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Differential Sandwich Theorems with Generalised Derivative Operator

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1. Introduction

Denote by $\mathbb{U}$ the unit disk of the complex plane:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the space of analytic function in $\mathbb{U}$. Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{U}),\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\},$$

for $(z \in \mathbb{U})$ with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{U}),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\},\ (z \in \mathbb{U}).$$

If f and g are analytic functions in $\mathbb{U}$, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{U}$ such that $f(z) = g(w(z))$ for $z \in \mathbb{U}$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) = g(\mathbb{U})$.
A function f analytic in $\mathbb{U}$, is said to be convex if it is univalent and $f(\mathbb{U})$ is convex.
Let $p, h \in \mathcal{H}(\mathbb{U})$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p(z)$ satisfies the (second-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad (z \in \mathbb{U}) \tag{1}$$

then $p(z)$ is called a solution of the differential supordination (1). (If $f(z)$ subordinate to $F(z)$, the $F(z)$ superordinate $f(z)$).

An analytic function q is called a subordinant of the differential superodination, or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1) is said to be the best subordinant . (Note that the best subordinant is unique up to a rotation of $\mathbb{U}$).

Recently Miller and Mocanu obtained conditions on *h,q* and Ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \Rightarrow q(z) \prec p(z) \quad (z \in \mathbb{U}) .$$

We now state the following definition.

Definition 1. (Al-Shaqsi & Darus, 2008). Let function f in $\mathcal{A}$, then for $j, \lambda \in \mathbb{N}_0$ and $\beta > 0$, we define the following differential operator

$$\mathfrak{D}^{j}_{\lambda,\beta} f(z) = z + \sum_{n=2}^{\infty} [1 + \beta(n-1)]^j C(\lambda, n) a_n z^n, \quad (z \in \mathbb{U}),$$

where $C(\lambda, n) = \begin{pmatrix} n + \lambda - 1 \\ \lambda \end{pmatrix}$.

Special cases of this operator includes the Ruscheweyh derivative operator $\mathfrak{D}^{o}_{\lambda,1} \equiv R_\lambda$ (Ruscheweyh, 1975), the Sălăgean derivative operator $\mathfrak{D}^{j}_{o,1} \equiv S_j$ (Sălăgean, 1983), the generalized Sălăgean derivative operator (or Al-Oboudi drivetive operator) $\mathfrak{D}^{j}_{o,\beta} \equiv F^{j}_{\beta}$ (Al-Oboudi, 2004) and the generalized Ruscheweyh derivative operator (or Al-Shaqsi-Darus drivative operator) $\mathfrak{D}^{o}_{\lambda,\beta} \equiv K^{\lambda}_{\beta}$ (Darus & Al-Shaqsi, 2006).

For $j, \lambda \in \mathbb{N}_0$ and $\beta > 0$, we obtain the following inclusion relations:

$$\mathfrak{D}^{j+1}_{\lambda,\beta} f(z) = (1-\beta)\mathfrak{D}^{j}_{\lambda,\beta} f(z) + \beta z(\mathfrak{D}^{j}_{\lambda,\beta} f(z))' \tag{2}$$

and

$$z(\mathfrak{D}^{j}_{\lambda,\beta} f(z))' = (1+\lambda)\mathfrak{D}^{j}_{\lambda+1,\beta} f(z) - \lambda \mathfrak{D}^{j}_{\lambda,\beta} f(z) . \tag{3}$$

In order to prove the original results we shall need the following definitions , lemma and theorems.

Definition 2: (Miller & Mocanu, 2000, Definition 2.2b p.21). Denote by Q, the set of all functions q that are analytic and injective on $\overline{\mathbb{U}} - E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ and $Q(1) = Q_1$.

Theorem 1: (Miller & Mocanu, 2000, Theorem 3.4h p.132). Let $q(z)$ be univalent in the unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set

$$\psi(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

1. $\psi(z)$ is starlike univalent in $\mathbb{U}$, and
2. $\operatorname{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0,$ for $z \in \mathbb{U}$.

