Chapter

Clairaut Submersion

Sanjay Kumar Singh and Punam Gupta

Abstract

In this chapter, we give the detailed study about the Clairaut submersion. The fundamental notations are given. Clairaut submersion is one of the most interesting topics in differential geometry. Depending on the condition on distribution of submersion, we have different classes of submersion such as anti-invariant, semi-invariant submersions etc. We describe the geometric properties of Clairaut anti-invariant submersions and Clairaut semi-invariant submersions whose total space is a Kähler, nearly Kähler manifold. We give condition for Clairaut anti-invariant submersion to be a totally geodesic map and also study Clairaut anti-invariant submersions with totally umbilical fibers. We also give the conditions for the semi-invariant submersions to be Clairaut map and also for Clairaut semi-invariant submersion to be a totally geodesic map. We also give some illustrative example of Clairaut anti-invariant and semi-invariant submersion.

Keywords: Riemannian submersion, nearly Kähler manifolds, Kähler manifolds, anti-invariant submersion, semi-invariant submersion, clairaut submersion, totally geodesic maps

1. Introduction

Riemannian submersion between two Riemannian manifolds was first introduced by O'Neill [1] and Gray [2]. After that Watson [3] introduced almost Hermitian submersions. Later, the notion of anti-invariant submersions and Lagrangian submersion from almost Hermitian manifolds onto Riemannian manifolds were introduced by Sahin [4] and studied by Taştan [5, 6], Gündüzalp [7], Beri et al. [8], Ali and Fatima [9], in which the fibers of submersion are anti-invariant with respect to the almost complex structure of total manifold. After that several new types of Riemannian submersions were defined and studied such as semi-invariant submersion [10, 11], slant submersion [12, 13], generic submersion [14–17], hemi-slant submersion [18], semi-slant submersion [19], pointwise slant submersion [20–22] and conformal semi-slant submersion [23]. Also, these kinds of submersions were considered in different kinds of structures such as nearly Kähler, Kähler, almost product, para-contact, Sasakian, Kenmotsu, cosymplectic and etc. In book [24], we find the recent developments in this field.

In 1735, A.C. Clairaut [25] obtained the very important result in the theory of surfaces, which is Clairaut's theorem and stated that for any geodesic α on a surface of revolution S, the function $r \sin\theta$ is constant along α, where r is the distant from a point on the surface to the rotation axis and θ is the angle between α and the meridian through α. Bishop [26] introduced the idea of Riemannian submersions and gave a necessary and sufficient conditions for a Riemannian submersion to be Clairaut. Allison [27] considered Clairaut semi-Riemannian submersions and showed that such submersions have interesting applications in the static space-times.

In [28], Tastan and Gerdan gave new Clairaut conditions for anti-invariant submersions whose total manifolds are Sasakian and Kenmotsu and got many interesting results. In [29], Tastan and Aydin studied Clairaut anti-invariant submersions whose total manifolds are cosymplectic. Gündüzalp [30] introduced Clairaut anti-invariant submersions from a paracosymplectic manifold and gave characterization theorems. In [31], Lee et al. studied Clairaut anti-invariant submersions whose total manifolds are Kähler.

Kähler manifolds [32, 33] have an especially rich geometric and topological structure because of Kähler identity. Kähler manifolds are very important in differential geometry, which has applications in several different fields such as supersymmetric gauge theory and superstring theory in theoretical physics, signal processing in information geometry. The simplest example of Kähler manifold is a complex Euclidean space $\mathbb{C}^n$ with the standard Hermitian metric.

Nearly Kähler manifolds introduced by Gray and Hervella [32], are the geometrically interesting class among the sixteen classes of almost Hermitian manifolds. The geometrical meaning of nearly Kähler condition is that the geodesics on the manifolds are holomorphically planar curves. Gray [2] studied nearly Kähler manifolds broadly and gave example of a non-Kählerian nearly Kähler manifold, which is 6-dimensional sphere.

Motivated by this, the authors [34] studied Clairaut anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds with some examples and obtained conditions for Clairaut Riemannian submersion to be totally geodesic map. The authors investigated conditions for the Clairaut anti-invariant submersions to be a totally umbilical map. The authors [34] studied Clairaut semi-invariant submersions from Kähler manifolds onto Riemannian manifolds with some examples. The authors also obtained conditions for Clairaut semi-invariant Riemannian submersion to be totally geodesic map and investigated conditions for the semi-invariant submersion to be a Clairaut map.

2. Almost complex manifold

An almost complex structure on a smooth manifold M is a smooth tensor field φ of type $(1, 1)$ such that $\varphi^2 = -I$. A smooth manifold equipped with such an almost complex structure is called an almost complex manifold. An almost complex manifold (M, φ) endowed with a chosen Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) \tag{1}$$

for all $X, Y \in TM$, is called an almost Hermitian manifold.

An almost Hermitian manifold M is called a nearly Kähler manifold [2] if

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0 \tag{2}$$

for all $X, Y \in TM$. If $(\nabla_X \varphi)Y = 0$ for all $X, Y \in TM$, then M is known as Kähler manifold [33]. Every Kähler manifold is nearly Kähler but converse need not be true.

3. Riemannian submersion

Definition 1.1 [1, 35] Let (M, g_m) and (N, g_n) be Riemannian manifolds, where $\dim(M) = m$, $\dim(N) = n$ and $m > n$. A Riemannian submersion $\pi : M \to N$ is a map of M onto N satisfying the following axioms:

i. π has maximal rank.

ii. The differential π_* preserves the lengths of horizontal vectors.

For each $q \in N$, $\pi^{-1}(q)$ is an $(m-n)$-dimensional Riemannian submanifold of M. The submanifolds $\pi^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and π-related to a vector field X' on N, that is, $\pi_* X_p = X'_{\pi_*(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker\pi_*$ and $(\ker\pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{V}$ and $\mathcal{H}$ are called the vertical vector fields and horizontal vector fields, respectively. So

$$\mathcal{V}_p = T_p\left(\pi^{-1}(q)\right), \qquad \mathcal{H}_p = T_p\left(\pi^{-1}(q)\right)^\perp.$$

The second fundamental tensors of all fibers $\pi^{-1}(q),\ q \in N$ gives rise to tensor field T and A in M defined by O'Neill [1] for arbitrary vector field E and F, which is

$$T_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F, \tag{3}$$

$$A_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{4}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections.

