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# Involuntary Unemployment in Diamond-Type Overlapping Generations Models

*Karl Farmer*

## Abstract

Thus far involuntary unemployment does not occur in Diamond-type Overlapping Generations models. In line with Keynesian macroeconomics, involuntary unemployment is traced back to aggregate demand failures. While macroeconomists majority refers aggregate demand failures to sticky prices, a minority attributes lacking aggregate demand to not perfectly flexible aggregate investment. The chapter investigates how an independent aggregate investment function causes involuntary unemployment under perfectly flexible competitive wage and interest rates in a Diamond-type neoclassical growth model with public debt and human capital accumulation. Moreover, it is shown that a higher public debt to output ratio enhances output growth and reduces involuntary unemployment.

**Keywords:** involuntary unemployment, overlapping generations models, inflexible aggregate investment, public debt, human capital accumulation

## 1. Introduction

Involuntary unemployment in Diamond-type Overlapping Generations (OLG) models seem to be a contradiction in terms. As in Solow [1]'s neoclassical growth model, Diamond assumed full employment of the workforce for the OLG economy with production and capital accumulation. Thus, in this economy, unemployment is purely voluntary. Moreover, fiscal policy does not impact the steady-state growth rate of gross domestic product (GDP) since the output growth is exogenously determined. Both, voluntary unemployment and growth ineffectiveness of fiscal policy, do not accord well with the current state of the pandemic-affected world economy which is characterized by high involuntary unemployment and enormous government expenditures to compensate lockdown-related private losses. To be capable to address the effectiveness of fiscal policy to reduce involuntary unemployment, an additional extension of Diamond [2]'s seminal OLG model towards endogenous growth becomes inevitable.

As is well-known, involuntary unemployment is usually associated with Keynesian macroeconomics [3, 4]. Involuntary unemployment is traced back to lacking aggregate demand. But on the reasons why aggregate demand remains below full employment output in a perfectly functioning market economy, there is no consensus among mainstream macro-economists to this date. The majority view

follows the New-Keynesian approach in which prices and wages adapt sluggishly to market imbalances due to imperfect competition and other market failures (for a survey see [5]). In contradistinction to the majority view, a macroeconomists' minority follows [6] and more recently [7] who trace back aggregate demand failures to inflexible aggregate investment demand governed by (pessimistic) "animal spirits" of investors independently from aggregate savings of households. In contrast to the imperfectly flexible-price approach, [6, 7] presume perfectly flexible and perfectly competitive output prices, wage rates and interest rates. Despite this perfect-market setting employees do not become fully employed because an independent investment function makes the general equilibrium equations' system over-determinate. Over-determinacy disappears only if at least one market-clearing condition is cancelled, and it is the labor market clearing condition that is deleted.

Magnani [7] without noting precursor Morishima [6] incorporates a macro-founded investment function into Solow [1]'s neoclassical growth model without public debt. Since the chapter intends to study the effects of public debt on private capital accumulation, GDP growth and unemployment in the long run, the present author switches to Diamond [2]'s OLG model with non-neutral internal public debt. Long-run GDP growth in Diamond [2]'s OLG model is, however, exogenous precluding the analysis of how larger public debt impacts GDP growth and unemployment. Hence, there is a need for a mechanism that endogenizes GDP growth. To this end, we stick to human capital accumulation à la [8, 9].

This chapter pursues several purposes: Firstly, it will be shown how in a log-linear utility and Cobb–Douglas production function version of Diamond [2]'s OLG model with internal public debt, the intertemporal equilibrium dynamics based on household's and firm's first-order conditions, on government's budget constraint and intertemporal market-clearing conditions is modified when aggregate investment demand is governed by a savings-independent investment function. Transcending [7] we secondly intend to rigorously prove the existence and dynamic stability of a steady-state of the equilibrium dynamics in a model which closely follows Farmer [10]'s model setting. Our third purpose is to investigate the effects of a higher public debt to output ratio on the output growth rate, on the capital-output ratio, on the interest factor and on the wage tax rate in a steady state of the Diamond OLG model extended by human capital accumulation which is financed by public human capital investment expenditures as in Farmer [11] and in Lin [9]. In extending [9, 11] by an independent aggregate investment function we are capable of exploring analytically and numerically the steady-state effects of a higher public debt to output ratio on the unemployment rate. In particular, we will demonstrate on which factors it depends whether a higher public debt to output ratio raises the output growth rate and decreases the unemployment rate. In contradistinction to the author's contributions in Farmer [10] and Farmer [12] this chapter exhibits the OLG model presented there more completely and succeeds in deriving the steady-state effects of larger public debt more succinctly. This chapter together with Farmer and Farmer [10, 12] can be seen as our contribution to the recent macroeconomic literature.

The structure of the chapter is as follows. In the next section (2.) the model set-up will be presented. In section 3, temporary equilibrium relations and the intertemporal equilibrium dynamics are derived from intertemporal utility maximization, atemporal profit maximization, government's budget constraint and the market-clearing condition in each period. In section 4, the existence of a steady-state solution of the equilibrium dynamics and its local dynamic stability is investigated. Section 5, is devoted to the analysis of the comparative steady-state effects of larger public debt. Section 6, concludes.

## 2. The model set-up

As in Farmer and Farmer [10, 12], we consider an economy of the infinite horizon which is composed of infinitely lived firms, finitely lived households and an infinitely lived government. In each period $t = 0, 1, 2, \ldots$ a new generation, called generation $t$, enters the economy. A continuum of $L_t > 0$ units of identical agents comprise generation $t$.

As mentioned above, to be able to address the question of how fiscal policy impacts long-run growth we extend Diamond [2]'s basic OLG model by introducing human capital accumulation. To point out the growth-enhancing effects of human capital accumulation most clearly, it is assumed here that there is no population growth $g^L$, i.e. $g^L = 0 \Leftrightarrow G^L = 1$, and no exogenous growth in labor efficiency denoted as $g^a$, i.e. $g^a = 0 \Leftrightarrow G^a = 1$. As a result of the first assumption, the number of households, $L_t$, remains constant over time: $L_t = L_{t-1} = L$.

Each household consists of one agent and the agent acts intergenerationally egoistic: The old agent does not take care of the young agent and the young agent does not take care of the old agent. They live two periods long, namely youth (adult) and old age. In youth age, each household starts with human capital $h_t$, accumulated by the household in period $t - 1$. Individual human capital is inelastically supplied to firms that remunerate the real wage rate $w_t$ in exchange for the labor supply. The former denotes the units of the produced good per efficiency unit of labor. In contradistinction to the original [2] OLG model, not the total labor supply is employed but only $(1 - u_t)L_t$, where $0 \leq u_t < 1$ denotes the unemployment rate. The number of unemployed people is thus $u_t L_t$. The government collects taxes on wages, quoted as a fixed proportion of wage income, $\tau_t w_t h_t$, $0 < \tau_t < 1$. The unemployed do not pay any taxes. Young, employed agents, denoted by superscript $E$, split the net wage income $(1 - \tau_t)w_t h_t$ each period between current consumption $c_t^{1,E}$ and savings $s_t^E$. Savings of the employed are invested in real capital in period $t$ per employed capita, $I_t^D / L_t(1 - u_t)$, which is demanded by employed households in youth, and in real government bonds per employed capita, $B_{t+1}^D / L_t(1 - u_t)$, which is also demanded by employed households in youth. For simplicity, we assume a depreciation rate of one for real capital.

