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Direct Design of Infinite Impulse Response Filters based on Allpole Filters

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This chapter presents a new framework to design different types of IIR filters based on the general technique for maximally flat allpole filter design. The resulting allpole filters have some desired characteristics, i.e., desired degree of flatness and group delay, and the desired phase response at any prescribed set of frequency points. Those characteristics are important to define the corresponding IIR filters. The design includes both real and complex cases.
In that way we develop a direct design method for linear-phase Butterworth-like filters, using the same specification as in traditional analog-based IIR filter design. The design includes the design of lowpass filters as well as highpass filters. The designed filters can be either real or complex. The design of liner-phase two-band filter banks is also discussed.
Additionally, we discussed the designs of some special filters such as Butterworth-like filters with improved group delay, complex wavelet filters, and fractional Hilbert transformers.
Finally, we addressed a new design of IIR filters based on three allpass filters. As a result we propose a new design of lowpass filters with a desired characteristic based on the complex allpole filters.
Closed form equations for the computation of the filter coefficients are provided. All design techniques are illustrated with examples.

1. Introduction

The design of allpole filters has been attractive in the last years due to some promising applications, like the design of allpass filters (Chan et al., 2005; Lang, 1998; Pun & Chan, 2003; Selesnick, 1999; Zhang & Iwakura, 1999), the design of orthogonal and biorthogonal IIR wavelet filters (Selesnick, 1998; Zhang et al., 2001; 2000; 2006), the design of complex wavelets (Fernandes et al., 2003), the design of half band filters (Zhang & Amaratunga, 2002), the filter bank design (Kim & Yoo, 2003; Lee & Yang, 2004; Saramaki & Bregovic, 2002), the fractional delay filter design (Laakso et al., 1996), the fractional Hilbert transform (Pei & Wang, 2002), notch filters (Joshi & Roy, 1999; Pei & Tseng, 1997; Tseng & Pei, 1998), among others. The majority of the methods use some approximation of the desired phase in the least square sense and minimax sense.

The allpole filters with maximally flat phase response characteristic have been specially attractive due to promising applications, like the design of IIR filters (Selesnick, 1999), the design

of orthogonal and biorthogonal IIR wavelet filters (Selesnick, 1998; Zhang et al., 2001; 2000; 2006), the design of complex wavelets (Fernandes et al., 2003), the design of half band filters (Zhang & Amaratunga, 2002), the fractional delay filter design (Laakso et al., 1996) and the fractional Hilbert transform design (Pei & Wang, 2002).

This chapter presents a new design of real and complex allpole filters with the given phase, group delay, and degree of flatness, at any desired set of frequency points. The main motivation of this work is to get some new promising cases related with the applications of maximally flat allpole filters. In that way, using the proposed extended allpole filter design, we introduced some new special cases.

The rest of the chapter is organized as follows. Section 2 establishes the general equations for maximally flat real and complex allpole filters. The discussion of the proposed design is given in Section 3 for both, real and complex cases. Different special cases of the general allpole filter design is discussed in Section 4. Finally, Section 5 presents some applications of the proposed allpole filter design, i.e., linear-phase Butterworth-like filter, Butterworth-like filters with improved group delay, complex wavelet filters, fractional Hilbert transformers, and new IIR filters based on three allpass filters.

2. Equations for Maximally Flat Allpole Filter

We derive here equations for real and complex allpole filters both of order N, delay τ, and degree of flatness K, at a given set of frequency points.

We consider that an allpole filter of order N is given by,

$$D(z) = \frac{\alpha}{F(z)}, \tag{1}$$

where α is a complex constant with unit magnitude, z is the complex variable, and $F(z)$ is a polynomial of degree N,

$$F(z) = 1 + \sum_{n=1}^{N} f_n z^{-n}. \tag{2}$$

In general, the filter coefficients f_n, $n = 1, \ldots, N$, are complex, i.e., $f_n = f_{Rn} + jf_{In}$ where f_{Rn} and f_{In} are the real and imaginary parts of f_n, respectively. Obviously, if $f_{In} = 0$, we obtain real coefficients.

The phase responses of $D(z)$ and $F(z)$ are related by

$$\phi_D(\omega) = \phi_\alpha - \phi_F(\omega), \tag{3}$$

where ϕ_α is the phase of α, and $\phi_D(\omega)$ and $\phi_F(\omega)$ are the phases of $D(z)$ and $F(z)$, respectively. The corresponding group delay is the negative derivative of the phase, as shown in (4).

$$G(\omega) = -\frac{d\phi_D(\omega)}{d\omega} = \frac{d\phi_F(\omega)}{d\omega}. \tag{4}$$

The conditions for the maximally flat group delay at the desired frequency point ω are

$$G(\omega) = \tau \tag{5a}$$

$$G^{(p)}(\omega) = 0, \qquad p = 1, \ldots, K, \tag{5b}$$

where τ is the desired group delay, K is the degree of flatness, and $G^{(p)}(\omega)$ indicates the pth derivative of $G(\omega)$.

By performing the Fourier transform, equation (2) can be written as

$$F(e^{j\omega}) = \left[F(e^{j\omega})F^*(e^{j\omega})\right]^{1/2} e^{j\phi_F(\omega)}, \tag{6}$$

where $F^*(e^{j\omega})$, is the complex conjugate of $F(e^{j\omega})$.

Using (4) and (6) the corresponding group delay $G(\omega)$ can be expressed as

$$G(\omega) = \frac{d\phi_F(\omega)}{d\omega} = \Im\left\{\frac{F^{(1)}(e^{j\omega})}{F(e^{j\omega})}\right\}, \tag{7}$$

where $F^{(1)}(e^{j\omega})$ is the first derivative of $F(e^{j\omega})$ and $\Im\{\cdot\}$ indicates the imaginary part of $\{\cdot\}$.

Combining (5) and (7), we arrive at

$$\Im\left\{\frac{F^{(1)}(e^{j\omega})}{F(e^{j\omega})}\right\} = \tau, \tag{8a}$$

$$\Im\left\{\frac{d^k}{d\omega^k}\left\{\frac{F^{(1)}(e^{j\omega})}{F(e^{j\omega})}\right\}\right\} = 0, \qquad l = 1,\ldots,K. \tag{8b}$$

The Fourier transform (6) can be rewritten as,

$$F(e^{j\omega}) = \sum_{n=0}^{N}\left(f_{Rn}\cos(\omega n) + f_{In}\sin(\omega n)\right) + j\sum_{n=1}^{N}\left(f_{In}\cos(\omega n) - f_{Rn}\sin(\omega n)\right). \tag{9}$$

Substituting (9) into (8), we find that that the conditions given in (8) result in the following set of linear equations:

$$\sum_{n=1}^{N}(n+\tau)^k\cos(\omega n + \phi_\alpha - \phi_D(\omega))f_{Rn}$$
$$+ \sum_{n=1}^{N}(n+\tau)^k\sin(\omega n + \phi_\alpha - \phi_D(\omega))f_{In} = -\tau^k\cos(\phi_D(\omega) - \phi_\alpha), \; k \text{ odd}, \tag{10a}$$

$$\sum_{n=1}^{N}(n+\tau)^k\sin(\omega n + \phi_\alpha - \phi_D(\omega))f_{Rn}$$
$$- \sum_{n=1}^{N}(n+\tau)^k\cos(\omega n + \phi_\alpha - \phi_D(\omega))f_{In} = \tau^k\sin(\phi_D(\omega) - \phi_\alpha), \; k \text{ even}. \tag{10b}$$

Equations (10a) and (10b) are the general set of equations, which includes desired phases, group delays and degrees of flatness at given frequency points for both real and complex cases.

Notice that for each frequency point ω_l, we have $K_l + 2$ equations (see (10)) and $2N$ unknown coefficients. A consistent set of linear equations (10) is obtained if the following condition is satisfied,

$$N = \left(\frac{K_1}{2} + 1\right) + \left(\frac{K_2}{2} + 1\right) + \cdots + \left(\frac{K_L}{2} + 1\right), \tag{11}$$

where L is the number of frequency points.

3. Description and discussion of the proposed allpole filter design

We describe the design procedure based on general equations for the allpole filter proposed in Section 2

The parameters of the design are the constant α, the number L, the corresponding frequency values ω_l, $l = 1, \dots, L$, phase values $\phi_D(\omega_l)$, $l = 1, \dots, L$, group delays $\tau(\omega_l)$, $l = 1, \dots, L$, and degrees of flatness K_l, $l = 1, \dots, L$.

For the real case, i.e., $f_{\mathrm{In}} = 0$ and α is a real constant, the relations (10a) and (10b) become

$$\sum_{n=1}^{N} (n + \tau)^k \cos(\omega n - \phi_D(\omega)) f_n = -\tau^k \cos(\phi_D(\omega)), \quad k \text{ odd}, \tag{12a}$$

$$\sum_{n=1}^{N} (n + \tau)^k \sin(\omega n - \phi_D(\omega)) f_n = \tau^k \sin(\phi_D(\omega)), \quad k \text{ even}. \tag{12b}$$

Similarly, the condition (11), for the real case becomes

$$N = (K_1 + 2) + (K_2 + 2) + \cdots + (K_L + 2). \tag{13}$$

The algorithm is described in the following steps:

Step 1. Compute the order of the allpole filter N, using (13) for the real case, and (11) for the complex case.

Step 2. Substitute the frequencies ω_l, $l = 1, \dots, L$, group delays $\tau(\omega_l)$ and phases $\phi_D(\omega_l)$ into (12), for the real case, or (10), for the complex case.

Step 3. Calculate the filter coefficients f_n solving the resulting set of equations.

The following example illustrates the design of real allpole filter $D(z)$, ($\alpha = 1$) using three desired frequency points, $L = 3$.

Example 1. The design parameters are shown in Table 1.

l	ω_l	$\phi_D(\omega_l)$	$\tau(\omega_l)$	K_l
1	$\pi/5$	$\pi/3$	3	5
2	$\pi/2$	$\pi/4$	3	7
3	$4\pi/5$	$\pi/5$	4	4

Table 1. Design parameters in Example 1, using $L = 3$ and $\alpha = 1$.

Step 1. From (13), the estimated value of N is 22.

Step 2. We substitute the frequencies ω_l, group delays $\tau(\omega_l)$ and phases $\phi_D(\omega_l)$, $l = 1, \dots, 3$ into (12).

Step 3. Solving the resulting linear equations, we get the filter coefficients f_n.

Figure 1a shows the corresponding group delay, while the phase response is presented in Fig. 1b. The desired phases at $\omega = \pi/5$, $\omega = \pi/2$ and $\omega = 4\pi/5$ are also indicated in Fig. 1b. The following example illustrates the complex case.

Example 2. We design the complex allpole filter with characteristics given in Table 2.

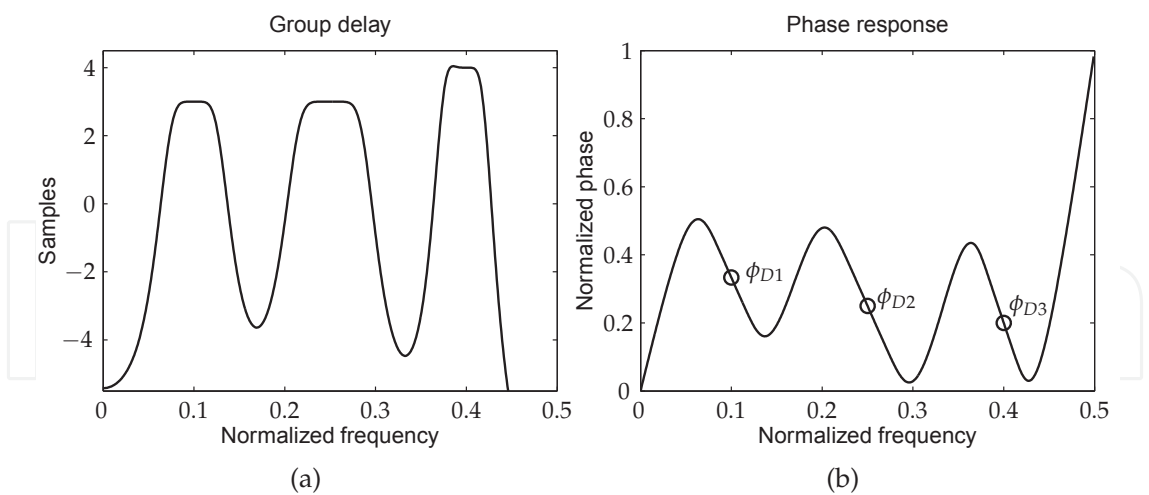*Step 1.* The order N of the allpole filter is 13 (see (11)).

Fig. 1. Phase response and group delay of the designed real allpole filter in Example 1.

l	ω_l	$\phi_D(\omega_l)$	$\tau(\omega_l)$	K_l
1	$\pi/3$	$\pi/6$	$1/2$	8
2	$4\pi/5$	$-\pi/20$	$1/2$	6
3	$8\pi/5$	$-3\pi/20$	$1/2$	6

Table 2. Design parameters in Example 2. The value L is 3 and $\alpha = 1$.