To discuss geodesics, we need a linear connection. We denote the Levi-Civita connection on M by $\hat{\nabla}$ and the adapted connection of the submersion by ∇. From Eqs. (3) and (4), we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W, \tag{5}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + T_V X, \tag{6}$$

$$\nabla_X V = A_X V + \mathcal{V}\nabla_X V, \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \tag{8}$$

for all $V, W \in \Gamma(\ker\pi_*)$ and $X, Y \in \Gamma(\ker\pi_*)^\perp$, where $\mathcal{V}\nabla_V W = \hat{\nabla}_V W$. If X is basic, then $A_X V = \mathcal{H}\nabla_V X$.

It is easily seen that for $p \in M,\ U \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$ the linear operators

$$T_U, A_X : T_pM \to T_pM$$

are skew-symmetric, that is,

$$g(A_X E, F) = -g(E, A_X F)\,\text{and}\,g(T_U E, F) = -g(E, T_U F), \tag{9}$$

for all $E, F \in T_pM$. We also see that the restriction of T to the vertical distribution $T|_{\ker\pi_* \times \ker\pi_*}$ is exactly the second fundamental form of the fibers of π. Since T_U is skew-symmetric, therefore π has totally geodesic fibers if and only if $T \equiv 0$.

Let $\pi : (M, g_m) \to (N, g_n)$ be a smooth map between Riemannian manifolds. Then the differential π_* of π can be observed as a section of the bundle $Hom(TM, \pi^{-1}TN) \to M$, where $\pi^{-1}TN$ is the bundle which has fibers $(\pi^{-1}TN)_x = T_{f(x)}N$. $Hom(TM, \pi^{-1}TN)$ has a connection ∇ induced from the Riemannian connection ∇^M and the pullback connection ∇^N [36, 37]. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(E,F) = \nabla^N_E \pi_* F - \pi_* \left(\nabla^M_E F\right), \quad \text{for all}\, E,F \in \Gamma(TM). \tag{10}$$

We also know that π is said to be totally geodesic map [36] if $(\nabla \pi_*)(E,F) = 0$, for all $E,F \in \Gamma(TM)$.

4. Clairaut submersion from Riemannian manifold

Let S be a revolution surface in $\mathbb{R}^3$ with rotation axis L. For any $p \in S$, we denote by $r(p)$ the distance from p to L. Given a geodesic $\alpha : J \subset \mathbb{R} \to S$ on S, let $\theta(t)$ be the angle between $\alpha(t)$ and the meridian curve through $\alpha(t), t \in I$. A well-known Clairaut's theorem [25] named after Alexis Claude de Clairaut, says that for any geodesic on S, the product $r \sin\theta$ is constant along α, i.e., it is independent of t. For proof, see [38, p.183]. In the theory of Riemannian submersions, Bishop [26] introduced the notion of Clairaut submersion in the following way:

Definition 1.2 [26] A Riemannian submersion $\pi : (M,g) \to \left(N, g_n\right)$ is called a Clairaut submersion if there exists a positive function r on M, which is known as the girth of the submersion, such that, for any geodesic α on M, the function $(r \circ \alpha) \sin\theta$ is constant, where, for any t, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

For further use, we are stating one important result of Bishop.

Theorem 1.1 [26] A curve h in M is a geodesic if and only if $\dot{X} + 2A_X U + T_U U = 0$ and $\nabla_E U + T_U X = 0$, where $\dot{h}(t) = E = X + U$, X is horizontal and U is vertical.

Bishop also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion, which is

Theorem 1.2 [26] Let $\pi : (M,g) \to \left(N, g_n\right)$ be a Riemannian submersion with connected fibers. Then, π is a Clairaut submersion with $r = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -gradf$, where $gradf$ is the gradient of the function f with respect to g.

Proof: Let $\pi : M \to N$ be a Riemannian submersion. For a geodesic h in M, we use $\dot{h}(s) = E = X + U$, where X is horizontal and U is vertical. and $\ell = \left\|\dot{h}(s)\right\|^2$. Let $\theta(s)$ be the angle between $\dot{h}(s)$ and the horizontal space at $h(s)$. Then

$$g(X(s), X(s)) = \ell \cos^2\theta(s), \tag{11}$$

$$g(U(s), U(s)) = \ell \sin^2\theta(s). \tag{12}$$

Differentiating (12), we get

$$g\left(\nabla_{\dot{h}(s)} U(s), U(s)\right) = \ell \sin\theta(s) \cos\theta(s) \frac{d\theta(s)}{ds}. \tag{13}$$

Using Theorem 1.1, (13) becomes

$$-g\left(T_{U(s)} X(s), U(s)\right) = \ell \sin\theta(s) \cos\theta(s) \frac{d\theta(s)}{ds}. \tag{14}$$

Since T_U is skew-symmetric, so form above equation, we have

$$g\left(T_{U(s)} U(s), X(s)\right) = \ell \sin\theta(s) \cos\theta(s) \frac{d\theta(s)}{ds}. \tag{15}$$

Now, π is a Clairaut submersion with $r = e^f$ if and only if $\frac{d}{ds}\left(e^{f \circ h} sin\theta\right) = 0$.
Using (12, 15) in $\frac{d}{ds}\left(e^{f \circ h} sin\theta\right) = 0$, we have

$$g(U(s), U(s))\frac{d}{ds}(f \circ h)(s) + g\left(T_{U(s)}U(s), X(s)\right) = 0, \tag{16}$$

$$g(U(s), U(s))g(gradf, E(s)) + g\left(T_{U(s)}U(s), X(s)\right) = 0. \tag{17}$$

Consider any geodesic h on M with initial vertical tangent vector, so $gradf$ turns out to be horizontal. Therefore, the function f is constant on any fiber, the fibers being connected. Therefore (17) reduces to

$$g(U(s), U(s))g(gradf, X(s)) + g\left(T_{U(s)}U(s), X(s)\right) = 0, \tag{18}$$

$$g(U(s), U(s))gradf + T_{U(s)}U(s) = 0. \tag{19}$$

Setting $U = U_1 + U_2$, where U_1, U_2 are vertical vector fields and using the fact that T is symmetric for vertical vector fields, we obtain

$$g(U_1, U_2)gradf + T_{U_1}U_2 = 0 \tag{20}$$

holds for all vertical vector fields U_1, U_2..

Since the restriction of T to the vertical distribution $T|_{\ker\pi_* \times \ker\pi_*}$ is exactly the second fundamental form of the fibers of π. It means that any fiber is totally umbilical with mean curvature vector field $H = -gradf$.

Conversely, suppose the fibers are totally umbilic with normal curvature vector field $H = -gradf$ so that we have

$$g(U, U)H + T_U U = 0. \tag{21}$$

Since $gradf$ is orthogonal to fibers, so

$$g(U, U)g(gradf, E) = -g(U, U)g(H, X) = -g(T_U U, X). \tag{22}$$

Since (18) holds. so $(r \circ h)sin\theta$ is constant along any geodesic h.