Any unemployed young-age household, denoted by superscript $U$, consumes $c_t^{1,U}$, saves $s_t^U$ and receives from the government an unemployment benefit $\varsigma_t > 0$ to be able to finance its consumption and savings: $c_t^{1,U} + s_t^U = \varsigma_t$.

In old age, the employed household supplies inelastically $K_{t+1}^S / L_t(1 - u_t) = I_t^D / L_t(1 - u_t)$ to firms, and $B_{t+1}^S / L_t(1 - u_t) = B_{t+1}^D / L_t(1 - u_t)$ to young households in period $t + 1$. Thus, the per capita savings of employed people are invested as follows: $s_t^E = K_{t+1}^S / L_t(1 - u_t) + B_{t+1}^S / L_t(1 - u_t)$. Similarly, the per capita savings of the unemployed household are invested as follows: $s_t^U = K_{t+1}^S / L_t u_t + B_{t+1}^S / L_t u_t$. In old age, both employed and unemployed households consume their gross return on assets: $c_{t+1}^{2,E} = q_{t+1} K_{t+1}^S / L_t(1 - u_t) + (1 + i_{t+1}) B_{t+1}^S / L_t(1 - u_t)$, respectively $c_{t+1}^{2,U} = q_{t+1} K_{t+1}^S / L_t u_t + (1 + i_{t+1}) B_{t+1}^S / L_t u_t$, where $c_{t+1}^{2,E}$, and $c_{t+1}^{2,U}$, represent the consumption of the employed, respectively unemployed, in old age, $q_{t+1}$ denotes the gross rental rate on real capital, and $i_{t+1}$ denotes the real interest rate on government bonds in period to remain as simple as possible, we assume that rental and interest income are not taxed.

A log-linear intertemporal utility function slightly generalized in comparison to Diamond ([2], p. 1134)'s leading example represents the intertemporal preferences of all two-period lived households. As usual, this simple specification aims at closed-form solutions for the intertemporal equilibrium dynamics (see e.g. [13, pp. 181–184]).

The typical younger, employed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (i) and the retirement period (ii):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,E} + \beta \ln c_{t+1}^{2,E}$$

subject to:

$$(i) \; c_t^{1,E} + I_t^{D,E}/L_t(1 - u_t) + B_{t+1}^{D,E}/L_t(1 - u_t) = w_t h_t(1 - \tau_t),$$

$$(ii) \; c_{t+1}^{2,E} = q_{t+1} K_{t+1}^{S,E}/L_t(1 - u_t) + (1 + i_{t+1}) B_{t+1}^{S,E}/L_t(1 - u_t),$$

$$K_{t+1}^{S,E} = I_t^{D,E}, B_{t+1}^{S,E} = B_{t+1}^{D,E}.$$

Here, $0 < \varepsilon \leq 1$ depicts the utility elasticity of employed household's consumption in youth, while $0 < \beta < 1$ denotes the subjective future utility discount factor. For the log-linear utility function above, a unique, interior solution of the optimization problem exists. Hence, one may solve the old-age budget constraint for $B_{t+1}^{S,E}/L_t(1 - u_t)$ and insert the result into the young-age, employed budget constraint of (i), and thus obtain:

$$c_t^{1,E} + c_{t+1}^{2,E}/(1 + i_{t+1}) + \left[1 - q_{t+1}/(1 + i_{t+1})\right] K_{t+1}^{S,E}/L_t(1 - u_t) = w_t h_t(1 - \tau_t). \quad (1)$$

A strictly positive and finite solution to maximizing the intertemporal utility function subject to the constraint (1) requires that the following no-arbitrage condition holds:

$$q_{t+1} = 1 + i_{t+1}. \quad (2)$$

The no-arbitrage condition (2) implies that $K_{t+1}^{S,E}/L_t(1 - u_t)$ is optimally indeterminate, and the first-order conditions for a maximum solution read as follows:

$$c_t^{1,E} + c_{t+1}^{2,E}/(1 + i_{t+1}) = (1 - \tau_t) w_t h_t, \quad (3)$$

$$(\beta/\varepsilon) c_t^{1,E} = c_{t+1}^{2,E}/(1 + i_{t+1}). \quad (4)$$

Solving equations (3) and (4) for $c_t^{1,E}$ and $c_{t+1}^{2,E}$ yields the following optimal consumption for employed people in youth and old age:

$$c_t^{1,E} = [\varepsilon/(\varepsilon + \beta)](1 - \tau_t) w_t h_t, \quad (5)$$

$$c_{t+1}^{2,E} = [\beta/(\varepsilon + \beta)](1 + i_{t+1})(1 - \tau_t) w_t h_t. \quad (6)$$

Since $s_t^E = K_{t+1}^{S,E}/L_t(1 - u_t) + B_{t+1}^{S,E}/L_t(1 - u_t)$, we find for the utility-maximizing savings:

$$s_t^E = [\beta/(\varepsilon + \beta)](1 - \tau_t) w_t h_t. \quad (7)$$

The typical younger, unemployed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (i) and the retirement period (ii):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,U} + \beta \ln c_{t+1}^{2,U}$$

subject to:

$$(i)\ c_t^{1,U} + I_t^{D,U}/L_t u_t + B_{t+1}^{D,U}/L_t u_t = \varsigma_t,$$

$$(ii)\ c_{t+1}^{2,U} = q_{t+1}K_{t+1}^{S,U}/L_t u_t + (1 + i_{t+1})B_{t+1}^{S,U}/L_t u_t,$$

$$K_{t+1}^{S,U} = I_t^{D,U}, B_{t+1}^{S,U} = B_{t+1}^{D,U}.$$

Again, $0 < \varepsilon \leq 1$ denotes the utility elasticity of consumption in unemployed youth, while $0 < \beta < 1$ depicts the subjective future utility discount factor and $\varsigma_t$ denotes the unemployment benefit per capita unemployed. As above, the log-linear intertemporal utility function ensures the existence of a unique, interior solution for the above optimization problem. Hence, one may again solve the old-age budget constraint for $B_{t+1}^{S,U}/L_t u_t$ and insert the result into the young-age, unemployed budget constraint of (i), and thus obtain:

$$c_t^{1,U} + c_{t+1}^{2,U}/(1 + i_{t+1}) + \left[1 - q_{t+1}/(1 + i_{t+1})\right]K_{t+1}^{S,U}/L_t u_t = \varsigma_t. \tag{8}$$

The no-arbitrage condition (2) implies that $K_{t+1}^{S,U}/L_t u_t$ is optimally indeterminate, and the first-order conditions for a maximum solution read as follows:

$$c_t^{1,U} + c_{t+1}^{2,U}/(1 + i_{t+1}) = \varsigma_t, \tag{9}$$

$$(\beta/\varepsilon)c_t^{1,U} = c_{t+1}^{2,U}/(1 + i_{t+1}). \tag{10}$$

Solving equations (9) and (10) for $c_t^{1,U}$ and $c_{t+1}^{2,U}$ yields the following optimal consumption in youth and old age:

$$c_t^{1,U} = [\varepsilon/(\varepsilon + \beta)]\varsigma_t, \tag{11}$$

$$c_{t+1}^{2,U} = [\beta/(\varepsilon + \beta)](1 + i_{t+1})\varsigma_t. \tag{12}$$

Since $s_t^U = K_{t+1}^{S,U}/L_t u_t + B_{t+1}^{S,U}/L_t u_t$, we find for the utility-maximizing savings:

$$s_t^U = [\beta/(\varepsilon + \beta)]\varsigma_t. \tag{13}$$

All firms are endowed with an identical (linear-homogeneous) Cobb–Douglas production function which reads as follows:

$$Y_t = M(h_t N_t)^{1-\alpha}(K_t)^{\alpha},\ 0 < \alpha < 1, M > 0. \tag{14}$$

Here, $Y_t$ denotes aggregate output or GDP, $M > 0$ stands for total factor productivity, $N_t$ represents the number of employed laborers, while $K_t$ denotes the input of capital services, all in period $t$, and $1 - \alpha$ ($\alpha$) depicts the production elasticity (= production share) of labor (capital) services.

