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Group Judgement with Ties.
Distance-Based Methods

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1. Introduction

Problems of determining group judgement have been widely explored and many methods have been proposed (Nurmi, 1987). It is due to the fact that there is no "ideal" method satisfying all the requirements formulated. Hence new methods – possessing desirable properties and avoiding deficiencies of the previous ones – have been developed.

To efficiently analyse and solve problems of determining group judgement usually some simplifying assumption are introduced.

Very often it is assumed that no equivalent alternatives can appear in experts' judgements. Such an assumption is adopted, despite of the fact that in real life problems experts are not always able to uniquely determine the order of alternatives with respect to the given criterion or set of criteria.

Almost all the methods of determining group judgement are based on the assumption that there are no ties in this judgement. This assumption seems to be more restrictive than the previous one, and may strongly influence the solution obtained. This is specially important in the case of distance-based methods i.e. making use of the concept of distance among preference orders (Cook, 2006; Cook et al., 1997; Cook & Seiford, 1978).

In further considerations experts' judgements given in the form of preference orders are taken into account only.

One of the methods making it possible to overcome the problem of ties is to use the approach proposed by Armstrong et al. (1982).

It will be shown that some distance-based methods of group judgement derived for the case of no ties can be extended to the case of ties in experts opinion as well as in group judgement.

2. Positions taken by alternatives and the table of structures

Assume that there is a set of n alternatives $O = \{O_1, \ldots, O_n\}$ and K experts, who are asked to order this set with respect to a given criterion (or criteria). Expert present their judgements in the form of preference orders

$$P^k = \{O_{i_1}, \ldots, O_{i_n}\}, \, k=1, \ldots, K, \tag{1}$$

where an alternative regarded as the best one takes the first position and that regarded as the worst one takes the last position.

This preference order can be also written as a ranking:

$$P^k = \{q_1^k, \ldots, q_n^k\},\tag{2}$$

where q_i^k denotes the position taken by the i-th alternative in k-th expert's judgement.

One can also assume that in an expert judgement as well as in the group judgement more than one alternative can be put in the same position, i.e. ties can occur. In this case a preference order may be given as follows

$$O_{i_1}, \ldots, (O_{i_p}, \ldots, O_{i_{p+r}}), \ldots, O_{i_n}, \text{ r tied alternatives are given in brackets.}\tag{3}$$

Example 1.
$P = \{O_2, (O_1, O_3, O_4), O_5\}$ may be written as $P = \{2, 1, 2, 2, 3\}$. We shall refer to this notation as to the classical one. It is sometimes called a dense ranking.
Cook & Seiford (1978) proposed to apply fractional ranking, i.e. when r alternatives are placed in the same – called it p – position, it is assumed that they take positions (p, p+1, …, p+r-1) and the fractional position assigned to them is the mean

$$t = \frac{p + (p+1) + \ldots + (p+r-1)}{r} = \frac{2p + (r-1)}{2r}r = p + \frac{r-1}{2}.\tag{4}$$

It should be emphasized that the expression obtained is of the form v+½ for any even r and is an integer otherwise; where p, r, v are integer numbers.
For the preference order given in Example 1 one has $P = \{3, 1, 3, 3, 5\}$.
Hence in the fractional notation alternatives can take positions from the following set:

$$\Psi = \{1, 1½, 2, 2½, 3, 3½, \ldots, n\text{-}1, n\text{-}½, n\}.\tag{5}$$

One should notice that there may be positions with no alternatives assigned to.
The number of positions (with respect to classical notation) is increased almost twice (2n-1), but for given n it is fixed. This property is important for construction of the table of structures described later on. An example explaining the use of this notation is given in Table 1 (Σ denotes the sum of the positions).