Step 2. Using (10a) and (10b), we obtain the set of linear equations with 26 unknowns coefficients; 13 for f_{Rn} and 13 for f_{In}.

Step 3. Solving the resulting set of equations, we get the coefficients of the complex allpole filter.

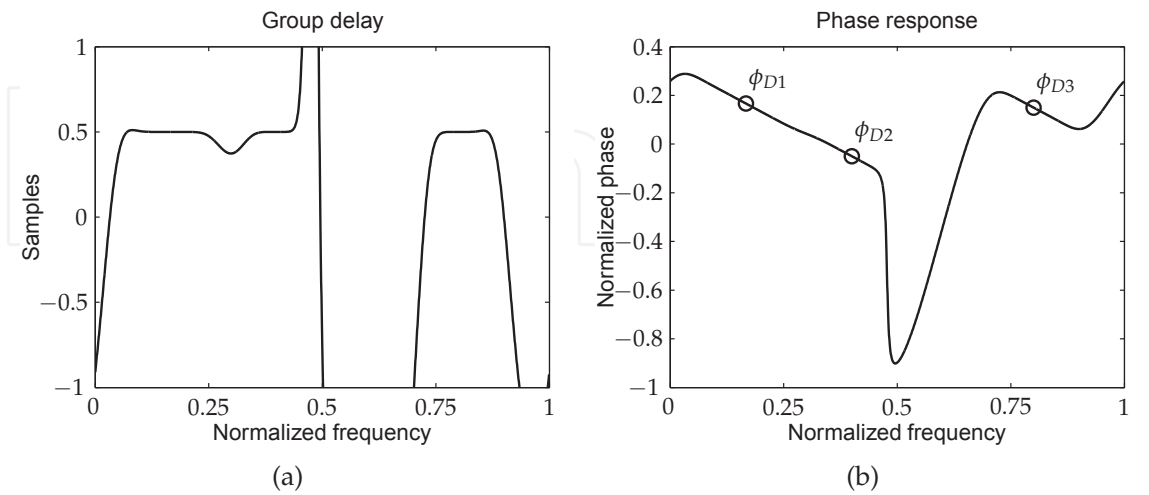Figure 2 illustrates the phase response and group delay of the designed allpole filter.

Fig. 2. Group delay and phase response of the complex allpole filter $D(z)$ in Example 2.

3.1 Relationships between allpole filters and allpass filters

We consider the relations between allpole filters of order N and allpass filters.

An allpass filter $A(z)$ is related with an allpole filter as follows (Selesnick, 1999),

$$A(z) = z^{-N} \frac{D(z)}{\widetilde{D}(z)} = z^{-N} \frac{\alpha \widetilde{F}(z)}{\alpha^* F(z)}, \tag{14}$$

where $\widetilde{D}(z)$ is the paraconjugate of $D(z)$, that is, it is generated by conjugating the coefficients of $D(z)$ and by replacing z by z^{-1}.

The phase $\phi_A(\omega)$ of $A(z)$ can be expressed as

$$\phi_A(\omega) = -\omega N + 2\phi_D(\omega), \tag{15}$$

where the desired phase $\phi_D(\omega)$ is given by

$$\phi_D(\omega) = \frac{\phi_A(\omega) + \omega N}{2}. \tag{16}$$

From (15), the group delay of the complex allpass filter $\tau_A(\omega)$ is given by

$$\tau_A(\omega) = N + 2\tau(\omega), \tag{17}$$

where $\tau(\omega)$ is the group delay of $D(z)$.

Using (17), it follows

$$\tau(\omega) = \frac{\tau_A(\omega) - N}{2}. \tag{18}$$

It is well known that the structures based on allpass filters exhibit a low sensitivity to the filter quantization and a low noise level (Mitra, 2005). Therefore, the relationship (14), between allpass and allpole filters, gives the possibility to use efficient allpass structures in the proposed design.

4. Promising special cases

The proposed allpole filters have desired phases, group delays and degrees of flatness at a specified set of frequency points. In this section we introduce some new special cases of the proposed design (10), which are used for the design of complex allpole filters, complex wavelet filters, and linear-phase IIR filters.

4.1 First order allpole filters

Using (12), the filter coefficient f_{R1} is computed as follows:

$$f_{R1} = \frac{\sin(\phi_{D_1})}{\sin(\omega_1 - \phi_{D_1})}, \tag{19}$$

where ϕ_{D_1} is the desired phase at $\omega = \omega_1$.

To ensure the stability of the allpole filter, we have

$$\tan(2\phi_{D_1}) > \frac{1 - \cos(2\omega_1)}{\sin(2\omega_1)}. \tag{20}$$

Similarly for the complex case, the filter coefficient f_1 is

$$f_1 = \frac{\sin(\phi_\alpha - \phi_{D_2})e^{j(\omega_1 + \phi_\alpha - \phi_{D_1})} - \sin(\phi_\alpha - \phi_{D_1})e^{j(\omega_2 + \phi_\alpha - \phi_{D_2})}}{\sin(\omega_1 - \omega_2 + \phi_{D_2} - \phi_{D_1})}, \tag{21}$$

where ϕ_{D_1} and ϕ_{D_2} are the phases of the allpole filter at the desired frequency points $\omega = \omega_1$ and $\omega = \omega_2$, respectively. The stability of the allpole filter is satisfied if the following equation holds

$$\tan(\phi_{D_2} - \phi_\alpha) < \frac{\cos(\omega_1 - \omega_2 + \phi_\alpha - \phi_{D_1}) - |\cos(\phi_{D_1} - \phi_\alpha)|}{\sin(\omega_1 - \omega_2 + \phi_\alpha - \phi_{D_1}) + \sin(\phi_{D_1} - \phi_\alpha)}. \tag{22}$$

4.2 Second order allpole filter

We consider the following two cases.

Case 1. For $\omega = \omega_1$, we specify the desired phase ϕ_{D_1} and group delay τ. Substituting these conditions into the general equations (12), the resulting filter coefficients are

$$f_{R1} = -\frac{(\tau + 1)\sin(2\omega_1) - \sin(2\omega_1 - 2\phi_{D_1})}{(\tau + 1)\sin\omega_1 - \sin(\omega_1 - \phi_{D_1})\cos(2\omega_1 - \phi_{D_1})}, \tag{23}$$

$$f_{R2} = \frac{\tau\sin\omega_1 + \sin(\phi_{D_1})\cos(\omega_1 - \phi_{D_1})}{(\tau + 1)\sin\omega_1 - \sin(\omega_1 - \phi_{D_1})\cos(2\omega_1 - \phi_{D_1})}. \tag{24}$$

Additionally, the condition for the stability of the allpole filter is

$$\tau > -1 + \frac{|\sin(2\omega_1 - 2\phi_{D_1})|}{2\sin\omega_1}. \tag{25}$$

Case 2. For two phases ϕ_{D_1} and ϕ_{D_2} at the frequencies ω_1 and ω_2, the filter coefficients are

$$f_{R1} = \frac{\sin(2\omega_1 - \phi_{D_1})\sin(\phi_{D_2}) - \sin(\phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}{\sin(\omega_2 - \phi_{D_2})\sin(2\omega_1 - \phi_{D_1}) - \sin(\omega_1 - \phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}, \tag{26}$$

$$f_{R2} = \frac{\sin(\phi_{D_1})\sin(\omega_2 - \phi_{D_2}) - \sin(\omega_1 - \phi_{D_1})\sin(\phi_{D_2})}{\sin(\omega_2 - \phi_{D_2})\sin(2\omega_1 - \phi_{D_1}) - \sin(\omega_1 - \phi_{D_1})\sin(2\omega_2 - \phi_{D_2})}. \tag{27}$$

Furthermore, the stability of the allpole filter is guaranteed if the equation

$$\tan(\omega_1 - \phi_{D_1}) < -\frac{\sin\omega_1 \sin\omega_2 \tan(\omega_2 - \phi_{D_2})}{\cos\omega_1 \cos\omega_2 - 1 + |\cos\omega_1 - \cos\omega_2|} \tag{28}$$

is satisfied.

4.3 Complex Thiran allpole filters

We generalize the result proposed by Thiran (Thiran, 1971), for the design of real allpole filters that are maximally flat at $\omega = 0$, to include both the real and complex cases. The required design specifications are the order of the allpole filter N, group delay $\tau(\omega)$ at $\omega = 0$, τ_0, degree of flatness K, and the phase value ϕ_α.

Consequently, the allpole filter must satisfy:

$\mathcal{A}.1$ The degree of flatness at $\omega = 0$ is K, where K can be either $2N - 2$ or $2N - 3$.

$\mathcal{A}.2$ The phase value $\phi_D(\omega)$ is equal to zero at $\omega = 0$.

4.3.1 Degree of flatness $K = 2N - 2$

Substituting conditions $\mathcal{A}.1$ and $\mathcal{A}.2$ into the set of equations (10), we compute the complex coefficients as follows

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n e^{j(\phi_\alpha - \pi/2)} \sin \phi_\alpha \right), \tag{29}$$

where $n = 1, \ldots, N$, the binomial coefficient is given by

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}, \tag{30}$$

and the Pochhammer symbol $(x)_m$ indicates the rising factorial of x, which is defined as (Andrews, 1998),

$$(x)_m = \begin{cases} (x)(x+1)(x+2)\cdots(x+m-1) & m > 0, \\ 1 & m = 0. \end{cases} \tag{31}$$

The expression in (29) is the extension of the result proposed in (Thiran, 1971), which includes both real and complex cases. If ϕ_α is 0 or π, the imaginary coefficients are zero, and the result is a real allpole filter, consistent with (Thiran, 1971). For $\phi_\alpha = \pm\pi/2$, the filter is a real allpole filter (this case is not included in (Thiran, 1971)). For all other phase values, the imaginary coefficients are strictly non-zero, i.e., the filter is complex.

4.3.2 Degree of flatness $K = 2N - 3$

In this case, in order to get a degree of flatness $K = 2N - 3$, we set $f_{\mathrm{IN}} = 0$. Consequently, the filter coefficients are

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n + n \frac{(n-N)e^{j\phi_\alpha} \cos \phi_\alpha}{2\tau_0 + N} \right), \tag{32}$$

where $n = 0, \ldots, N$.

In contrast with (32), to obtain a different solution, we now set $f_{\mathrm{RN}} = 0$. Therefore, we have

$$f_n = (-1)^n \binom{N}{n} \frac{2(2\tau_0 + 1)_{n-1}}{(2\tau_0 + N + 1)_n} \left(\tau_0 + n - \frac{n e^{j\phi_\alpha}}{N \cos \phi_\alpha} \left(\tau_0 + n + \frac{(N-n)(\tau_0 + N \cos^2 \phi_\alpha)}{2\tau_0 + N} \right) \right), \tag{33}$$

where $n = 0, \ldots, N$.

We illustrate the design with one example.

Example 3. The desired phase ϕ_α, and the group delay τ_0 at $\omega = 0$, are $-\pi/6$, and $7/3$, respectively. The order N of the filter is 5.

We compute the corresponding filter coefficients using (29), (32), and (33). The resulting group delays of $D(z)$ are shown in Fig. 3a, while the phase responses of the designed filters are shown in Fig. 3b.

4.4 Complex allpole filter with flatness at $\omega = 0$ and $\omega = \pi$

Now, we present the design of complex allpole filters of order N (any positive integer) with flatness at $\omega = 0$ and $\omega = \pi$.

The design conditions are: (More detailed explanation is given in Section 5.1.)

$\mathcal{B}.1$ The phase response of $D(z)$ is flat at the frequency points $\omega = 0$ and $\omega = \pi$ with group delays $\tau(0) = \tau(\pi) = -N/2$.
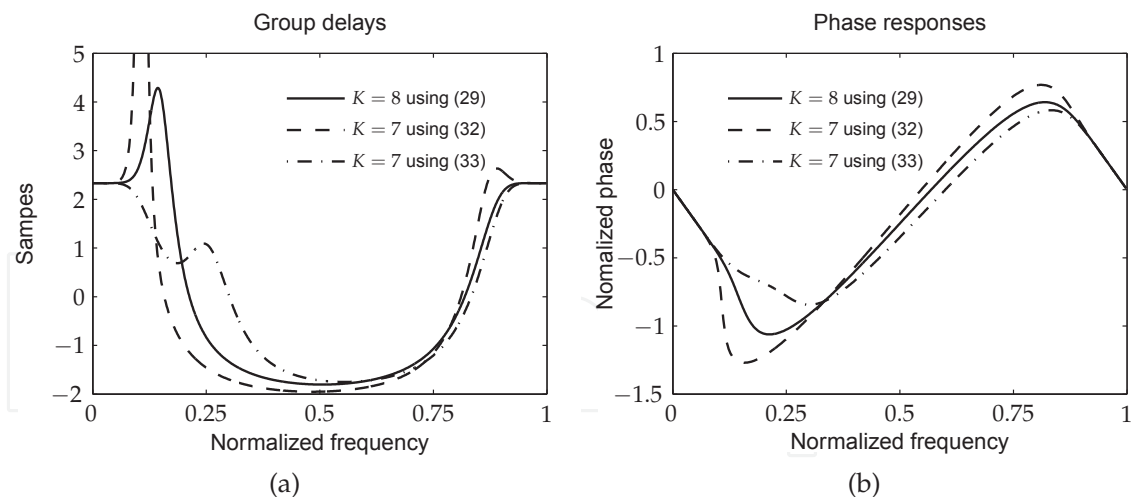
Fig. 3. Group delays and phase responses of the complex allpole filters in Example 3.