Maximization of $Y_t - w_t h_t N_t - q_t K_t$ subject to Cobb–Douglas production (14) implies the following first-order conditions:

$$(1 - \alpha)M[K_t/(h_t N_t)]^{\alpha} = w_t, \tag{15}$$

$$\alpha M[K_t/(h_t N_t)]^{(\alpha-1)} = q_t. \tag{16}$$

However, since the number of employed workers is $N_t = L(1 - u_t)$, we can rewrite the profit maximization conditions (15) and (16) as follows:

$$(1 - \alpha)M[K_t/(h_t L(1 - u_t))]^{\alpha} = w_t, \tag{17}$$

$$\alpha M[K_t/(h_t L(1 - u_t))]^{(\alpha - 1)} = q_t. \tag{18}$$

Finally, the GDP function can be rewritten as follows:

$$Y_t = M(h_t L(1 - u_t))^{1-\alpha}(K_t)^{\alpha}. \tag{19}$$

As in Diamond [2], the government does not optimize, but is subject to the following constraint period by period:

$$B_{t+1} = (1 + i_t)B_t + \Delta_t + L_t u_t \varsigma_t + \Gamma_t - \tau_t(1 - u_t)w_t h_t L, \tag{20}$$

where $B_t$ denotes the aggregate stock of real public debt at the beginning of period $t$, $\Gamma_t$ denotes human capital investment (HCI) expenditures, and $\Delta_t$ denotes all non-HCI expenditures of the government exclusive of government's unemployment benefits $L_t u_t \varsigma_t$ per period.

In line with Glomm and Ravikumar [8] human capital in period $t$ is determined by the human capital of the generation entering the economy in period $t - 1$, and by the government's HCI spending in period $t - 1$, $\Gamma_{t-1}$:

$$h_t = H_0(h_{t-1})^{\mu}(\Gamma_{t-1}/L)^{1-\mu}, H_0 = \underline{H} > 0, 0 < \mu < 1, \tag{21}$$

whereby $\underline{H}$ indicates a level parameter, $\mu$ depicts the production elasticity of human capital, and $1 - \mu$ denotes the production elasticity of public HCI spending. The macroeconomic version of equation (21) is obtained by multiplying it on both sides by $L$:

$$Lh_t \equiv H_t = H_0(Lh_{t-1})^{1-\mu}(\Gamma_{t-1})^{\mu} \equiv H_0(H_{t-1})^{1-\mu}(\Gamma_{t-1})^{\mu}. \tag{22}$$

The economy grows, even in the absence of population growth and exogenous progress in labor efficiency. Using the GDP growth factor $G_t^Y \equiv Y_{t+1}/Y_t$ as well as equations (19) and (22), the GDP growth factor can be written as follows:

$$G_{t+1}^Y = \frac{H_{t+1}}{H_t}\frac{(1 - u_{t+1})^{1-\alpha}}{(1 - u_t)^{1-\alpha}}\frac{(k_{t+1})^{\alpha}}{(k_t)^{\alpha}}, k_t \equiv \frac{K_t}{H_t}. \tag{23}$$

As Magnani [7], Morishima [6], Salotti and Trecroci [14] rightly states, aggregate investment in Solow [1]'s neoclassical growth model is not micro- but macro-founded since it is determined by aggregate savings. The same holds in Diamond [2]'s OLG model of neoclassical growth where perfectly flexible aggregate investment is also determined by aggregate savings of households. Deviating from those neoclassical growth models, Morishima [6] and more recently Magnani [7] and Salotti [14] claim that "investments are determined by an independent investment function." This function is specified in discrete time as follows:

$$I_t^D = \phi H_t(1 + i_t)^{-\theta}, \phi > 0, \theta \geq 0. \tag{24}$$

The positive parameter $\phi$ reflects "Keynesian investors' animal spirits" [7, 14] while $\theta$ denotes the interest-factor elasticity of aggregate investment demand $I_t^D$.

In addition to the restrictions imposed by household and firm optimizations and the government budget constraint, markets for labor, capital services and assets, ought to clear in all periods (the market for the output of production is cleared using Walras' Law[1]).

$$L_t(1 - u_t) = N_t, \forall t. \tag{25}$$

$$K_t^{S,E} + K_t^{S,U} = K_t^S = K_t, \forall t. \tag{26}$$

$$B_t^{D.E} + B_t^{D,U} = B_t^{S,E} + B_t^{S,U} = B_t, \forall t. \tag{27}$$

## 3. Temporary equilibrium and intertemporal equilibrium dynamics

As a first step, the unemployment rate in period $t$ (= temporary unemployment rate) is derived. To this end, we use the output market-clearing identity:

$$P_t L_t(1 - u_t)c_t^{1,E} + P_t L_{t-1}(1 - u_{t-1})c_t^{2,E} + P_t L_t u_t c_t^{1,U} + P_t L_{t-1} u_{t-1} c_t^{2,U}$$
$$+ P_t I_t^{D,E} + P_t I_t^{D,U} + P_t \Gamma_t + P_t \Delta_t = P_t Y_t. \tag{28}$$

Starting with identity (28), we insert equations (19), equation (5) and constraint (ii) from the employed household's optimization problem for period $t$ as well as equation (11) and constraint (ii) from unemployed household's optimization problem for period $t$. In addition, we also insert equation (24), with $I_t^D = I_t^{D,E} + I_t^{D,U}$ and add the market clearing conditions (26) and (27). In this way, the following equation for $P_t = 1, \forall t$ is obtained:

$$M(h_t L(1 - u_t))^{1-\alpha}(K_t)^\alpha = \varepsilon/(\varepsilon + \beta)(1 - \tau_t)w_t h_t L_t(1 - u_t)$$

$$+ \varepsilon/(\varepsilon + \beta)\varsigma_t L_t u_t + q_t K_t + (1 + i_t)B_t + \phi H_t(1 + i_t)^{-\theta} + \Gamma_t + \Delta_t. \tag{29}$$