Example 1.1 [24] Consider the warped product manifold $M_1 \times_f M_2$ of Riemannian manifolds (M_1, g_1) and (M_2, g_2), where $f : M_1 \to (0, \infty)$. The fibers of the first projection $p_1 : M_1 \times_f M_2 \to M_1$ are totally umbilical with mean curvature vector field $H = -grad\left(logf^{1/2}\right)$. Thus, if M_2 is connected, p_1 is a Clairaut submersion with $r = f^{1/2}$.

5. Anti-invariant Riemannian submersion

Definition 1.3 [39] Let (M, φ, g) be an almost Hermitian manifold and N be a Riemannian manifold with Riemannian metric g_n. Suppose that there exists a Riemannian submersion $\pi : M \to N$, such that the vertical distribution $\ker\pi_*$ is anti-invariant with respect to φ, i.e., $\varphi\ker\pi_* \subseteq \ker\pi_*^\perp$. Then, the Riemannian submersion π is called an anti-invariant Riemannian submersion. We will briefly call such submersions as anti-invariant submersions.

Let π be an anti-invariant Riemannian submersion from nearly Kähler manifold (M, φ, g_m) onto Riemannian manifold (N, g_n). For any arbitrary tangent vector fields U and V on M, we set

$$(\nabla_U \varphi)V = P_U V + Q_U V \tag{23}$$

where $P_U V, Q_U V$ denote the horizontal and vertical part of $(\nabla_U \varphi)V$, respectively. Clearly, if M is a Kähler manifold then $P = Q = 0$.

If M is a nearly Kähler manifold then P and Q satisfy

$$P_U V = -P_V U, \qquad Q_U V = -Q_V U. \tag{24}$$

Consider

$$(\ker \pi_*)^{\perp} = \varphi \ker \pi_* \oplus \mu,$$

where μ is the complementary distribution to $\varphi \ker \pi_*$ in $(ker\pi_*)^{\perp}$ and $\varphi\mu \subset \mu$.

For $X \in \Gamma(ker\pi_*)^{\perp}$, we have

$$\varphi X = \alpha X + \beta X, \tag{25}$$

where $\alpha X \in \Gamma(\ker \pi_*)$ and $\beta X \in \Gamma(\mu)$. If $\mu = 0$, then an anti-invariant submersion is known as Lagrangian submersion.

5.1 Anti-invariant Clairaut submersions from nearly Kähler manifolds

In this section, we give new Clairaut conditions for anti-invariant submersions from nearly Kähler manifolds after giving some auxiliary results.

Theorem 1.3 [34] Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). If $h : J \subset \mathbb{R} \to M$ is a regular curve and $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, respectively, then h is a geodesic if and only if along h

$$A_X \varphi U + A_X \beta X + T_U \beta X + \mathcal{V} \nabla_X \alpha X + T_U \varphi U + \hat{\nabla}_U \alpha X = 0, \tag{26}$$

$$\mathcal{H}\left(\nabla_{\dot{h}} \varphi U + \nabla_{\dot{h}} \beta X\right) + A_X \alpha X + T_U \alpha X = 0. \tag{27}$$

Proof: Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). Since $\varphi^2 \dot{h} = -\dot{h}$. Taking the covariant derivative of this and using (2), we have

$$(\nabla_{\dot{h}} \varphi)\varphi \dot{h} + \varphi\left(\nabla_{\dot{h}} \varphi \dot{h}\right) = -\nabla_{\dot{h}} \dot{h}. \tag{28}$$

Since $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, that is, $\dot{h} = U + X$. So (28) becomes

$$\begin{aligned} -\nabla_{\dot{h}} \dot{h} &= \varphi(\nabla_{U+X} \varphi(U + X)) + P_{\dot{h}} \varphi \dot{h} + Q_{\dot{h}} \varphi \dot{h} \\ &= \varphi(\nabla_U \varphi U + \nabla_X \varphi U + \nabla_U \varphi X + \nabla_X \varphi X) + P_{\dot{h}} \varphi \dot{h} + Q_{\dot{h}} \varphi \dot{h} \\ &= \varphi(\nabla_U \varphi U + \nabla_X \varphi U + \nabla_U (\alpha X + \beta X) + \nabla_X (\alpha X + \beta X)) \\ &\quad + P_{\dot{h}} \varphi \dot{h} + Q_{\dot{h}} \varphi \dot{h}. \end{aligned} \tag{29}$$

Using (5)–(8) in (29), we get

$$-\nabla_{\dot{h}}\dot{h} = \varphi\big(\mathcal{H}\big(\nabla_{\dot{h}}\varphi U + \nabla_{\dot{h}}\beta X\big) + A_X\alpha X + A_X\beta X + A_X\varphi U \\ + T_U\beta X + T_U\alpha X + \mathcal{V}\nabla_X\alpha X + T_U\varphi U + \hat{\nabla}_U\alpha X\big) + P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}. \tag{30}$$

Let $Y, Z \in TM$. Since $\varphi^2 Z = -Z$, on differentiation, we have

$$\varphi(\nabla_Y\varphi Z) + (\nabla_Y\varphi)\varphi Z = -\nabla_Y Z,$$
$$\varphi^2(\nabla_Y Z) + \varphi(\nabla_Y\varphi)Z + (\nabla_Y\varphi)\varphi Z = -\nabla_Y Z,$$

using (23) in above, we obtain

$$\varphi(P_Y Z + Q_Y Z) = -P_Y\varphi Z - Q_Y\varphi Z. \tag{31}$$

By (31), we have

$$\varphi\Big(P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}\Big) = P_{\dot{h}}\dot{h} + Q_{\dot{h}}\dot{h},$$

since P and Q are skew-symmetric, so

$$\varphi\Big(P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}\Big) = 0. \tag{32}$$

Using (32) and equating the vertical and horizontal part of (30), we obtain

$$\mathcal{V}\varphi\nabla_{\dot{h}}\dot{h} = A_X\varphi U + A_X\beta X + T_U\beta X + \mathcal{V}\nabla_X\alpha X + T_U\varphi U + \hat{\nabla}_U\alpha X,$$
$$\mathcal{H}\varphi\nabla_{\dot{h}}\dot{h} = \mathcal{H}\big(\nabla_{\dot{h}}\varphi U + \nabla_{\dot{h}}\beta X\big) + A_X\alpha X + T_U\alpha X.$$

By using above equations, we can say that h is geodesic if and only if (26, 27) hold.
Theorem 1.4 [34] Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). Also, let $h : J \subset \mathbb{R} \to M$ be a regular curve and $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$. Then π is a Clairaut submersion with $r = e^f$ if and only if along h

$$g(gradf, X)g(U, U) = g\Big(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U\Big).$$