If p is analytic with $p(0) = q(0), p(\mathbb{U}) \subseteq \mathcal{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{4}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

Definition 3:(Miller & Mocanu, 2000, Definition 2.3a p.27). Let Ω be a set in $\mathbb{C}$, $q(z) \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega$ whenever $r = q(\zeta),\ s = k\zeta q'(\zeta)$, and

$$\operatorname{Re}\left\{\frac{t}{s}+1\right\} \geq k \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}, \quad (z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} - E(q), k \geq n).$$

We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\overline{a}z}$ with $M > 0$ and $|a| < M$, then $q(\mathbb{U}) = \mathbb{U}_M = \{w : |w| < M\}$ $q(0) = a,\ E(q) = \varnothing$ and $q \in Q(a)$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega,q]$, and in the special case when the set $\Omega = \mathbb{U}_M$ the class is simply denote by $\Psi_n[M,a]$.

Theorem 2:(Miller & Mocanu, 2000, Theorem 2.3b p.28). Let $\psi \in \Psi_n[\Omega,q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, satisfies

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

Lemma 1: (Bulboacă, 2002). Let $q(z)$ be convex univalent in the unit disk $\mathbb{U}$ and ϑ and φ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$.
Suppose that

1. $\operatorname{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} > 0$, for $z \in \mathbb{U}$ and
2. $\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ with $p(\mathbb{U}) \subseteq \mathcal{D}$ and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{5}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.
Definition 4:(Miller & Mocanu, 2003, Definition 3 p.817). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\zeta) \in \Omega$ whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$, and

$$\operatorname{Re}\left\{\frac{t}{s}+1\right\} \geq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\}, \quad (z \in \mathbb{U}, \zeta \in \partial\mathbb{U}, 1 \leq n \leq m).$$

In particular, we write $\Psi'_1[\Omega,q]$ as $\Psi'[\Omega,q]$.
Theorem 3:(Miller & Mocanu, 2003, Theorem 1 p.818). Let $\psi \in \Psi'_n[\Omega,q]$ with $q(0) = a$. If the $p(z) \in Q(a)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\right\}.$$

implies $p(z) \prec q(z)$.

2. Subordination Results

Using Theorem 1, we first prove the following theorem.
Theorem 4: Let $j, \lambda \in \mathbb{N}_0, \beta > 0, \delta, \alpha \in \mathbb{C}$ and $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$. Further, assume that

$$\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + \frac{2\delta q(z)}{\alpha} + 1\right\} > 0 \qquad (z \in \mathbb{U}). \tag{6}$$

Let

$$\Psi(j,\lambda,\beta,\delta,\alpha;z) = \frac{\delta[2-\beta(2+\lambda)]}{\beta}\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} + \delta\beta(\lambda+2)(\lambda+1)\frac{\mathfrak{D}^{j}_{\lambda+2,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} - \delta\beta(\lambda+1)^2\frac{\mathfrak{D}^{j}_{\lambda+1,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}$$
$$+[\alpha+\delta(1-\frac{1}{\beta})]\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\right)^2 - \frac{\delta(1-\beta)[1-\beta(\lambda+1)]}{\beta}. \tag{7}$$

If $f(z)\in\mathcal{A}$ satisfies

$$\Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \delta zq'(z) + (\delta+\alpha)(q(z))^2 \tag{8}$$

Then

$$\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.
Proof. Define the function p by

$$p(z) = \frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} \qquad (z\in\mathbb{U})\,. \tag{9}$$

Then the function p is analytic in $\mathbb{U}$ and $p(0)=1$. Therfore, by making use of (2), (3) and (9). we obtain

$$\frac{\delta[2-\beta(2+\lambda)]}{\beta}\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} + \delta\beta(\lambda+2)(\lambda+1)\frac{\mathfrak{D}^{j}_{\lambda+2,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)} - \delta\beta(\lambda+1)^2\frac{\mathfrak{D}^{j}_{\lambda+1,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}$$
$$+[\alpha+\delta(1-\frac{1}{\beta})]\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\right)^2 - \frac{\delta(1-\beta)[1-\beta(\lambda+1)]}{\beta} \tag{10}$$
$$= \delta zp'(z) + (\delta+\alpha)(p(z))^2.$$

By using (10) in (8), we have

$$\delta zp'(z) + (\delta+\alpha)(p(z))^2 \prec \delta zq'(z) + (\delta+\alpha)(q(z))^2\,.$$

By sitting $\theta(w)=\delta w^2$ and $\phi(w)=\alpha$, it can be easily observed that $\theta(w)$ and $\phi(w)$ are anlytic in $\mathbb{C}-\{0\}$ and that $\phi(w)\neq 0$. Hence the result now follows by an application of Theorem 1.
Corollary 1: Let $q(z)=\dfrac{1+Az}{1+Bz}(-1\le B<A\le 1)$ in Theorem 4. Further assuming that (6) holds. If $f(z)\in\mathcal{A}$ then,

$$\Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2 \Rightarrow \frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Also, let $q(z) = \frac{1+z}{1-z}$, then for $f(z) \in \mathcal{A}$ we have,

$$\Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2 \Rightarrow \frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

By taking $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$ $(0 < \mu \le 1)$, then for $f(z) \in \mathcal{A}$ we have,

$$\Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^{2\mu} \Rightarrow \frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu},$$

and $\left(\frac{1+z}{1-z}\right)^{\mu}$ is the best dominant.

