|^m] \leq \frac{b}{a} E[V_{\sigma(t_1)}(x(t_2))] \quad (26)$$

From (25) and (26) follows

$$E[V_p(x(t_2))] \leq b\left(\frac{b}{a}\right)^2 \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_2 - t_0)\right\} \quad (27)$$

The maximum possible value of the function V_σ occurs when the switching signal σ takes the every element from finite set P^S . Let us suppose that t_j^* is the first switching instant after all subsystems have become active at least once since initialization at t_0 .

From the above analysis follows

$$E[V_p(x(t_j^*))] \leq b\left(\frac{b}{a}\right)^{N-1} \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t_j^* - t_0)\right\} \quad (28)$$

Let us define the

$$\begin{aligned} \gamma = \max & \left(b, b\left(\frac{b}{a}\right) \exp\left\{-\frac{c}{b}(t_1 - t_0)\right\} \right), \dots, \\ & \dots b\left(\frac{b}{a}\right)^{N-1} \exp\left\{-\frac{c}{b}(t_j^* - t_0)\right\} \end{aligned} \quad (29)$$

From condition 7° of theorem it is possible to conclude

$$E[V_{\sigma(t)}(x(t))] \leq \gamma \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t - t_0)\right\} \quad (30)$$

If a σ become constant at same index (the case of switching stops in finite times) the switched system, according with condition 5° and 6° of theorem and Lemma 1, is exponentially m-stable.

The second situation is that exists at least one index $p \in P^S$ such that positive subsequence

$$E[V_{\sigma(t_i)}(x(t_i))] \quad i \geq 0, \quad \sigma(t_i) = p \quad (31)$$

is infinite in length and exponentially fast decreasing according with hypothesis 7° of theorem.

From (30) and (31) it is possible to conclude that for $\exists p \in P^S$

$$E[V_p(x(t))] \leq \gamma \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t-t_0)\right\} \quad (32)$$

Using condition 5° of theorem we, finally, have

$$E[\|x(t)\|^m] \leq \frac{\gamma}{a} \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t-t_0)\right\} \quad (33)$$

Theorem is proved ■

Corollary 1. If solution of system (4) is exponentially m-stable then for $\forall m_1, m_1 \in (0, m)$ the solution is m_1 -stable ■

Proof: Using Holder inequality we have

$$E[\|x(t)\|^{m_1}] \leq \left[E[\|x(t)\|^m]\right]^{\frac{m_1}{m}} \quad (34)$$

From (33) and (34) follows

$$E[\|x(t)\|^{m_1}] \leq \gamma^{\frac{m_1}{m}} \|x(t_0)\|^{m_1} \exp\left\{-\frac{cm_1}{bm}(t-t_0)\right\} \quad (35)$$

Corollary is proved ■

Now we will show that from results of Theorem 1 follows stability in probability. For that to us need generalized Chebyshev's inequality (Gihman et al., 1988).

Generalized Chebyshev's inequality: Let us suppose that $g(x)$ nonnegative and nondecreasing function on set of random variables ξ and $E[g(\xi)]$ exists. Then for $\forall \varepsilon > 0$

$$P[\xi > \varepsilon] < \frac{E[g(\xi)]}{g(\varepsilon)} \quad \blacksquare$$

Corollary 2. If solution of (4) is exponentially m-stable then, also, is stable in probability ■

Proof: From generalized Chebyshev's inequality follows that for function $g(x) = \|x\|^m$ ($m > 0$)

$$P[\|x(t)\| > \varepsilon] \leq \frac{E[\|x(t)\|^m]}{\varepsilon^m} \quad (36)$$

Using (33) and (36) we have

$$P[\|x(t)\| > \varepsilon] \leq \frac{\gamma}{\varepsilon^m a} \|x(t_0)\|^m \exp\left\{-\frac{c}{b}(t-t_0)\right\} \quad (37)$$

From last inequality we have

$$\lim_{\|x(t_0)\| \rightarrow 0} P[\|x(t)\| > \varepsilon] = 0 \quad (38)$$

Corollary is proved ■

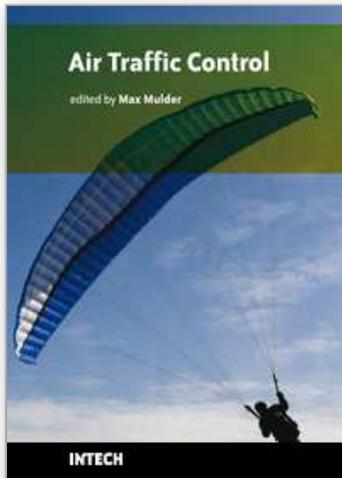
7. Conclusion

In this chapter the exponential m-stability of stochastic switched system is proved. The models, in a set of models, are nonlinear stochastic autonomous systems. For stability analysis it is used a multiple Lyapunov functions. The further possibility of investigations is consideration of stochastic switched system with control input.

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Improving air traffic control and air traffic management is currently one of the top priorities of the global research and development agenda. Massive, multi-billion euro programs like SESAR (Single European Sky ATM Research) in Europe and NextGen (Next Generation Air Transportation System) in the United States are on their way to create an air transportation system that meets the demands of the future. Air traffic control is a multi-disciplinary field that attracts the attention of many researchers, ranging from pure mathematicians to human factors specialists, and even in the legal and financial domains the optimization and control of air transport is extensively studied. This book, by no means intended to be a basic, formal introduction to the field, for which other textbooks are available, includes nine chapters that demonstrate the multi-disciplinary character of the air traffic control domain.

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