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[1] The proof of Walras' law proceeds as follows: Denote by $P_t > 0$ the nominal price (level) of production output (GDP). Then, the current period budget constraint of employed households in youth can be rewritten as follows: $P_t L_t(1 - u_t)c_t^{1,E} + P_t I_t^{D,E} + P_t B_{t+1}^{D,E} = (1 - \tau_t)P_t w_t h_t(1 - u_t)L_t$. (F.1) The budget constraint of households in old age employed in youth reads as follows: $P_t L_{t-1}(1 - u_{t-1})c_t^{2,E} = P_t q_t K_t^{S,E} + P_t(1 + i_t)B_t^{S,E}$. (F.2) The budget constraint of young unemployed households in period $t$ is as follows: $P_t L_t u_t c_t^{1,U} + P_t I_t^{D,U} + P_t B_{t+1}^{D,U} = L_t u_t \varsigma_t$. (F.3) Moreover, the budget constraint of the household in old age in period $t$, which was unemployed in youth, reads as follows: $P_t L_{t-1} u_{t-1} c_t^{2,U} = P_t q_t K_t^{S,U} + P_t(1 + i_t)B_t^{S,U}$. (F.4) In addition, maximum profits are zero, which implies: $P_t Y_t = P_t w_t N_t + P_t q_t K_t$. (F.5) Finally, government's budget constraint is rewritten as follows: $P_t B_{t+1} = P_t(1 + i_t)B_t + P_t \Delta_t + P_t \Gamma_t + P_t L_t u_t \varsigma_t - P_t \tau_t w_t h_t L_t(1 - u_t)$. (F.6) Adding up the left- and right-hand side of equations (F.1), (F.2), (F.3) and (F.4) yields: $P_t L_t(1 - u_t)c_t^{1,E} + P_t I_t^{D,E} + P_t L_t u_t c_t^{1,U} + P_t I_t^{D,U} + P_t L_{t-1}(1 - u_{t-1})c_t^{2,E} + P_t L_{t-1} u_{t-1} c_t^{2,U} = (1 - \tau_t)P_t w_t h_t \times \times (1 - u_t)L_t - P_t B_{t+1}^{D,E} - P_t B_{t+1}^{D,U} + L_t u_t \varsigma_t + P_t q_t K_t^{S,E} + P_t(1 + i_t)B_t^{S,E} + P_t q_t K_t^{S,U} + P_t(1 + i_t)B_t^{S,U}$. (F.7) Considering (25), (26) and (27) in (F.7), gives: $P_t L_t(1 - u_t)c_t^{1,E} + P_t L_{t-1}(1 - u_{t-1})c_t^{2,E} + P_t I_t^{D,E} + P_t L_t u_t c_t^{1,U} + P_t L_{t-1} u_{t-1} c_t^{2,U} + P_t I_t^{D,U} = P_t w_t N_t - \tau_t P_t w_t N_t - P_t B_{t+1} + P_t q_t K_t + P_t(1 + i_t)B_t$. (F.8) Considering labor market clearing condition (24) when inserting (F.6) into (F.8), and taking account of (F.5) in (F.8) yields: $P_t L_t(1 - u_t)c_t^{1,E} + P_t L_{t-1}(1 - u_{t-1})c_t^{2,E} + P_t L_t u_t c_t^{1,U} + P_t L_{t-1} u_{t-1} c_t^{2,U} + P_t I_t^{D,E} + P_t I_t^{D,U} + P_t \Gamma_t = P_t Y_t$, which is production-output market clearing. Since this equation is always true, $P_t$ is indeterminate and can be fixed as $P_t = 1$.

On dividing equation (29) into both sides by $Y_t$, this equation turns into the following equation:

$$1 = \varepsilon/(\varepsilon + \beta)(1 - \tau_t)w_t h_t L_t(1 - u_t)/Y_t + \varepsilon/(\varepsilon + \beta)\varsigma_t L_t u_t/Y_t$$

$$+q_t K_t/Y_t + (1 + i_t)B_t/Y + \phi H_t(1 + i_t)^{-\theta}/Y + \Gamma_t/Y + \Delta_t/Y. \tag{30}$$

Rewriting profit maximization condition (15) as

$$w_t h_t L(1 - u_t) = (1 - \alpha)(K_t)^\alpha(h_t L(1 - u_t))^{1-\alpha} = (1 - \alpha)Y_t, \tag{31}$$

and using the definitions $v_t \equiv K_t/Y_t, b_t \equiv B_t/Y_t, \delta_t \equiv \Delta_t/Y_t, \gamma_t \equiv \Gamma_t/Y_t$ and $\xi_t \equiv (\varsigma_t u_t L_t)/Y_t$, equation (30) can be rewritten as follows:

$$1 = \varepsilon/(\varepsilon + \beta)(1 - \tau_t)(1 - \alpha) + \varepsilon/(\varepsilon + \beta)\xi_t$$

$$+q_t v_t + (1 + i_t)b_t + \phi(1/k_t)(1 + i_t)^{-\theta}v_t + \gamma_t + \delta_t. \tag{32}$$

The capital-output ratio $v_t$ is related to the real-capital to human-capital ratio $k_t$ as follows:

$$v_t = \frac{K_t}{M(H_t)^{1-\alpha}(1 - u_t)^{1-\alpha}(K_t)^\alpha} = \frac{(K_t)^{1-\alpha}}{M(H_t)^{1-\alpha}(1 - u_t)^{1-\alpha}} = \frac{(k_t)^{1-\alpha}}{M(1 - u_t)^{1-\alpha}}, \tag{33}$$

which implies:

$$q_t = \alpha/v_t. \tag{34}$$

By use of the no-arbitrage condition (2) as well as of equations (33) and (34) equation (32) turns out to be:

$$1 = (\varepsilon/(\varepsilon + \beta)(1 - \tau_t)(1 - \alpha) + (\varepsilon/(\varepsilon + \beta)\xi_t + \alpha[1 + b_t/v_t]$$

$$+\phi(\alpha/v_t)^{-\theta}M^{-1/(1-\alpha)}(1 - u_t)^{-1}v_t^{-\alpha/(1-\alpha)} + \gamma_t + \delta_t. \tag{35}$$

Next, it is apt to specify how the government determines its intertemporal policy profile. To this end, we assume that government consumption expenditures per GDP, $\delta_t$, government human capital investment expenditures per GDP, $\gamma_t$, and unemployment benefits per GDP, $\xi_t$, are time-stationary, i.e., $\delta_t = \delta_{t+1} = \delta, \gamma_t = \gamma_{t+1} = \gamma, \forall t$ and $\xi_t = \xi_{t+1} = \xi, \forall t$. As in Diamond ([2], p. 1137) we furthermore assume that the government runs a 'constant-stock' fiscal policy: $b_{t+1} = b_t = b$. The budget constraint of the government written in per GDP terms reads as follows:

$$\frac{B_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} \equiv bG_t^Y = (1 + i_t)B_t/Y_t + \Gamma_t/Y_t + \frac{\Delta_t}{Y_t} + \frac{L_t u_t \varsigma_t}{Y_t} - \frac{\tau_t(1 - u_t)w_t h_t L}{Y_t}$$

$$\equiv \alpha b/v_t + \gamma + \delta + \xi - \tau_t(1 - \alpha). \tag{36}$$

Equation (36) implies that the wage-tax rate ought to become endogenous and is determined by the following equation:

$$\tau_t = [(\alpha/v_t - G_t^Y)b + \gamma + \delta + \xi]/(1 - \alpha). \tag{37}$$