Preference orders		Positions			
		classical notation	Σ	fractional notation	Σ
P^1:	O_2, O_3, O_4, O_1, O_5	4, 1, 2, 3, 5	15	4, 1, 2, 3, 5	15
P^2:	$(O_2,O_3), O_4, (O_1, O_5)$	3, 1, 1, 2, 3	10	4½, 1½, 1½, 3, 4½	15
P^3:	$O_2, (O_1,O_3,O_4),O_5$	2, 1, 2, 2, 3	10	3, 1, 3, 3, 5	15
P^4:	$O_2, (O_1, O_3, O_4, O_5)$	2, 1, 2, 2, 2	9	3½, 1, 3½, 3½, 3½	15

Table 1. Classical and fractional notation of positions of alternatives.

It can be easily shown that the number of equivalent alternatives taking a given position t can be different.
Assume the number of position is t=4.
Let's consider the following preference orders of seven elements:

1. $(O_1, O_2, O_3, O_4, O_5, O_6, O_7)$ $t_1 = (1 + 2 + 3 + 4 + 5 + 6 + 7)/7 = 4$
2. $O_1, (O_2, O_3, O_4, O_5, O_6), O_7$ $t_2 = (2 + 3 + 4 + 5 + 6)/5 = 4$
3. $O_1, O_2, (O_3, O_4, O_5), O_6, O_7$ $t_3 = (3 + 4 + 5)/3 = 4$.
4. $O_1, O_2, O_3, O_4, O_5, O_6, O_7$ $t_4 = 4$.

Hence for all the preference orders tied alternatives take the fourth position. In order to precise the number of equivalent alternatives taking a given position one has to provide an additional information on a position (in the sense of (4)) taken by the first alternative from the group of equivalent ones. This position is called *level* and is denoted by ℓ . It is evident that ℓ = 1, …,n.

The level ℓ =1 is assigned to the group of equivalent alternatives such that the first alternative is located in the first position. The level ℓ =2 defines such groups of equivalent alternatives for which the first alternative takes the second position, etc.

This approach makes it possible to define the concept of structure.

A group of positions (in the sense of (4)) taken by equivalent alternatives is called structure and is denoted by $S^{\ell t}$, because it depends on the position t as well as on the level ℓ . The number of positions corresponding to a given structure is denoted as $s_{\ell t}$. It is evident that $1 \le s_{\ell t} \le n$. The concept introduced is, in the authors opinion, more useful for solving the problem under consideration than that given in (Armstrong et al., 1982).

The application of structures $S^{\ell t}$ enables one to define a table whose rows correspond to structures related to a given level ℓ =1, …, n and columns to a given position t=1; 1½,..., n.

In a preference order with ties positions taken by alternatives are of the form "an integer or an integer + ½". Hence it is easy to transform them into the integer domain (needed to formulate integer programming problem (Section 5); the numbers of positions are doubled (T=2t). The numbers in the shaded area denote positions (in the classical sense) that can be taken by alternatives.

T	2	3	4	5	6	7	8	9	10
t	1	1,5	2	2,5	3	3,5	4	4,5	5
ℓ =1	1	(1,2)	(1,2,3)	(1,2,3,4)	(1,2,3,4,5)	(1,2,3,4,5,6)	(1,2,3,4,5,6,7)	(1,2,3,4,5,6,7,8)	(1,2,3,4,5,6,7,8,9)
ℓ =2			2	(2,3)	(2,3,4)	(2,3,4,5)	(2,3,4,5,6)	(2,3,4,5,6,7)	(2,3,4,5,6,7,8)
ℓ =3					3	(3,4)	(3,4,5)	(3,4,5,6)	(3,4,5,6,7)
ℓ =4							4	(4,5)	(4,5,6)
ℓ =5									5

Table 2 . The table of structures $S^{\ell t}$ for ℓ =5 and t=1, 1½,...,10.

The bottom edges of shaded area, for t being integer, correspond to preference orders with no ties. The detailed description of the table of structures introduced is given in (Bury & Wagner, 2007a).

