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Reduced-Order LQG Controller Design by Minimizing Information Loss*

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Introduction

The problem of controller reduction plays an important role in control theory and has attracted lots of attentions[1-10] in the fields of control theory and application. As noted by Anderson and Liu[2], controller reduction could be done by either direct or indirect methods. In direct methods, designers first constrain the order of the controller and then seek for the suitable gains via optimization. On the other hand, indirect methods include two reduction methodologies: one is firstly to reduce the plant model, and then design the LQG controller based on this model; the other is to find the optimal LQG controller for the full-order model, and then get a reduced-order controller by controller reduction methods. Examples of direct methods include optimal projection theory[3-4] and the parameter optimization approach[5] . Examples of indirect methods include LQG balanced realization[6-8], stable factorization[9] and canonical interactions[10].

In the past, several model reduction methods based on the information theoretic measures were proposed, such as model reduction method based on minimal K-L information distance[11], minimal information loss method(MIL)[12] and minimal information loss based on cross-Gramian matrix(CGMIL)[13]. In this paper, we focus on the controller reduction method based on information theoretic principle. We extend the MIL and CGMIL model reduction methods to the problem of LQG controller reduction.

The proposed controller reduction methods will be introduced in the continuous-time case. Though, they are applicable for both of continuous- and discrete-time systems.

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LQG Control

LQG is the most fundamental and widely used optimal control method in control theory. It concerns uncertain linear systems disturbed by additive white noise. LQG compensator is an optimal full-order regulator based on the evaluation states from Kalman filter. The LQG control method can be regarded as the combination of the Kalman filter gain and the optimal control gain based on the separation principle, which guarantees the separated components could be designed and computed independently. In addition, the resulting closed-loop is (under mild conditions) asymptotically stable[14]. The above attractive properties lead to the popularity of LQG design.
The LQG optimal closed-loop system is shown in Fig. 1

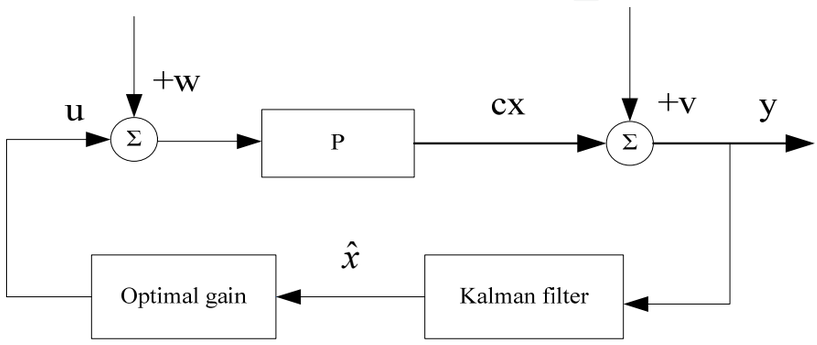


Fig. 1. LQG optimal closed-loop system

Consider the *n*th-order plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + w(t)), x(t_0) = x_0 \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $w(t) \in R^m$, $y(t), v(t) \in R^p$. A, B, C are constant matrices with appropriate dimensions. $w(t)$ and $v(t)$ are mutually independent zero-mean white Gaussian random vectors with covariance matrices Q and R ,respectively, and uncorrelated with x0. The performance index is given by

$$J = \lim_{t \to \infty} E\left\{ x^T R_1 x + u^T R_2 u \right\}, R_1 \geq 0, R_2 \geq 0. \tag{2}$$

While in the latter part, the optimal control law u would be replaced with the reduced-order suboptimal control law, such as u_r and u_G .

The optimal controller is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) = (A - BK - LC)\hat{x} + Ly, \tag{3}$$

$$u = -K\hat{x}. \tag{4}$$

where L and K are Kalman filter gain and optimal control gain derived by two Riccati equations, respectively.

Model Reduction via Minimal Information Loss Method (MIL)[12]

Different from minimal K-L information distance method, which minimizes the information distance between outputs of the full-order model and reduced-order model, the basic idea of MIL is to minimize the state information loss caused by eliminating the state variables with the least contributions to system dynamics.
Consider the *n*-order plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), x(t_0) = x_0 \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{5}$$

where $x(t) \in R^n$, $w(t) \in R^m$, $y(t), v(t) \in R^p$. A, B, C are constant matrices with appropriate dimensions. $w(t)$ and $v(t)$ are mutually independent zero-mean white Gaussian random vectors with covariance matrices Q and R ,respectively, and uncorrelated with x0.
To approximate system (5), we try to find a reduced-order plant

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r w(t), x(t_0) = x_0 \\ y(t) &= C_r x_r(t) + v(t), \end{aligned} \tag{6}$$

where $x_r(t) \in R^l, l < n$, $y_r(t) \in R^p$, A_r, B_r, C_r are constant matrices.
Define

$$x_r(t) = \Lambda x(t), \tag{7}$$

where $x_r(t)$ is the aggregation state vector of $x(t)$ and $\Lambda \in R^{l \times n}$ is the aggregation matrix. From (5), (6) and (7), we obtain

$$A_r = \Lambda A \Lambda^+, B_r = \Lambda B, C_r = C\Lambda^+. \tag{8}$$

In information theory, the information of a stochastic variable is measured by the entropy function[15]. The steady-state entropy of system (5) and (6) are

$$H(x) = \frac{n}{2}\ln(2\pi e) + \frac{1}{2}\ln\det\Pi, \tag{9}$$

$$H(x_r) = \frac{l}{2}\ln(2\pi e) + \frac{1}{2}\ln\det\Pi_r. \tag{10}$$

where

$$\Pi_r = \Lambda\Pi\Lambda^+ \tag{11}$$

The steady-state information loss from (5) and (6) is defined by

$$IL(x; x_r) = H(x) - H(x_r). \tag{12}$$

From (11), (12) can be transformed to

$$H(x) - H(x_r) = \frac{n-l}{2}\ln(2\pi e) + \frac{1}{2}\ln\det(\Pi - \Lambda\Pi\Lambda^+). \tag{13}$$