Now, the following class of admissile functions is required in our following result.

Defintion 5: Let Ω be a set in $\mathbb{C}$ and $q(z) \in Q_1$. The class of admissible functions $\Phi_{n,1}[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), \qquad v = q(\zeta) + \frac{\beta k \zeta q'(\zeta)}{q(\zeta)} \quad q(\zeta) \neq 0,$$

$$\mathrm{Re}\left\{\frac{(w-v)v}{\beta(v-u)} - \frac{u(\beta+1)-v}{\beta^2}\right\} \ge k\,\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \quad (z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} - E(q), k \ge 1).$$

Theorem 5: Let $\phi \in \Phi_{n,1}[\Omega,q]$. If $f \in \mathcal{A}$ satisfies

$$\left\{\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+3}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}; z\right) : z \in \mathbb{U}\right\} \subset \Omega \tag{11}$$

then $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \prec q(z)$.

Proof. Define function p given by (9). Then by using (2), we get

$$\frac{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)} = p(z) + \frac{\beta zp'(z)}{p(z)} \tag{12}$$

Differenitating logarithmically (12), further computations show that

$$\frac{\mathfrak{D}_{\lambda,\beta}^{j+3}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)} = p(z) + \frac{\beta zp'(z)}{p(z)} + \frac{\beta\left[\left(p(z)+\beta\right)\frac{zp'(z)}{p(z)} + \frac{\beta z^2 p''(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2\right]}{p(z) + \frac{\beta zp'(z)}{p(z)}} \tag{13}$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ \ v = r + \frac{\beta s}{r},\ \ w = r + \frac{\beta s}{r} + \frac{\beta\left[(r+\beta)\frac{s}{r} + \frac{\beta t}{r} - \left(\frac{s}{r}\right)^2\right]}{r + \frac{\beta s}{r}} \tag{14}$$

Let

$$\begin{aligned}&\psi(r,s,t;z)\\ &= \phi(u,v,w;z)\\ &= \phi\left(r, r + \frac{\beta s}{r}, r + \frac{\beta s}{r} + \frac{\beta\left[(r+\beta)\frac{s}{r} + \frac{\beta t}{r} - \left(\frac{s}{r}\right)^2\right]}{r + \frac{\beta s}{r}}; z\right).\end{aligned} \tag{15}$$

Using (9), (12) and (13), from (15), it follows that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) = \phi\left(\frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)}, \frac{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}, \frac{\mathfrak{D}_{\lambda,\beta}^{j+3}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}; z\right). \tag{16}$$

Hence (11) implies $\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega$. The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Defintion 3. For this purpose note that

$$\frac{s}{r} = \frac{(v-u)}{\beta},\ \ \frac{t}{r} = \frac{(w-v)v - (v-u)(u+\beta) + \frac{(v-u)^2}{\beta}}{\beta^2}$$

and thus

$$\frac{t}{s}+1=\frac{(w-v)v}{\beta(v-u)}-\frac{u(\beta+1)-v}{\beta^2}.$$

Hence $\psi\in\Psi[\Omega,q]$ and by Theorem 2, $p(z)\prec q(z)$ or $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\prec q(z)$.

In the case $\Omega\neq\mathbb{C}$ is a simply connected domain with $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, the class $\Phi_{n,1}[h(\mathbb{U}),q]$ is written as $\Phi_{n,1}[h,q]$. The following result is immediate consequence of Theorem 5.

Theorem 6: Let $\phi\in\Phi_{n,1}[h,q]$ with $q(0)=1$. If $f\in\mathcal{A}$ satisfies

$$\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+3}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)};z\right)\prec h(z), \tag{17}$$

then $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\prec q(z)$.