Proof: Let $h : J \subset \mathbb{R} \to M$ be a geodesic on M and $\ell = \left\|\dot{h}(s)\right\|^2$. Let $\theta(s)$ be the angle between $\dot{h}(s)$ and the horizontal space at $h(s)$. Then

$$g(X(s), X(s)) = \ell\cos^2\theta(s), \tag{33}$$
$$g(U(s), U(s)) = \ell\sin^2\theta(s). \tag{34}$$

Differentiating (34), we get

$$2g\Big(\nabla_{\dot{h}(s)}U(s), U(s)\Big) = 2\ell\sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}. \tag{35}$$

Using (1) in (35), we have

$$g\Big(\mathcal{H}\nabla_{\dot{h}(s)}\varphi U(s), \varphi U(s)\Big) - g\Big(\Big(\nabla_{\dot{h}(s)}\varphi\Big)U(s), \varphi U(s)\Big) = \ell\sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Now by use of (23), we have

$$g\left(\mathcal{H}\nabla_{\dot{h}(s)}\varphi U(s), \varphi U(s)\right) - g\left(P_{\dot{h}(s)}U + Q_{\dot{h}(s)}U, \varphi U(s)\right) = \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Along the curve h, using Theorem 1.3, we obtain

$$-g\left(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U(s)\right) = \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Now, π is a Clairaut submersion with $r = e^f$ if and only if $\frac{d}{ds}\left(e^f \sin\theta\right) = 0$. Therefore

$$e^f\left(\frac{df}{ds}\sin\theta + \cos\theta\frac{d\theta}{ds}\right) = 0,$$

$$e^f\left(\frac{df}{ds}\ell\sin^2\theta + \ell\sin\theta\cos\theta\frac{d\theta}{ds}\right) = 0.$$

So, we obtain

$$\frac{df}{ds}(h(s))g(U(s), U(s)) = g\left(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U(s)\right). \tag{36}$$

Since $\frac{df}{ds}(h(s)) = g\left(gradf, \dot{h}(s)\right) = g(gradf, X)$. Therefore by using (36), we get the result.

Theorem 1.5 [34] Let π be an Clairaut anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. Then

$$A_{\varphi W}\varphi X + Q_W\varphi X = X(f)W$$

for $X \in (\ker\pi_*)^\perp$, $W \in \ker\pi_*$ and φW is basic.

Proof: Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. We know that any fiber of Riemannian submersion π is totally umbilical if and only if

$$T_VW = g(V, W)H, \tag{37}$$

for all $V, W \in \Gamma(\ker\pi_*)$, where H denotes the mean curvature vector field of any fiber in M. By using Theorem 1.2 and (37), we have

$$T_VW = -g(V, W)gradf. \tag{38}$$

Let $X \in \mu$ and $V, W \in \Gamma(\ker\pi_*)$, then by using (1) and (2), we have

$$g(\nabla_V\varphi W, \varphi X) = g(\varphi\nabla_VW + (\nabla_V\varphi)W, \varphi X) = g(\nabla_VW, X) + g(P_VW + Q_VW, \varphi X). \tag{39}$$

By using (1), we have

$$g(\varphi Y, Z) = -g(Y, \varphi Z),$$

where $Y, Z \in TM$. Taking covariant derivative of above, we get

$$g((\nabla_X\varphi)Y, Z) = -g(Y, (\nabla_X\varphi)Z),$$

using (23), we get

$$\begin{aligned} g(P_XY + Q_XY, Z) &= -g(Y, P_XZ + Q_XZ) \\ &= g(Y, P_ZX + Q_ZX). \end{aligned} \tag{40}$$

Using (40), we have

$$g(P_W\varphi X + Q_W\varphi X, V) = g(\varphi X, P_VW + Q_VW). \tag{41}$$

Using (5), (38), (41) in (39), we have

$$g(\nabla_V\varphi W, \varphi X) = -g(V, W)(gradf, X) + g(V, Q_W\varphi X).$$

Since φW is basic, so $\mathcal{H}\nabla_V\varphi W = A_{\varphi W}V$, therefore we have

$$\begin{aligned} g\left(A_{\varphi W}V, \varphi X\right) &= -g(V, W)(gradf, X) + g(V, Q_W\varphi X), \\ g\left(V, A_{\varphi W}\varphi X\right) + g(V, Q_W\varphi X) &= g(V, W)(gradf, X) \end{aligned} \tag{42}$$

because A is skew-symmetric. By using (42), we get the result.

Theorem 1.6 [34] Let π be a Clairaut anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold $\left(N, g_n\right)$ with $r = e^f$ and $gradf \in \varphi \ker\pi_*$. Then either f is constant on $\varphi\ker\pi_*$ or the fibers of π are 1-dimensional.

Proof: Using (5) and (38), we have

$$g(\nabla_VW, \varphi U) = -g(V, W)g(gradf, \varphi U),$$

where $U, V, W \in \Gamma(\ker\pi_*)$. Since $g(W, \varphi U) = 0$. therefore we have

$$g(W, \nabla_V\varphi U) = g(V, W)g(gradf, \varphi U). \tag{43}$$

By use of (1) and (23) in (43), we get

$$g(W, Q_VU) - g(\varphi W, \nabla_VU) = g(V, W)g(gradf, \varphi U).$$

By using (5), we obtain

$$g(W, Q_VU) - g(\varphi W, T_VU) = g(V, W)g(gradf, \varphi U).$$

Now, using (38), we get

$$g(W, Q_VU) + g(V, U)g(gradf, \varphi W) = g(V, W)g(gradf, \varphi U) \tag{44}$$

Take $V = U$ in (44), we have

$$g(V, V)g(gradf, \varphi W) = g(V, W)g(gradf, \varphi V). \tag{45}$$

Interchange V with W in (45), we have

$$g(W, W)g(gradf, \varphi V) = g(V, W)g(gradf, \varphi W). \tag{46}$$

By (45) and (46), we have

$$g^2(V,W)g(gradf,\varphi V) = g(V,V)g(W,W)g(gradf,\varphi V).$$

Therefore either f is constant on $\varphi\ker\pi_*$ or $V = aW$, where a is constant (by using Schwarz's Inequality for equality case).

Corollary 1.1 [34] Let π be a Clairaut anti-invariant submersion from a nearly Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$ and $gradf \in \varphi\ker\pi_*$. If $\dim(\ker\pi_*) > 1$, then the fibers of π are totally geodesic if and only if $A_{\varphi W}\varphi X + Q_W\varphi X = 0$ for $W \in \ker\pi_*$ such that φW is basic and $X \in \mu$.

Proof: By Theorem 1.5 and Theorem 1.6, we get the result.

