Chapter

Theory of Control Stochastic Systems with Unsolved Derivatives

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Abstract

Various types of stochastic differential systems with unsolved derivatives (SDS USD) arise in problems of analytical modeling and estimation (filtering, extrapolation, etc.) for control stochastic systems, when it is possible to neglect higher-order time derivatives. Methodological and algorithmic support of analytical modeling, filtering, and extrapolation for SDS USD is developed. The methodology is based on the reduction of SDS USD to SDS by means of linear and nonlinear regression models. Two examples that are illustrating stochastic aspects of methodology are presented. Special attention is paid to SDS USD with multiplicative (parametric) noises.

Keywords: analytical modeling, estimation (filtering, extrapolation), normal approximation method (NAM), regression (linear, nonlinear), stochastic differential systems with unsolved derivatives (SDS USD)

1. Introduction

Approximate methods of analytical modeling (MAM) of the wideband stochastic processes (StP) in stochastic differential systems with unsolved derivatives (SDS USD) based on normal approximate method (NAM), orthogonal expansions method, and quasi moment methods are developed in [1, 2]. For stochastic integrodifferential systems with unsolved derivatives reducible to SDS corresponding equations for MAM are given in [3, 4]. In [3, 4], problems of mean square (m.s.) synthesis of normal (Gaussian) estimators (filters, extrapolators, etc.) were firstly stated and solved in [1–4]. Results presented in [1–4] are valid for smooth (in m.s. sense) functions in SDS USD. For unsmooth functions in SDS USD theory of normal filtering and extrapolation is developed in [5].

Let us present an overview and generalization of [1–5] for linear and nonlinear regression models. Section 2 is developed to normal analytical modeling algorithms. Normal linear filtering and extrapolation algorithms are given in Sections 3 and 4. Linear modeling and estimation algorithms for SDS USD with multiplicated (parametric) noises are presented in Section 5. Normal nonlinear algorithms for filtering and extrapolation are described in Section 6. Section 7 contains two illustrative examples. In Section 8, main conclusions and some generalizations are given.

2. Normal modeling

Different types of SDS USD arise in problems of analytical modeling and estimator design for stochastic nonlinear dynamical systems when it is possible to neglect higher-order time derivatives [1–3].

First-order SDS USD is described by the following scalar equation:

$$\varphi = \varphi(t, X_t, \dot{X}_t, U_t) = 0, \tag{1}$$

where X_t and $\dot{X}_t$ are scalar state variable and its time derivative; U_t is noise vector StP $(\dim U_t = n^U)$; nonlinear function φ admits regression approximation [6–8].

For vector SDS USD, we have the following vector equation:

$$\overline{\varphi} = \overline{\varphi}(t, X_t, \overline{X}_t, U_t) = 0. \tag{2}$$

Here $\overline{X}_t$ being vector of derivatives till l order

$$\overline{X}_t = \left[\dot{X}_t^T \ldots X_t^{(l-1)T}\right]^T; \tag{3}$$

U_t being autocorrelated noise vector defined by linear vector equation:

$$\dot{U}_t = a_{0t}^U + a_{1t} U_t + b_t^U V_t, \tag{4}$$

where $\dim X_t = n^X$; $\dim U_t = n^U$; V_t is white noise, $\dim V_t = n^V$; $\dim a_{0t}^U = n^U \times 1$; $\dim a_t^U = n^U \times n^U$; $\dim b_t^U = n^U \times n^V$. Further, we consider the Wiener white noise W_{0t} with matrix intensity $v_0 = v_0(t)$ and the mixed Wiener-Poisson white noise [9–13]:

$$V_t = \dot{W}_t, \quad W_t = W_{0t} + \int_{R_0^q} c(\rho) P^0(t, d\rho), \tag{5}$$

$$v_t = v_{0t}^W + \int_{R_0^q} c(\rho)[c(\rho)]^T v_P(t, \rho) d\rho. \tag{6}$$

Here, $\dim c(\rho) = \dim W_{0t} = n^V$; stochastic Ito integrals are taken in R_0^q (R_0^q with pricked origin).

As it is known [6–8], a deterministic model for real StP defined by $Y = \varphi(Z)$ at $Z = \left[X^T \overline{X}^T U^T\right]^T$ in (2) is given by the formula

$$\hat{y}(z) = E[Y|z], \quad \hat{y}(z) \in \Psi \tag{7}$$

at accuracy criterion

$$\varepsilon(z) = \sum_{p=1}^{n^Y} E\left[\left|\hat{y}_p - Y_p\right|^2 |z\right], p = \left[1, \ldots, n^Y\right]. \tag{8}$$

Class of functions $\psi \in \Psi$ represents linear functional space satisfying the following necessary and sufficient conditions:

$$\operatorname{tr} E\left\{\left[|\hat{y}(z) - Y|\right] \psi(z)^T\right\} = 0. \tag{9}$$

For linear shifted and unshifted regression models, we have two known models:

$$\hat{y}(z) = g^B z, \quad g^B = \Gamma_{yz}\Gamma_z^{-1} \tag{10}$$

(Booton [6–8]),

$$\hat{y}(z) = a + g^K z^0, \quad g^K = K_{yz}K_z^{-1}, \quad a = E^Y - g^K E^z \tag{11}$$

(Kazakov [6–8]),

where E^z, Γ_z, K_z being first and second moments for given one-dimensional distribution.