B.2 The degree of flatness at these frequency points is the same, i.e., $K = N - 2$.

B.3 The phase values of the allpole filter $\phi_D(\omega)$ at $\omega = 0$ and $\omega = \pi$, are 0 and $\pi(2N + (2l + 1))/4$, respectively, where l is an integer.

B.4 The desired phase value $\phi_D(\omega)$ at the given frequency $\omega = \omega_p$ is ϕ_p, i.e., $\phi_p = \phi_D(\omega_p)$.

Substituting conditions B.1–B.4 into (10a) and (10b) and solving the resulting set of linear equations, we arrive at

$$f_n = \begin{cases} \dbinom{N}{n} & n \text{ even,} \\ \dbinom{N}{n}\left(\sqrt{2}e^{j(2\phi_\alpha + \frac{\pi}{4})} - j\right) & n \text{ odd,} \end{cases} \tag{34}$$

where

$$\phi_\alpha = \angle\left\{-j - 1 - (-1)^{\lceil N/2 \rceil}\left(\cot\left(\phi_p - \frac{\omega_p N}{2}\right) - 1\right)\tan^N\left(\frac{\omega_p}{2}\right)\right\}, \tag{35}$$

and $\angle\{\cdot\}$ indicates the angle of $\{\cdot\}$, while $\lceil \cdot \rceil$ stands for the floor function.

Next example illustrates the proposed design where the parameters of the design are the filter order N and the phase value ϕ_p at the frequency point ω_p.

Example 4. We design a complex allpole filter using the following specifications: the order of the allpole filter is $N = 7$ and the phase value $\phi_D(\omega)$ at ω_p is 1.2π, where $\omega_p = 0.3\pi$. The group delay and phase response of the designed filter are presented in Fig. 4a and 4b, respectively.

4.4.1 Closed form equations for the singularities of the allpole filter

In the following, we consider the computation of the poles of $D(z)$.

Using (34), we obtain the z-transform of the denominator of $D(z)$ defined in (1) as,

$$F(z) = \sum_{n \text{ even}}\binom{N}{n}z^{-n} + (\sqrt{2}e^{j(2\phi_\alpha + \pi/4)} - j)\sum_{n \text{ odd}}\binom{N}{n}z^{-n}. \tag{36}$$
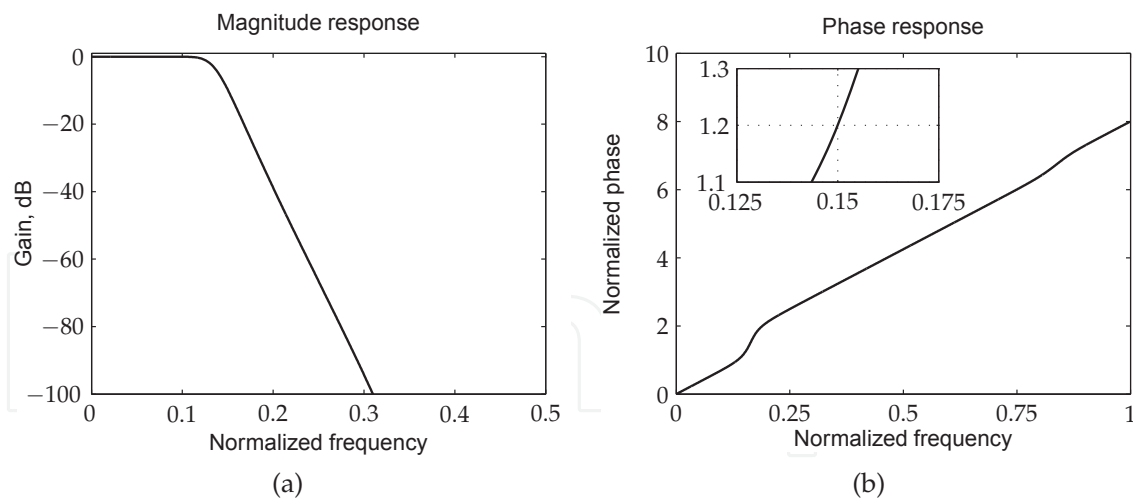
Fig. 4. Group delay and phase response and of the complex allpole filter in Example 4.

After some computations, we get

$$F(z) = \frac{e^{j\phi_\alpha}}{\sqrt{2}}\left[(\cos\phi_\alpha - \sin\phi_\alpha)(1 + z^{-1})^N - (j - 1)\sin\phi_\alpha(1 - z^{-1})^N\right]. \tag{37}$$

Therefore, the corresponding poles are

$$p_k = \frac{\gamma_k + 1}{\gamma_k - 1}, \tag{38}$$

where $k = 0, \ldots, N - 1$, and

$$\gamma_k = \left(\frac{\sqrt{2}}{1 - \cot\phi_\alpha}\right)^{\frac{1}{N}} e^{-j\frac{8k+1}{4N}\pi}. \tag{39}$$

4.5 Complex allpole filters with flatness at $\omega = 0$, and $\omega = \pm\omega_r$

In this section, we design a complex allpole filter with the following characteristics:

$\mathcal{C}$.1 The order N is even.

$\mathcal{C}$.2 The allpole filter has flat group delay at the frequency points $\omega = 0$, $\omega = -\omega_r$, and $\omega = \omega_r$. The degrees of flatness are $K_1(\omega = 0) = N - 2$, $K_2(\omega = \pm\omega_r) = N/2 - 2$. The group delay at those frequency points is $\tau(0) = \tau(\pm\omega_r) = -N/2$.

$\mathcal{C}$.3 The desired allpole phase value $\phi_D(\omega)$ at the given frequency $\omega = \omega_p$ is ϕ_p, i.e., $\phi_p = \phi_D(\omega_p)$.

$\mathcal{C}$.4 The phase values of the allpole filter $\phi_D(\omega)$ at $\omega = 0$, $\omega = -\omega_r$, and $\omega = \omega_r$ are 0, $\pi/3 + \omega_r N/2$, and $\pi/3 - \omega_r N/2$, respectively.

Substituting conditions $\mathcal{C}$.1–$\mathcal{C}$.4 into (10a) and (10b) and solving the resulting set of linear equations, we have

$$f_n = (-1)^n\left[\binom{N}{n} - \frac{4e^{j\phi_\alpha}}{\sqrt{3}}\binom{N/2}{n}c_{N,n}(\omega_r)\cos\left(\phi_\alpha + \pi/6\right)\right], \tag{40}$$

where $n = 0, \ldots, N/2$,

$$\phi_\alpha = \angle \left\{ \sqrt{3} R_\mathrm{p} \cot(\phi_\mathrm{p} - \omega_\mathrm{p} N/2) + 1 + \mathrm{j}\sqrt{3}(R_\mathrm{p} + 1) \right\}, \tag{41}$$

and

$$R_\mathrm{p} = \frac{-2^{N-1} \sin^N \left(\frac{\omega_\mathrm{p}}{2} \right)}{c_{N,N/2}(\omega_\mathrm{r}) + 2C_N(\omega_\mathrm{r}, \omega_\mathrm{p})}, \tag{42}$$

where

$$C_N(\omega_\mathrm{r}, \omega_\mathrm{p}) = \sum_{n=1}^{N/2-1} (-1)^{N/2+n} \binom{N/2}{n} c_{N,n}(\omega_\mathrm{r}) \cos\left((N/2 - n)\omega_\mathrm{p} \right). \tag{43}$$

The function $c_{N,n}(\omega_\mathrm{r})$ for different values of N is given in Table 3. Moreover, we have $c_{N,0}(\omega_\mathrm{r}) = 0$ and $f_n = f_{N-n}$.

Example 5. The desired design specification is as follows: the allpole filter order is equal to 8, $\omega_\mathrm{p} = 0.35\pi$, $\omega_\mathrm{r} = 0.75\pi$, and $\phi_\mathrm{p} = 1.5\pi$. The resulting group delay and phase response of the designed filter are shown in Fig. 5.
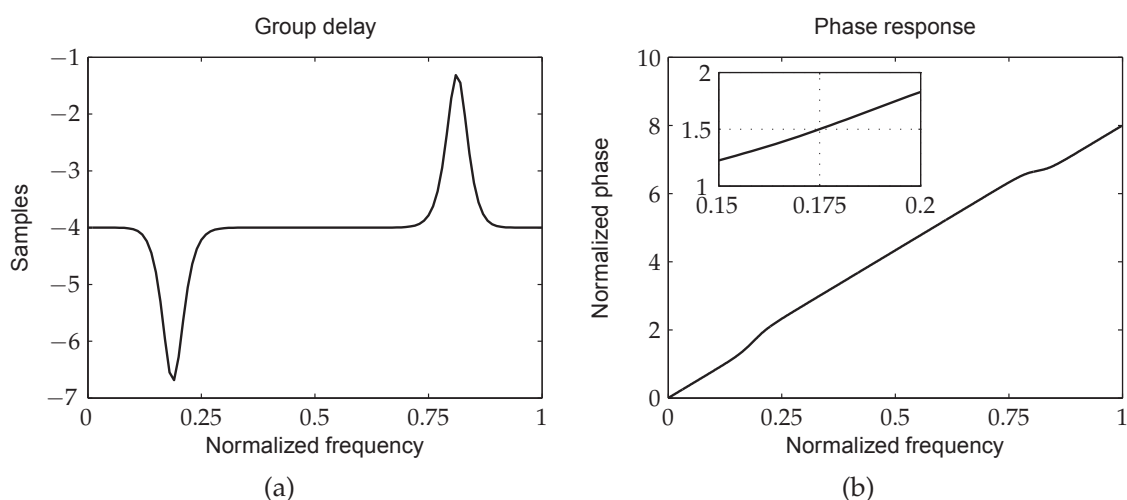

Fig. 5. Group delay and phase response and of the designed complex allpole filter in Example 5.

5. Design of IIR filters based on allpole filters

5.1 Direct design of linear-phase IIR Butterworth filters

A filter $H(z)$ has linear-phase if,

$$H(z) = cz^{-k}\widetilde{H}(z), \tag{44}$$

where $H(z)$ is not necessary causal, z^{-k} is the delay, the complex constant c has unit magnitude and $\widetilde{H}(z)$ is the paraconjugate of $H(z)$, that is, it is generated by conjugating the coefficients of $H(z)$ and by replacing z by z^{-1}.

It has been shown that causal Finite Impulse Response (FIR) filters can be designed to have linear-phase. However, Infinite Impulse Response (IIR) filters can have linear-phase property only in the noncausal case (Vaidyanathan & Chen, 1998), (the phase response is either zero or π). It has been recently shown that filters with the linear-phase property are useful in the filter

N	n	$c_{N,n}(\omega_{\mathrm{r}}) = c_{N,N-n}(\omega_{\mathrm{r}})$
2	1	$1 - \cos(\omega_{\mathrm{r}})$
4	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
6	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$10 - 9\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
8	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$7 - 6\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
	4	$17 - 16\cos(2\omega_{\mathrm{r}}) - \cos(4\omega_{\mathrm{r}})$
10	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$6 - 5\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
	4	$11 - 10\cos(2\omega_{\mathrm{r}}) - \cos(4\omega_{\mathrm{r}})$
	5	$126 - 100\cos(\omega_{\mathrm{r}}) - 25\cos(3\omega_{\mathrm{r}}) - \cos(5\omega_{\mathrm{r}})$
12	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$11/2 - 9/2\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
	4	$9 - 8\cos(2\omega_{\mathrm{r}}) - \cos(4\omega_{\mathrm{r}})$
	5	$66 - 50\cos(\omega_{\mathrm{r}}) - 15\cos(3\omega_{\mathrm{r}}) - \cos(5\omega_{\mathrm{r}})$
	6	$262 - 225\cos(2\omega_{\mathrm{r}}) - 36\cos(4\omega_{\mathrm{r}}) - \cos(6\omega_{\mathrm{r}})$
14	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$26/5 - 21/5\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
	4	$8 - 7\cos(2\omega_{\mathrm{r}}) - \cos(4\omega_{\mathrm{r}})$
	5	$143/3 - 35\cos(\omega_{\mathrm{r}}) - 35/3\cos(3\omega_{\mathrm{r}}) - \cos(5\omega_{\mathrm{r}})$
	6	$127 - 105\cos(2\omega_{\mathrm{r}}) - 21\cos(4\omega_{\mathrm{r}}) - \cos(6\omega_{\mathrm{r}})$
	7	$1761 - 1225\cos(\omega_{\mathrm{r}}) - 441\cos(3\omega_{\mathrm{r}}) - 49\cos(5\omega_{\mathrm{r}}) - \cos(7\omega_{\mathrm{r}})$
16	1	$1 - \cos(\omega_{\mathrm{r}})$
	2	$1 - \cos(2\omega_{\mathrm{r}})$
	3	$5 - 4\cos(\omega_{\mathrm{r}}) - \cos(3\omega_{\mathrm{r}})$
	4	$37/5 - 32/5\cos(2\omega_{\mathrm{r}}) - \cos(4\omega_{\mathrm{r}})$
	5	$39 - 28\cos(\omega_{\mathrm{r}}) - 10\cos(3\omega_{\mathrm{r}}) - \cos(5\omega_{\mathrm{r}})$
	6	$87 - 70\cos(2\omega_{\mathrm{r}}) - 16\cos(4\omega_{\mathrm{r}}) - \cos(6\omega_{\mathrm{r}})$
	7	$715 - 490\cos(\omega_{\mathrm{r}}) - 196\cos(3\omega_{\mathrm{r}}) - 28\cos(5\omega_{\mathrm{r}}) - \cos(7\omega_{\mathrm{r}})$
	8	$3985 - 3136\cos(2\omega_{\mathrm{r}}) - 784\cos(4\omega_{\mathrm{r}}) - 64\cos(6\omega_{\mathrm{r}}) - \cos(8\omega_{\mathrm{r}})$

Table 3. Function $c_{N,n}(\omega_{\mathrm{r}})$ for different values of N.

bank design and the Nyquist filter design and different methods have been proposed for this design (Djokic et al., 1998; Powell & Chau, 1991; Surma-aho & Saramaki, 1999).