The aggregation matrix Λ minimizing (13) consists of l eigenvectors corresponding to the l largest eigenvalues of the steady-state covariance matrix Π.

MIL-RCRP: Reduced-order Controller Based-on Reduced-order Plant Model

The basic idea of this method is firstly to find a reduced-order model of the plant, then design the suboptimal LQG controller according to the reduced-order model.
We have obtained the reduced-order model as (6). The LQG controller of the reduced-order model is given by

$$\dot{\hat{x}}_{r1} = A_{c1}\hat{x}_{r1} + B_{c1}y, \tag{14}$$

$$u_{r1} = C_{c1}\hat{x}_{r1}, \tag{15}$$

where $A_{c1} = A_{r1} - B_{r1}K_{r1} - L_{r1}C_{r1}$, $B_{c1} = L_{r1}$, $C_{c1} = -K_{r1}$ .The l-order suboptimal filter gain L_{r1} and suboptimal control gain K_{r1} are given by

$$L_{r1} = S_{r1}(\Lambda_r^+)^T C^T V^{-1},\ K_{r1} = -R^{-1}\Lambda^T B^T P_{r1}, \tag{16}$$

where S_{r1} and P_{r1} are respectively the non-negative definite solutions to two certain Riccati equations as following:

$$P_{r1}A_{r1} + A_{r1}{}^T P_{r1} - P_{r1}B_{r1}R^{-1}B_{r1}{}^T P_{r1} + Q = 0, \tag{17}$$

$$A_{r1}S_{r1} + S_{r1}A_{r1}{}^T - S_{r1}C_{r1}{}^T V^{-1}C_{r1}S_{r1} + W = 0. \tag{18}$$

The stability of the closed-loop system is not guaranteed and must be verified.

MIL-RCFP: Reduced-order Controller Based on Full-order Plant Model

In this method , the basic idea is first to find a full-order LQG controller based on the full-order plant model, then get the reduced-order controller by minimizing the information loss between the states of the closed-loop systems with full-order and reduced-order controllers.

The full-order LQG controller is given by as (3) and (4). Then we use MIL method to obtain the reduced-order controller, which approximates the full-order controller.

The *l*-order Kalman filter is given by

$$\dot{\hat{x}}_{r2} = A_{c2}\hat{x}_{r2} + B_{c2}y, \tag{19}$$

where $A_{c2} = \Lambda_c A \Lambda_c^{+} - \Lambda_c B K \Lambda_c^{+} - \Lambda_c L C \Lambda_c^{+}$, $B_{c2} = L_{r2} = \Lambda_c L = \Lambda_c S C^T V^{-1}$.

And the *l*-order control gain is given by

$$u_{r2} = C_{c2}\hat{x}_{r2}, \tag{20}$$

where $C_{c2} = -K_{r2} = -K\Lambda_c^{+} = -R^{-1}B^T P \Lambda_c^{+}$. Λ_c is the aggregation matrix consists of the *l* eigenvectors corresponding to the *l* largest eigenvalues of the steady-state covariance matrix of the full-order LQG controller.

In what follows, we will propose an alternative approach, the CGMIL method, to the LQG controller-reduction problem. This method is based on the information theoretic properties of the system cross-Gramian matrix[16]. The steady-state entropy function corresponding to the cross-Gramian matrix is used to measure the information loss of the plant system. The two controller-reduction methods based on CGMIL, called CGMIL-RCRP and CGMIL-RCFP, respectively, possess the similar manner as MIL controller reduction methods.

Model Reduction via Minimal Cross-Gramian Information Loss Method (CGMIL)[16]

In the viewpoint of information theory, the steady state information of (5) can be measured by the entropy function $H(x)$, which is defined by the steady-state covariance matrix Π. Let $\tilde{\Pi}$ denote the steady-state covariance matrix of the state $\tilde{x}$ of the dual system of (5). When Q, the covariance matrix of the zero-mean white Gaussian random noise $w(t)$ is unit matrix I, Π and $\tilde{\Pi}$ are the unique definite solutions to

$$\begin{aligned} A\Pi + \Pi A^T + BB^T &= 0, \\ A^T\tilde{\Pi} + \tilde{\Pi}A + C^T C &= 0, \end{aligned} \tag{21}$$

respectively.

From Linear system theory, the controllability matrix and observability matrix satisfy the following Lyapunov equation respectively:

$$\begin{aligned} AW_C + W_C A^T + BB^T &= 0 \\ A^T W_O + W_O A + C^T C &= 0. \end{aligned} \tag{22}$$

By comparing the above equations, we observe that the steady-state covariance matrix is equal to the controllability matrix of (5), and the steady-state covariance matrix of the dual system is equal to the observability matrix. We called $H(x)$ and $H(\tilde{x})$ the "controllability information" and "observability information", respectively. In MIL method, only "controllability information" is involved in deriving the reduced-order model, while the "observability information" is not considered.