Following similar arguments as in (Miller & Mocanu 2000, Theorem 2.3d, page 30), Theorem 6 can be extended to the following theorem where the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Theorem 7: Let h and q be univalent in $\mathbb{U}$ with $q(0)=1$, and set $q_\rho(z)=q(\rho z)$ and $h_\rho(z)=h(\rho z)$. Let $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ satisfy one of the following conditions:

1. $\phi\in\Phi_{n,1}[h,q_\rho]$ for some $\rho\in(0,1)$, or
2. there exists $\rho_0\in(0,1)$ such that $\phi\in\Phi_{n,1}[h_\rho,q_\rho]$ for all $\rho\in(\rho_0,1)$.

If $f\in\mathcal{A}$ satisfies (17), then $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\prec q(z)$.

The next theorem yields the best dominant of the differential subordination (17).

Theorem 8: Let h be univalent in $\mathbb{U}$, and $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$. Suppose that the differential equation

$$\phi\left(q(z),q(z)+\frac{\beta zq'(z)}{q(z)},q(z)+\frac{\beta zq'(z)}{q(z)}+\frac{\beta\left[\left(q(z)+\beta\right)\frac{zq'(z)}{q(z)}+\frac{\beta z^2q''(z)}{q(z)}-\left(\frac{zq'(z)}{q(z)}\right)^2\right]}{q(z)+\frac{\beta zq'(z)}{q(z)}};z\right)=h(z) \tag{18}$$

has a solution $q(z)$ with $q(0)=1$ and one of the following conditions is satisfied:

1. $q \in Q_1$ and $\phi \in \Phi_{n,1}[h,q]$,
2. q is univalent in $\mathbb{U}$ and $\phi \in \Phi_{n,1}[h,q_\rho]$ for some $\rho \in (0,1)$, or
3. q is univalent in $\mathbb{U}$ there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_{n,1}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (17), then $\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \prec q(z)$, and $q(z)$ is the best dominant.

Proof. Applying the same arguments as in (Miller & Mocanu 2000, Theorem 2.3e, page 31), we first note that $q(z)$ is a dominant from Theorems 6 and 7. Since $q(z)$ satisfies (18), it is also a solution of (17), and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the particular case $q(z) = 1 + Mz,\ M > 0$, the class of admissible functions $\Phi_{n,1}[\Omega, q]$, is simply denoted by $\Phi_{n,1}[\Omega, M]$.

Theorem 9: Let Ω be a set in $\mathbb{C}$, and $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfy the admissibility condition

$$\phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{k\beta Me^{i\theta}}{1 + Me^{i\theta}}, L; z\right) \notin \Omega$$

whenever $z \in \mathbb{U}, \theta \in \mathbb{R}$,with

$$\operatorname{Re}\left\{\begin{array}{l}\beta L\left[(1 + Me^{i\theta})(e^{-i\theta} + M) + \beta kM\right] \\ -\left[3\beta^2 kM(1 + Me^{i\theta}) + \dfrac{\beta(1 + Me^{i\theta})^2(e^{-i\theta} + M)^2 + (\beta kM)^2(\beta - 1)}{e^{-i\theta} + M}\right]\end{array}\right\} \geq k^2\beta^3 M,$$

for all real θ and $k \geq 1$.

If $f(z) \in \mathcal{A}$ satisfies

$$\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+3}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} - 1\right| < M.$$

Proof. Let $q(z) = 1 + Mz,\ M > 0$. A computation shows that the conditions on ϕ implies that it belongs to the class of admissible functions $\Phi_{n,1}[\Omega, M]$. The result follows immediately from Theorem 5.

In the special case $\Omega = q(\mathbb{U}) = \{w : |w - 1| < M\}$, the conclusion of Theorem 9 can be written as

$$\left|\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+3}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)};z\right)-1\right|<M\Rightarrow\left|\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}-1\right|<M.$$

3. Superordination and Sandwich Results

Now, by applying Lemma 1, we prove the following theorem.
Theorem 10: Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0)=1$. Assume that

$$\operatorname{Re}\left\{\frac{2\,(\delta+\alpha)q(z)q'(z)}{\delta}\right\}>0. \tag{19}$$

Let $f(z)\in\mathcal{A}$, $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Further, Let $\Psi(j,\lambda,\beta,\delta,\alpha;z)$ given by (7) be univalent in $\mathbb{U}$ and

$$(\delta+\alpha)(q(z))^2+\delta zq'(z)\prec\Psi(j,\lambda,\beta,\delta,\alpha;z)$$

then

$$q(z)\prec\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},$$

and $q(z)$ the best subordinant.
Proof. Theorem 10 follows by using the same technique to prove Theorem 4 and by an application of Lemma 1.
By using Theorem 10, we have the following corollary.
Corollary 2: Let $q(z)=\frac{1+Az}{1+Bz}(-1\le B<A\le 1)$, $f(z)\in\mathcal{A}$, and $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Further, assuming that (19) satisfies. If

$$\frac{\delta(A-B)z}{(1+Bz)^2}+(\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2\prec\Psi(j,\lambda,\beta,\delta,\alpha;z)\Rightarrow\frac{1+Az}{1+Bz}\prec\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},$$

and $\frac{1+Az}{1+Bz}$, is the best subordinant.