It should be emphasized that one of the structures from the level 1 must appear in the preference order. Moreover, it should be noted that for any level ℓ >1 only one structure, if

any, can be taken into account. The latter holds true also for any position t. It is worth to note that at a given level ℓ the alternatives can take positions T=2ℓ, ..., n+ℓ and the number of positions $s_{\ell T}$ to be taken into account for given ℓ and T is equal to

$$s_{\ell T} = T-2\ell +1, \text{ for } \underline{\ell}_T \leq \ell \leq \overline{\ell}_T, \tag{6}$$

where

$$\underline{\ell}_T = \max(1, T - n), \quad \overline{\ell}_T = [T/2]. \tag{7}$$

The number of levels at which alternatives can be placed depends on the number of alternatives considered n and position T.

T	2	3	4	5	6	7	8	9	10
$\underline{\ell}_T$	1	1	1	1	1	2	3	4	5
$\overline{\ell}_T$	1	1	2	2	3	3	4	4	5

Table 3. The levels that should be taken into account for subsequent positions T for n=5.

For given T, ℓ and n one can determine structures and number of positions to be considered.

T	2	3	4	5	6	7	8	9	10
t	1	1,5	2	2,5	3	3,5	4	4,5	5
ℓ =1	1	2	3	4	5				
ℓ =2			1	2	3	4			
ℓ =3					1	2	3		
ℓ =4							1	2	
ℓ =5									1

Table 4. Values of $s_{\ell T}$ for n=5.

For the preference order P^1={O_2, O_3, O_4, O_1, O_5} the table of structures is as follows

t	1	1,5	2	2,5	3	3,5	4	4,5	5
ℓ =1	O_2								
ℓ =2			O_3						
ℓ =3					O_4				
ℓ =4							O_1		
ℓ =5									O_5

Table 5. Positions and levels of alternatives for the preference order P^1.

3. Determining group opinion

Methods of group judgement based on the concept of distance consist in determining a preference order $\hat{P}$ that is the closest one – in the sense of a distance assumed - to the

given set of expert judgements presented in the form of preference orders. In general one has to solve the following problem

$$\min_{P} \sum_{k=1}^{K} d(P^k, P) \to \hat{P} . \tag{8}$$

This problem can be solved by means of searching over the set of all the preference orders that can be regarded as a group judgement and choosing one (ones) that minimizes the distance assumed. However, this approach is limited by the total number s_n of preference orders to be considered. It is growing fast with n and equals to

$$s_n = \sum_{k=1}^{n} k! S_{n,k} , \tag{9}$$

where $S_{n,k}$ is the Stirling number of the second kind (Lipski & Marek, 1985).

This number can be estimated (Bailey, 1998) as

$$\bar{s}_{n+1} = \frac{(n+1)}{\log 2} \bar{s}_n = 1,442695(n+1)\bar{s}_n . \tag{10}$$

No of alternatives	No of preference orders - no ties	s_n - total number of preference orders
3	6	13
4	24	75
5	120	541
6	720	4 683
7	5 040	47 293
8	40 320	545 835
9	362 880	7 087 261
10	3 628 800	102 247 563

Table 6. Number of preference orders to be considered with respect to the number of alternatives with no ties or with ties allowed.

For n=17 the total number of the preference orders to be considered is expressed by 18-digits number. Hence instead of searching through all the preference orders, it is worth to apply more specialized approach, e.g. to formulate a constrained optimization problem which can be regarded as a modification of the linear assignment problem.

If ties are allowed in the group judgement then the problem is to be modified according to the table of structures related to the problem under consideration. Moreover, some additional constraints are to be added. The optimization problem with ties was investigated by Cook & Seiford (1978), Armstrong et al. (1982) and Bury and Wagner (2007a, b).

The framework described in Section 4 was applied to formulate and solve the problem of determining the Cook-Seiford median, the Litvak median as well as the Kemeny median. The results are presented in Section 5. For the first two problems the distance is defined with respect to the positions of alternatives. For the Kemeny median the distance is based on pairwise comparisons of alternatives and differs significantly from the previous ones. However a general framework formulated can be applied in all the cases mentioned.