Corollary 1.2 [34] Let π be an Clairaut Lagrangian submersion from a nearly Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$. Then either the fibers of π are 1-dimensional or they are totally geodesic.

Proof: Let π be an Clairaut Lagrangian submersion from a Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$. Then $\mu = \{0\}$, so $A_{\varphi W}\varphi X + Q_W\varphi X = 0$ always.

Now, we discuss some examples for Clairaut anti-invariant submersions from a nearly Kähler manifold.

Example 1.2 [34] Let $(\mathbb{R}^4,\varphi,g)$ be a nearly Kähler manifold endowed with Euclidean metric g on $\mathbb{R}^4$ given by

$$g = \sum_{i=1}^{4} dx_i^2$$

and canonical complex structure

$$\varphi(x_j) = \begin{cases} -x_{j+1} & j = 1,3 \\ x_{j-1} & j = 2,4 \end{cases}.$$

The φ-basis is $\left\{e_i = \frac{\partial}{\partial x_i} | i = 1,2,3,4\right\}$. Let $(\mathbb{R}^3,g_1)$ be a Riemannian manifold endowed with metric $g_1 = \sum_{i=1}^{3} dy_i^2$.

i. Consider a map $\pi : (\mathbb{R}^4,\varphi,g) \to (\mathbb{R}^3,g_1)$ defined by

$$\pi(x_1,x_2,x_3,x_4) = \left(\frac{x_1+x_2}{\sqrt{2}},x_3,x_4\right).$$

Then by direct calculations, we have

$$\ker\pi_* = span\left\{X_1 = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\right\},$$

$$(\ker\pi_*)^\perp = span\left\{X_2 = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right), X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4}\right\}$$

and $\varphi X_1 = -X_2$, therefore $\varphi(\ker\pi_*) \subset (\ker\pi_*)^\perp$. Thus, we can say that π is an anti-invariant Riemannian submersion. Since the fibers of π are 1-dimensional, therefore fibers are totally umbilical.

Consider the Koszul formula for Levi-Civita connection ∇ for $\mathbb{R}^4$

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([Y, Z], X) - g([X, Z], Y) + g([X, Y], Z)$$

for all $X, Y, Z \in \mathbb{R}^4$. By simple calculations, we obtain

$$\nabla_{e_i} e_j = 0 \quad \text{for all}\, i, j = 1, 2, 3, 4.$$

Hence $T_X Y = T_Y X = T_X X = 0$ for all $X, Y \in \Gamma(\ker\pi_*)$. Therefore fibers of π are totally geodesic. Thus π is Clairaut trivially.

ii. Consider a map $\pi : (\mathbb{R}^4, \varphi, g) \to (\mathbb{R}^3, g_1)$ defined by

$$\pi(x_1, x_2, x_3, x_4) = \left(\sqrt{x_1^2 + x_2^2}, x_3, x_4\right).$$

Then by direct calculations, we have

$$\ker\pi_* = span\left\{X_1 = \left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_1} - \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_2}\right)\right\},$$

$$(\ker\pi_*)^\perp = span\left\{X_2 = \left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_2}\right), X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4}\right\}$$

and $\varphi X_1 = -X_2$, therefore $\varphi(\ker\pi_*) \subset (\ker\pi_*)^\perp$. Thus, we can say that π is an anti-invariant Riemannian submersion. Since the fibers of π are 1-dimensional, therefore fibers are totally umbilical. By using Koszul formula, we obtain

$$\nabla_{e_i} e_j = 0 \quad \text{for all}\, i, j = 1, 2, 3, 4.$$

Hence

$$T_{X_1} X_1 = -\left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_2}\right).$$

Now, for the function $f = \ln\left(\sqrt{x_1^2 + x_2^2}\right)$ on $(\mathbb{R}^4, \varphi, g)$, the gradient of f with respect to g is given by

$$gradf = \sum_{i,j=1}^{4} g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_2}.$$

Therefore for $X_1 \in \Gamma(\ker\pi_*)$, $T_{X_1} X_1 = -gradf$. Since $\|X_1\| = 1$, so $T_{X_1} X_1 = -\|X_1\|^2 gradf$. By using Theorem 1.2, we can say that π is an proper Clairaut anti-invariant submersion with $r = e^f$ for $f = \ln\left(\sqrt{x_1^2 + x_2^2}\right)$.

Remark: From all results of this section, we can easily find conditions for anti-invariant Clairaut Submersions from Kähler manifolds.

6. Semi-invariant Riemannian submersion

Definition 1.4 Let (M, φ, g) be an almost Hermitian manifold and N be a Riemannian manifold with Riemannian metric g_n. A Riemannian submersion

$\pi : M \to N$ is called a semi-invariant Riemannian submersion [11] if there is a distribution $\mathcal{D}_1 \subseteq \ker\pi_*$ such that

$$\ker\pi_* = \mathcal{D}_1 \oplus \mathcal{D}_2 \text{ and } \varphi(\mathcal{D}_1) = \mathcal{D}_1, \quad \varphi(\mathcal{D}_2) \subseteq (\ker\pi_*)^\perp,$$

where $\mathcal{D}_2$ is orthogonal complementary to $\mathcal{D}_1$ in $\ker\pi_*$. For $V \in \Gamma(ker\pi_*)$, we have

$$\varphi V = \phi V + \omega V, \tag{47}$$

where $\phi V \in \Gamma(\mathcal{D}_1)$ and $\omega V \in \Gamma(\varphi\mathcal{D}_2)$.

Definition 1.5 A semi-invariant Riemannian submersion π is said to be a Lagrangian Riemannian submersion [4] if $\varphi(\ker\pi_*) = (\ker\pi_*)^\perp$. Hence, if π is a Lagrangian Riemannian submersion then for any $V \in \Gamma(ker\pi_*)$, $\varphi V = \omega V$, $\phi V = 0$ and for $X \in \Gamma(ker\pi_*)^\perp$, $\varphi X = \alpha X$, $\beta X = 0$.

6.1 Semi-invariant Clairaut submersions from Kähler manifolds

In this section, we give new Clairaut conditions for semi-invariant submersions from Kähler manifolds after giving some auxiliary results.