In order to improve MIL model reduction method, CGMIL model reduction method was proposed in [13]. By analyzing the information theoretic description of the system, a definition of system "cross-Gramian information" (CGI) was defined based on the information properties of the system cross-Gramian matrix. This matrix indicates the "controllability information" and "observability information" comprehensively.

Fernando and Nicholson first define the cross-Gramian matrix by the step response of the controllability system and observability system. The cross-Gramian matrix of the system is defined by the following equation:

$$\boldsymbol{G}_{\text{cross}} = \int_0^\infty (\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{b})(\mathrm{e}^{\boldsymbol{A}^{\mathrm{T}}t}\boldsymbol{c}^{\mathrm{T}})^{\mathrm{T}}\,dt = \int_0^\infty \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{b}\boldsymbol{c}\mathrm{e}^{\boldsymbol{A}t}\,dt, \tag{23}$$

which satisfies the following Sylvester equation:

$$\boldsymbol{A}\mathbf{G}_{\text{cross}} + \mathbf{G}_{\text{cross}}\boldsymbol{A} + \boldsymbol{b}\boldsymbol{c} = 0. \tag{24}$$

From [16], the cross-Gramian matrix satisfies the relationship between the controllability matrix and the observability matrix as the following equation:

$$\boldsymbol{G}_{\text{cross}}^2 = W_C W_O. \tag{25}$$

As we know that, the controllability matrix W_C corresponds to the steady-state covariance matrix of the system, while the observability matrix W_O corresponds to the steady-state covariance matrix of the dual system, which satisfy the following equations:

$$\boldsymbol{W}_C = \lim_{t\to\infty} E\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{T}}(t)\}, \tag{26}$$

$$\boldsymbol{W}_O = \lim_{t\to\infty} E\{\tilde{\boldsymbol{x}}(t)\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\}. \tag{27}$$

Combine equation (25)、(26) and (27), we obtain:

$$\boldsymbol{G}^2_{\text{cross}} = W_C W_O = \lim_{t\to\infty} E\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{T}}(t)\}E\{\tilde{\boldsymbol{x}}(t)\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\}. \tag{28}$$

The cross-Gramian matrix corresponds to the steady-state covariance information of the original system and the steady-state covariance information of the dual system. Here we define a new stochastic state vector $\boldsymbol{\xi}(t)$, and the relationship among $\boldsymbol{\xi}(t)$, $x(t)$ and $\tilde{x}(t)$ satisfies the following equation:

$$\begin{aligned}\lim_{t\to\infty} E\{\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathrm{T}}(t)\} &= \lim_{t\to\infty} f(\boldsymbol{x}(t),\tilde{\boldsymbol{x}}(t)) \\ &= \lim_{t\to\infty} E\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{T}}(t)\}E\{\tilde{\boldsymbol{x}}(t)\tilde{\boldsymbol{x}}^{\mathrm{T}}(t)\} = \boldsymbol{G}^2_{\text{cross}}.\end{aligned} \tag{29}$$

We called $\boldsymbol{\xi}(t)$ as "cross-Gramian stochastic state vector", which denotes the cross-Gramian information of the system.

From the above part, we know that the steady-state covariance matrix of $\boldsymbol{\xi}(t)$ is the cross-Gramian matrix $\boldsymbol{G}^2_{\text{cross}}$, the steady information entropy is called cross-Gramian information $I_{\text{cross}}(\boldsymbol{G}^2_{\text{cross}})$, which satisfies the following equation:

$$I_{\text{cross}}(\boldsymbol{G}^2_{\text{cross}}) = H(\boldsymbol{\xi}). \tag{30}$$

where $\boldsymbol{\xi}$ is the steady form of the stochastic state vector $\boldsymbol{\xi}(t)$, that is $\boldsymbol{\xi} = \lim_{t\to\infty}\boldsymbol{\xi}(t)$, and the information entropy of the steady-state $\boldsymbol{\xi}$ is defined as follows:

$$I_{\text{cross}}(\boldsymbol{G}^2_{\text{cross}}) = H(\boldsymbol{\xi}) = \frac{n}{2}\ln(2\pi\mathrm{e}) + \frac{1}{2}\ln\det\boldsymbol{G}^2_{\text{cross}}. \tag{31}$$

And the following equation can be obtained:

$$I_{\text{cross}}(\boldsymbol{G}^2_{\text{cross}}) = \frac{n}{2}\ln(2\pi\mathrm{e}) + \frac{1}{2}\ln\det\boldsymbol{PQ}. \tag{32}$$

$$I_{\text{cross}}(\boldsymbol{G}^2_{\text{cross}}) = \frac{H(\boldsymbol{x}) + H(\tilde{\boldsymbol{x}})}{2}. \tag{33}$$

From the above, we get that the cross-Gramian matrix indicates the controllability matrix and observability matrix comprehensively.