For Eq. (2), linear regression model takes the Booton form

$$\hat{\varphi} = \hat{\varphi}_0 + k_1^\varphi X_t + k_2^\varphi \overline{X}_t + k_3^\varphi U_t = 0, \tag{12}$$

where $\hat{\varphi}_0, k_{1,2,3}^\varphi$ being regressors depending on φ and joint distribution of StP $X_t, \overline{X}_t, U_t$. After Eq. (12) differentiation till the $(l-1)$ order, we get the following set of $(l-1)$ equations:

$$\dot{\hat{\varphi}}_t = 0, \dots, \hat{\varphi}_t^{(l-1)} = 0. \tag{13}$$

At algebraic solvability condition of linear Eqs. (12) and (13), we reduce SDS USD to SDS of the following form:

$$\dot{X}_t = A_0 + A_1 X_t + A_2 U_t, \tag{14}$$

where A_0, A_1, A_2 are expressed in terms $\hat{\varphi}_0, k_{1,2,3}^\varphi \left(\det\left(k_2^\varphi\right)^{-1} \neq 0\right)$ and indirectly depends on statistical characteristics of X_t, its derivatives and noise U_t. For combined vector $\left[X_t^T \, U_t^T\right] = \tilde{Y}_t$ we have equation:

$$\dot{\tilde{Y}}_t = B_0 + B_1\tilde{Y}_t + B_2 V_t, \quad Y_{t0} = Y_0, \tag{15}$$

Its one and second probabilistic moments satisfy the following equations [12–14]:

$$\dot{\tilde{Y}}_t = B_0 + B_1\tilde{Y}_t + B_2 V_t, \quad Y_{t0} = Y_0, \tag{16}$$

$$\dot{E}_t^{\tilde{Y}} = B_0 + B_1 E_t^{\tilde{Y}}, \quad E_{t0}^{\tilde{Y}} = E_0^{\tilde{Y}}, \tag{17}$$

$$\dot{K}_t^{\tilde{Y}} = B_1 K_t^{\tilde{Y}} + K_t^{\tilde{Y}} B_1^T + B_2 \upsilon B_2^T, \quad \dot{K}_{t0}^{\tilde{Y}} = K_0^{\tilde{Y}}, \tag{18}$$

$$\frac{\partial K^{\tilde{Y}}(t_1, t_2)}{\partial t_2} = K^{\tilde{Y}}(t_1, t_2) B_{1t_2}^T, \quad K^{\tilde{Y}}(t_1, t_1) = K_{t_1}^{\tilde{Y}} \tag{19}$$

where $E_t^{\tilde{Y}} = E\left[\tilde{Y}_t\right], K_t^{\tilde{Y}} = E\left[\left(\tilde{Y}_t - \tilde{E}_t^Y\right)\left(\tilde{Y}_t - \tilde{E}_t^Y\right)^T\right]$, $(t_1 > t_2)$. So, we get two proposals.

Proposal 1. *Let vector non-Gaussian SDS USD (2) satisfy conditions:*

i. *vector functions φ in Eq. (2) admit m.s. regression of linear class Ψ;*

ii. *linear Eqs. (12) and (13) are solvable regards all derivatives till $(l-1)$ order.*

Then SDS USD may be reduced to parametrized SDE. First and second moments of joint vector $\tilde{Y}_t = \left[X_t^T U_t^T\right]^T$ satisfy Eqs. (16)–(19).

Proposal 2. *For normal joint distribution* $\mathcal{N} = \mathcal{N}\left(E_t^Y, K_t^Y\right)$ *of vector variables in Eqs. (16)–(19) it is necessary in equations of Theorem 1 to put*

$$\begin{aligned} &W_t = W_{0t}, \quad E_t^{\tilde{Y}} = E_{\mathcal{N}}^{\tilde{Y}}, \quad K_t^{\tilde{Y}} = E_{\mathcal{N}}^{\tilde{Y}}\left[\left(\tilde{Y}_t - E_{\mathcal{N}}^{\tilde{Y}}\right)\left(\tilde{Y}_t - E_{\mathcal{N}}^{\tilde{Y}}\right)^T\right], \\ &K^{\tilde{Y}}(t_1, t_2) = E_{\mathcal{N}}\left[\left(\tilde{Y}_{t_1} - E_{\mathcal{N}t_1}^{\tilde{Y}}\right)\left(\tilde{Y}_{t_2} - E_{\mathcal{N}t_2}^{\tilde{Y}}\right)^T\right]. \end{aligned} \tag{20}$$

For Eq. (2) using Kazakov form

$$\overline{\varphi} = \overline{\varphi}_0 + \overline{\varphi}^0 = 0 \tag{21}$$

where

$$\overline{\varphi}^0 = k_1^{\overline{\varphi}} X_t^0 + k_2^{\overline{\varphi}} \overline{X}_t^0 + k_3^{\overline{\varphi}} U_t^0, \tag{22}$$

we have two sets of equations for mathematical expectations and centered variables:

$$\dot{\overline{\varphi}}_0 = 0, \ \dots, \overline{\varphi}_0^{(l-1)} = 0 \tag{23}$$

$$\dot{\overline{\varphi}}_0 = 0, \ \dots, \overline{\varphi}_0^{0(l-1)} = 0. \tag{24}$$

So, we reduce SDS USD to two sets of equations for E_t^X and $X_t^0 = X_t - E_t^X$

$$\dot{E}_t^X = \overline{A}_0 + \overline{A}_1 E_t^X + \overline{A}_2 E_t^U, \tag{25}$$

$$\dot{X}_t^0 = A_1 X_t^0 + A_2 U_t^0. \tag{26}$$

For the composed vector $\overline{Y}_t^0 = \left[X_t^{0T} U_t^{0T}\right]^T$ its probabilistic one and second moments satisfy the following equations:

$$E_t^{\overline{Y}} = \overline{B}_0 + \overline{B}_1 E_t^{\overline{Y}}, \quad \overline{Y}_{t0} = \overline{Y}_0, \tag{27}$$

$$\dot{K}^{\overline{Y}} = \overline{B}_1 K_t^{\overline{Y}} + K^{\overline{Y}} \overline{B}_1^T + \overline{B}_2 v \overline{B}_2^T, \quad K_{t0}^{\overline{Y}} = K_0^{\overline{Y}}, \tag{28}$$

$$\frac{\partial K^{\overline{Y}}(t_1, t_2)}{\partial t_2} = K^{\overline{Y}}(t_1, t_2) \overline{B}_{1t_2}^T, \quad K^{\overline{Y}}(t_1, t_1) = K_{t_1}^{\overline{Y}}, \quad t_2 > t_1. \tag{29}$$

Here $v = v_0$ being defined by Eq. (6).

So for Kazakov regression, Eqs. (21)–(24) are the basis of Proposal 3.