A linear-phase lowpass IIR filter $H(z)$ can be expressed in terms of complex allpass filters as (Zhang et al., 2001),

$$H(z) = \frac{1}{2}\left[A(z) + \widetilde{A}(z) \right], \tag{45}$$

where $A(z)$ is a complex allpass of order N (see (14)).

We can note that the filter defined in (45) satisfies the relation (44) if $k = 0$ and $c = 1$.

The main goal is to propose a new technique to design real and complex IIR filters with linear-phase, based on general design of Section 3, where the design specification is same as in traditional IIR filters design based on analog filters, i.e., the passband and stopband frequencies, ω_p and ω_s, the passband droop A_p, and the stopband attenuation A_s, shown in Fig. 6.
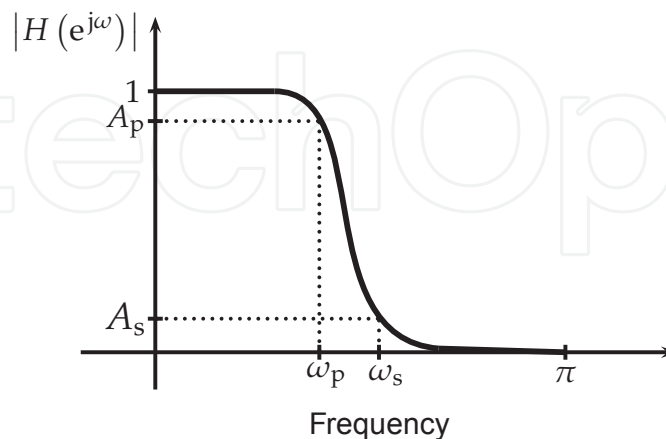
Fig. 6. Design parameters for low pass filter.

We relate the design of linear-phase IIR filter with allpass filter and in the next section we use the general approach to design the corresponding allpole filter.
First, we establish the conditions which the auxiliary complex allpass filters from (45) has to satisfy.
From (45), the magnitude response of $H(z)$ can be expressed as,

$$|H(e^{j\omega})| = \left|\cos\left(\phi_A(\omega)\right)\right|, \quad \text{for all } \omega. \tag{46}$$

The magnitude responses of $|H(e^{j\omega})|$ at $\omega = 0$, and $\omega = \pi$ are 1 and 0, respectively (see Fig. 6). Therefore, the values of $\phi_A(\omega)$ at these frequency points are 0 and $(2l + 1)\pi/2$, respectively, where l is an integer. Since the magnitude response of $H(z)$ decreases monotonically, relation (46) can be rewritten as,

$$|H(e^{j\omega})| = \cos\left(\phi_A(\omega)\right), \quad 0 \le \omega \le \pi. \tag{47}$$

Note that $|H(e^{j\omega})|$ has a flat magnitude response at $\omega = 0$ and $\omega = \pi$, and that the filter $A(z)$ has a flat phase response at the same frequency points. As a consequence, the corresponding group delays $\tau_A(0)$ and $\tau_A(\pi)$ are equal to 0.
Considering the value A_p in dB we write

$$20 \log_{10} |H(e^{j\omega})|_{\omega=\omega_\mathrm{p}} = -A_\mathrm{p}. \tag{48}$$

From (47) it follows,

$$\phi_{\mathrm{p}A} = \phi_A(\omega_\mathrm{p}) = \arccos\left(10^{-A_\mathrm{p}/20}\right). \tag{49}$$

In summary, the conditions that the auxiliary complex allpass filter in (45) needs to satisfy are the following:

$\mathcal{D}$.1 The phase values of $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pi$ are 0 and $(2l + 1)\pi/2$, respectively.

$\mathcal{D}$.2 The phase response of $A(z)$ is flat at $\omega = 0$ and $\omega = \pi$. Therefore, $\tau_A(0) = \tau_A(\pi) = 0$.

$\mathcal{D}$.3 The phase value $\phi_{\mathrm{p}A}$ is controlled by A_{p} (see (49)).

In the following, we use the results from Section 3.1 and the Conditions $\mathcal{D}$.1–$\mathcal{D}$.3 in order to obtain the corresponding conditions for the allpole filter $D(z)$.

5.1.1 Design of flat linear-phase IIR filters based on complex allpole filters

We relate the allpass filter from (45) with the corresponding allpole filter.
Using (16) and the phase values $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pi$ (see Condition $\mathcal{D}$.1), we get
$\phi(0) = 0$ and $\phi(\pi) = \pi(2N + (2l+1))/4$.
Now, from (18) and Condition $\mathcal{D}$.2, we have $\tau(0) = \tau(\pi) = -N/2$.
Finally, the following relation is obtained using Condition $\mathcal{D}$.3 and (16),

$$\phi_D(\omega_{\mathrm{p}}) = \phi_{\mathrm{p}} = \frac{\arccos\left(10^{-A_{\mathrm{P}}/20}\right) + \omega_{\mathrm{p}}N}{2}. \tag{50}$$

As a consequence, the corresponding conditions that the allpole filter $D(z)$ has to satisfy are:

$\mathcal{E}$.1 The phase values of $D(z)$ at $\omega = 0$ and $\omega = \pi$ are 0 and $\pi(2N + (2l+1))/4$, respectively.

$\mathcal{E}$.2 The group delay $\tau(\omega)$ of $D(z)$ at $\omega = 0$ and $\omega = \pi$ are $-N/2$.

$\mathcal{E}$.3 The phase value of $D(z)$ at ω_{p}, $\phi_D(\omega_{\mathrm{p}})$, is given by (50).

For a filter having coefficients given in (34) the Conditions $\mathcal{E}$.1 and $\mathcal{E}$.2 are satisfied.
From the Condition $\mathcal{E}$.3 and (35), the corresponding value of $\phi_\alpha(N, \omega_{\mathrm{p}}, A_{\mathrm{p}})$ is equal to

$$\phi_\alpha(N, \omega_{\mathrm{p}}, A_{\mathrm{p}}) = \angle\left\{ -\mathrm{j} - 1 - (-1)^{\lceil N/2 \rceil} A_{\mathrm{p}}' \tan^N\left(\frac{\omega_{\mathrm{p}}}{2}\right) \right\}, \tag{51}$$

where

$$A_{\mathrm{p}}' = \sqrt{\frac{10^{A_{\mathrm{P}}/20} + 1}{10^{A_{\mathrm{P}}/20} - 1}} - 1. \tag{52}$$

We note that the resulting allpole filter has a causal and an anticausal parts. The causal part can be implemented with the well known structures for allpass filters while the anticausal part can be implemented with the structures proposed in (Vaidyanathan & Chen, 1998).
The degree of flatness of the allpass filter $A(z)$ at $\omega = 0$ and $\omega = \pi$ is equal to $N - 2$. Based on this result it can be shown that we have $2N - 1$ null derivatives in the square magnitude response $|H(\mathrm{e}^{\mathrm{j}\omega})|^2$ at $\omega = 0$ and $\omega = \pi$.

5.1.2 Closed form equations for the singularities of $H(z)$

It follows from (37) and (45) that the transfer function $H(z)$ is given as,

$$H(z) = \frac{(1 + z^{-1})^N E(z)}{2z^{-N}F(z)\widetilde{F}(z)}, \tag{53}$$

where

$$E(z) = (1 - \sin(2\phi_\alpha))(1 + z^{-1})^N + ((\mathrm{j}+1) - (\mathrm{j}-1))\sin\phi_\alpha(\cos(\phi_\alpha) - \sin(\phi_\alpha))(1 - z^{-1})^N. \tag{54}$$

We note that the transfer function $H(z)$ has N zeros at $z = -1$ and the other zeros are at (see (54)),

$$z_k = \frac{\beta_k + 1}{\beta_k - 1}, \tag{55}$$

where $k = 0, \ldots, N-1$, and the parameter β_k is given by,

$$\beta_k = \begin{cases} \left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}} e^{j\frac{2\pi}{N}k} & N \text{ even}, \\[4mm] \left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}} e^{j\frac{4k-1}{2N}\pi} & N \text{ odd}. \end{cases} \tag{56}$$

It is easily shown that the absolute values of z_k in (55), for even values of N, are always different than 1. However, there also exists one absolute value of z_k, for N odd, which is equal to 1, i.e., there is a zero on the unit circle. The corresponding frequency ω_0 is expressed as,

$$\omega_0 = \pi + 2\arctan\left(2\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}}. \tag{57}$$

As a consequence, the frequency at which $H(e^{j\omega})$ is equal to -1 is given by

$$\omega_1 = \pi + 2\arctan\left(\frac{1}{2}\frac{1-\cos(2\phi_\alpha)}{1-\sin(2\phi_\alpha)}\right)^{\frac{1}{2N}}. \tag{58}$$

Finally, the transfer function $H(z)$ has $2N$ poles which are poles of the corresponding complex allpole filters $D(z)$ and $\widetilde{D}(z)$ (see Section 4.4.1).

5.1.3 Description of the algorithm

The proposed algorithm is described in the following steps:

Step 1. Estimate the order of the allpole filter using the following equation, which can be obtained by solving $\phi_\alpha(N, \omega_p, A_p) = \phi_\alpha(N, \omega_s, A_s)$,

$$N = \left\lceil \frac{\left|\log_{10}\left(\frac{A'_p}{A'_s}\right)\right|}{\log_{10}\left(\frac{\omega_s'}{\omega_p'}\right)} \right\rceil, \quad A_p' = \sqrt{\frac{10^{A_p/20}+1}{10^{A_p/20}-1}} - 1, \quad A_s' = \sqrt{\frac{10^{A_s/20}+1}{10^{A_s/20}-1}} - 1, \tag{59}$$

where $\lceil \cdot \rceil$ is the ceiling function.

Step 2. From the values N, ω_p and A_p, compute the phase value $\phi_\alpha(N, \omega_s, A_s)$, using (51).

Step 3. Using (34), compute the filter coefficients f_n.

Step 4. Calculate the filter coefficients of $H(z)$ using (45).

We illustrate the procedure with the following example.

Example 6. We design the IIR linear-phase lowpass filter with the passband and stopband frequencies $\omega_p = 0.25\pi$ and $\omega_s = 0.5\pi$, respectively. The passband droop is $A_p = 1$ dB, while the stopband attenuation is $A_s = 65$ dB.

Step 1. Using (59), we estimate $N = 10$. As a consequence, the filter $H(z)$ is real.

Step 2. We calculate the phase value $\phi_\alpha(N, \omega_s, A_s)$, to be $\phi_\alpha(N, \omega_s, A_s) = -0.749925\pi$.

Step 3. The filter coefficients f_n are computed from (34).