Also, by let $q(z)=\frac{1+z}{1-z}$, $f(z)\in\mathcal{A}$, and $\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Furhter, assuming that (19) satisfies. If

$$\frac{2\delta z}{(1-z)^2}+(\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2\prec\Psi(j,\lambda,\beta,\delta,\alpha;z)\Rightarrow\frac{1+z}{1-z}\prec\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},$$

and $\dfrac{1+z}{1-z}$, is the best subordinant.

Finally, by taking $q(z)=\left(\dfrac{1+z}{1-z}\right)^{\mu}$ $(0<\mu\le 1)$, $f\in\mathcal{A}$, and $\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Furhter, assuming that (19) satisfies. If

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1}+(\delta+\alpha)\left(\frac{1+z}{1-z}\right)^{2\mu}\prec\Psi(j,\lambda,\beta,\delta,\alpha;z)\Rightarrow\left(\frac{1+z}{1-z}\right)^{\mu}\prec\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},\ \text{and}\ \left(\frac{1+z}{1-z}\right)^{\mu}$$

is the best subordinant.

Now we will give the dual result of Theorem 5 for differential superordination.

Definition 5: Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}[q(0),1]$ with $zq'(z)\neq 0$. The class of admissible functions $\Phi'_{n,1}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;\zeta)\in\Omega$$

Whenever

$$u=q(z),\qquad v=q(z)+\frac{\beta zq'(z)}{mq(z)}\quad\left(q(z)\neq 0,\ zq'(z)\neq 0\right),$$

$$\operatorname{Re}\left\{\frac{(w-v)v}{\beta(v-u)}-\frac{u(\beta+1)-v}{\beta^2}\right\}\le\frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},\quad (z\in\mathbb{U},\ \zeta\in\partial\mathbb{U},m\ge 1).$$

Theorem 11: Let $\phi\in\Phi'_{n,1}[\Omega,q]$. If $f(z)\in\mathcal{A}$, $\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}\in Q_1$ and

$$\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+3}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)};z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega\subset\left\{\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)},\frac{\mathfrak{D}^{j+3}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta}f(z)};z\right)z\in\mathbb{U}\right\},\tag{20}$$

implies $q(z)\prec\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{j}_{\lambda,\beta}f(z)}$.

Proof. Let $p(z)$ be defined by (9) and ψ by (15). Since $\phi\in\Phi'_{n,1}[\Omega,q]$, (16) and (20) yield

$$\Omega\subset\left\{\psi(p(z),zp'(z),z^2p''(z);z):z\in\mathbb{U}\right\}.$$

From (14), the admissibility condition for $\phi \in \Phi'_{n,1}[\Omega,q]$ is univalent to the admissibility condition for ψ as given in Definition 4. Hence $\psi \in \Phi'_{n}[\Omega,q]$, and by Theorem 3, $q(z) \prec p(z)$ or $q(z) \prec \dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω ,the the class $\Phi'_{n,1}[h(\mathbb{U}),q]$ is written as $\Phi'_{n,1}[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 11.

Theorem 12: Let $q \in \mathcal{H}[q(0),1]$, $h(z)$ be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{n,1}[h,q]$. If $f \in \mathcal{A}$, $\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \in Q_1$ and

$$\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+3}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+3}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}; z\right), \tag{21}$$

implies $q(z) \prec \dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}$.

Theorems 11 and 12 can only be used to obtain subordinants of differential superordinations of the form 20 or 21. The following theorem proves the existence of the best subordinant of 21 for an appropriate ϕ .