CGMIL model reduction method is suit for SISO system. The basic idea of the algorithm is

presented as follows, for continuous-time linear system.
The cross-Gramian matrix of the full-order system and the reduced-order system are as follows:

$$\boldsymbol{A}\mathbf{G}_{\text{cross}} + \mathbf{G}_{\text{cross}}\boldsymbol{A} + \boldsymbol{bc} = 0, \tag{34}$$

$$\boldsymbol{A}\mathbf{G}^{r}_{\text{cross}} + \mathbf{G}^{r}_{\text{cross}}\boldsymbol{A} + \boldsymbol{bc} = 0. \tag{35}$$

When the system input is zero mean Gaussian white noise signal, the cross-Gramian information of the two systems can be obtained as:

$$I_{\text{cross}}(\boldsymbol{G}^{2}{}_{\text{cross}}) = H(\boldsymbol{\xi}) = \frac{n}{2}\ln(2\pi\text{e}) + \frac{1}{2}\ln\det\boldsymbol{G}^{2}{}_{\text{cross}}, \tag{36}$$

$$I^{\text{r}}_{\text{cross}}(\boldsymbol{G}^{2}{}_{\text{cross}}{}^{\text{r}}) = H(\boldsymbol{\xi}_{\text{r}}) = \frac{l}{2}\ln(2\pi\text{e}) + \frac{1}{2}\ln\det\boldsymbol{G}^{2}{}_{\text{cross}}{}^{\text{r}}. \tag{37}$$

The cross-Gramian information loss is:

$$\begin{aligned}\Delta I_{\text{cross}} &= I_{\text{cross}}(\boldsymbol{G}^{2}{}_{\text{cross}}) - I^{\text{r}}_{\text{cross}}(\boldsymbol{G}^{2}{}_{\text{cross}}{}^{\text{r}}) = H(\boldsymbol{\xi}) - H(\boldsymbol{\xi}_{\text{r}}) \\ &= \frac{n-l}{2}\ln(2\pi\text{e}) + \frac{1}{2}[\ln\det\boldsymbol{G}^{2}{}_{\text{cross}} - \ln\det\boldsymbol{G}^{2}{}_{\text{cross}}{}^{\text{r}}].\end{aligned} \tag{38}$$

In order to minimize the information loss, we use the same method with the MIL method:

$$G^{2}{}_{cross}{}^{r} = \Lambda G^{2}{}_{cross}\Lambda^{+}. \tag{39}$$

where the aggregation matrix Λ is adopted as the l ortho-normal eigenvectors corresponding to the l th largest eigenvalues of the cross-Gramian matrix, then the information loss is minimized.
Theoretical analysis and simulation verification show that, cross-Gramian information is a good information description and CGMIL algorithm is better than the MIL algorithm in the performance of model reduction.

CGMIL-RCRP: Reduced-order Controller Based-on Reduced-order Plant Model By CGMIL

In this section, we apply the similar idea as method 1 of MIL model reduction to obtain the reduced-order controller.
The LQG controller of the reduced-order model consists of Kalman filter and control law as follows:

$$\dot{\hat{x}}_{GC1} = A_{GC1}\hat{x}_{GC1} + B_{GC1}y, \tag{40}$$

$$u_{G1} = C_{GC1}\hat{x}_{G1}. \tag{41}$$

where $A_{GC1} = A_{G1} - B_{G1}K_{G1} - L_{G1}C_{G1}$, $B_{GC1} = L_{G1}$, $C_{GC1} = -K_{G1}$.

The r-order filer gain and control gain are obtained:

$$L_{G1} = S_{G1}C_{G1}{}^{T}V^{-1} = S_{G1}(\Lambda_{G1}{}^{+})^{T}C^{T}V^{-1}, \tag{42}$$

$$K_{G1} = -R^{-1}B_{G1}{}^{T}P_{G1} = -R^{-1}\Lambda_{G1}{}^{T}B^{T}P_{G1}. \tag{43}$$

where S_{G1} and P_{G1} satisfy the following Riccati equations

$$P_{G1}A_{G1} + A_{G1}{}^{T}P_{G1} - P_{G1}B_{G1}R^{-1}B_{G1}{}^{T}P_{G1} + Q = 0, \tag{44}$$

$$A_{G1}S_{G1} + S_{G1}A_{G1}{}^{T} - S_{G1}C_{G1}{}^{T}V^{-1}C_{G1}S_{G1} + W = 0. \tag{45}$$

And the state space equation of the r -order closed-loop system is as follow:

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\hat{x}}_{G1}\end{bmatrix} &= \begin{bmatrix}A & BC_{GC1}\\ B_{GC1}C & A_{G1} + B_{G1}C_{GC1} - B_{GC1}C_{G1}\end{bmatrix}\begin{bmatrix}x\\ \hat{x}_{G1}\end{bmatrix} + \begin{bmatrix}w\\ L_{G1}v\end{bmatrix}\\ &= \begin{bmatrix}A & -BK_{G1}\\ L_{G1}C & A_{G1} - B_{G1}K_{G1} - L_{G1}C_{G1}\end{bmatrix}\begin{bmatrix}x\\ \hat{x}_{G1}\end{bmatrix} + \begin{bmatrix}w\\ L_{G1}v\end{bmatrix},\end{aligned} \tag{46}$$

$$y_{G1} = \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ \hat{x}_{G1}\end{bmatrix} + v. \tag{47}$$