The regression $E^y(z)$ and its m.s. estimator $\hat{y}(z)$ represent deterministic regression model. So to obtain a stochastic regression model, it is sufficient to represent Y in the form $Y = E^y(z) + Y'$ or $Y = \hat{y}(z) + Y''$, where Y', Y'' being some random variables. For finding a deterministic linear regression model, it is sufficient to know the mathematical expectations E^z, E^y and covariance matrices K^z, K^{yz}. In the case of a stochastic linear regression model, it is necessary to know the distribution of Y for any z or at list its regression $\hat{y}(z)$ and covariance matrix $K^y(z)$ (coinciding with the covariance matrices $K^{Y'}(z)$ or $K^{Y''}(z)$). A more general problem of the best m.s. approximation of the regression by a finite linear combination of given functions $\chi_1(z), \dots, \chi_N(z)$ is reduced to the problem of the best approximation to the regression, as any linear combination of the functions $\chi_1(z), \dots, \chi_N(z)$ represents a linear function of variables $z_1 = \chi_1(z), \dots, z_N(z) = \chi_N(z)$. Corresponding models based on m.s. optimal regression are given in [7].

In the general case, we have the following vector equation:

$$\dot{Z}_t = a^z(Z_t, t) + b^z(Z_t, t)V_t, \tag{30}$$

where V_t being defined by Eqs. (5) and (6). Functions $a^z = a^z(Z_t, t)$ and $b^z = b^z(Z_t, t)$ are composed on the basis of Eq. (2) after nonlinear regression approximation $\hat{\varphi}_t = \sum_j c_j \chi(Z_t)$ and Eq. (13).

According to normal approximation method (NAM), we have for Eq. (30) the following equations for normal modeling [9–12]:

$$\dot{E}_t^z = F_1(E_t^z, K_t^z, t), \tag{31}$$

$$\dot{K}_t^z = F_2(E_t^z, K_t^z, t), \tag{32}$$

$$\frac{\partial K_t^z(t_1, t_2)}{\partial t_2} = F_3(E_t^z, K^z(t_2), K^z(t_1, t_2), t_1, t_2). \tag{33}$$

Here

$$F_1(E_t^z, K_t^z, t) = E_N a^z(Z_t, t), \tag{34}$$

$$F_2(E_t^z, K_t^z, t) = F_{21}(E_t^z, K_t^z, t) + F_{21}(E_t^z, K_t^z, t)^T + F_{22}(E_t^z, K_t^z, t), \tag{35}$$

$$F_{21}(E_t^z, K_t^z, t) = E_N a^z(Z_t, t)(Z_t - E_t^z), \tag{36}$$

$$F_{22}(E_t^z, K_t^z, t) = E_N b^z(Z_t, t) v b^z(Z_t, t)^T, \tag{37}$$

$$\begin{gathered} F_3\left(E_{t_2}^z, K_{t_2}^z, K^z(t_1, t_2), t\right) = K^z(t_1, t_2)\left(K_{t_2}^z\right)^{-1} F_{21}\left(E_{t_2}^z, K_{t_2}^z, t_2\right)^T, \\ E_{t_0}^z = E^Z(t_0), \quad K_{t_0}^z = K^Z(t_0), \quad K^z(t_1, t_2) = K_{t_1}^z \end{gathered} \tag{38}$$

where E_N being symbol of normal mathematical expectation.

3. Normal linear filtering

In filtering SDS USD problems, we use two types of equations: reduced SDE USD for vector state variables X_t and equation for vector observation variables Y_t and $\dot{Y}_t \equiv Z_t$.

Consider SDS USD Eq. (2) reducible to SDE Eq. (3.9) at conditions of Theorem 1. We introduce new variables putting $X_t \equiv \tilde{Y}_t$,

$$\dot{X}_t = A_{0t} + A_{1t} X_t + A_{2t} V_{1t}. \tag{39}$$

Let the observation vector variable Y_t satisfy the following linear equations:

$$Z_t = \dot{Y}_t = B_{0t} + B_{1t} X_t + B_{2t} V_{2t}. \tag{40}$$

where V_{1t} and V_{2t} are normal white noises with matrix $v_{1t} = v_{01}$ and $v_{2t} = v_{02}$ intensities.

Equations of Kalman-Bucy filter in case of Eqs. (39) and (40) for the Gaussian white noises are as follows [12–14]:

$$\dot{\hat{X}}_t = A_0 + A_1 \hat{X}_t + \beta_t \left[Z_t - \left(B_0 + B_1 \hat{X}_t\right)\right]. \tag{41}$$

$$\beta_t = R_t B_{1t}^T v_{2t}^{-1}, \quad \det v_{2t} \neq 0. \tag{42}$$

$$\dot{R}_t = A_{1t} R_t + R_t A_{1t}^T + v_{1t} - \beta_t v_{2t} \beta_t^T \tag{43}$$

at corresponding initial conditions. R_t being m.s. covariance matrix error, β_t being gain coefficient. So, we have the following result.

Proposal 4. *Let:*

i. *USD are reducible to SDS according to Proposal 2 or Proposal 3;*

ii. *observations are performed according to Eq.* (40).

Then equations for m.s. normal filtering have the generalized Kalman-Bucy filter of the form (41)–(43).

4. Normal linear extrapolation

Using equations of linear m.s. extrapolation for time interval Δ [12–14] we get the following equations for the generalized Kalman–Bucy extrapolator:

$$\dot{\hat{X}}_{t+\Delta t|t} = A_1 \hat{X}_{t+\Delta t|t} \quad (\Delta > 0) \tag{44}$$

with initial condition

$$\left[\hat{X}_{t+\Delta t|t}\right]_{\Delta=0} = \hat{X}_t. \tag{45}$$

For the initial time moment t and for the final time moment $t + \Delta$ according to Eq. (44), we get

$$X_{t+\Delta t|t} = u(t+\Delta, t) X_t + \int_t^{t+\Delta} u(t+\Delta, \tau) a_0(\tau) d\tau + \int_t^{t+\Delta} u(t+\Delta, \tau) \psi(\tau) dW(\tau). \tag{46}$$

where $u(t, \tau)$ being the fundamental solution of equation $\dot{u}_t = A_{1t} u_t$ at condition $u(t, t) = I$. For conditional mathematical expectation relatively $Y_{t_0}^t$ in Eq. (46), we get m.s. estimate future state $X_{t+\Delta}$

$$\hat{X}_{t+\Delta t|t} = \mathrm{E}\left[X_{t+\Delta|t} | Y_{t_0}^t\right] = u(t+\Delta, t) \hat{X}_{t|t} + \int_t^{t+\Delta} u(t+\Delta, \tau) a_0(\tau) d\tau. \tag{47}$$

In this case, error covariance matrix $R_{t+\Delta t|t}$ satisfies the following equation:

$$\dot{R}_{t+\Delta t|t} = a_1 R_{t+\Delta|t} + R_{t+\Delta|t} a_1^T + \psi v_0 \psi^T. \tag{48}$$

At initial condition

$$\left[R_{t+\Delta|t}\right]_{\Delta=0} = R_t. \tag{49}$$

Hence, the error matrix R_t is known from Proposal 4. So, we have the following proposition.