Theorem 1.7 [40] Let π be a semi-invariant submersion from a Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). If $h : J \subset \mathbb{R} \to M$ is a regular curve and $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, respectively, then h is a geodesic if and only if along h

$$\mathcal{V}\nabla_X\phi U + \mathcal{V}\nabla_X\alpha X + A_X\omega U + A_X\beta X + \hat{\nabla}_U\phi U + T_U\beta X + T_U\omega U + \hat{\nabla}_U\alpha X = 0, \tag{48}$$

$$A_X\phi U + A_X\alpha X + \mathcal{H}\left(\nabla_{\dot{h}}\omega U + \nabla_{\dot{h}}\beta X\right) + T_U\phi U + T_U\alpha X = 0. \tag{49}$$

Proof: Let π be a semi-invariant submersion from a Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). Since $\varphi^2\dot{h} = -\dot{h}$. Taking the covariant derivative of this and using (2), we have

$$\varphi\left(\nabla_{\dot{h}}\varphi\dot{h}\right) = -\nabla_{\dot{h}}\dot{h}. \tag{50}$$

Since $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, that is, $\dot{h} = U + X$. So (50) becomes

$$\begin{aligned}
-\nabla_{\dot{h}}\dot{h} &= \varphi(\nabla_{U+X}\varphi(U+X)) \\
&= \varphi(\nabla_U\varphi U + \nabla_X\varphi U + \nabla_U\varphi X + \nabla_X\varphi X) \\
&= \varphi(\nabla_U(\phi U + \omega U) + \nabla_X(\phi U + \omega U) + \nabla_U(\alpha X + \beta X) + \nabla_X(\alpha X + \beta X)).
\end{aligned} \tag{51}$$

Using (5)–(8) in (51), we get

$$\begin{aligned}
-\nabla_{\dot{h}}\dot{h} = \varphi\big(&\mathcal{H}\left(\nabla_{\dot{h}}\varphi U + \nabla_{\dot{h}}\beta X\right) + A_X\alpha X + A_X\beta X + A_X\varphi U \\
&+ T_U\beta X + T_U\alpha X + \mathcal{V}\nabla_X\alpha X + T_U\varphi U + \hat{\nabla}_U\alpha X\big).
\end{aligned} \tag{52}$$

Equating the vertical and horizontal part of (52), we obtain

$$\mathcal{V}\varphi\nabla_{\dot{h}}\dot{h} = \mathcal{V}\nabla_X\phi U + \mathcal{V}\nabla_X\alpha X + A_X\omega U + A_X\beta X + \hat{\nabla}_U\phi U + T_U\beta X + T_U\omega U + \hat{\nabla}_U\alpha X,$$

$$\mathcal{H}\varphi\nabla_{\dot{h}}\dot{h} = A_X\phi U + A_X\alpha X + \mathcal{H}\left(\nabla_{\dot{h}}\omega U + \nabla_{\dot{h}}\beta X\right) + T_U\phi U + T_U\alpha X.$$

By using above equations, we can say that h is geodesic if and only if (48) and (49) hold.

Theorem 1.8 [40] Let π be a semi-invariant submersion from a Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n). Also, let $h : J \subset \mathbb{R} \to M$ be a regular curve. $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$. Then π is a Clairaut submersion with $r = e^f$ if and only if along h

$$\begin{aligned} g(gradf, X)g(U,U) = {} & g\left(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X, \omega U\right) \\ & + g\left(\mathcal{V}\nabla_X\alpha X + A_X\beta X + T_U\beta X + \hat{\nabla}_U\alpha X, \phi U\right). \end{aligned}$$

Proof: Let $h : J \subset \mathbb{R} \to M$ be a geodesic on M and $\ell = \left\|\dot{h}(s)\right\|^2$. Let $\theta(s)$ be the angle between $\dot{h}(s)$ and the horizontal space at $h(s)$. Then

$$g(X(s), X(s)) = \ell \cos^2\theta(s), \tag{53}$$

$$g(U(s), U(s)) = \ell \sin^2\theta(s). \tag{54}$$

Differentiating (54), we get

$$2g\left(\nabla_{\dot{h}(s)} U(s), U(s)\right) = 2\ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}. \tag{55}$$

Using (1) in (55), we have

$$g\left(\nabla_{\dot{h}(s)}\varphi U(s), \varphi U(s)\right)) = \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Now by use of (47), we have

$$g\left(\nabla_{\dot{h}(s)}\phi U(s), \varphi U(s)\right) + g\left(\nabla_{\dot{h}(s)}\omega U(s), \varphi U(s)\right) = \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}$$

Along the curve h, using Theorem 1.7 and (5)–(8), we obtain

$$\begin{aligned} \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds} = {} & -g\left(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X, \omega U\right)(s) \\ & -g\left(\mathcal{V}\nabla_X\alpha X + A_X\beta X + T_U\beta X + \hat{\nabla}_U\alpha X, \phi U\right)(s) \end{aligned}$$

Now, π is a Clairaut submersion with $r = e^f$ if and only if $\frac{d}{ds}\left(e^f \sin\theta\right) = 0$. Therefore

$$e^f\left(\frac{df}{ds}\sin\theta + \cos\theta\frac{d\theta}{ds}\right) = 0,$$

$$e^f\left(\frac{df}{ds}\ell \sin^2\theta + \ell \sin\theta\cos\theta\frac{d\theta}{ds}\right) = 0.$$

So, we obtain

$$\frac{df}{ds}(h(s))g(U(s),U(s)) = g\left(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X, \omega U\right)(s) \\ +g\left(\mathcal{V}\nabla_X\alpha X + A_X\beta X + T_U\beta X + \hat{\nabla}_U\alpha X, \phi U\right)(s), \tag{56}$$

Since $\frac{df}{ds}(h(s)) = \dot{h}[g](s) = g\left(gradf, \dot{h}(s)\right) = g(gradf, X)$. Therefore by using (56), we get the result.

Theorem 1.9 [40] Let π be a Clairaut semi-invariant submersion from a Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$. Then

$$g(A_{\omega W}V, \beta X) + g(\mathcal{H}\nabla_V\phi W, \beta X) + g(\mathcal{V}\nabla_V\omega W, \alpha X) + g\left(\hat{\nabla}_V\phi W, \alpha X\right) \\ = -g(V,W)(Xf)$$

for $X \in \Gamma\mu$, $V, W \in \Gamma(\mathcal{D}_2)$ and ωW is basic.