Inserting $\tau_t$ from equation (37) into equation (32), leads to the following result:

$$
\begin{aligned}
1 &= (\varepsilon/(\varepsilon+\beta))\left[1 - \alpha(1 + b/v_t) - \gamma - \delta - \xi + bG_t^Y\right] + (\varepsilon/(\varepsilon+\beta))\xi \\
&\quad + \alpha[1 + b_t/v_t] + \gamma + \delta + \phi(\alpha/v_t)^{-\theta}M^{-1/(1-\alpha)}(1 - u_t)^{-1}v_t^{-\alpha/(1-\alpha)}.
\end{aligned}
\tag{38}
$$

Collecting terms and simplifying the resulting expression yields the following equation for $(1 - u_t)$:

$$
\begin{aligned}
(1 - u_t) &= \left\{(\varepsilon + \beta)\phi\alpha^{-\theta}M^{-1/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\right\} \\
&\quad / \left\{\beta[1 - \gamma - \delta - \alpha(1 + b/v_t)] - \varepsilon b G_t^Y\right\}.
\end{aligned}
\tag{39}
$$

In terms of the transformed variables, the growth factor of human capital reads as follows:

$$
\begin{aligned}
\frac{H_{t+1}}{H_t} &= H_0(H_t)^{\mu-1}(\gamma)^{1-\mu}(Y_t)^{1-\mu} \\
&= H_0(H_t)^{\mu-1}(\gamma)^{1-\mu}M^{(1-\mu)}(K_t)^{\alpha(1-\mu)}(H_t)^{(1-\alpha)(1-\mu)}(1 - u_t)^{(1-\alpha)(1-\mu)} \\
&= H_0 M^{(1-\mu)/(1-\alpha)}(\gamma)^{1-\mu}(v_t)^{\alpha(1-\mu)/(1-\alpha)}(1 - u_t)^{(1-\mu)}.
\end{aligned}
\tag{40}
$$

The GDP growth factor in terms of the capital-output ratio can be rewritten as follows:

$$
\begin{aligned}
G_t^Y &= \frac{H_{t+1}}{H_t}\left(\frac{v_{t+1}}{v_t}\right)^{\alpha/(1-\alpha)}\frac{(1 - u_{t+1})}{(1 - u_t)} \\
&= H_0 M^{(1-\mu)/(1-\alpha)}(\gamma)^{1-\mu}(v_{t+1})^{\alpha/(1-\alpha)}(v_t)^{-\alpha\mu/(1-\alpha)}(1 - u_{t+1})(1 - u_t)^{-\mu}.
\end{aligned}
\tag{41}
$$

By using the intertemporal equilibrium condition $K_{t+1}^S = I_t^D$, one obtains the following equation for the dynamics of the capital-output ratio:

$$
v_{t+1}G_t^Y = \phi\frac{H_t}{K_t}\frac{K_t}{Y_t}q_t^{-\theta} = \phi\frac{1}{k_t}v_t q_t^{-\theta} = \alpha^{-\theta}\phi M^{\frac{-1}{(1-\alpha)}}(v_t)^{\frac{\theta(1-\alpha)-\alpha}{(1-\alpha)}}(1 - u_t)^{-1}.
\tag{42}
$$

The final steps needed to arrive at the equation of motion for the capital-output ratio entail; first, inserting the GDP growth factor equation (41) into equation (42). This procedure yields:

$$
v_{t+1}^{1/(1-\alpha)}(1 - u_{t+1}) = \left[\phi/(\alpha^\theta H_0)\right]\gamma^{\mu-1}M^{(\mu-2)/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1 - u_t)^{\mu-1}.
\tag{43}
$$

Next, after inserting the growth factor equation (41) into equation (39) and rearranging, we arrive at the following intermediate result:

$$
\begin{aligned}
v_{t+1}^{\alpha/(1-\alpha)}(1 - u_{t+1}) &= (\varepsilon b H_0)^{-1}\gamma^{\mu-1}M^{(1-\mu)/(\alpha-1)}(1 - u_t)^{\mu-1}v_t^{(\alpha\mu)/(1-\alpha)} \\
&\quad \times \left\{\beta(1 - u_t)[1 - \alpha(1 + b/v_t) - \gamma - \delta] - (\beta + \varepsilon)\phi M^{-1/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\right\}.
\end{aligned}
\tag{44}
$$

Solving equation (44) for $1 - u_{t+1}$ and inserting the result into equation (43) then yields, after re-arranging, the first equation of motion:

$$
v_{t+1} = \frac{\varepsilon b}{\beta\phi^{-1}\alpha^\theta M^{1/(1-\alpha)}[1 - \alpha(1 + b/v_t) - \gamma - \delta](1 - u_t)v_t^{[\alpha-\theta(1-\alpha)]/(1-\alpha)} - \beta - \varepsilon}.
\tag{45}
$$

Reinserting the dynamic equation (45) into equation (43) and solving for $1 - u_{t+1}$ generate the second equation of motion:

$$u_{t+1} = 1 - \phi(\alpha^\theta H_0)^{-1} \gamma^{\mu-1}(\varepsilon b)^{-1/(1-\alpha)} M^{(2-\mu)/(\alpha-1)} v_t^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1-u_t)^{\mu-1}$$

$$\times \left\{ \alpha^\theta \beta \phi^{-1} M^{1/(1-\alpha)}(1-u_t)[1-\alpha(1+b/v_t)-\gamma-\delta]v_t^{[\alpha-\theta(1-\alpha)]/(1-\alpha)} - (\beta+\varepsilon) \right\}^{1/(1-\alpha)}.$$

(46)

## 4. Existence and dynamic stability of steady states

The steady states of the equilibrium dynamics depicted by the difference equations (45) and (46) are defined as $\lim_{t->\infty} v_t = v$ and $\lim_{t->\infty} u_t = u$. Explicit steady state solutions are not possible. Thus, we are in need to resort to an intermediate value theorem to prove the existence of at least one feasible steady-state solution $v_{\min} < v < \infty$ and $0 < u < 1$.

To this end, for given structural and policy parameters (except $b$), the maximally sustainable debt to GDP parameter is defined as $b^{\max}$, and the minimal capital-output ratio as $v_{\min}$ which ensure full employment. On inserting $G^Y$ from the steady-state version of equation (41) into the steady-state version of equation (42) with $u = 0$, $v_{\min}$ can then explicitly be determined as follows:

$$v_{\min} = \left[ \alpha^{-\theta} \gamma^{\mu-1} \phi (H_0)^{-1} M^{(\mu-2)/(1-\alpha)} \right]^{(1-\alpha)/[1+\alpha(1-\mu)-\theta(1-\alpha)]}.$$

(47)

Using the steady-state version of equation (39) with $u = 0$, $b = b^{\max}$ can be calculated as follows:

$$b^{\max} = \left[ \beta(1-\alpha-\gamma-\delta) - \alpha^{-\theta}(\beta+\varepsilon)\phi M^{-1/(1-\alpha)} v_{\min}^{[\theta(1-\alpha)-\alpha]/(1-\alpha)} \right]$$
$$/ \left[ \alpha\beta(v_{\min})^{-1} + \varepsilon H_0 M^{(1-\mu)/(1-\alpha)} \gamma^{1-\mu}(v_{\min})^{\alpha(1-\mu)/(1-\alpha)} \right],$$

(48)

Whereby, to ensure a strictly larger than zero $b^{\max}$, it is assumed that:
$\beta(1-\alpha-\gamma-\delta) > \alpha^{-\theta}(\beta+\varepsilon)\phi M^{-1/(1-\alpha)} v_{\min}^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}$.