Proposal 5. *At conditions of Proposal 4 m.s. normal extrapolation $\hat{X}_{t+\Delta t|t}$ is defined by Eqs. (47)–(49).*

This extrapolator presents a sequel connection m.s. filter with gain $u(t+\Delta,t)$, summator $u(t+\Delta,t)\hat{X}_{t|t}$ and integral term $\int_t^{t+\Delta} u(t+\Delta,\tau)A_{0\tau}d\tau$. The accuracy of extrapolation is estimated according to Eqs. (48) and (49).

5. Linear modeling and estimation in SDS USD with multiplicated noises

Let us consider vector Eqs. (2)–(6) for the multiplicative Gaussian noises:

$$\varphi = \varphi(\dot{X}_t, X_t, V_t) = \varphi_1(\dot{X}_t, t) + \left[\varphi_{20}(t) \sum_{h=1}^{n^X} \varphi_{2h}(t) X_h\right] V_t = 0. \tag{50}$$

Here, $\dim X_t = \dim \dot{X}_t = n^X$, $\dim \varphi = n^X$, φ_1 being nonlinear vector function of vector argument $\dot{X}_t$ admitting linear regression

$$\varphi_1(\dot{X}_t, t) \approx \varphi_{11} \dot{X}_t, \quad \varphi_{11} = \varphi_{11}\left(E_t^{\dot{X}}, K_t^{\dot{X}}, t\right). \tag{51}$$

Here, φ_{11} being matrix of regressors; V_{1t} being vector Gaussian white noise, $\dim V_t = n^V$ with matrix intensity $v = v_0(t)$. In this case, Eqs. (50) and (51) at condition $\det \varphi_{11} \neq 0$ may be resolved relatively $\dot{X}_t$

$$\dot{X}_t = B_0 + B_1 X_t + \left(B_2 + \sum_{r=1}^{n^X} B_{3r} X_{rt}\right) V_t, \tag{52}$$

where B_0, B_1, B_2, B_{3r} depend upon regressors φ_{11}. Using [9–12], we get equations for mathematical expectations. E_t^X, covariance matrix K_t^X, and matrix of covariance functions $K^X(t_1, t_2)$:

$$\dot{E}_t^X = B_0 + B_1 E_t^X, \quad E_{t0}^X = E_0^X, \tag{53}$$

$$\begin{aligned} \dot{K}_t^X &= B_1 K_t^X + K_t^X B_1^T + B_2 v_0 B_2^T + \sum_{r=1}^{n^X} \left(B_{3r} v_0 B_2^T + B_2 v_0 B_{3r}^T\right) \\ & E_{rt}^X \sum \sum_{r,s=1}^{n^X} B_{3r} v_0 B_{3s}^T \left(E_{rt}^X E_{st}^X + K_{rst}^X\right), \quad K_{t0}^X = K_0^X, \end{aligned} \tag{54}$$

$$\frac{\partial K^X(t_1, t_2)}{\partial t_2} = K^X(t_1, t_2) B_{1t_2}^T, \quad K^X(t_1, t_1) = K_{t_1}^X, \quad t_2 > t_1. \tag{55}$$

Here $K_t^X = \left[K_{rst}^X\right]$; $K^X(t_1, t_2) = \left[K_{rs}^X(t_1, t_2)\right]$. So for MAM in nonstationary regimes, we have Eqs. (54) and (55) Proposal 6. In stationary case Eqs. (54) and (55) we get the following finite set of equations for E_* and K_* (Proposal 7):

$$B_0^* + B_1^* E_*^X = 0, \tag{56}$$

$$\begin{aligned} & B_1^* K_*^X + K_*^X B_1^* + B_2^* v_0^* B_2^{*T} + \sum_{r=1}^{n^X} \left(B_{3r}^* v_0^* B_2^{*T} + B_2^* v_0^* B_{3r}^{*T}\right) E_*^X \\ & + \sum \sum_{r,s=1}^{n^X} B_{3r}^* v_0^* B_{3s}^{*T} \left(E_{r*}^X E_{s*}^X + K_{rs*}^X\right) = 0 \end{aligned} \tag{57}$$

and ordinary differential equation for $k^X(\tau)$, $(\tau = t_2 - t_1)$:

$$\frac{dk^X(\tau)}{d\tau} = B_1^* k^X(\tau), \quad k^X(0) = K_*. \tag{58}$$

Applying linear theory Pugachev (conditionally m.s. optimal) filtering [9–12] to equations

$$\dot{X}_t = A_0 + A_1 X_t + \left(A_2 + \sum_{r=1}^{n^X} A_{3r} X_{rt}\right) V_t, \tag{59}$$

$$Z_t = \dot{Y}_t = B_0 + B_1 X_t + B_2 V_t, \tag{60}$$

We get the following normal filtering equations:

$$\dot{\hat{X}}_t = A_0 + A_1 \hat{X}_t + \beta_t \left[Z_t - \left(B_{0t} + B_1 \hat{X}_t\right)\right], \tag{61}$$

$$\beta_t = \left[R_t B_1 + \left(A_2 + \sum_{r=1}^{n^X+n^Y} A_{3r} E_r^X\right) v_0 B_2\right] \kappa_{11}^{-1}, \tag{62}$$

$$\dot{R}_t = A_1 R_t + R_t A_1^T - \left[R_t B_1^T + \left(A_2 + \sum_{r=1}^{n^X+n^Y} A_{3r} E_r^X\right) v_0 B_2\right] \kappa_{11}^{-1} \times$$
$$\times \left[B_1 + B_2 v_0 \left(A_2^T + \sum_{r=1}^{n^X+n^Y} A_{3r}^T E_r^X\right)\right] + \left(A_2 + \sum_{r=1}^{n^X+n^Y} A_{3r} E_r^X\right) v_0 \left(A_2^T + \sum_{r=1}^{n^X+n^Y} A_{3r}^T E_r^X\right)$$
$$+ \sum \sum\nolimits_{r,s=1}^{n^X+n^Y} A_{3r} v_0 A_{3s}^T K_{rs}. \tag{63}$$