Proof: Let π be a Clairaut semi-invariant submersion from a Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$. We know that any fiber of Riemannian submersion π is totally umbilical if and only if

$$T_V W = g(V,W)H, \tag{57}$$

for all $V, W \in \Gamma(\ker\pi_*)$, where H denotes the mean curvature vector field of any fiber in M. By using Theorem 1.2 and (57), we have

$$T_V W = -g(V,W)gradf. \tag{58}$$

Let $X \in \mu$ and $V, W \in \Gamma(\ker\pi_*)$, then by using (1) and (2), we have

$$g(\nabla_V\varphi W, \varphi X) = g(\varphi\nabla_V W + (\nabla_V\varphi)W, \varphi X) = g(\nabla_V W, X). \tag{59}$$

Using (5), (58) in (59), we have

$$g(\nabla_V\varphi W, \varphi X) = -g(V,W)(gradf, X).$$

Since ωW is basic, so $\mathcal{H}\nabla_V\omega W = A_{\omega W}V$, therefore we have

$$g(A_{\omega W}V, \beta X) + g(\mathcal{H}\nabla_V\phi W, \beta X) + g(\mathcal{V}\nabla_V\omega W, \alpha X) + g\left(\hat{\nabla}_V\phi W, \alpha X\right) \\ = -g(V,W)(gradf, X). \tag{60}$$

Theorem 1.10 [40] Let π be a Clairaut semi-invariant submersion from a Kähler manifold (M,φ,g) onto a Riemannian manifold (N,g_n) with $r = e^f$ and $V, W \in \Gamma(\mathcal{D}_1)$ Then $gradf \in \Gamma(\varphi\ker\pi_*)$.

Proof: Let $V, W \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma(\mu)$. Using (5), (47) and (58) in

$$\nabla_V\varphi U = \varphi\nabla_V U + (\nabla_V\varphi)U,$$

we have

$$T_V\phi U + \mathcal{V}\nabla_V\phi U = \alpha T_V U + \beta T_V U + \phi\mathcal{V}\nabla_V U + \omega\mathcal{V}\nabla_V U,$$

which gives

$$-g(V,\phi U)g(gradf, X) = g(V,U)g(gradf, \varphi X). \tag{61}$$

By interchanging U and V in (61) and adding the resulting equation with (61), we get

$$g(V, U)g(gradf, \varphi X) = 0,$$

which gives $g(gradf, \varphi X) = 0$. Therefore $gradf \in \Gamma(\varphi \ker \pi_*)$.

Theorem 1.11 [40] Let π be a Clairaut semi-invariant submersion from a Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$ and $gradf \in \Gamma(\varphi \ker \pi_*)$. Then either f is constant on $\varphi \ker \pi_*$ or the fibers of π are 1-dimensional.

Proof: Let $U, V \in \Gamma(\mathcal{D}_2)$. Using (5) and (58), we have

$$g(\nabla_V U, \varphi U) = -g(V, U)g(gradf, \varphi U),$$

which gives

$$g(\varphi \nabla_V U, U) = g(V, U)g(gradf, \varphi U),$$

since $\ker \pi_*$ is integrable, so we have

$$g(\varphi \nabla_U V, U) = g(V, U)g(gradf, \varphi U),$$

which equals to

$$g(\nabla_U \varphi V - (\nabla_U \varphi)V, U) = g(V, U)g(gradf, \varphi U).$$

Since $g(\varphi V, U) = 0$. therefore we have

$$g(\varphi V, \nabla_U U) = -g(V, U)g(gradf, \varphi U). \tag{62}$$

By using (5) in (62), we obtain

$$g(\varphi V, T_U U) = -g(V, U)g(gradf, \varphi U).$$

Now, using (58), we get

$$g(U, U)g(gradf, \varphi V) = g(V, U)g(gradf, \varphi U). \tag{63}$$

Interchanging V and U in (63), we have.

$$g(V, V)g(gradf, \varphi U) = g(V, U)g(gradf, \varphi V). \tag{64}$$

By (63) and (64), we have

$$g^2(V, U)g(gradf, \varphi U) = g(V, V)g(U, U)g(gradf, \varphi U).$$

Therefore either f is constant on $\varphi \ker \pi_*$ or $V = aU$, where a is constant (by using Schwarz's Inequality for equality case).

Since $\ker \pi_*$ is CR-submanifold of Kähler manifold (M, φ, g), therefore by using [41], Theorem 6.1, p. 96], we can state that.

Theorem 1.12 [40] Let π be a Clairaut semi-invariant submersion from a Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. If $\dim \mathcal{D}_2 > 1$, then fibers are totally geodesic.

Corollary 1.3 [40] Let π be a Clairaut Lagrangian submersion from a Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. Then either the fibers of π are 1-dimensional or they are totally geodesic.

Lastly, we discuss some examples for Clairaut semi-invariant submersions [40] from a Kähler manifold.

Example 1.3 Every Clairaut anti-invariant submersion from a Kähler manifold onto a Riemannian manifold is a Clairaut semi-invariant submersion with $\mathcal{D}_1 = \{0\}$.

Example 1.4 Let $(\mathbb{R}^6, \varphi, g)$ be a Kähler manifold endowed with Euclidean metric g on $\mathbb{R}^6$ given by

$$g = \sum_{i=1}^{6} dx_i^2$$

and canonical complex structure

$$\varphi(x_j) = \begin{cases} -x_{j+1} & j = 1, 3, 5 \\ x_{j-1} & j = 2, 4, 6 \end{cases}.$$

The φ-basis is $\left\{e_i = \frac{\partial}{\partial x_i} | i = 1, \dots, 6\right\}$. Let $(\mathbb{R}^3, g_1)$ be a Riemannian manifold endowed with metric $g_1 = \sum_{i=1}^{3} dy_i^2$.

Consider a map $\pi : (\mathbb{R}^6, \varphi, g) \to (\mathbb{R}^3, g_1)$ defined by

$$\pi(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\frac{x_1 + x_2}{\sqrt{2}}, x_3, x_4\right).$$

Then by direct calculations, we have

$$\ker\pi_* = span\left\{X_1 = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right), X_2 = \frac{\partial}{\partial x_5}, X_3 = \frac{\partial}{\partial x_6}\right\},$$
$$(\ker\pi_*)^\perp = span\left\{V_1 = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right), V_2 = \frac{\partial}{\partial x_3}, V_3 = \frac{\partial}{\partial x_4}\right\}$$

and $\varphi X_1 = -V_1$, $\varphi X_2 = -X_3$, $\varphi X_3 = -X_2$ therefore $\mathcal{D}_1 = span\{X_2, X_3\}$ and $\mathcal{D}_2 = span\{X_1\}$. Thus, we can say that π is a semi-invariant Riemannian submersion.

Consider the Koszul formula for Levi-Civita connection ∇ for $\mathbb{R}^6$

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([Y, Z], X) - g([X, Z], Y) + g([X, Y], Z)$$

for all $X, Y, Z \in \mathbb{R}^6$. By simple calculations, we obtain

$$\nabla_{e_i} e_j = 0 \quad \text{for all}\, i, j = 1, \dots, 6.$$

Hence $T_X Y = T_Y X = T_X X = 0$ for all $X, Y \in \Gamma(\ker\pi_*)$. Therefore fibers of π are totally geodesic. Thus π is Clairaut trivially.