CGMIL-RCFP: Reduced-order Controller Based on Full-order Plant Model By CGMIL

Similar to the second method of MIL controller reduction method, the reduced-order controller obtained by the full-order controller using CGMIL method is:

$$\dot{\hat{x}}_{G2} = A_{GC2}\hat{x}_{G2} + B_{GC2}y, \tag{48}$$

$$u_{G2} = C_{GC2}\hat{x}_{G2}. \tag{49}$$

where $A_{GC2} = \Lambda_{G2}A_c\Lambda_{G2}{}^{+}$, $B_{GC2} = L_{G2}$, $C_{GC2} = -K_{G2}$, Λ_{G2} is the aggregation matrix consists of the l largest eigenvalues corresponding to the l th largest eigenvectors of

the cross-Gramian matrix of the full-order controller. The r-order filter gain and control gain is obtained:

$$L_{G2} = \Lambda_{G2} L = \Lambda_{G2} S C^T V^{-1}, \tag{50}$$

$$K_{G2} = K \Lambda_{G2}{}^{+} = R^{-1} B^T P \Lambda_{G2}{}^{+}. \tag{51}$$

The state space equation of the reduced-order controller is then given by:

$$\begin{aligned} \dot{\hat{x}}_{G2} &= A_{GC2} \hat{x}_{G2} + B_{GC2} y = (\Lambda_{G2} A \Lambda_{G2}{}^{+} - \Lambda_{G2} B K \Lambda_{G2}{}^{+} - \Lambda_{G2} L C \Lambda_{G2}{}^{+}) \hat{x}_{G2} + \Lambda_{G2} L y \\ u_{G2} &= C_{GC2} \hat{x}_{G2} = -K \Lambda_{G2}{}^{+} \hat{x}_{G2}. \end{aligned} \tag{52}$$

Stability Analysis of the Reduced-Order Controller

Here we present our conclusion in the case of discrete systems.
Suppose the full-order controller is stable, and we analyze the stability of the reduced-order controller obtained by method MIL-RCFP.
Conclusion 1.1 [Lyapunov Criterion] The discrete-time time-invariant linear autonomous system, when the state $x_e = 0$ is asymptotically stable, that is the amplitude of all of the eigenvalues of G $\lambda_i(G)$ $(i = 1, 2, ..., n)$ less than 1. If and only if for any given positive definite symmetric matrix Q, the discrete-time Lyapunov equation:

$$G^T P G + Q = P, \tag{53}$$

has the uniquely positive definite symmetric matrix P.
The system parameter of the full-order controller is: $A_c = A - BK - LC$. From Lyapunov Criterion, the following equation is obtained:

$$A_c \; P A_c{}^T + Q = P. \tag{54}$$

Multiplying leftly by the aggregation matrix Λ_c and rightly by $\Lambda_c{}^T$, we get:

$$\Lambda_c A_c P (\Lambda_c A_c)^T + \Lambda_c Q \Lambda_c{}^T = \Lambda_c P \Lambda_c{}^T. \tag{55}$$

Because $\Lambda_c A_c = A_{c2} \Lambda_c$, the following equation is obtained:

$$A_{c2} \Lambda_c P \Lambda_c{}^T A_{c2} + \Lambda_c Q \Lambda_c{}^T = \Lambda_c P \Lambda_c{}^T. \tag{56}$$

When $\Lambda_c^{'} = [\Lambda_c^T, \eta_{l+1}, ..., \eta_n]^T$ is assumed, where $\eta_{l+1}, ..., \eta_n$ is the n-l smallest eigenvectors corresponding to the n-l smallest eigenvalues of the steady-state covariance matrix Π_c. The aggregation matrix $\Lambda_c^{'}$ consists of the orthogonal eigenvectors, when P and Q are positive definite matrix, $\Lambda_c^{'} P (\Lambda_c^{'})^T$ and $\Lambda_c^{'} Q (\Lambda_c^{'})^T$ are positive definite. The matrix $\Lambda_c P (\Lambda_c)^T$ consists of the first $l \times l$ main diagonal elements of matrix $\Lambda_c^{'} P (\Lambda_c^{'})^T$; similarly, the matrix $\Lambda_c Q (\Lambda_c)^T$ consists of the first $l \times l$ main diagonal elements of matrix $\Lambda_c^{'} Q (\Lambda_c^{'})^T$. If $\Lambda_c^{'} P (\Lambda_c^{'})^T$ and $\Lambda_c^{'} Q (\Lambda_c^{'})^T$ are positive definite, then $\Lambda_c P (\Lambda_c)^T$ and $\Lambda_c Q (\Lambda_c)^T$ are positive definite. As a result, the reduced-order controller obtained from method MIL-RCFP is stable.

Illustrative Example

1. Lightly Damped Beam

We applied these two controller-reduction methods to the lightly damped, simply supported beam model described in [11] as (5).
The full-order Kalman filter gain and optimal control gain are given by

$$L = [2.0843 \;\; 2.2962 \;\; 0.1416 \;\; 0.1774 \;\; -0.2229 \;\; -0.4139 \;\; -0.0239 \;\; -0.0142 \;\; 0.0112 \;\; -0.0026]^T, \tag{57}$$

$$K = [0.4143 \;\; 0.8866 \;\; 0.0054 \;\; 0.0216 \;\; -0.0309 \;\; -0.0403 \;\; 0.0016 \;\; -0.0025 \;\; -0.0016 \;\; 0.0011]. \tag{58}$$