For the proof of the existence of at least one $0 < u < 1$ and $v_{\min} < v < \infty$ the steady-state versions of equations (45) and (46) are used. This results in:

$$v = \frac{\varepsilon b}{\alpha^\theta \beta \phi^{-1} M^{1/(1-\alpha)}[1-\alpha(1+b/v)-\gamma-\delta](1-u)v^{[\alpha-\theta(1-\alpha)]/(1-\alpha)} - \beta - \varepsilon}.$$

(49)

$$1 - u = \phi(\alpha^\theta H_0)^{-1}\gamma^{\mu-1}(\varepsilon b)^{-1/(1-\alpha)} M^{(2-\mu)/(\alpha-1)} v^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1-u)^{\mu-1}$$
$$\times \{ \alpha^\theta \beta \phi^{-1} M^{1/(1-\alpha)}(1-u)[1-\alpha(1+b/v)-\gamma-\delta]v^{[\alpha-\theta(1-\alpha)]/(1-\alpha)}$$
$$-(\beta+\varepsilon)\}^{1/(1-\alpha)}.$$

(50)

By substituting $\alpha^\theta \beta \phi^{-1} M^{1/(1-\alpha)}(1-u)[1-\alpha(1+b/v_t)-\gamma-\delta]v^{\alpha/(1-\alpha)} - \beta - \varepsilon$ in (50) for $\varepsilon b/v$, equation (50) can be reduced to the following simpler equation:

$$1 - u = \alpha^{-\theta}\gamma^{\mu-1}\phi H_0^{-1} M^{(\mu-2)/(1-\alpha)}(1-u)^{\mu-1}v^{[\theta(1-\alpha)-\alpha(1-\mu)-1]/(1-\alpha)}$$

(51)

Using the short cut $1 - u \equiv w$, the two equations (49) and (50) can be explicitly solved for $w$ as follows:

$$w_1 = \frac{\alpha^{-\theta}\phi M^{-1/(1-\alpha)}v^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}[\beta + \varepsilon(1+b/v)]}{\beta[1-\alpha(1+b/v)-\gamma-\delta]}, \tag{52}$$

$$w_2 = \left[\alpha^{-\theta}\gamma^{\mu-1}\phi H_0^{-1}M^{(\mu-2)/(1-\alpha)}v^{[\theta(1-\alpha)-\alpha(\mu-1)-1]/(1-\alpha)}\right]^{1/(2-\mu)}. \tag{53}$$

Hereby, $w_1$ represents the solution of equation (49) for $w$, while $w_2$ exhibits the solution of equation (50) for $w$. A steady state solution exists if $w_1(v) = w_2(v)$ for at least one $v_{\min} < v < \infty$.

**Proposition 1**. Suppose there exist $\eta > 0$ and $v^{\max} > 0$ such that $w_1(v^{\max}) = w_2(v^{\max}) + \eta$. Then, the solution of $w_1(v) = w_2(v)$ for at least one $v_{\min} < v < v^{\max}$ exists and represents a steady state of the equilibrium dynamics (45) and (46) with $0 < w < 1$, $(0 < u < 1)$.

**Proof**. For $b = b^{\max}$ it is known from above that $v = v_{\min}$ and $u = 0$. Thus, let be $b < b^{\max}$. Using this assumption, we may then show that $0 < u < 1$ and $v_{\min} < v < v^{\max}$. From both $b < b^{\max}$, $dw_1/db = \{[\varepsilon(1-\gamma-\delta)+\alpha\beta] \times \alpha^{-\theta}\phi M^{-1/(1-\alpha)}v^{[1-2\alpha+\theta(1-\alpha)]/(1-\alpha)}/\{\beta[\alpha b-(1-\alpha-\gamma-\delta)]v\}^2 > 0$ and from equations (52) and (53), it follows that $0 < w < 1$, $(0 < u < 1)$. Now an intermediate value theorem is applied to demonstrate that $w_1(v) = w_2(v)$ for $v_{\min} < v < v^{\max}$. To this end, notice first that for $b = b^{\max}$, $w_1(v_{\min}) = w_2(v_{\min})$. Under $b < b^{\max}$ and $dw_1/db > 0$ it is clear that $w_1(v_{\min};b) < w_1(v_{\min};b^{\max})$. Since $w_2(v)$ does not depend on $b$, it follows that $w_2(v_{\min};b) = w_2(v_{\min};b^{\max})$ for all feasible $b$. Hence, $w_1(v_{\min};b) < w_2(v_{\min};b), \forall b < b^{\max}$. At the upper boundary of feasible values for $v$, at $v = v^{\max}$, the assumption employed in Proposition 1 above ensures that $w_1(v^{\max};b) > w_2(v^{\max};b)$. Since the two functions $w_1(v;b)$ and $w_2(v;b)$ are continuous on $v_{\min} < v < v^{\max}$, the intermediate value theorem implies at least one $v_{\min} < v < v_{\max}$ such that $w_1(v) = w_2(v)$ and $0 < w < 1$, $(0 < u < 1)$.. Moreover, for a broad set of feasible parameters the solution is unique. Q.E.D.

The next step is to investigate the local dynamic stability of the unique steady-state solution. To this end, the intertemporal equilibrium equations (39), (41), and (42) are totally differentiated with respect to $v_{t+1}, w_{t+1}, G_t^Y, v_t, w_t$. Then, the Jacobian matrix $J(v,w)$ of all partial differentials with respect to $v_t$ and $w_t$ is formed as follows:

$$J(v,w) \equiv \begin{bmatrix} \dfrac{\partial v_{t+1}}{\partial v_t}(v,w) & \dfrac{\partial v_{t+1}}{\partial w_t}(v,w) \\[2mm] \dfrac{\partial w_{t+1}}{\partial v_t}(v,w) & \dfrac{\partial w_{t+1}}{\partial w_t}(v,w) \end{bmatrix}, \tag{54}$$

with

$$\frac{\partial v_{t+1}}{\partial v_t} \equiv j_{11} = -\frac{\varepsilon b G^Y[\alpha-\theta(1-\alpha]+(1-\alpha)\beta bq + [\alpha-\theta(1-\alpha](\beta+\varepsilon)G^Y v}{(1-\alpha)\varepsilon b\, G^Y},$$

$$\frac{\partial v_{t+1}}{\partial w_t} \equiv j_{12} = -\frac{v[\varepsilon b + (\beta+\varepsilon)v]}{\varepsilon bw} < 0,$$

$$\frac{\partial w_{t+1}}{\partial v_t} \equiv j_{21} =$$

$$\frac{w\{[\alpha-\theta(1-\alpha)](\beta+\varepsilon)G^Y v + (1-\alpha)\beta bq + \alpha\varepsilon b G^Y[\alpha-\theta(1-\alpha)+\mu(1-\alpha)]\}}{(1-\alpha)^2\varepsilon b G^Y v},$$

$$\frac{\partial w_{t+1}}{\partial w_t} \equiv j_{22} = \frac{\varepsilon b[\alpha+\mu(1-\alpha)]+(\beta+\varepsilon)v}{\varepsilon b(1-\alpha)} > 0.$$