Here

$$\kappa_{11} = B_2 v_0 B_0^T, \quad \kappa_{22} = B_2 v_0 B_2^T. \tag{64}$$

For calculating (62) we need to find mathematical expectation E_t^Q, covariance matrix K_t^Q of combined vector $Q_t = [X_1, \dots, X_{n^X}, \ Y_1, \dots, Y_{n^Y}]^T$ and error $\tilde{X}_t$, $\tilde{X}_t = \tilde{X}_t - X_t$ covariance matrix R_t using equations

$$\dot{E}_t^Q = a^Q E_t^Q + a_0^Q, \tag{65}$$

$$\dot{K}_t^Q = a^Q K_t^Q + K_t^Q \left(a_0^Q\right)^T + c^Q v_0 \left(c^Q\right)^T + \sum_{r=1}^{n^X+n^Y} \left[c^Q v_0 \left(c_r^Q\right)^T + c_r^Q v_0 \left(c_0^Q\right)^T\right] E_r^Q +$$
$$+ \sum \sum_{r,s=1}^{n^X+n^Y} c_r^Q v_0 \left(c_s^Q\right)^T \left(E_r^Q E_s^Q + K_{rs}^Q\right), \tag{66}$$

where

$$a^Q = \begin{bmatrix} 0 & B_1 \\ 0 & A_1 \end{bmatrix}, \quad a_0^Q = \begin{bmatrix} B_0 \\ A_0 \end{bmatrix}, \quad c_r^Q = \begin{bmatrix} B_2 \\ A_{1r} \end{bmatrix} \quad \left(r = 0, 1, \dots, n^X + n^Y\right). \tag{67}$$

So, Eqs. (61)–(67) define linear Pugachev filter for SDS USD with multiplicative noises reduced to SDS (59) and (60) (Proposal 8).

At last following [9–12] let us consider linear Pugachev extrapolator for reduced SDS USD. Taking into account equations

$$\dot{X}_t = A_0 + A_1 X_t + \left(A_2 + \sum_{r=1}^{n^X} A_{3,n^Y+r} X_{rt} \right) V_1, \tag{68}$$

$$Z_t = \dot{Y}_t = B_0 + B_1 X_t + B_2 V_2 \tag{69}$$

($V_{1,2}$ being independent normal white noises with $v_{1,2}$ intensities) and the corresponding result (Section 5) we come to the following equation:

$$\dot{\hat{X}}_t = A_0(t+\Delta) + A_1(t+\Delta)\hat{X}_t + \beta_t \left[Z_t - \left(B_0 + B_1 \varepsilon_t^{-1} \hat{X}_t - B_1 \varepsilon_t^{-1} h_t \right) \right]. \tag{70}$$

Here $\varepsilon_t = u(t+\Delta, t)$, $u(s,t)$ being fundamental solution of equation $du/ds = A_1(s)$,

$$h_t = h(t) = \int_t^{t+\Delta} u(t+\Delta, \tau) A_0(\tau) d\tau. \tag{71}$$

Accuracy of linear Pugachev extrapolator (70) is performed by integration of the following equation:

$$\begin{aligned} \dot{R}_t = {} & A_1(t+\Delta)R_t + R_t A_1(t+\Delta)^T - \beta_t \left(B_2 v_1 B_2^T \right) \beta_t^T + \Bigg[A_2(t+\Delta) + \\ & + \sum_{r=n^Y+1}^{n^X+n^Y} A_{3r}(t+\Delta) E_r(t+\Delta) \Bigg] v_2(t+\Delta) \Bigg[A_2(t+\Delta)^T + \sum_{r=n^Y+1}^{n^X+n^Y} A_{3r}(t+\Delta)^T E_r(t+\Delta) \Bigg] + \\ & + \sum \sum_{r,s=n^Y+1}^{n^X+n^Y} A_{3r}(t+\Delta) v_2(t+\Delta) A_{3r}^T (t+\Delta)^T K_{rs}. \end{aligned} \tag{72}$$

Equations (70)–(72) define normal linear Pugachev extrapolator for SDS USD reduced to SDS (Proposal 9).

6. Normal nonlinear filtering and extrapolation

Let us consider SDS (2) reducible to SDS and fully observable measuring system described by the following equations:

$$\dot{X}_t = a(X_t, Y_t, \alpha, t) + b(X_t, Y_t, \alpha, t) V_0, \tag{73}$$

$$Z_t = \dot{Y}_t = a_1(X_t, Y_t, t) + b_1(X_t, Y_t, t) V_0. \tag{74}$$

Here, a, a_1, b, b_1 being known functions of mentioned variable; α being vector of parameters in Eq. (73); V_0 being normal white noise with intensity matrix $v_0 = v_0(t)$.

Using the theory of normal nonlinear suboptimal filtering [10–12], we get the following equations for $\hat{X}_t$ and R_t:

$$\dot{\hat{X}}_t = f\left(\hat{X}_t, Y_t, R_t, t\right) dt + h\left(\hat{X}_t, Y_t, R_t, t\right) dt \left[dY_t - f^{(1)}\left(\hat{X}_t, Y_t, R_t, t\right) dt \right], \tag{75}$$

$$\dot{R}_t = \left\{ f^{(2)}\left(\hat{X}_t, Y_t, R_t, t\right) - h\left(\hat{X}_t, Y_t, R_t, t\right) b_1 \nu_0 b_1^T (Y_t, t) h\left(\hat{X}_t, Y_t, R_t, t\right)^T \right\} dt + \\ + \sum_{r=1}^{n_y} \rho_r\left(\hat{X}_t, Y_t, R_t, t\right) \left[dY_r - f_r^{(1)}\left(\hat{X}_t, Y_t, R_t, t\right) dt \right]. \tag{76}$$