4. Framework for preference structure description

The following binary variables are to be introduced:

$$y_{iT}^{\ell} = \begin{cases} 1 & \text{if } O_i \text{ takes the position T at the level } \ell \text{ in the preference order P} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$$y_{iT} = \begin{cases} 1 & \text{if } O_i \text{ takes the position T in the preference order P} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$\gamma_{\ell T} = \begin{cases} 1 & \text{if the structure } S^{\ell T} \text{ appears in the preference order P} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$$\Lambda_{\ell} = \begin{cases} 1 & \text{if a structure from the level } \ell \text{ appears in the preference order P} \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

It follows from table of structures that for a given position T structures can be chosen from levels ℓ such that $\underline{\ell_T} \le \ell \le \overline{\ell_T}$, $\underline{\ell_T}, \overline{\ell_T}$ are given by (7) and

$$y_{iT} = \sum_{\ell = \underline{\ell_T}}^{\overline{\ell_T}} y_{iT}^{\ell}. \tag{15}$$

The constraints are as follows:

$$\bigvee_{i=1,\dots,n} \sum_{T=2}^{2n} y_{iT} = 1, \tag{16}$$

i.e. a given alternative can be placed in one position only.

$$\bigvee_{T=2,\dots,2n} \bigvee_{\ell = \underline{\ell_T}\dots\overline{\ell_T}} \sum_{i=1}^{n} y_{iT}^{\ell} = s_{\ell T}\gamma_{\ell T}, \tag{17}$$

i.e. the number of alternatives that can be placed in a given position T on a given level ℓ is equal to zero or $s_{\ell T}$, where $s_{\ell T}$ (6) is the number of positions corresponding to a given structure $S^{\ell t}$.

$$\bigvee_{i=1,\dots,n} \bigvee_{T=2,\dots,2n} \sum_{\ell = \underline{\ell_T}}^{\overline{\ell_T}} y_{iT}^{\ell} = y_{iT}, \tag{18}$$

i.e. for a given position an alternative can be placed at one level at least.

$$\bigvee_{\ell=1,\dots,n} \Lambda_{\ell} = \sum_{T=2\ell}^{n+\ell} \gamma_{\ell T}, \tag{19}$$

i.e. for a given level ℓ, $\ell > 1$ only one structure, if any, can be taken into account.
For one of the structures from level $\ell = 1$ must appear in the preference order, then $\Lambda_1 = 1$ and

$$\bigvee_{\ell=2,\dots,n} \quad \Lambda_\ell = \sum_{\lambda=1}^{\ell-1} \gamma_{\lambda,\lambda+\ell-1} \ \text{ for } \ \ell=2, \dots, n, \tag{20}$$

i.e. the possibility of the occurrence of ℓ-level structure is determined by the structures chosen on preceding levels.

It is worth to note that if for any reason no tied alternatives should occur in group judgement it suffices to impose $\Lambda_\ell =1$ for $\ell =1, \dots, n$ in (20).

5. Optimization problem

As it was mentioned in Section 3 the group judgement based on the concept of distance is to be derived as the solution of an optimization problem (8). A solution obtained depends upon the assumed definition of the distance as well as of the form of the preference order $\hat{P}$ searched for.

The median distance is understood as a l^1 norm defined in an adequate space. Three cases are to be considered:

•   the space of positions taken by alternatives for the Cook-Seiford median,

•   the space of preference vectors for the Litvak median

•   the space of pairwise comparisons for the Kemeny median.

This problem will be analyzed for two cases:

a) there are no ties in expert as well as in group judgement *(complete ordering)*,

b) tied alternatives can occur both in expert and in group judgement *(weak ordering)*.

The CPLEX optimization software was used to solve discrete optimization problems formulated.