The sign of $j_{12}$ is unambiguously smaller than zero, while the sign of $j_{22}$ is always larger than zero. The signs of $j_{11}$ and $j_{21}$ depend on whether $\theta > \alpha/(1-a)$ or

$\theta \leq \alpha/(1-a)$. In the former case the sign of $j_{11}$ is smaller than zero, whereby the sign of $j_{21}$ is larger than zero. In the latter case, the signs of these entries of the Jacobian (54) are in general ambiguous. To evaluate the dynamic stability of the equilibrium dynamics in the neighborhood of the steady-state solution, the eigenvalues of the Jacobian matrix (54) are needed. To this end, the trace $\mathrm{Tr}J(v,w)$, the determinant $\mathrm{Det}J(v,w)$ and $1 - \mathrm{Tr}J(v,w) + \mathrm{Det}J(v,w)$ need to be calculated.

$$\mathrm{Tr}J(v,w) = \theta + \mu - \frac{q\beta}{\varepsilon G^Y} + \frac{(\beta+\varepsilon)(1+\theta)v}{\varepsilon b}, \tag{55}$$

$$\mathrm{Det}J(v,w) = \frac{\beta(1-\mu)q}{\varepsilon G^Y} + \theta\mu\left[1 + \frac{(\beta+\varepsilon)v}{\varepsilon b}\right] > 0, \tag{56}$$

$$1 - \mathrm{Tr}J(v,w) + \mathrm{Det}J(v,w) = (1-\theta)(1-\mu) + \frac{q\beta(2-\mu)}{\varepsilon G^Y} - \frac{(\beta+\varepsilon)[1+\theta(1-\mu)]v}{\varepsilon b}. \tag{57}$$

The sign of the trace turns out to be, in general, indeterminate, while the determinant of the Jacobian (54) is larger than zero. Moreover, the sign of $1 - \mathrm{Tr}J(v,u) + \mathrm{Det}J(v,u)$ is in general ambiguous. However, for a broad set of feasible parameters, all of which are following the assumptions used thus far, the trace is larger than zero (larger than 2) and $1 - \mathrm{Tr}J(v,u) + \mathrm{Det}J(v,u) < 0$.

**Proposition 2**. Suppose the assumptions of Proposition 1 hold. Then, the calculation of the eigenvalues $\lambda_1$ and $\lambda_2$ of Jacobian (54) at the steady-state solution $0 < w < 1$, $(0 < u < 1)$ and $v_{\min} < v < v^{\max}$ shows that for a broad set of feasible parameter combinations with $b < b^{\max}$, $\lambda_1 > 1$ and $0 < \lambda_2 < 1$.

In other words, the steady-state solution in the present endogenous growth model with involuntary unemployment represents a non-oscillating, monotone saddle point with $v_t$ as a slowly moving variable and $w_t(u_t)$ as jump variables. With $v_0 = \underline{v} > 0$ historically given, $w_0(u_0)$ jumps onto the saddle-path along which both variables converge monotonically towards the steady-state solution.

## 5. Comparative steady-state effects of a higher public debt to GDP ratio

Being assured of the existence and dynamic stability of a steady-state solution it is now apt to investigate how a larger government debt to GDP ratio impacts the steady-state GDP growth rate and the steady-state unemployment rate. A comparable OLG model with endogenous growth and full employment [9] finds that the GDP growth is raised by a higher government debt to GDP ratio if the GDP growth rate is larger than the real interest rate in the initial steady state. If the real interest rate is higher than the GDP growth rate in the initial steady-state, larger government debt to GDP ratio lowers the GDP growth rate. Because of these interesting results, it will be expedient to explore whether in our model with an independent aggregate investment function the GDP growth rate effect of more government debt will also depend on the difference between the initial GDP growth rate and the initial interest rate. In addition, of particular interest is how a larger public debt to GDP ratio affects the unemployment rate which could not be investigated by Lin [9].

To proceed, we now consider the intertemporal equilibrium equations (37), (39), (41) and (42) in a steady-state and differentiate the resulting static equation system totally with respect to $\tau, G^Y, v, w$ and $b$. The following linear equation system with respect to the total differentials $d\tau, dG^Y, dv, dw, db$ is then obtained:

$$d\tau(1-\alpha) = (q - G^Y)db - bdG^Y - qb\,dv/v, \tag{58}$$

$$\frac{dG^Y}{G^Y} = (1-\mu)\left(\frac{\alpha}{(1-\alpha)}\frac{dv}{v} + \frac{dw}{w}\right), \tag{59}$$

$$\frac{dG^Y}{G^Y} + \frac{dw}{w} + \frac{[1-\theta(1-\alpha)]}{(1-\alpha)}\frac{dv}{v} = 0, \tag{60}$$

$$G^Y v\frac{dw}{w} + \left[\frac{\alpha - \theta(1-\alpha)vG^Y + (1-\alpha)bq}{(1-\alpha)}\right]\frac{dv}{v} + \frac{(1-\alpha)\varepsilon}{(\beta+\varepsilon)}d\tau = qdb. \tag{61}$$

Solving simultaneous equations (58) and (61) for $d\tau$ and $dG^Y$, and inserting the result for $dG^Y$ into equations (59) and (60) we obtain a two-dimensional linear equation system comprising the variables $dw$ and $dv$. The solution of this equation system for $dv/db$ and $dw/db$ reads as follows:

$$\frac{dv}{db} = \frac{(2-\mu)\big[\varepsilon(G^Y - q) + (\beta+\varepsilon)q\big]v}{\varepsilon G^Y b(1 - \mathrm{Tr}J + \mathrm{Det}J)}, \tag{62}$$

$$\frac{dw}{db} = -\frac{[1-\theta(1-\alpha) + \alpha(1-\mu)]\big[\varepsilon(G^Y - q) + (\beta+\varepsilon)q\big]w}{(1-\alpha)\varepsilon G^Y b(1 - \mathrm{Tr}J + \mathrm{Det}J)}. \tag{63}$$

The right-hand side of the differential quotient (62) shows that a higher public debt to GDP ratio affects the capital-output ratio unambiguously negatively if dynamic inefficiency prevails, i.e. the GDP growth factor is larger than the real interest factor since $1 - \mathrm{Tr}J + \mathrm{Det}J < 0$ for a broad set of admissible parameters. Under dynamic efficiency, i.e. the real interest factor is larger than the GDP growth factor the response of the capital-output ratio to higher public debt to GDP ratio becomes in general ambiguous.

The term on the right-hand side of equation (63) shows the response of one minus the unemployment rate to a higher government debt to GDP ratio. It transpires that when dynamic inefficiency prevails and moreover $\alpha > \theta(1-\alpha)$ holds, the response of one minus the unemployment rate is unambiguously positive and thus a higher public debt to GDP ratio reduces the unemployment rate. A glance at the output market equilibrium equation (38) makes clear why this is so. It shows that one minus the unemployment rate balances the inflexible investment to GDP ratio with the other aggregate demand to GDP ratios. E.g., with a higher debt to GDP ratio wealth of old consumers rises which induces a higher consumption demand of older consumers. With a relatively inflexible investment to GDP ratio, a higher consumption demand to GDP ratio can be maintained only if the unemployment rate falls. Moreover, higher public debt raises the real interest rate which necessitates a decline of the capital-output ratio due to profit maximization which additionally increases labor demand and thus diminishes unemployment.