Step 4. We compute the coefficients of the designed filter $H(z)$. The magnitude response of the designed filter is given in Fig. 7.
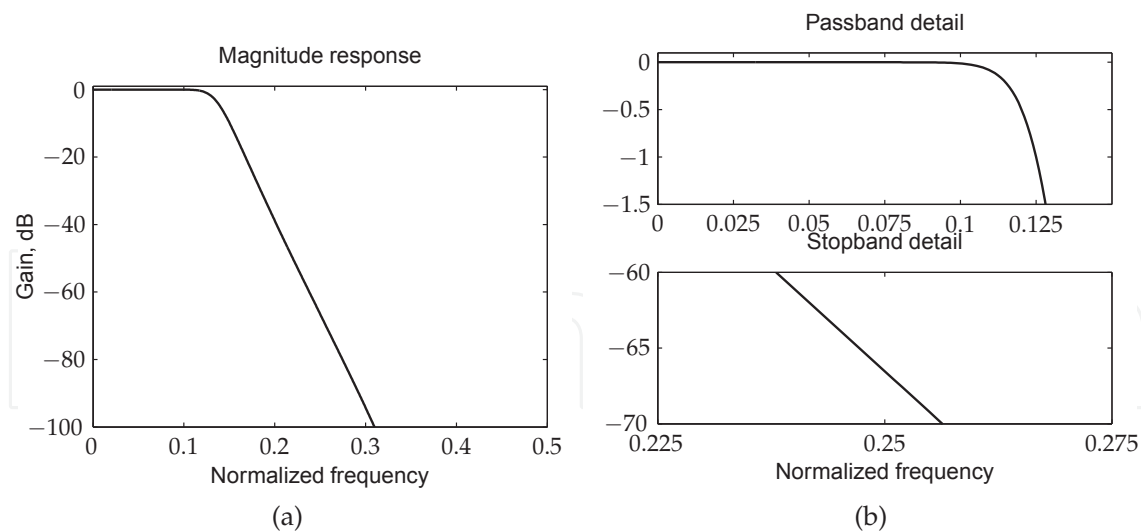
Fig. 7. Example 6.

5.1.4 Linear-phase IIR highpass filter design

Now, we extend the proposed algorithm for lowpass filter to highpass filter design.
Using the power-complementary property (Vaidyanathan et al., 1987), it can be shown that
the corresponding complementary filter of $H(z)$, defined in (45), is given by

$$H_1(z) = \frac{1}{2j} \left[A(z) - \widetilde{A}(z) \right],\tag{60}$$

where $H_1(z)$ is a highpass filter.
Using (60), the phase value ϕ_{pA} is expressed as,

$$\phi_{pA} = \arcsin\left(10^{-A_p/20} \right).\tag{61}$$

Similarly, the phase value $\phi_\alpha(N, \omega_p, A_p)$ is given by,

$$\phi_\alpha(N, \omega_p, A_p) = \angle \left\{ -(j+1) \cot^N \left(\frac{\omega_p}{2} \right) - \frac{2(-1)^{\lfloor N/2 \rfloor}}{A_{p'}} \right\}.\tag{62}$$

Finally, the filter coefficients of $H_1(z)$ are computed using (60).
The following example illustrates the procedure.
Example 7. The parameters of the design of the highpass filter are: the passband and stopband
frequencies are $\omega_p = 0.75\pi$ and $\omega_s = 0.4\pi$, respectively. The stopband attenuation and
passband droop are 50 dB and 1 dB, respectively.
The resulting filter order is equal to 6 and $\phi_\alpha(N, \omega_p, A_p) = -0.002569\pi$. The magnitude
response, the passband and stopband details of the designed filter are shown in Fig. 8.

5.2 Direct design of linear-phase IIR filter banks

The modified two-band filter bank (Galand & Nussbaumer, 1984), is shown in Fig. 9. The
analysis filter $H_0(z)$ and the synthesis filter $G_0(z)$ are lowpass filters, while the analysis filter
$H_1(z)$ and the synthesis filter $G_1(z)$ are highpass filters. However, both the analysis and the
synthesis filters are not causal. As a difference with traditional structure, in this structure
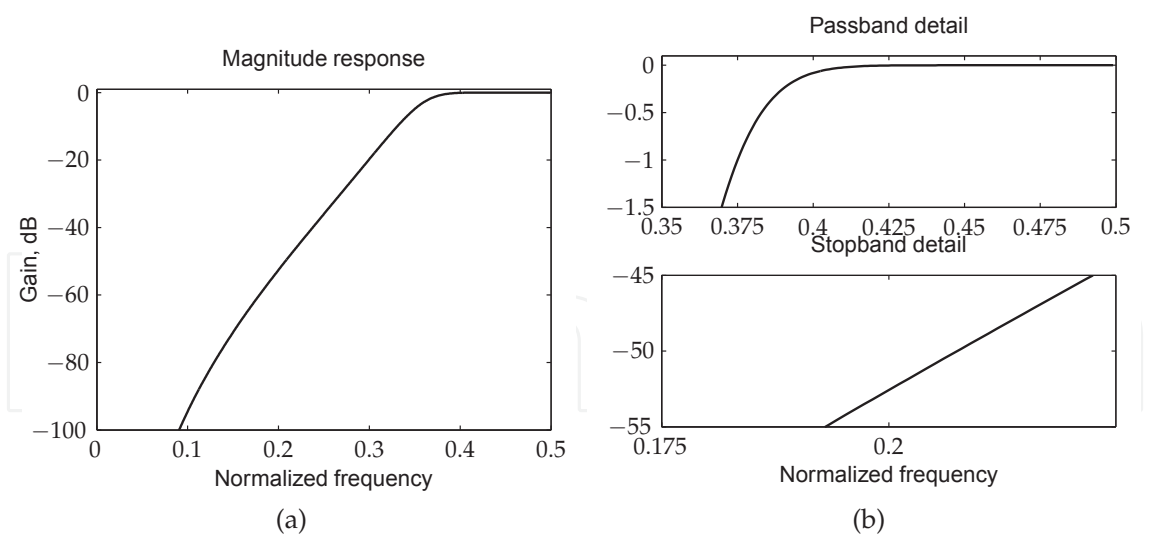
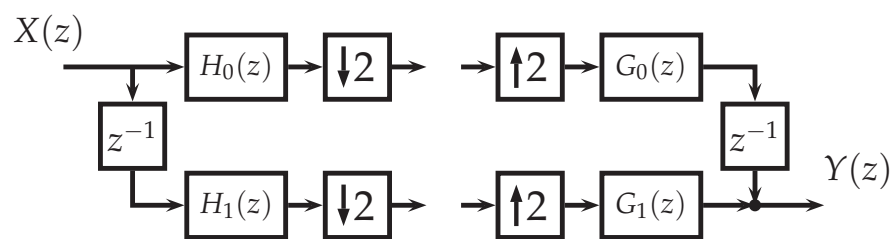Fig. 8. Magnitude response of $H_1(z)$ in Example 7.

Fig. 9. Modified two-band filter bank.

there are two extra delays, one in the highpass analysis filter and another one in the lowpass synthesis filter (see Fig. 9).

The output $Y(z)$ is obtained using some multirate computations (Jovanovic-Dolecek, 2002), i.e.,

$$Y(z) = \frac{z^{-1}}{2}\left(X(z)\Big(G_0(z)H_0(z) + G_1(z)H_1(z)\Big) + X(-z)\Big(G_0(z)H_0(-z) - G_1(z)H_1(-z)\Big)\right).$$
(63)

The output of the filter bank (63) suffers from three types of errors, i.e., aliasing, amplitude distortion and phase distortion.

To avoid aliasing, the synthesis filters are related to the analysis filter $H_0(z)$ in the following form (Vaidyanathan et al., 1987),

$$G_0(z) = \widetilde{H}_0(z), \quad G_1(z) = H_0(-z),$$
(64)

where $\widetilde{H}_0(z)$ is the paraconjugate of $H_0(z)$ and $H_1(z) = \widetilde{H}_0(-z)$.

The amplitude and phase distortions are eliminated if the analysis filters are chosen to satisfy

$$H_0(z)\widetilde{H}_0(z) + H_0(-z)\widetilde{H}_0(-z) = 1.$$
(65)

From (65), the following relation holds,

$$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega-\pi)})|^2 = 1.$$
(66)

The relationship between the passband frequency ω_p and stopband frequency ω_s of $H_0(z)$, is given by,

$$\omega_p + \omega_s = \pi. \tag{67}$$

Additionally, using (66) and (67) we have

$$10^{-A_p/10} + 10^{-A_s/10} = 1, \tag{68}$$

where A_p and A_s are the passband droop and the stopband attenuation in dB.

According to (Zhang et al., 2001), the analysis filters are given by,

$$H_0(z) = \frac{1}{2}\left[A(z) + \widetilde{A}(z)\right], \tag{69}$$

$$H_1(z) = \frac{1}{2j}\left[A(z) - \widetilde{A}(z)\right], \tag{70}$$

where $A(z)$ is a complex allpass filter and $\widetilde{A}(z)$ is its paraconjugate.

From (69) and (70), we can see that the design of perfect reconstruction filter banks is reduced to the design of the complex allpass filter $A(z)$. In the following, we present one method for the modified two-band filter bank design based on the results obtained in Section 5.1.

The perfect reconstruction condition for the modified two-band IIR filter banks is established in (Vaidyanathan et al., 1987; Zhang et al., 2001), which implies that the poles of $H_0(z)$ and $H_1(z)$ must appear on the imaginary axis and in pairs jp and $1/jp$, where p is a pole. From this condition, it follows that the filter coefficients given in (34) must be imaginary for even values of n (Vaidyanathan et al., 1987).

Consequently, the values of ϕ_α in (34) for an even N, must be

$$\phi_\alpha = \begin{cases} -\frac{7}{8}\pi & \text{for } \frac{N}{2} \text{ even,} \\ -\frac{3}{8}\pi & \text{for } \frac{N}{2} \text{ odd.} \end{cases} \tag{71}$$

Similarly, the values of ϕ_α when N is odd must be,

$$\phi_\alpha = \begin{cases} -\frac{3}{8}\pi & \text{for } \frac{N+1}{2} \text{ even,} \\ -\frac{7}{8}\pi & \text{for } \frac{N+1}{2} \text{ odd.} \end{cases} \tag{72}$$

5.2.1 Description of the algorithm

In the following, we describe the proposed algorithm for a linear-phase IIR filter banks. The IIR filters are real if N is even, otherwise they are complex.

The steps of the algorithm are described in the following

Step 1. Calculate the order N of the allpole filter using (68), (67) and (59). (Note that the filter $H_0(z)$ has order $2N$.)

Step 2. If N is even compute the filter coefficients (34) using (71), otherwise use (72).

We illustrate the method with the following examples.

Example 8. Stopband frequency ω_s of the analysis filter $H_0(z)$ is 0.65π, while the stopband attenuation A_s is 45 dB.

Step 1. From (68) and (67), it follows that $A_p = 1.373381 \times 10^{-4}$ and $\omega_p = 0.35\pi$. Using (59), the order of the complex allpole filter is 12. From (71), $\phi_\alpha = -\frac{7}{8}\pi$.

Step 2. We compute the filter coefficients using (34) and (71).

Figure 10a shows the corresponding magnitude responses of $H_0(z)$ and $H_1(z)$.

The following example illustrates the design of IIR filter banks using complex IIR filters. **Example 9.** We design the IIR filter bank with the following specifications $\omega_s = 0.75\pi$ and $A_s = 60$ dB for the analysis filter $H_0(z)$.

Step 1. The estimated value of N is 9 since $\omega_p = 0.25\pi$ and $A_p = 4.342946 \times 10^{-6}$. From (72), $\phi_\alpha = -\frac{3}{8}\pi$.

Step 2. Using (34) and (71), we compute the allpole filter coefficients.

The magnitude responses of $H_0(z)$ and $H_1(z)$ are shown in Fig. 10b. From Fig. 10b, (57), and (58), we note that both filters $H_0(z)$ and $H_1(z)$ have notch frequencies at $\omega_0 = 1.512254\pi$ and $\omega_1 = 1.487745\pi$, respectively.

In general, for N odd, both analysis filters have notch frequencies in the vicinity of $\omega = 3\pi/2$.

(a) Example 8   (b) Example 9
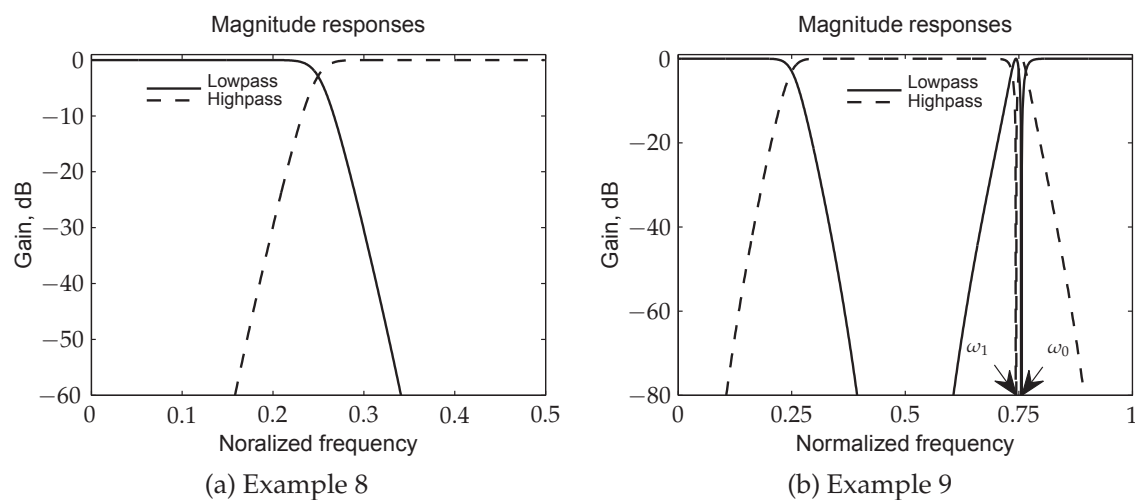
Fig. 10. Magnitude responses of the analysis filters in Examples 8 and 9.