The proposed methods are compared with that given in [11], which will be noted by method 3 later. The order of the reduced controller is 2. We apply the two CGMIL controller reduction methods and the first MIL controller reduction method (MIL-RCRP) to this model. The reduced-order Kalman filter gains and control gains of the reduced-order closed-loop systems are given as follows:

MIL-RCRP: $L_{r1} = [-1.5338; -2.6951]^T, K_{r1} = [-0.1767 \;\; -0.9624]$

CGMIL-RCRP: $L_{G1} = [-3.0996 \;\; -0.0904]^T, K_{G1} = [-0.9141 \;\; -0.3492]$

CGMIL-RCFP: $L_{G2} = [0.4731 \;\; 0.9706]^T, K_{G2} = [0.4646 \;\; -0.9785]$

Method 3: $L_{r3} = [2.1564 \;\; 2.2826]^T, K_{r3} = [0.3916 \;\; 0.8752].$

Three kinds of indices are used to illustrate the performances of the reduced-order controllers.

a) We define the output mean square errors to measure the performances of the reduced-order controllers

$$E_a^* = \int_0^T y_*^2(t)dt / T, \tag{59}$$

where $*=1,2,3$ indicates the closed-loop systems obtained from method 1,2,3, respectively. T is the simulation length.

b) We compare the reduced-order controllers with the full-order one by using relative error indices

$$E_b^* = \int_0^T (y(t) - y_*(t))^2 dt / T, \tag{60}$$

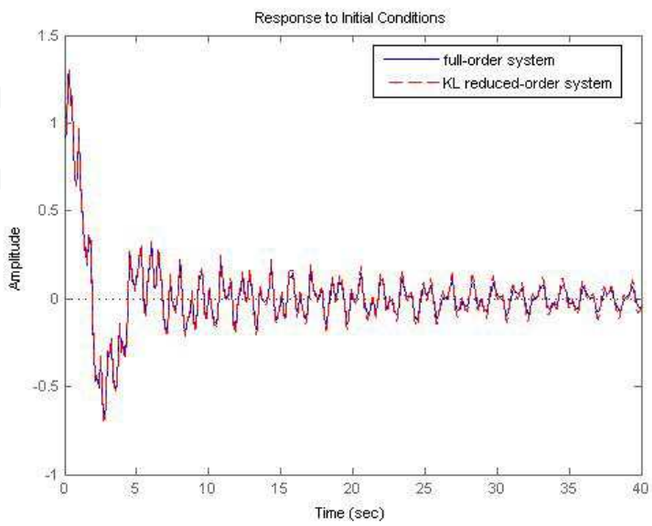
where $y(t)$ is the system output of the full-order closed-loop system.

c) We also use the LQG performance indices given by following equations, to illustrate the controller performances

$$J^* = \frac{1}{T}\int_0^T \left\{ x^T(t)Qx(t) + u_*^T(t)Ru_*(t) \right\} dt \ . \tag{61}$$

The performances of the reduced-order controllers are illustrated by simulating the responses of the zero-input and Gaussian white noise, respectively. The simulation results are shown in the following figures and diagrams.

As shown in Fig. 1 (Response to initial conditions), when input noise and observation noise are zero, the system initial states are set as $x_i(0) = 1/i, i = 1,...10$ .The reduced-order closed-loop system derived by method 3 is close to the full-order one.



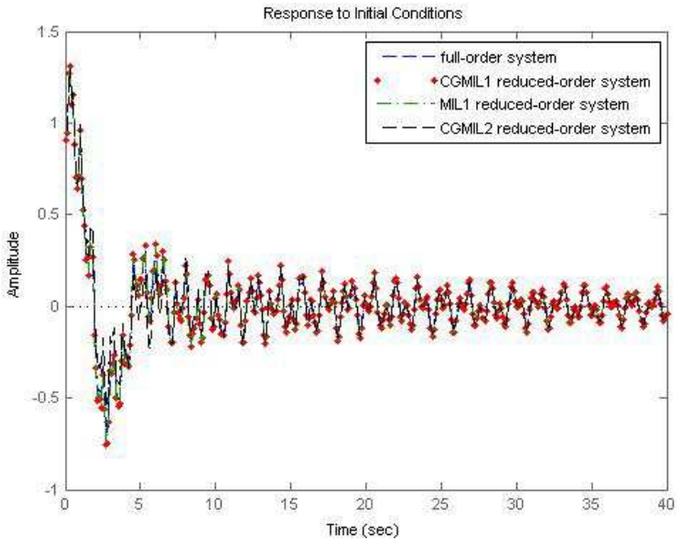


Fig. 1. Zero-input response for full-order system and reduced-order system

In Fig. 2 (Response of Gaussian white noise), almost all the reduced-order closed-loop system are close to the full-order one except the reduced-order system obtained by CGMIL 2.