Here

$$f\left(\hat{X}_t, Y_t, R_t, t\right) = \left[(2\pi)^n \, |R_t|\right]^{-1/2} \int_{-\infty}^{\infty} a(Y_t, x, t) \exp\left\{ -\left(x^T - \hat{X}_t^T\right) R_t^{-1} \left(x - \hat{X}_t\right)/2 \right\} dx, \tag{77}$$

$$f^{(1)}\left(\hat{X}_t, Y_t, R_t, t\right) = \left\{ f_r^{(1)}\left(\hat{X}_t, Y_t, R_t, t\right) \right\} \\ = \left[(2\pi)^{n_x} |R_t| \right\}^{-1/2} \int_{-\infty}^{\infty} a_1(Y_t, x, t) \exp\left\{ -\left(x^T - \hat{X}_t^T\right) R_t^{-1} \left(x - \hat{X}_t\right)/2 \right\} dx, \tag{78}$$

$$h\left(\hat{X}_t, Y_t, R_t, t\right) = \left\{ \left[(2\pi)^{n_x} |R_t|\right]^{-1/2} \int_{-\infty}^{\infty} \left[x a_1(Y_t, x, t)^T + b\nu_0 b_1^T(Y_t, x, t) \right] \times \right. \\ \left. \times \exp\left\{ -\left(x^T - \hat{X}_t^T\right) R_t^{-1} \left(x - \hat{X}_t\right)/2 \right\} dx - \hat{X}_t f^{(1)}\left(\hat{X}_t, Y_t, R_t, t\right)^T \right\} \left(b_1 \nu_0 b_1^T\right)^{-1} (Y_t, t), \tag{79}$$

$$f^{(2)}\left(\hat{X}_t, Y_t, R_t, t\right) = \left[(2p)^{n_x} |R_t|\right\}^{-1/2} \int_{-\infty}^{\infty} (x - \hat{X}_t) a(Y_t, x, t)^T \\ + a(Y_t, x, t)\left(x^T - \hat{X}_t^T\right) + b\nu_0 b_1^T(Y_t, x, t) \times \\ \times \exp\left\{ -(x^T - \hat{X}_t^T) R_t^{-1} (x - \hat{X}_t)/2 \right\} dx, \tag{80}$$

$$\rho_r\left(\hat{X}_t, Y_t, R_t, t\right) = \left[(2p)^{n_x} \int_{-\infty}^{\infty} \left\{ \left(x - \hat{X}_t\right)\left(x^T - \hat{X}_t^T\right) a_r(Y_t, x, t) + \right. \right. \\ \left. + \left(x - \hat{X}_t\right) b_r(Y_t, x, t)^T \left(x^T - \hat{X}_t^T\right) + b_r(Y_t, x, t)\left(x^T - \hat{X}_t^T\right) \right\} \\ \times \exp\left\{ -\left(x^T - \hat{X}_t^T\right) R_t^{-1} \left(x - \hat{X}_t\right)/2 \right\} dx \quad \left(r = \overline{1, n_y}\right), \tag{81}$$

$$\hat{X}_0 = \mathrm{E}_N[X_0 | Y_0], \quad R_0 = \mathrm{E}_N\left[\left(X_0 - \hat{X}_0\right)\left(X_0^T - \hat{X}_0^T\right) | Y_0 \right], \tag{82}$$

where a_r being r th element of line-matrix $\left(a_1^T - \hat{a}_1^T\right)\left(b_1 \nu_0 b_1^T\right)^{-1}$; b_{kr} being element of k th line and r th column of the matrix $b_1 \nu_0 b_1^T$; b_r being the rth column of the matrix $b_1 \nu_0 b_1^T \left(b_1 \nu_0 b_1^T\right)^{-1}$, $b_r = \left[b_{1r} \dots b_{pr}\right]$ $\left(r = \overline{1, n_1}\right)$.

Proposal 10. *If vector SDS USD (2) is reducible to Eqs. (73) and (74) then Eqs. (75)–(81) at conditions (82) define normal filtering algorithm. The number of equations is equal to*

$$Q_{NAM} = n_x + \frac{n_x(n_x + 1)}{2} = \frac{n_x(n_x + 3)}{2}. \tag{83}$$

Hence, if the function a_1 is linear in X_t and function b does not depend on X_t all matrices $\rho_r = 0$ and Eq. (76) does not contain $\dot{Y}_t$ (Section 3).

Analogously Section 6 we get from [12] corresponding equations of normal conditionally optimal (Pugachev) extrapolator for reduced equations

$$\dot{X}_t = a(X_t, Y_t, t) + b(X_t, t)V_1, \tag{84}$$

$$Z_t = \dot{Y}_t = a_1(X_t, Y_t, t) + b_1(X_t, Y_t, t)V_2, \tag{85}$$

where V_1 and V_2 are normal independent white noises.

7. Examples

Let us consider scalar system

$$\varphi(\dot{X}_t, X_t) \equiv \varphi_1(\dot{X}_t) + \varphi_2(X_t) + U_{1t} = 0 \tag{86}$$

$$\dot{U}_{1t} = \alpha_{10} + \alpha_{11}U_{1t} + \beta_1 V_{1t}. \tag{87}$$

Here, $X_t, \dot{X}_t$ being state variable and its time derivative; U_{1t} being scalar stochastic disturbance; V_{1t} being scalar normal white noise with intensity v_{1t}; φ_1 and φ_2 being nonlinear functions; $\alpha_{10}, \alpha_{11}, \beta_1$ being constant parameters. After regression linearization of nonlinear functions, we have

$$\varphi_1 \approx \varphi_{10} + k_{\dot{X}}^{\varphi_1}\dot{X}_t^0, \varphi_2 \approx \varphi_{20} + k_X^{\varphi_2}X_t^0. \tag{88}$$