5.3 Butterworth filters with an improved group delay
5.3.1 Linear-phase Butterworth filters

We relate the linear-phase Butterworth filter given in Section 5.1 with the corresponding stable and causal IIR filter.

We remember that a linear phase filter $H(z)$ can be expressed as

$$H(z) = cz^{-k}\widetilde{H}(z), \tag{73}$$

where z^{-k} is the delay, c is a rear or complex constant with unit magnitude and $\widetilde{H}(z)$ is the paraconjugate of $H(z)$.

Using $k = 0$ and $c = 1$, the linear-phase IIR filter $H(z)$ can be expressed as (Powell & Chau, 1991),

$$H(z) = H_c(z)\widetilde{H}_c(z). \tag{74}$$

where $H_c(z)$ is a causal and stable IIR filter. Consequently, the corresponding Fourier transform $H(e^{j\omega})$ is real and positive for all ω.

We note that the linear-phase filter $H(e^{j\omega})$ given in (53) takes both positive and negative values for N odd (see (58)). However, when N is even, the function $H(e^{j\omega})$ take only positives values. Therefore, the condition (74) is satisfied.

In the following we design the filter $H_c c(z)$ from (53).

From (53)–(54), it is easily shown that the polynomials $E(z)$ and $F(z)$ are symmetric for even values of N. Consequently, they can be expressed as,

$$E(z) = z^{-N/2} E_0(z) E_0(z^{-1}),\tag{75}$$

$$F(z) = z^{-N/2} F_0(z) F_0(z^{-1}),\tag{76}$$

where $E_0(z)$ and $F_0(z)$ are subfilters of $E(z)$ and $F(z)$, respectively.

From (75) and (76), the transfer function $H(z)$ (see (83)) can be rewritten as

$$H(z) = \frac{z^{-N/2}(1+z^{-1})^N E_0(z) E_0(z^{-1})}{2z^{-N} F_0(z) \widetilde{F}_0(z) F_0(z^{-1}) \widetilde{F}_0(z^{-1})}.\tag{77}$$

Using (77) and (74), it follows

$$H_c(z) = \frac{(1+z^{-1})^{N/2} E_0(z)}{\sqrt{2} F_0(z) \widetilde{F}_0(z^{-1})}.\tag{78}$$

Therefore, the transfer function $H_c(z)$ has $N/2$ zeros at $z = -1$.

There exist many polynomials $E_0(z)$ and $F_0(z)$ satisfying (78). In order that $H_c(z)$ be stable, all zeros of $F_0(z)$ must be inside the unit circle. Moreover, it can be shown that for the given value of N, there exists

$$N_{\text{poly}} = 2^{\lfloor N/4 \rfloor},\tag{79}$$

polynomials $E_0(z)$.

In the following example we design IIR filter $H(z)$ with linear-phase and the corresponding filter $H_c(z)$ using (77) and (78).

Example 10. We design an IIR filter $H(z)$ with the passband frequency, $\omega_p = 0.3\pi$, and passband droop, $A_p = 1$ dB. The allpole filter order is 8.

Consequently, $H(z)$ has 8 zeros at $z = -1$, while the remaining zeros z_k, $k = 0, \dots, 7$, and poles p_k, $k = 0, \dots, 2N - 1$, are calculated using (55) and (38), respectively.

According to (79), there are four different polynomials for $E_0(z)$. The zeros of the first polinomial $E_0^{(1)}(z)$ are z_0, z_0^*, z_1, and z_1^*. Similarly, we can obtain the polynomials $E_0^{(l)}(z)$, $l = 2, 3, 4$, which shown in Table 4.

	$z_0, z_7 = 1/z_3 = z_0^*$	$z_1, z_6 = 1/z_2 = z_1^*$	$z_2, z_5 = 1/z_1 = z_2^*$	$z_3, z_4 = 1/z_0 = z_3^*$
$E_0^{(1)}(z)$	×	×		
$E_0^{(2)}(z)$	×		×	
$E_0^{(3)}(z)$		×		×
$E_0^{(4)}(z)$			×	×

Table 4. Different polynomials for $E_0(z)$.

The group delays of $H_c(z)$ for all $E_0^{(l)}(z)$, $l = 1, \dots, 4$, are shown in Fig. 11.
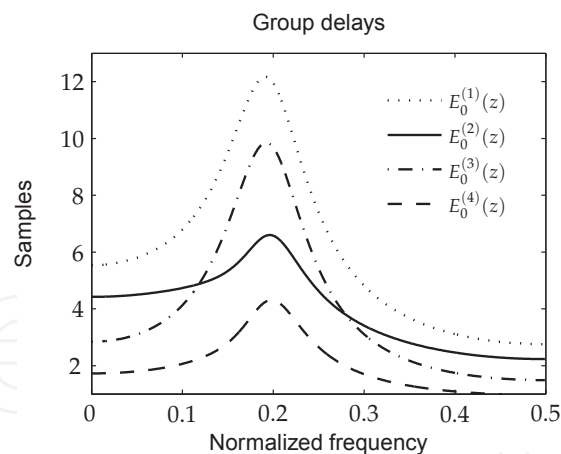
Fig. 11. Different group delays for the IIR filter $H_c(z)$ in Example 10.

We have different degrees of nonlinearity as illustrated in Fig. 11 for different $E_0^{(l)}, l = 1, \ldots, 4$. In this example, the group delay for $E_0^{(2)}(z)$ is more linear than the others. Therefore, for this example the best polynomial is $E_0^{(2)}(z)$.

The next issue is how to select in general case the best polynomial for $E_0(z)$.

Numerous examples indicate that it is necessary to satisfy the following two conditions:

1. The number of zeros of $H_c(z)$ inside and outside the unit circle, N_i and N_o, respectively, are related as

$$N_i \geq N_o, \tag{80}$$

   where

$$N_o = \begin{cases} \left\lceil \frac{N}{2} \right\rceil & \text{if } \left\lceil \frac{N}{2} \right\rceil \text{ has the same parity of } N, \\ \left\lceil \frac{N}{2} \right\rceil - 1 & \text{if } \left\lceil \frac{N}{2} \right\rceil - 1 \text{ has the same parity of } N. \end{cases} \tag{81}$$

2. For each zero z_m inside, and each zero z_l outside of the unit circle, we have

$$\left| \frac{1}{z_l} \right| < |z_m|. \tag{82}$$

5.3.2 Description of the algorithm

The design parameters of the algorithm are passband and stopband frequencies, ω_p and ω_s, respectively, the passband droop A_p and the stopband attenuation A_s.

The algorithm has the following design steps:

Step 1. We estimate the order of the allpole filter $D(z)$ using results from Section 5.1,

$$N = \left\lfloor \frac{\log_{10} \left(\frac{A_p{}''}{A_s{}''} \right)}{\log_{10} \left(\frac{\omega_s{}'}{\omega_p{}'} \right)} \right\rfloor, \quad A_p{}'' = \sqrt{\frac{10^{A_p/10} + 1}{10^{A_p/10} - 1}} - 1, \quad A_s{}'' = \sqrt{\frac{10^{A_s/10} + 1}{10^{A_s/10} - 1}} - 1. \tag{83}$$

If the estimation filter order N is odd, increase it by one.

Step 2. Using the estimated value of N, we calculate the value of the phase $\phi_\alpha(N, \omega_p, A_p)$ as,

$$\phi_\alpha(N, \omega_p, A_p) = \angle \left\{ -j - 1 - (-1)^{N/2} \tan^N \left(\frac{\omega_p}{2} \right) A_p{}'' \right\}. \tag{84}$$

Step 3. Compute poles and zeros of $H_c(z)$ as indicated in the following

1. The zeros and poles of $H(z)$ are obtained using (55), (56), (38) and (39), respectively.

2. The N poles of $H(z)$, which are inside the unit circle, become poles of $H_c(z)$.

3. We select $N/2$ zeros of $H(z)$ which satisfy conditions (80)–(82) to be zeros of the filter $H_c(z)$ and the others $N/2$ zeros are at $z = -1$.

Step 4. Using the MATLAB function poly.m, we find the transfer function $H_c(z)$.

Example 11. We design the IIR filter with the following specifications: the passband and stopband frequencies are 0.25π and 0.55π, respectively; the passband droop and stopband attenuation are 1 dB and 50 dB, respectively.

Step 1. From (83), it follows that $N = 12$.

Step 2. Using the estimated value N and (84) we have, $\phi_\alpha(N, \omega_p, A_p) = -0.75\pi$.

Step 3. The resulting pole-zero pattern of $H_c(z)$ is shown in Fig. 12a.

Step 4. We compute the filter coefficients of the transfer function $H_c(z)$.
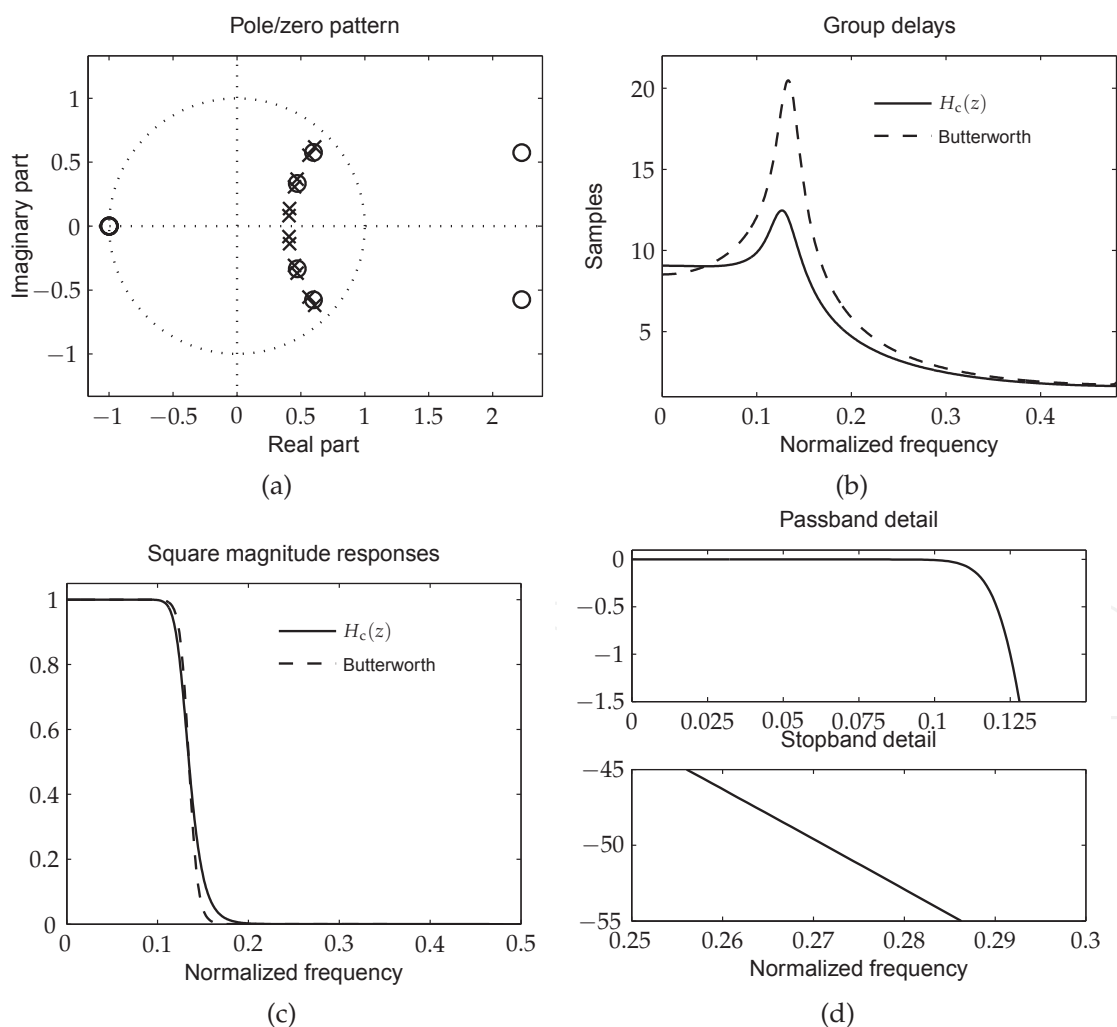
Fig. 12. Example 11.