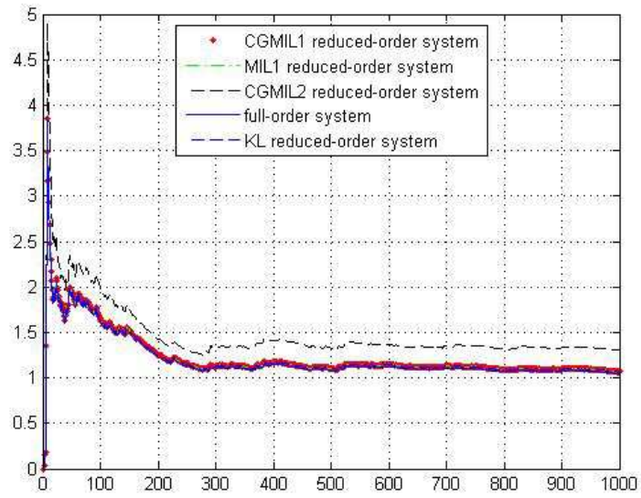


Fig. 2. Gaussian white noise response for full-order system and reduced-order system

As illustrated in Fig. 3 (Bode Plot), the reduced-order closed-loop systems obtained from method 1 and 3 are close to the full-order closed-loop system.

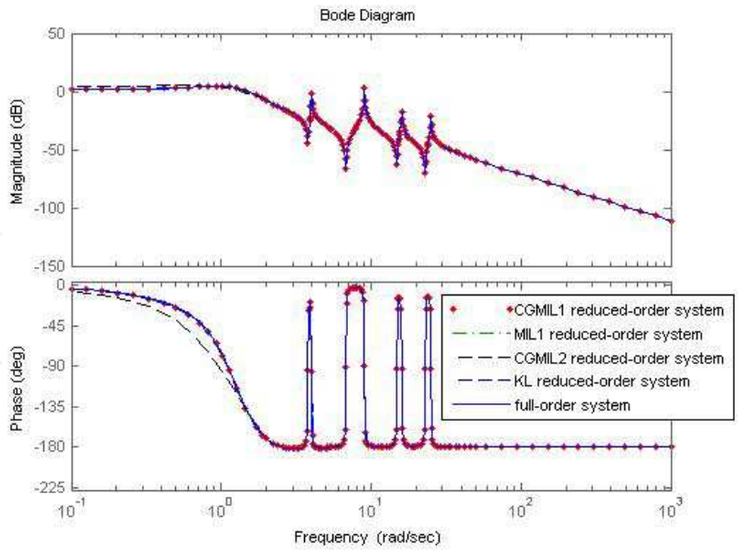


Fig. 3. Bode plots for full-order system and reduced-order system

	CGMIL-RCRP	CGMIL-RCFP	Method 3	MIL-RCRP
E_a^* of the zero-input	0.4139	0.3694	0.3963	0.4139
E_b^* of the zero-input	0.0011	0.0088	9.69e-05	0.0011
E_a^* of the Gaussian white noise	1.0867	1.2382	1.0693	1.0867
E_b^* of the Gaussian white noise	7.7550e-004	0.1367	6.88e-04	7.7550e-004
The LQG performance index J^*	12.5005	16.1723	12.5749	12.5005

Diagram.1 Performances of the reduced-order controllers

2. Deethanizer Model

Distillation column is a common operation unit in chemical industry. We apply these two MIL controller-reduction methods to a 30th-order deethanizer model.

The order of the reduced-order controller is 2. The reduced-order Kalman filter gains and control gains of the reduced-order closed-loop systems are given as follows:

MIL-RCRP: $L_{r1} = [\text{-}0.0031\ 0.0004]^T$, $K_{r1} = [\text{-}0.2289 \quad \text{-}0.1007; \text{-}0.3751 \quad \text{-}0.5665]^T$;

MIL-RCFP: $L_{r1} = [\text{-}0.0054\ \text{-}0.0082]^T$, $K_{r2} = [32.8453 \quad 2.0437; \text{-}9.4947 \quad 6.6710]^T$;

We use the same performances as example 1 to measure the reduced-order controller.

Fig. 4 (Impulse Response): When the system input is impulse signal, the reduced-order closed-loop system is close to the full-order system.

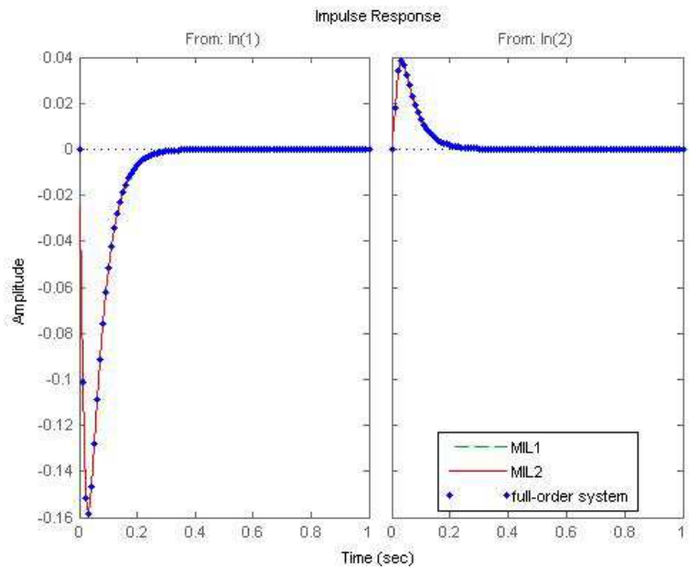


Fig. 4. Impulse response for full-order system and reduced-order system

Fig. 5 (Step Response): When the system input is step signal, the reduced-order closed-loop system is close to the full-order system.