At condition $k_{\dot{X}}^{\varphi} \neq 0$ we get from (86) and (88) equations for mathematical expectation $m_t^X = EX_t$ and centered $X_t^0 = X_t - m_t^X$:

$$\varphi_{10} + \varphi_{20} + m_{1t}^U = 0, \tag{89}$$

$$\dot{X}_t^0 = a_t X_t^0 + b_t U_{1t}^0, \tag{90}$$

where

$$\varphi_{10} = \varphi_{10}\left(m_t^{\dot{X}}, D_t^{\dot{X}}\right), \quad \varphi_{20} = \varphi_{20}\left(m_t^X, D_t^X\right), \tag{91}$$

$$a_t = a_t\left(m_t^X, m_t^{\dot{X}}, D_t^X, D_t^{\dot{X}}, D_t^{U_1}, D_t^{XU_1}\right) = -k_X^{\varphi_2}\left(k_{\dot{X}}^{\varphi_1}\right)^{-1}, \quad b_t = -\left(k_{\dot{X}}^{\varphi_1}\right)^{-1}. \tag{92}$$

Equations (87) and (90), for $U_{1t}^0 = U_{1t} = m_t^{U_1}$ $\left(m_t^{U_1} = EU_{1t}\right)$ and $\overline{X}_t = [X_t \;\; U_{1t}]^T$ may be presented in vector form

$$\dot{m}_t^{\overline{X}} = A_{0t} + A_t m_t^{\overline{X}}, \tag{93}$$

$$\dot{\overline{X}}_t^0 = A_t \overline{X}_t^0 + B_t V_{1t}, \tag{94}$$

$$A_t = \begin{bmatrix} a_t & b_t \\ 0 & \alpha_1 \end{bmatrix}, \quad B_t = \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix}. \tag{95}$$

Covariance matrix

$$K_t^{\overline{X}} = \begin{bmatrix} D_t^X & K^{XU_1} \\ K_t^{\dot{X}U_1} & D_t^{U_1} \end{bmatrix} \tag{96}$$

and matrix of covariance functions

$$K^{\overline{X}}(t_1,t_2) = \begin{bmatrix} K_{11}^{\overline{X}}(t_1,t_2) & K_{12}^{\overline{X}}(t_1,t_2) \\ K_{21}^{\overline{X}}(t_1,t_2) & K_{22}^{\overline{X}}(t_1,t_2) \end{bmatrix} \tag{97}$$

satisfy to linear equations for correlation theory (Section 3)

$$\dot{K}_t^{\overline{X}} = A_t K_t^{\overline{X}} + K_t^{\overline{X}} A_t^T + B_t \nu_{1t} B_t^T, \quad K_{t_0}^{\overline{X}} = K_0^{\overline{X}}, \tag{98}$$

$$\frac{\partial K^{\overline{X}}(t_1,t_2)}{\partial t_2} = K^{\overline{X}}(t_1,t_2) A_{t_2}^T, \quad K^{\overline{X}}(t_1,t_1) = K_{t_1}^{\overline{X}}. \tag{99}$$

Vector Eq. (98) is equal to the following scalar equations:

$$\dot{D}_t^X = 2\left(a_t D_t^X + b_t K_t^{XU_1}\right), \quad \dot{D}_t^{U_1} = 2\,\alpha_1 D_t^{U_1} + \beta_1^2 \nu_{1t}, \quad \dot{K}_t^{XU_1} = a_t K_t^{XU_1} + b_t D_t^{U_1} + \alpha_1 K_t^{XU_1}. \tag{100}$$

From Eq. (90) we calculate variance

$$D_t^{\dot{X}} = a_t^2 D_t^X + b_t^2 D_t^{U_1} + 2 a_t b_t K_t^{XU_1}. \tag{101}$$

Thus, for MAM algorithm we use Eqs. (89), (91), (98)–(101).
Let system (86) and (87) is observable so that

$$Z_t = \dot{Y}_t = X_t + V_{2t}. \tag{102}$$

Then for Kalman-Bucy filter equations (Proposal 4), we have

$$\dot{\hat{X}}_t = A_t \hat{X}_t + \beta_t\left(Z_t - \hat{X}_t\right), \quad \beta_t = R_t \nu_{2t}^{-1} \quad (\det \nu_{2t} \neq 0), \quad \dot{R}_t = 2 A_t R_t + \nu_1 - \nu_2 \beta_t^2. \tag{103}$$

For Kalman-Bucy extrapolator equations are defined by Proposal 5 at $u(t,\tau) = e^{-a(t-\tau)}$.

In **Table 1**, the coefficients of statistical linearization of for typical nonlinear function are given.

Let us consider normal scalar system

$$F \equiv \Phi\left(\dot{X}_t\right) + a_t X_t + u_t = 0. \tag{104}$$

Here random function admits Pugachev normalization

$$\Phi\left(\dot{X}_t\right) \approx \Phi_0 + k^{\Phi} \dot{X}_t^0 + \Delta\Phi_t^0, \tag{105}$$

where $\Delta\Phi_t^0$ being normal StP satisfying equation of forming filter

$$\dot{\Delta\Phi}_t^0 = a_t^{\Delta\Phi} \Delta\Phi_t^0 + b_t^{\Delta\Phi} V_t. \tag{106}$$

Note that functions Φ_t^0 and k^{Φ} depend on $E_t^{\dot{\Phi}}$ and $D_t^{\dot{\Phi}}$. Equations (104) and (105) are decomposing on two equations. First equation at condition $k^{\Phi} \neq 0$ is as follows:

$$\Phi_0 + a_t E_t^X + u_t = 0, \quad \Phi_0 = k^{\Phi_0} E_t^{\dot{X}}. \tag{107}$$