The group delay of the designed filter is shown in Fig. 12b, while Figs. 12c and 12d present the magnitude response.

We compare our result with the traditional IIR Butterworth filter using the following specification: the filter order is equal to 12 and $\omega_c = 0.2689\pi$. Figure 12b and 12c show the group delays and magnitude responses of the Butterworth filter and the proposed one.

Notice that the proposed filter $H_c(z)$ has a better group delay than the traditional Butterworth filter.

5.4 Complex wavelet IIR filters

The main idea is to generalize the design of real IIR wavelets presented in (Phoong et al., 1995; Selesnick, 1998) and (Zhang et al., 2006) in a way that the complex case is also included. To this end we use the general approach for the complex allpole filter design from Section 4.3.

We generalize the result in (Selesnick, 1998), replacing the real allpass filter $A(z^{-2})$ with the complex allpass filter $\widetilde{A}(z^2)$, i.e.,

$$H_0(z) = \frac{1}{2}\left[A(z^2) + z^{-2M+1}\widetilde{A}(z^2)\right],\tag{85}$$

$$H_1(z) = \frac{1}{2}\left[A(z^2) - z^{-2M+1}\widetilde{A}(z^2)\right],\tag{86}$$

where $H_0(z)$ and $H_1(z)$ are lowpass and highpass filters, respectively, and M is arbitrary integer.

Knowing that $A(z) = 1/\widetilde{A}(z)$ (see (14)), it is easy to verify that $H_0(z)$ can be rewritten as,

$$H_0(z) = \frac{\widetilde{A}(z^2)}{2}\left[z^{-k} + A^2(z^2)\right],\tag{87}$$

where

$$k = 2M - 1.\tag{88}$$

Now, the problem to design complex filters (85) and (86) is reduced to the design of an allpass filter $A(z)$, which has the phase response equal to $-k\omega/4$ near to $\omega = 0$.

The group delay of $A(z)$ at $\omega = 0$ is equal to $k/4$. Then, the corresponding group delay $\tau_A(0)$ of $A(z)$ is expressed as,

$$\tau_{A0} = \frac{2M - 1}{4},\tag{89}$$

where $\tau_{A0} = \tau_A(0)$.

Using (17), the corresponding group delay of the complex allpole filter $D(z)$ is

$$\tau_0 = \frac{2M - 4N - 1}{8}.\tag{90}$$

The design of biorthogonal wavelet filter based on real allpass filter is proposed in (Phoong et al., 1995). The generalization of this result is written in the form

$$H_0(z) = \frac{1}{2}\left[z^{-2M} + z^{-1}A(z^2)\right],\tag{91}$$

$$H_1(z) = -A(z^2)H_0(z) + z^{-4M+1},\tag{92}$$

where $A(z)$ is a complex allpass filter and M is any integer.

In this case, we first design the corresponding allpass filter $A(z)$ having the group delay at $\omega = 0$ equal to,

$$\tau_{A0} = \frac{2M - 1}{2}. \tag{93}$$

The corresponding group delay τ_0 of $D(z)$ is given by, (See (17)),

$$\tau_0 = \frac{2M - 2N - 1}{4}. \tag{94}$$

Finally, the generalization of the orthogonal wavelet filters proposed in (Zhang et al., 2006) is given as,

$$H_0(z) = \frac{1}{2} \left[1 + z^{-2M+1} \widetilde{A}(z^2) \right], \tag{95}$$

$$H_1(z) = \frac{1}{2} \left[z^{-1} - z^{2M} A(z^2) \right]. \tag{96}$$

The group delays of the complex allpass filter $A(z)$ and allpole filter $D(z)$ at $\omega = 0$ are the same as for the filter (91) (see (93) and (94)).

We design the complex allpass filter $A(z)$ using complex Thiran allpole filter $D(z)$ given in Section 4.3, i.e.,

$$A(z) = z^{-N} \frac{D(z)}{\widetilde{D}(z)} == \frac{e^{j\phi_\alpha}}{e^{-j\phi_\alpha}} \frac{f_N^* + f_{N-1}^* z^{-1} + \cdots + z^{-N}}{1 + f_1 z^{-1} + \cdots + f_N z^{-N}}. \tag{97}$$

We can notice that by setting different values of the phase ϕ_α of the corresponding complex allpole filter $D(z)$, we can obtain different types of complex wavelet filters.

The following example illustrates the proposed method.

Example 12. We consider the design of complex wavelet filters using the methods proposed in (Selesnick, 1998), (Phoong et al., 1995) and (Zhang et al., 2006). We design a complex allpass filter of order $N = 6$, the phase value at $\omega = 0$ is equal to $\phi_\alpha = -\pi/5$, and delay $k = 1$. Therefore, according to (88), $M = 1$. Additionally, the degree of flatness K in this example is 9 and the filter coefficients are computed using (33).

Substituting the values of M and N into (90) and (94) we compute different group delays of the allpole filter $D(z)$. In particular, we denote the group delay of $D(z)$ based on equations (85) and that based on (86) as τ_1, and on (91) and (92) as τ_2. For the design based on (95) and (96) we have $\tau_3 = \tau_2$. The values of τ_1, τ_2 and τ_3 are -2.875, -2.75 and -2.75 samples, respectively. Substituting the values of τ_1 and τ_2 and the value of ϕ_α into (29), we compute the corresponding filter coefficients.

The magnitude responses of the complex wavelet filters based on (85) and (86) are shown in Fig. 13a, while the magnitude responses of the complex wavelet filters based on (91) and (92), and (95) and (96) are shown in Fig. 14b and 14c, respectively.

5.5 Fractional Hilbert transformers

Fractional Hilbert transform has applications in digital communications and signal processing (Pei & Yeh, 2000; Tseng & Pei, 2000). There exist different techniques for designing fractional Hilbert transformers (FHT) (Pei & Wang, 2002; Tseng & Pei, 2000). Here, we describe a direct method for the design of FHT. The method is based on the design of an allpass filter with desired characteristic.
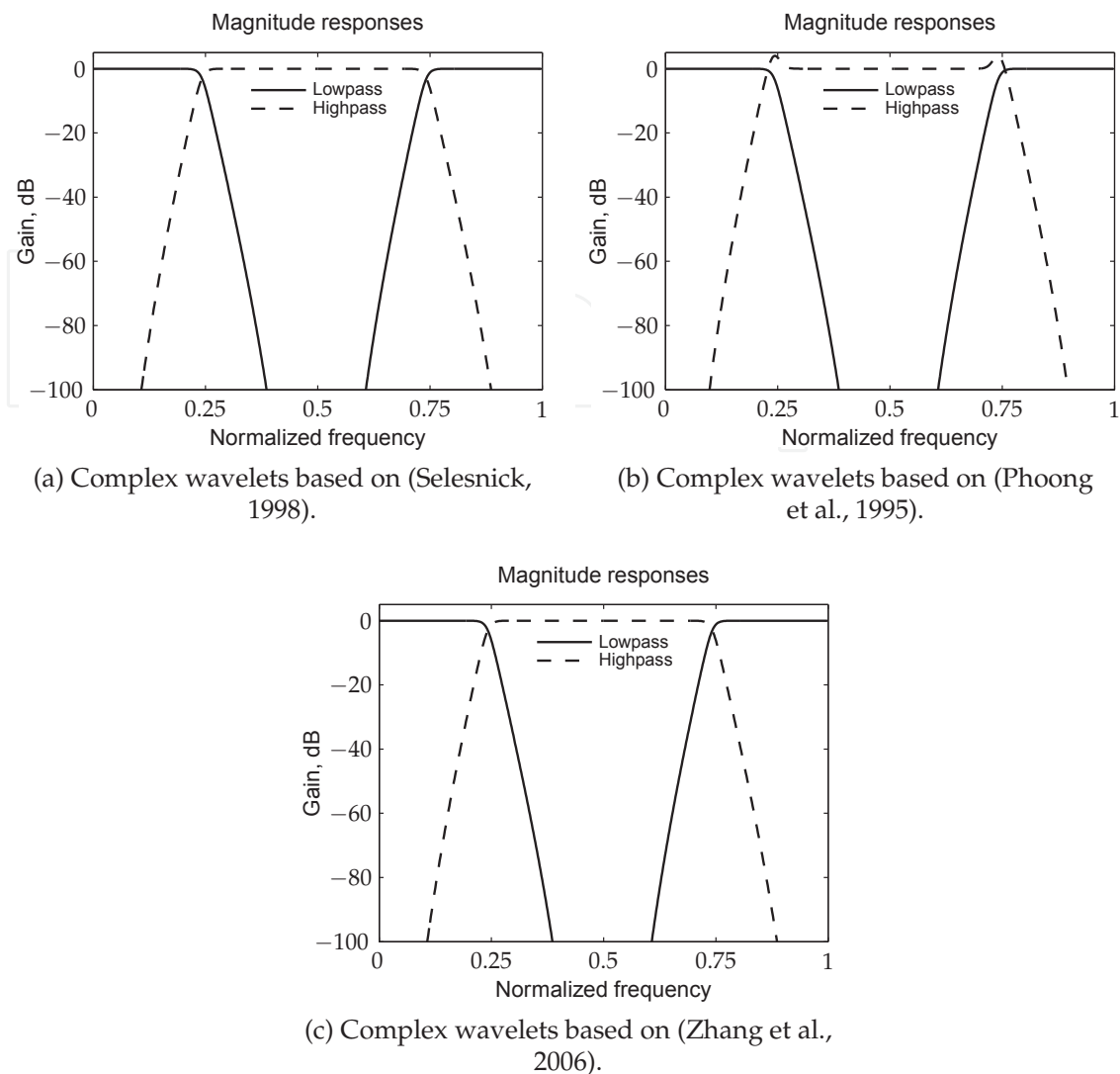
(a) Complex wavelets based on (Selesnick, 1998).

(b) Complex wavelets based on (Phoong et al., 1995).

(c) Complex wavelets based on (Zhang et al., 2006).

Fig. 13. Magnitude response of the designed wavelet filters in Example 12.

The fractional Hilbert transformer is defined as (Pei & Wang, 2002),

$$
H_\beta(\omega) = \begin{cases} e^{-j\frac{\beta\pi}{2}} & 0 \leq \omega < \pi, \\ e^{j\frac{\beta\pi}{2}} & -\pi \leq \omega < 0, \end{cases} \tag{98}
$$

where β, satisfying $0 \leq \beta \leq 1$, is the fraction of the Hilbert transformer.
We can see from (98) that the magnitude response of $H_\beta(\omega)$ is 1 and the phase response is given by,

$$
\angle H_\beta(\omega) = \begin{cases} -\frac{\beta\pi}{2} & 0 \leq \omega < \pi, \\ \frac{\beta\pi}{2} & -\pi \leq \omega < 0. \end{cases} \tag{99}
$$

Therefore, the design of FHT is reduced to the design of an allpass filter $A(z)$ with the phase response given in (99). If the allpass filter has real coefficients, it is well known that, its phase response is an odd function of ω, (Mitra, 2005). Consequently, the allpass filter needs to satisfy (99) only $0 \leq \omega < \pi$.

We use a second order allpass filter and $\omega = \pi/2$ to design the FHT. Therefore, for an stable allpass filter, the phase $\phi_A(\omega)$ and group delay $\tau_A(\omega)$ must be $-\ell\omega - \beta\pi/2$ and ℓ, respectively, where ℓ is a positive integer.

Consequently, from (16) and (18), the design parameters of the corresponding allpole filter are

$$\phi_D(\pi/2) = \frac{(2 - \ell - \beta)\pi}{4}, \tag{100}$$

$$\tau(\pi/2) = \ell/2 - 1. \tag{101}$$

To ensure stability, from (25), we have

$$\ell > |\sin((\ell + \beta)\pi/2)|. \tag{102}$$

Substituting (100) and (101) into (23) and (24), we get

$$f_{R1} = \frac{2\sin((\ell + \beta)\pi/2)}{\ell + 1 - \cos((\ell + \beta)\pi/2)}, \tag{103}$$

$$f_{R2} = \frac{\ell - 1 + \cos((\ell + \beta)\pi/2)}{\ell + 1 - \cos((\ell + \beta)\pi/2)}. \tag{104}$$

Example 13. The design parameter for the fractional Hilbert transformer are $\beta = 0.2, 0.4, 0.6, 0.8, 1$ and $\ell = 2$. The resulting phase responses are shown in Fig. 14.
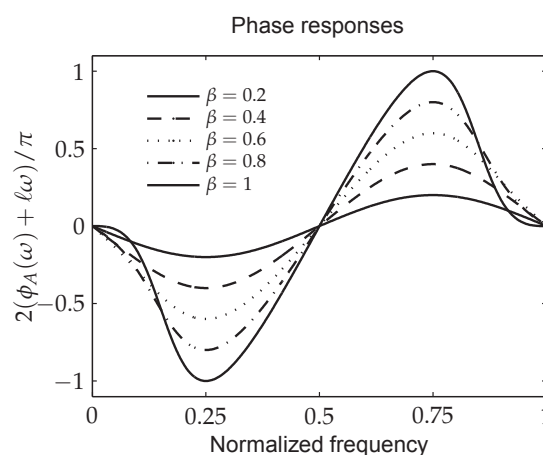

Fig. 14. Designed Fractional Hilbert transformers.