5.1 No ties case

For the case of no ties in expert opinion as well as in group judgement the optimization problem (8) can be formulated as follows (Cook & Seiford, 1978; Hwang & Lin, 1987; Nurmi, 1987; Bury & Wagner, 2000):

minimize the distance defined subject to following constraints:

•   any given alternative O_i (i=1, …, n) can be placed in one position only,   (21)

•   the number of alternatives that can be placed in a given position is equal to one.   (22)

To avoid ambiguity it is assumed that positions the alternatives can take in experts' judgements are denoted as j, j=1, …, n.

The assumption of no ties in expert judgement will be relaxed for the Litvak and for the Kemeny median.

5.1.1 The Cook–Seiford median

Definition (Cook & Seiford, 1978)

The distance between two preference orders is defined in terms of positions taken by alternatives

$$d(P^{k_1},P^{k_2}) = \sum_{i=1}^{n} \left| q_i^{k_1} - q_i^{k_2} \right| \tag{23}$$

where q_i^k denotes the position taken by the i-th alternative in k-th expert's opinion.

It can be shown that the distance defined in such a way satisfies all the axioms describing the measure of closeness (Cook & Seiford, 1978; Cook, 2006).

The distance of a preference vector P from the set of experts' judgements {P^k}is as follows

$$d(P, P^{(k)}) = \sum_{k=1}^{K} d(P, P^k) = \sum_{k=1}^{K} \sum_{i=1}^{n} \left| q_i^k - q_i \right| \tag{24}$$

where q_i denotes the position of an alternative O_i in the preference order P.

The optimization problem can be formulated as follows.

Find such a preference order $\hat{P}$ that

$$d(\hat{P}, P^{(k)}) = \min_{P} \sum_{k=1}^{K} d(P^k, P) = \min_{P} \sum_{k=1}^{K} \sum_{i=1}^{n} \left| q_i^k - q_i \right| . \tag{25}$$

If one assumes that the alternative O_i can take the j-th position in the preference order P, (j = 1, …, n), the distance (24) can be written in the form

$$d(P^k, P) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} y_{ij} . \tag{26}$$

where

$$d_{ij} = \sum_{k=1}^{K} \left| q_i^k - j \right| \tag{27}$$

and y_{ij} is defined as (12).

The matrix of d_{ij} coefficients is denoted as D. d_{ij} expresses the aggregated difference between the position of the i-th alternative in the preference order P and its positions in the preference orders P^k, k=1, …, K. The distance matrix D is of the form

$$D = \begin{array}{c} \\ O_1 \\ O_2 \\ \vdots \\ O_n \end{array} \begin{array}{cccc} j=1 & j=2 & \cdots & j=n \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} \end{array} . \tag{28}$$

The lower bound of the distance (26) is equal

$$C = \sum_{i=1}^{n} d_{i\,min} , \text{ where } d_{i\,min} = \min_{j}[d_{i1},....,d_{in}] . \tag{29}$$

The value of C is of importance for estimation how close is the group judgement derived to an "ideal" judgement.

The minimization problem (25) can be rewritten as a linear assignment problem

$$\min_{y_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} y_{ij} , \tag{30}$$

subject to

$$\underset{i=1,\ldots,n}{\forall} \quad \sum_{j=1}^{n} y_{ij} = 1, \tag{31}$$

i.e. an alternative can be placed at one position only and

$$\underset{j=1,\ldots,n}{\forall} \quad \sum_{i=1}^{n} y_{ij} = 1, \tag{32}$$

i.e. only one alternative can be placed at a given position.

The latter constraint can be easily derived from *(17)* in the structure framework.

Summing (17) over ℓ, $\ell = \underline{\ell}_T, \ldots, \overline{\ell_T}$ and taking into account (15) one obtains

$$\underset{T=2,\ldots,2n}{\forall} \quad \sum_{\ell=\underline{\ell}_T}^{\overline{\ell_T}} \sum_{i=1}^{n} y_{iT}^{\ell} = \sum_{i=1}^{n} \sum_{\ell=\underline{\ell}_T}^{\overline{\ell_T}} y_{iT}^{\ell} = \sum_{i=1}^{n} y_{iT} = \sum_{\ell=\underline{\ell}_T}^{\overline{\ell_T}} s_{\ell T} \gamma_{\ell T}. \tag{33}$$

For the case of no ties one has T=2, 4, ..., 2n (j=1, 2, ...n).