The calculation of the marginal change of the steady-state wage tax rate and the GDP growth factor from equations (62) and (63) brings forth the following result:

$$\frac{d\tau}{db} = \frac{(\beta+\varepsilon)[1 + \theta(1-\mu)]\big[G^Y v - q(b+v)\big]}{(1-\alpha)\varepsilon b(1 - \mathrm{Tr}J + \mathrm{Det}J)}, \tag{64}$$

$$\frac{dG^Y}{db} = -\frac{(1-\theta)(1-\mu)G^Y\big(\varepsilon G^Y + \beta q\big)}{(1-\alpha)\varepsilon G^Y b(1 - \mathrm{Tr}J + \mathrm{Det}J)}. \tag{65}$$

| Debt to GDP ratio | Capital-output ratio | GDP growth factor | One minus unemployment rate | Wage tax rate | Interest factor |
|---|---|---|---|---|---|
| $b = 0.024$ | 0.14450 | 1.51422 | 0.905413 | 0.44784 | 2.07608 |
| $b = 0.03$ | 0.134872 | 1.5212 | 0.941195 | 0.45870 | 2.22433 |

**Table 1.**
*Steady solutions before and after the policy shock.*

A glance on the right-hand side of equation (64) shows that the response of the wage tax rate to a higher debt to GDP ratio is in general ambiguous. If, however, the GDP growth factor is sufficiently larger than the interest factor (precisely if $G^Y > (1 + b/v)q$) then higher public debt decreases the wage tax rate. On the other hand, the right-hand side of equation (65) reveals that a larger public debt to GDP ratio is conducive for GDP growth provided that $\theta < 1$. Interestingly, for $\theta = 1$ more public debt does not enhance GDP growth.

Because in the case of dynamic efficiency the response of the unemployment rate to higher public debt is in general ambiguous, we use a numerical parameter set that implies dynamic efficiency before the policy shock and which is in line with the assumptions of Proposition 1. To this end, the following parameter combination not untypical because of medium-term econometric parameter estimations is assumed: $\beta = 0.6, \varepsilon = 0.45, \alpha = 0.3, A = 7, H_0 = 3, \mu = 0.5, \gamma = 0.04, \delta = 0.2, \xi = 0.06, \phi = 2.5, \theta = 0.8$. For the policy shock, it is assumed that $b$ is raised from 0.024 (= 72% p. a.) towards 0.03 (= 90% p. a.). The calculation of the steady-state solutions for the capital-output ratio $v$, the GDP growth factor $G^Y$, the wage tax rate $\tau$, 1 minus the unemployment rate $w$ and the real interest factor $q$ before and after the policy shock is depicted in the following **Table 1**.

Although under the present parameter set dynamic efficiency prevails (= the interest factor is larger than the GDP growth factor in **Table 1** with $b = 0.024$), a larger public debt to GDP ratio, i.e. $b = 0.03$ diminishes the capital-output ratio, enhances the GDP growth factor, reduces the unemployment rate and raises the wage tax rate and the interest factor. Thus, the qualitative responses of main macroeconomic variables to higher public debt are similar to those under dynamic inefficiency. As can be shown by variations of main parameters (see Farmer [10] and Farmer [12] in a similar model context), these results are not constrained to the specific parameter set presented above but hold for a broader set of structural and policy parameters.

## 6. Conclusions

This chapter aims to incorporate involuntary unemployment in an OLG growth model with internal public debt and human capital accumulation. Deviating from new-Keynesian macro models in which involuntary unemployment is traced back to inflexible wages, output prices and interest rates vis-à-vis market imbalances, real wages and real interest rates are perfectly flexible in our Diamond-type growth model with involuntary unemployment. Involuntary unemployment occurs in line with [6, 7] aggregate investment is inflexible due to investors' animal spirits.

After presenting the model set-up temporary equilibrium relations and the intertemporal equilibrium dynamics are derived from intertemporal utility maximization, atemporal profit maximization, government's budget constraint and the market-clearing conditions in each period. To arrive at determinate equilibrium

dynamics, it is assumed that the government holds constant over time: the public debt to GDP ratio, the HCI-expenditure ratio, the non-HCI expenditure ratio and the unemployment benefit to GDP ratio. As a consequence, the wage tax rate becomes endogenous.

Due to the complexity of the intertemporal equilibrium relations, an explicit steady-state solution is not possible. Thus, the simplest mathematical existence theorem, the intermediate value theorem is applied to prove the existence of a steady-state solution with a strictly positive capital-output ratio and an unemployment rate larger than zero and smaller than one. Contrary to intuitive expectations, there exists a finite limit to the public debt to GDP ratio even in the economy with involuntary unemployment. Public debt to GDP ratios higher than that limit implies negative unemployment rates which are infeasible. As Farmer [10] shows in a similar model context maximum public debt in a growth model with involuntary unemployment is not a purely theoretical notion but turns out to be empirically relevant in a numerically specified growth model with involuntary unemployment.

Besides the existence of a steady-state solution for the intertemporal equilibrium dynamics, its dynamic stability was shown. It turns out that for a broad set of feasible structural and policy parameters the dynamics is saddle-point stable. With the capital-output ratio as a sluggish variable historically given, the unemployment rate jumps initially suddenly onto the saddle-path along which both variables converge monotonically (non-oscillating) towards their steady-state values.

Being assured of the existence and dynamic stability of the unique steady-state solution we shocked it by a higher public debt to GDP ratio mimicking the pandemic-related larger public debt to GDP ratios in almost all countries of the world economy. In line with Keynesian policy expectations, we were able to show analytically that in case of dynamic inefficiency, i.e., the GDP growth rate is larger than the real interest rate, a higher public debt to GDP ratio (below the maximum debt to GDP ratio) unambiguously reduces both the capital-output ratio and the unemployment rate while raising the GDP growth rate in a dynamic market economy with perfectly flexible real wage and interest rates. In the case of dynamic efficiency, the responses to the policy shock become in general ambiguous. However, in a numerically specified version of the presented model, it was shown that qualitatively similar comparative steady-state effects occur even in the case of dynamic efficiency. The main reason for these results is that under inflexible aggregate investment higher public debt creates a positive wealth effect with old-age consumers which raises aggregate demand and hence reduces unemployment.

The limitations of the present research are obvious: First, micro-foundations for the aggregate investment function are lacking. Here, stock-market foundations for the aggregate investment function in line with Farmer [15]'s investor's belief function should be provided to overcome the purely macro-foundation of the aggregate investment function. Second, there is no impact of larger public debt to GDP ratios on aggregate investment. Here, Salotti [14]'s an empirical specification of a negative relationship between public debt and aggregate investment may be incorporated in a future version of the present model. Both subjects are left to future research.

## Author details

Karl Farmer[1,2]

1 Carl-Francis-University Graz, Graz, Austria

2 Babes-Bolyai-University Cluj-Napoca, Cluj-Napoca, Romania

*Address all correspondence to: karl.farmer@uni-graz.at

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