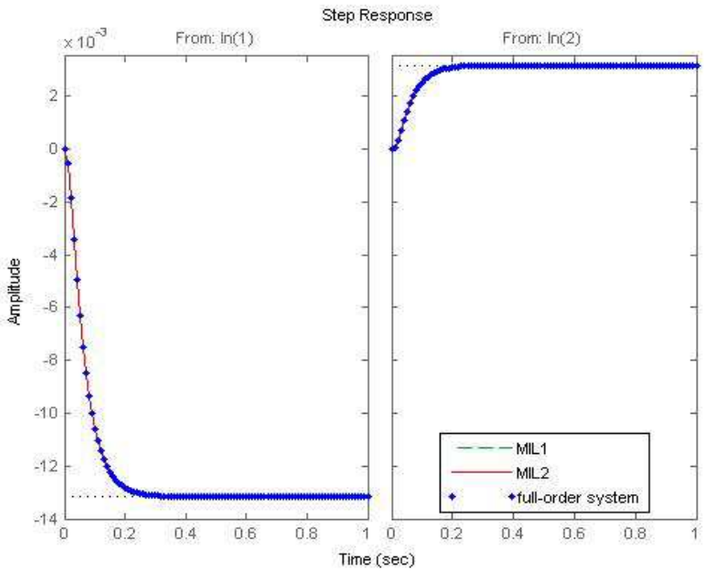


Fig. 5. Step response for full-order system and reduced-order system

Fig. 6 (Gaussian white noise Response): When the system input is Gaussian white noise, the reduced-order closed-loop system is close to the full-order system and outputs are near zero.

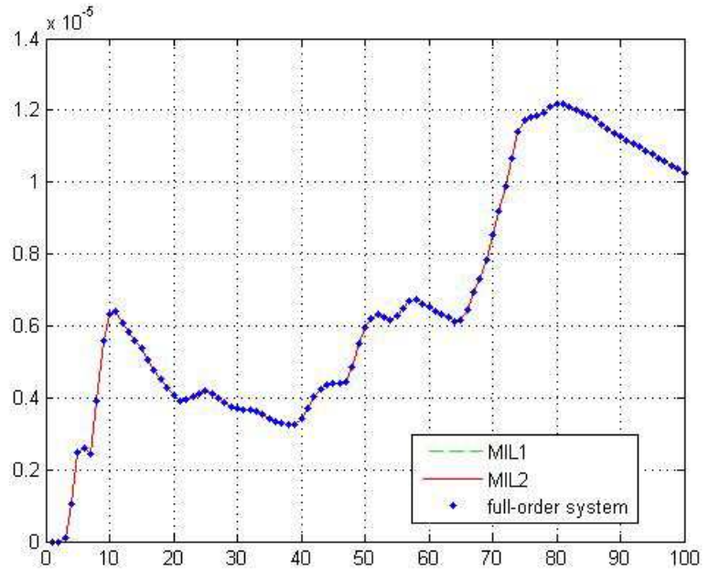


Fig. 6. Gaussian white response for full-order system and reduced-order system

Fig. 7 (Bode Plot):

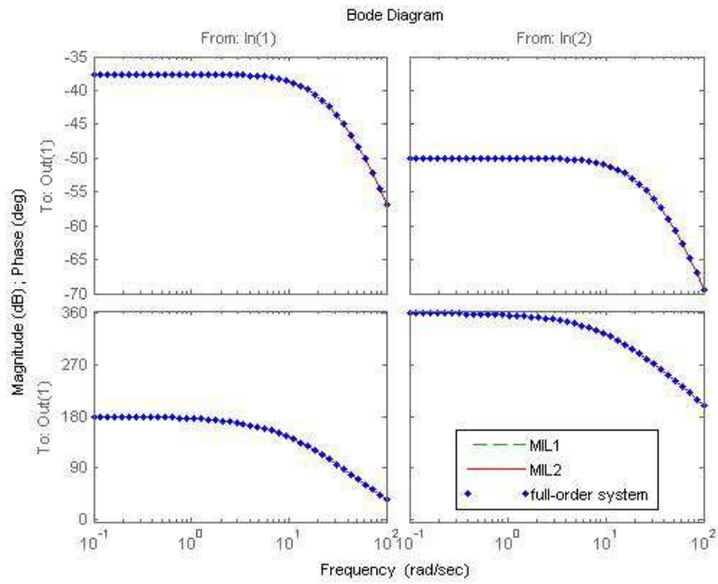


Fig. 7. Bode plots for full-order system and reduced-order system

	MIL-RCRP	MIL-RCFP	Full-order system
E_a	3.0567e-019	2.4160e-022	0
E_b	2.1658e-005	2.1658e-005	2.1658e-005
J	2.1513e-005	2.1513e-005	2.1513e-005

Diagram.2 Performances of the reduced-order controllers

Conclusion

1. This paper proposed two controller-reduction methods based on the information principle—minimal information loss(MIL). Simulation results show that the reduced-order controllers derived from the proposed two methods can approximate satisfactory performance as the full-order ones.
2. According to the conclusion of literature [17], the closed-loop system with optimal LQG controller is stable. However, its own internal stability can not be guaranteed. If the full-order controller is internal stability, the reduced-order controller is generally stable. We would modify the parameters such as the weighting matrix or noise intensity to avoid the instability of the controller.
3. The performances of the two reduced-order controllers obtained by CGMIL method approximate the full-order one satisfactorily and under certain circumstances. CGMIL method is a better information interpretation instrument of the control system relative to the MIL method, while it is only suit for single-variable stable system.

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