Theorem 13: Let $h(z)$ be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation

$$\phi\left(q(z), q(z)+\frac{\beta z q'(z)}{q(z)}, q(z)+\frac{\beta z q'(z)}{q(z)}+\frac{\beta\left[\left(q(z)+\beta\right)\dfrac{zq'(z)}{q(z)}+\dfrac{\beta z^2 q''(z)}{q(z)}-\left(\dfrac{zq'(z)}{q(z)}\right)^2\right]}{q(z)+\dfrac{\beta z q'(z)}{q(z)}}; z\right)=h(z)$$

has a solution $q \in Q_1$. If $\phi \in \Phi'_{n,1}[h,q]$, $f(z) \in \mathcal{A}$, $\dfrac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)} \in Q_1$, and

$$\phi\left(\frac{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+1}_{\lambda,\beta} f(z)}, \frac{\mathfrak{D}^{j+3}_{\lambda,\beta} f(z)}{\mathfrak{D}^{j+2}_{\lambda,\beta} f(z)}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)}, \frac{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}, \frac{\mathfrak{D}_{\lambda,\beta}^{j+3}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}; z\right),$$

implies $q(z) \prec \dfrac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)}$ and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 8, and is therefore omitted.

Combining Theorems 4 and 10, we obtain the following sandwich-type theorem.

Theorem 14: Let $q_1(z)$ and $q_2(z)$ be convex univalent in $\mathbb{U}$ and satisfies (19) and (6), respectively. If $f(z) \in \mathcal{A}$, $\dfrac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\Psi(j,\lambda,\beta,\delta,\alpha;z)$ given by (7) be univalent in $\mathbb{U}$ and $\delta z q'_1(z) + (\delta+\alpha)(q_1(z))^2 \prec \Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \delta z q'_2(z) + (\delta+\alpha)(q_2(z))^2$ then

$$q_1(z) \prec \frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)} \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.

For $q_1(z) = \dfrac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \dfrac{1+A_2 z}{1+B_2 z}$ where $(-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$, we have the following corollary.

Corollary 3: If $f \in \mathcal{A}$, $\dfrac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and

$$\Psi_1(A_1,B_1,j,\lambda,\beta,\delta,\alpha;z) \prec \Psi(j,\lambda,\beta,\delta,\alpha;z) \prec \Psi_2(A_2,B_2,j,\lambda,\beta,\delta,\alpha;z)$$

then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$$

where

$$\Psi_1(A_1,B_1,j,\lambda,\beta,\delta,\alpha;z) = \frac{\delta(A_1-B_1)z}{(1+B_1 z)^2} + (\delta+\alpha)\left(\frac{1+A_1 z}{1+B_1 z}\right)^2,$$

$$\Psi_2(A_2,B_2,j,\lambda,\beta,\delta,\alpha;z) = \frac{\delta(A_2-B_2)z}{(1+B_2 z)^2} + (\delta+\alpha)\left(\frac{1+A_2 z}{1+B_2 z}\right)^2.$$

Hence $\dfrac{1+A_1 z}{1+B_1 z}$ and $\dfrac{1+A_2 z}{1+B_2 z}$ respectively the best subordinant and the best dominant.

Also, by combining Theorems 6 and 12, we state the following sandwich-type theorem.

Theorem 15: Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, let $h_2(z)$ be an analytic univalent function in $\mathbb{U}$, $q_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{n,1}[h_2,q_2] \cap \Phi'_{n,1}[h_1,q_1]$. If $f \in \mathcal{A}$, $\dfrac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)} \in \mathcal{H}[q(0),1] \cap Q_1$ and

$$\phi\left(\frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)},\frac{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)},\frac{\mathfrak{D}_{\lambda,\beta}^{j+3}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)};z\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z)\prec\phi\left(\frac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)},\frac{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)},\frac{\mathfrak{D}_{\lambda,\beta}^{j+3}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j+2}f(z)};z\right)\prec h_2(z),$$

implies $q_1(z)\prec\dfrac{\mathfrak{D}_{\lambda,\beta}^{j+1}f(z)}{\mathfrak{D}_{\lambda,\beta}^{j}f(z)}\prec q_2(z)$.

Remark 1 : By using the same techniques to prove the earlier results and by using the relation (3), the new resuts will be obtained.

Acknowledgment

The work presented here was fully supported by eSciencefund: 04-01-02-SF0425, MOSTI, Malaysia.

7. Conclusion

There are many other results can be obtained by using the operator studied earlier by the authors (Darus & Al-Shaqsi, 2006).

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Advanced Technologies

Edited by Kankesu Jayanthakumaran

ISBN 978-953-307-009-4
Hard cover, 698 pages
Publisher InTech
Published online 01, October, 2009
Published in print edition October, 2009

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