φ	φ_0
$\dot{Y}^3$	$m(m^2+3D)$
$\sin\omega\dot{Y}$	$\exp\left(-\frac{\omega^2 D}{2}\right)\sin\omega m$
$\cos\omega\dot{Y}$	$\exp\left(-\frac{\omega^2 D}{2}\right)\cos\omega m$
$\dot{Y}\exp(\alpha\dot{Y})$	$(m+\alpha D)\exp\left(\alpha m+\frac{\alpha^2 D}{2}\right)$
$\dot{Y}\sin\omega\dot{Y}$	$(m\sin\omega m-\omega D\cos\omega m)\exp\left(-\frac{\omega^2 D}{2}\right)$
$\dot{Y}\cos\omega\dot{Y}$	$(m\cos\omega m-\omega D\sin\omega m)\exp\left(-\frac{\omega^2 D}{2}\right)$
$\operatorname{sgn}\dot{Y}$	$2\Phi\left(\frac{m}{\sqrt{D}}\right)$
$\dot{Y}^2\operatorname{sgn}\dot{Y}$	$2D\left\{\left(\frac{m^2}{D}+1\right)\Phi\left(\frac{m}{\sqrt{D}}\right)+\frac{1}{\sqrt{2\pi D}}\exp\left(-\frac{m^2}{2D}\right)\right\}\quad(m=m_{\dot{Y}},\quad D=D_{\dot{Y}})$
$\begin{cases}\frac{l}{D}\dot{Y}, & \lvert\dot{Y}\rvert\le d;\\ l, & \dot{Y}<d;\\ -l & \dot{Y}<-d\end{cases}$	$l\left\{(1+m_1)\Phi\left(\frac{1+m_1}{\sigma_1}\right)-(1-m_1)\Phi\left(\frac{1-m_1}{\sigma_1}\right)+\right.$ $\left.+\frac{\sigma_1}{\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2\right\}-\exp\left\{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2\right\}\right]\right\}$
$\begin{cases}\gamma(\dot{Y}+d), & \lvert\dot{Y}\rvert<-d;\\ 0, & \lvert\dot{Y}\rvert\le d;\\ \gamma(\dot{Y}-d) & \dot{Y}>d\end{cases}$	$\gamma\left\{1-\frac{1}{m_1}\left[(1+m_1)\Phi\left(\frac{1+m_1}{\sigma_1}\right)-(1-m_1)\Phi\left(\frac{1-m_1}{\sigma_1}\right)\right]+\right.$ $\left.+\frac{\sigma_1}{m_1\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{2}\left(\frac{1-m_1}{\sigma_1}\right)^2\right\}-\exp\left\{-\frac{1}{2}\left(\frac{1+m_1}{\sigma_1}\right)^2\right\}\right]\right\}$
$\begin{cases}-l, & \dot{Y}<-d;\\ 0, & \lvert\dot{Y}\rvert\le d;\\ l & \dot{Y}>d\end{cases}$	$l\left[\Phi\left(\frac{1+m_1}{\sigma_1}\right)-\Phi\left(\frac{1-m_1}{\sigma_1}\right)\right]$
$\dot{Y}_1\dot{Y}_2$	$m_1 m_2+K_{12}$
$\dot{Y}_1^2\dot{Y}_2$	$(m_1^2+K_{11})m_2+2m_1K_{12}$
$\sin(\omega_1\dot{Y}_1+\omega_2\dot{Y}_2)$	$\exp\left[\frac{\omega_1^2K_{11}+2\omega_1\omega_2K_{12}+\omega_2^2K_{22}}{2}\right]\sin(\omega_1m_1+\omega_2m_2)$
$\operatorname{sgn}(\dot{Y}_1+\dot{Y}_2)$	$2\Phi(\zeta_{1,2}),\zeta_{1,2}=\frac{m_1+m_2}{\sqrt{D}},\quad D=K_{11}+2K_{12}+K_{22}$

Table 1.
Coefficients of statistical linearization for typical nonlinear functions [12–14].

Second equation at condition $k^\Phi\neq 0$ is as follows: $k^\Phi\dot{X}_t^0+\Delta\Phi_t^0+a_tX_t^0=0$ may be presented as

$$\dot{X}_t^0=a_t\left(k^\Phi\right)^{-1}X_t^0-\left(k^\Phi\right)^{-1}\Delta\Phi_t^0. \tag{108}$$

Equations (106) and (108) for $Z_t^0=\left[X_t^0\ \ \Delta\Phi_t^0\right]^T$ leads to the following vector equation for covariance matrix

$$\dot{K}_t^Z=AK_t^Z+K_t^ZA^T+B\nu^VB^T, \tag{109}$$

where $A=\begin{bmatrix}-a_t\left(k^\Phi\right)^{-1} & -\left(k^\Phi\right)^{-1}\\ 0 & a^{\Delta\Phi}\end{bmatrix},\quad B=\begin{bmatrix}0\\ b^{\Delta\Phi}\end{bmatrix}$. Eqs. (107) and (109) give the following final relations:

$$E_t^{\dot{X}}=\left(a_tE_t^X+u_t\right)\left(k^{\Phi_0}\right)^{-1}, \tag{110}$$

$$
\begin{aligned}
D_t^{\dot{X}} &= a_t^2 \left(k^\Phi\right)^{-2} D_t^X + \left(k^\Phi\right)^{-2} D_t^{\Delta\Phi} - 2a_t \left(k^\Phi\right)^{-2} K_t^{X\Delta\Phi}, \\
\dot{D}_t^X &= -2\left(a_t D_t^X + K_t^{X\Delta\Phi}\right), \\
\dot{D}_t^{\Delta\Phi} &= 2a_t^{\Delta\Phi} D_t^{\Delta\Phi} + \left(b_t^{\Delta\Phi}\right)^2 \nu^V, \\
\dot{K}_t^{X\Delta\Phi} &= a_t^{\Delta\Phi} K_t^{\Delta\Phi} - \left(a_t K_t^{X\Delta\Phi} + D_t^{\Delta\Phi}\right)\left(k^\Phi\right)^{-1}.
\end{aligned}
\tag{111}
$$

8. Conclusion

Models of various types of SDS USD arise in problems of analytical modeling and estimation (filtering, extrapolation, etc.) for control stochastic systems, when it is possible to neglect higher-order time derivatives. Linear and nonlinear methodological and algorithmic support of analytical modeling, filtering, and extrapolation for SDS USD is developed. The methodology is based on the reduction of SDS USD to SDS by means of linear and nonlinear regression models. Special attention is paid to SDS USD with multiplicative (parametric) noises. Examples illustrating methodology are presented. The described results may be generalized for systems with stochastically unsolved derivatives and stochastic integrodifferential systems reducible to the differential.

Acknowledgements

The author is grateful to experts for their appropriate and constructive suggestions to improve this template. Research is supported by the Russian Academy of Sciences (Project-AAAA-A19-119001990037-5). Also, the author is much obliged to Mrs. Irina Sinitsyna and Mrs. Helen Fedotova for translation and manuscript preparation.

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