Classification

2010 mathematics subject classification. 53C12, 53C15, 53C20, 53C55

Author details

Sanjay Kumar Singh[1*] and Punam Gupta[2]

1 Indian Institute of Science Education and Research, Bhopal, India

2 Dr. Harisingh Gour University, Sagar, India

*Address all correspondence to: sanjayks@iiserb.ac.in

References

[1] O'Neill B. The fundamental equations of a submersion. Michigan Mathematical Journal. 1966;**13**:458-469

[2] Gray A. Nearly Kähler manifolds. Journal of Differential Geometry. 1970; **6**:283-309

[3] Watson B. Almost Hermitian submersions. Journal of Differential Geometry. 1976;**11**:147-165

[4] Sahin B. Anti-invariant Riemannian submersion from almost Hermitian manifolds. Central Europian Journal of Mathematics. 2010;**8**(3):437-447

[5] Tastan HM. On Lagrangian submersions. Hacettepe Journal of Mathematics and Statistics. 2014;**43**(6): 993-1000

[6] Tastan HM, Özdemir F, Sayar C. On anti-invariant Riemannian submersions whose total manifolds are locally product Riemannian. Journal of Geometry. 2017;**108**:411-422

[7] Gündüzalp Y. Anti-invariant Riemannian submersions from almost product Riemannian manifolds. Mathematical Sciences and Applications E-Notes. 2013;**1**:58-66

[8] Beri A, Kupeli EI, Murathan C. Anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. Turkish Journal of Mathematics. 2016;**40**(3): 540-552

[9] Ali S, Fatima T. Anti-invariant Riemannian submersions from nearly Kähler manifolds. Univerzitet u Nišu. 2013;**27**:1219-1235

[10] Özdemir F, Sayar C, Tastan HM. Semi-invariant submersions whose total manifolds are locally product Riemannian. Quaestiones Mathematicae. 2017;**40**(7):909-926

[11] Sahin B. Semi-invariant submersion from almost Hermitian manifolds. Canadian Mathematical Bulletin. 2011; **56**:173-182

[12] Küpeli Erken I, Murathan C. On slant submersions for cosymplectic manifolds. Bulletin of the Korean Mathematical Society. 2014;**51**(6):1749-1771

[13] Sahin B. Slant submersions from almost Hermitian manifolds. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie. 2011;**54** **(102)**(1):93-105

[14] Akyol MA. Generic Riemannian submersions from almost product Riemannian manifolds. GUJ Sci. 2017; **30**(3):89-100

[15] Ali S, Fatima T. Generic Riemannian submersions. Tamkang Journal of Mathematics. 2013;**44**(4):395-409

[16] Chen BY. Differential geometry of real submanifolds in Kähler manifold. Monatshefte für Mathematik. 1981;**91**: 257-274

[17] Ronsse GB. Generic and skew CR-submanifolds of a Kähler manifold. Bulletin of the Institute of Mathematics, Academia Sinica. 1990;**18**:127-141

[18] Tastan HM, Sahin B, Yanan S. Hemi-slant submersions. Mediterranean Journal of Mathematics. 2016;**13**(4):2171-2184

[19] Park KS, Prasad R. Semi-slant submersions. Bulletin of the Korean Mathematical Society. 2013;**50**(3):951-962

[20] Aykurt SA, Ergüt M. Pointwise slant submersions from cosymplectic manifolds. Turkish Journal of Mathematics. 2016;**40**(3):582-593

[21] Chen BY, Garay O. Pointwise slant submanifolds in almost Hermitian

manifolds. Turkish Journal of Mathematics. 2012;**364**:630-640

[22] Lee JW, Sahin B. Pointwise slant submersions. Bulletin of the Korean Mathematical Society. 2014;**51**:1115-1126

[23] Akyol MA. Conformal semi-slant submersions. International Journal of Geometric Methods in Modern Physics. 2017;**14**(7):1750114

[24] Sahin B. Riemannian Submersions, Riemannian Maps in Hermitian Geometry and their Applications. Cambridge: Elsevier Academic Press; 2017

[25] Clairaut AC. Mémoires de l'Académie royale des sciences. Mem. Acad. Paris Pour. 1735;**1733**:86

[26] Bishop RL. Clairaut submersions, Differential Geometry (in Honor of Kentaro Yano). Tokyo: Kinokuniya; 1972. pp. 21-31

[27] Allison D. Lorentzian Clairaut submersions. Geometriae Dedicata. 1996;**63**(3):309-319

[28] Tastan HM, Gerdan S. Clairaut anti-invariant submersions from Sasakian and Kenmotsu manifolds. Mediterranean Journal of Mathematics. 2017;**14**(6):235. 17 pp

[29] Tastan HM, Aydin SG. Clairaut anti-invariant submersions from cosymplectic manifolds. Honam Mathematical Journal. 2019;**41**(4): 707-724

[30] Gündüzalp Y. Anti-invariant pseudo-Riemannian submersions and Clairaut submersions from paracosymplectic manifolds. Mediterranean Journal of Mathematics. 2019;**16**(4):94. 18 pp

[31] Lee J, Park JH, Shahin B, Song DY. Einstein conditions for the base of anti-invariant Riemannian submersions and Clairaut submersions. Taiwanse Journal of Mathematics. 2015;**19**(4): 1145-1160

[32] Gray A, Hervella M. The sixteen classes of almost Hermitian manifolds and their linear invariants. Annali di matematica pura ed applicata. 1980;**123**: 35-58

[33] Yano K. Structures on manifolds, Series in Pure Mathematics, 3. Singapore: World Scientific; 1984

[34] Gupta P, Rai AK. Clairaut anti-invariant submersion from nearly Kähler manifold. arXiv. Mat. Vesnik, Accepted. 2020. 2010.05552

[35] O'Neill B. Semi-Riemannian Geometry with Application to Relativity. New York: Academic Press; 1983

[36] Baird P, Wood JC. Harmonic Morphisms Between Riemannian Manifolds, London Mathematical Society Monographs. Vol. 29. Oxford: Oxford University Press, The Clarendon Press; 2003

[37] Bejancu A. Geometry of CR-Submanifolds. Dordrecht: Kluwer Academic Publishers; 1986

[38] Pressley A. Elementary Differential Geometry. Springer-Verlag London: Springer Undergraduate Mathematics Series; 2001

[39] Lee JW. Anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact manifolds. Hacettepe Journal of Mathematics and Statistics. 2013;**42**(3): 231-241

[40] Gupta P, Singh SK. Clairaut semi-invariant submersion from Kähler manifold. Afrika Mat. Accepted. 2020. DOI: 10.13140/RG.2.2.30240.30725

[41] Yano K, Kon M. CR-Submanifolds of Kählerian and Sasakian Manifolds. Progress in Mathematics, 30. Birkhäuser, Boston: Mass; 1983