5.6 New design of IIR Butterworth-like filters based on three allpass filters

We address the magnitude approximation of real-valued lowpass Butterworth-like filters based on a new parallel connection of three allpass filters, that is, the proposed structure is composed by one real- and two complex-valued allpass filters. The design problem of lowpass filter is reduced further to design one complex-valued allpole filter with desired characteristics.

The proposed IIR filter is given by

$$H(z) = \frac{1}{3}\left[A_0(z) + A_1(z) + \widetilde{A}_1(z^{-1})\right]. \tag{105}$$

The allpass filters $A_0(z)$ and $A_1(z)$ must be stable in order that the filter $H(z)$ be stable.

We show that the problem of designing lowpass and stable IIR filter is reduced to designing a complex-valued allpole filter with desired characteristics.

At first, notice that (105) can be rewritten as

$$H(z) = \frac{A_0(z)}{3} \left[1 + A(z) + \widetilde{A}(z^{-1}) \right],$$ (106)

where $A(z) = A_1(z)/A_0(z)$ is a complex allpass filter, which can have poles outside the unit circle due to the zeros of $A_0(z)$. The complex allpass filter is defined by (14).

In the following, some characteristics of $A(z)$ are described.

From (106), the corresponding magnitude response of $H(z)$ is

$$|H(\mathrm{e}^{\mathrm{j}\omega})| = \frac{1}{3} \left| 1 + \mathrm{e}^{\mathrm{j}\phi_A(\omega)} + \mathrm{e}^{-\mathrm{j}\phi_A(-\omega)} \right|,$$ (107)

where $\phi_A(\omega)$ is the phase response of $A(z)$.

In order that $|H(\mathrm{e}^{\mathrm{j}\omega})|$ has the value 1 in the passband and the value 0 in the stopband (ideal case), the condition $\phi_A(\omega) = \phi_A(-\omega)$ must be satisfied, that is, the phase response must be an even function of ω.

Using this property, it follows that $A(z) = A(z^{-1})$. Consequently, the magnitude response $|H(\mathrm{e}^{\mathrm{j}\omega})|$ becomes

$$|H(\mathrm{e}^{\mathrm{j}\omega})| = \frac{1}{3} |1 + 2\cos(\phi_A(\omega))|.$$ (108)

Considering the passband edge frequency ω_{p} and the attenuation in dB at this frequency point A_{p}. From (108) we define

$$\phi_{pA} = \cos^{-1}\left(\frac{3 \cdot 10^{-A_{\mathrm{p}}/20} - 1}{2} \right),$$ (109)

which gives the desired phase $\phi_A(\omega)$ at ω_{p}, i.e., $\phi_{pA} = \phi_A(\omega_{\mathrm{p}})$.

In order to achieve the condition $A(z) = A(z^{-1})$, the corresponding filter coefficients f_n, $n = 0, \ldots, N$, need to satisfy $f_n = f_{N-n}$, i.e., they must be a symmetric sequence. Generally, there are two cases that should be considered: N odd and N even. However, one can verify that N odd implies that at least one pole of $A(z)$ must be on the unit circle. As a consequence, in our design, we only consider the case where N is even.

Based on (108), We define the following properties for $A(z)$:

$\mathcal{F}$.1 We select three frequency points where the phase response $\phi_A(\omega)$ is flat, i.e., $\omega = 0$ for the passband and $\omega = \pm\omega_{\mathrm{r}}$ for the stopband. Furthermore, $\phi_A(0) = 0$ and $\phi_A(\pm\omega_{\mathrm{r}}) = 2\pi/3$. This condition ensures that the filter $H(z)$ has flat magnitude response at $\omega = 0$ and $\omega = \omega_{\mathrm{r}}$.

$\mathcal{F}$.2 The group delay τ_A at $\omega = 0$ and $\omega = \omega_{\mathrm{r}}$ is 0.

$\mathcal{F}$.3 The phase value ϕ_{pA} is controlled by A_{p} (see (109)).

We relate the allpass filter $A(z)$ with the corresponding allpole filter.

Using (16) and the phase values $\phi_A(\omega)$ at $\omega = 0$ and $\omega = \pm\omega_{\mathrm{r}}$ (see Condition $\mathcal{F}$.1), we get $\phi_D(0) = 0$ and $\phi_D(\pm\omega_{\mathrm{r}}) = \pi/3 \pm \omega_{\mathrm{r}}N/2$.

From (18) and Condition $\mathcal{F}$.2, we have $\tau(0) = \tau(\pm\omega_{\mathrm{r}}) = -N/2$.

Finally, the following relation is obtained using Condition $\mathcal{F}$.3 and (16),

$$\phi_D(\omega_{\mathrm{p}}) = \phi_{\mathrm{p}} = \frac{\cos^{-1}\left(\frac{3 \cdot 10^{-A_{\mathrm{p}}/20} - 1}{2}\right) + \omega_{\mathrm{p}} N}{2}. \tag{110}$$

As a consequence, the allpole filter $D(z)$ has to satisfy the following conditions:

$\mathcal{G}$.1 The phase values of $D(z)$ at $\omega = 0$ and $\omega = \pm\omega_{\mathrm{r}}$ are 0 and $\pi/3 \pm \omega_{\mathrm{r}} N/2$, respectively.

$\mathcal{G}$.2 The group delay $\tau(\omega)$ of $D(z)$ at $\omega = 0$ and $\omega = \pm\omega_{\mathrm{r}}$ are $-N/2$.

$\mathcal{G}$.3 The phase value of $D(z)$ at ω_{p}, $\phi_D(\omega_{\mathrm{p}})$, is given by (110).

For a filter having coefficients given in (40) the Conditions $\mathcal{G}$.1 and $\mathcal{G}$.2 are satisfied. From the Condition $\mathcal{G}$.3 and (41), the corresponding value of ϕ_α becomes

$$\phi_\alpha(\omega_{\mathrm{p}}, A_{\mathrm{p}}, \omega_{\mathrm{r}}) = \angle\left\{R_{\mathrm{p}} A_{\mathrm{p}}' + 1 + \mathrm{j}\sqrt{3}(R_{\mathrm{p}} + 1)\right\}, \tag{111}$$

where

$$A_{\mathrm{p}}' = \sqrt{\frac{1 + 3 \cdot 10^{-A_{\mathrm{p}}/20}}{1 - 10^{-A_{\mathrm{p}}/20}}}. \tag{112}$$

In order to find the value ω_{r} and the order of the allpole filter N, we solve the following set of nonlinear equations:

$$\phi_\alpha(\omega_{\mathrm{p}}, A_{\mathrm{p}}, \omega_{\mathrm{r}}) - \phi_\alpha(\pi, A_{\mathrm{s}}, \omega_{\mathrm{r}}) = 0, \tag{113}$$

$$\phi_\alpha(\omega_{\mathrm{p}}, A_{\mathrm{p}}, \omega_{\mathrm{r}}) - \phi_\alpha(\omega_{\mathrm{s}}, A_{\mathrm{s}}, \omega_{\mathrm{r}}) = 0. \tag{114}$$

Finally, we wish to find the allpass filters $A_0(z)$ and $A_1(z)$. First note that $F(z)$, the z-transform of f_n, can be rewritten as $F(z) = z^{-N/2} F_2(z^{-1}) F_2(z)/\beta$, where $F_2(z)$ is a polynomial with all zeros inside the unit circle, i.e., $F_2(z) = 1 + f_{2,1} z^{-1} + \cdots + f_{2,N/2} z^{-N/2}$, and $\beta = f_{2,N/2}$. Accordingly, the corresponding allpass filters are expressed as,

$$A_0(z) = z^{-N} \frac{F_2(z^{-1})\widetilde{F}_2(z)}{\widetilde{F}_2(z^{-1})F_2(z)}, \quad A_1(z) = z^{-N} \frac{\alpha}{\alpha^*} \frac{\beta}{\beta^*} \frac{\widetilde{F}_2^2(z)}{F_2^2(z)}. \tag{115}$$

Example 14. We design the IIR filter based on three allpass filter using the following specification: $\omega_{\mathrm{p}} = 0.3\pi$, $\omega_{\mathrm{s}} = 0.55\pi$, $A_{\mathrm{p}} = 0.5$ dB, and $A_{\mathrm{s}} = 45$ dB.
From (111)–(114), it follows that $N = 6$, $\omega_{\mathrm{r}} = 0.641272\pi$, and $\phi_\alpha(\omega_{\mathrm{p}}, A_{\mathrm{p}}, \omega_{\mathrm{r}}) = 0.407889\pi$. Figure 15 shows the magnitude response of the designed IIR filter.

6. Conclusions

In this chapter, we have proposed a new general framework to designing real and complex allpole filters with given degree of flatness, and with phase and group delays at any desired set of frequency points. The filter coefficients are obtained by solving a set of linear equations. In the proposed allpole filter design, we can control the phase, group delay, and degree of flatness at different frequency points. Consequently, as demonstrated here, our proposal is useful for special IIR filter designs, i.e., linear-phase Butterworth-like filter, Butterworth-like filters with improved group delay, complex wavelet filters, fractional Hilbert transformers, and new IIR filters based on three allpass filters.
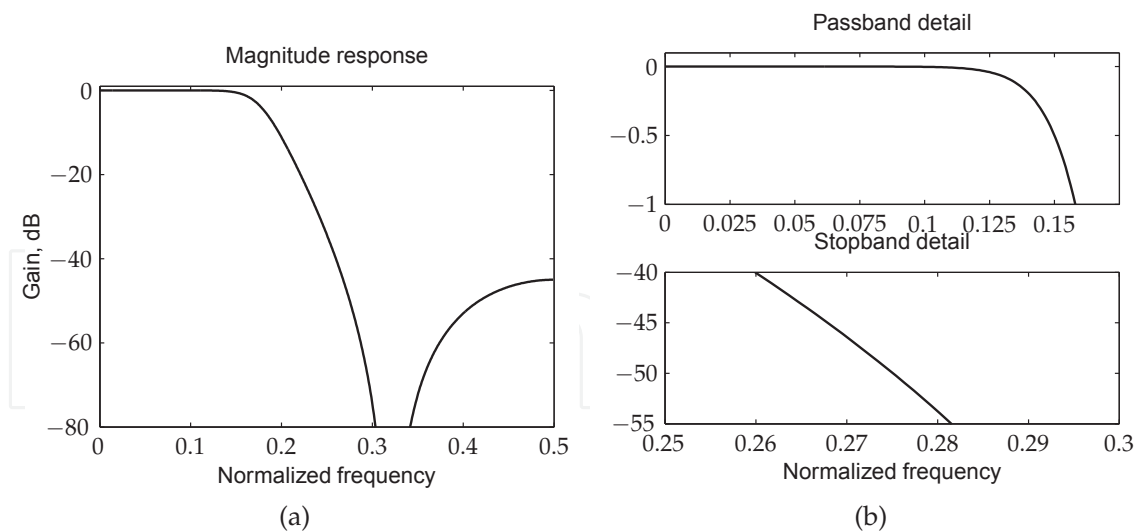
Fig. 15. Magnitude response of the designed IIR filter in Example 14.

Our approach is also useful for the direct design of causal Butterworth filters (Fernandez-Vazquez & Jovanovic-Dolecek, 2006) and higher order digital audio equalizers (Fernandez-Vazquez et al., 2007).

As a future work, we will turn our attention to other interesting applications of our proposed design.

7. Acknowledgments

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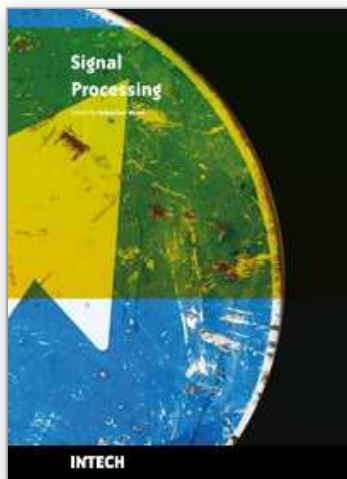
Signal Processing

Edited by Sebastian Miron

This book intends to provide highlights of the current research in signal processing area and to offer a snapshot of the recent advances in this field. This work is mainly destined to researchers in the signal processing related areas but it is also accessible to anyone with a scientific background desiring to have an up-to-date overview of this domain. The twenty-five chapters present methodological advances and recent applications of signal processing algorithms in various domains as telecommunications, array processing, biology, cryptography, image and speech processing. The methodologies illustrated in this book, such as sparse signal recovery, are hot topics in the signal processing community at this moment. The editor would like to thank all the authors for their excellent contributions in different areas of signal processing and hopes that this book will be of valuable help to the readers.

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