From the table of structures (Table 5) it can be seen that $s_{jj}=1$ and $\gamma_{jj}=1$. Finally one gets $\underset{j=1,\ldots,n}{\forall} \quad \sum_{i=1}^{n} y_{ij} = 1$.

This problem was described in details e.g. in Bury & Wagner, (2000) and Cook, (2006).

5.1.2 The Litvak median

Litvak (1982) introduced the notion of so called preference vector. For the preference order given by k-th expert the preference vector is defined as follows:

$$\pi^k = [\pi_1^k, \ldots, \pi_n^k], \tag{34}$$

where π_i^k is equal to the number of alternatives preceding the i-th one in this preference order.

Under assumptions taken

$$\pi_i^k = q_i^k - 1, \tag{35}$$

where q_i^k denotes the position taken by the i-th alternative in the k-th expert's opinion.

Example 3.

For a preference order P^1 from Table 1, $P^1 = \{O_2, O_3, O_4, O_1, O_5\}$ one has $\pi^1 = [3, 0, 1, 2, 4]$.

The distance between preference orders is expressed in terms of corresponding preference vectors.

Definition (Litvak, 1982).

Given two preference vectors π^{k_1} i π^{k_2} of the preference orders P^{k_1} and P^{k_2} respectively. The distance between these two preference orders is defined as

$$d(P^{k_1}, P^{k_2}) = \sum_{i=1}^{n} \left| \pi_i^{k_1} - \pi_i^{k_2} \right|. \tag{36}$$

It can be shown that the distance defined in such a way satisfies all the axioms describing the measure of closeness (Litvak, 1982).

Definition (Litvak, 1982).

The distance of a preference order P from the set of experts' preference orders {P^k} is as follows

$$d(P, P^{(k)}) = \sum_{k=1}^{K} \sum_{i=1}^{n} \left| \pi_i^P - \pi_i^k \right|. \tag{37}$$

$$\text{Let } h_i^{k(j)} = \left| \pi_i^{P(j)} - \pi_i^k \right| \quad i=1,\ldots,n; \ j=1,\ldots,n; \ k=1,\ldots K, \tag{38}$$

where $\pi_i^{P(j)}$ is the number of alternatives preceding O_i if O_i takes the j-th position in the preference order P.

Summing $h_i^{k(j)}$ (38) over k (k=1,...,K) one obtains $h_i^{(j)}$

$$h_i^{(j)} = \sum_{k=1}^{K} h_i^{k(j)} \tag{39}$$

The distance (37) may be written as follows

$$d(P, P^{(k)}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{K} \left| \pi_i^{P(j)} - \pi_i^k \right| y_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i^{(j)} y_{ij} \tag{40}$$

where y_{ij} is defined by (12).

The matrix of $h_i^{(j)}$ coefficients is denoted as H, $h_i^{(j)}$ expresses the aggregated difference between the position of the i-th alternative in the preference order P and its positions in the preference orders P^k, k=1, …, K.

The distance matrix H is of the form

$$H = \begin{matrix} & \begin{matrix} j=1 & j=2 & \cdots & j=n \end{matrix} \\ \begin{matrix} O_1 \\ O_2 \\ \vdots \\ O_n \end{matrix} & \begin{bmatrix} h_1^{(1)} & h_1^{(2)} & \cdots & h_1^{(n)} \\ h_2^{(1)} & h_1^{(2)} & \cdots & h_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ h_n^{(1)} & h_1^{(2)} & \cdots & h_n^{(n)} \end{bmatrix} \end{matrix}. \tag{41}$$

The lower bound of the distance (40) is equal (Litvak (1982))

$$G = \sum_{i=1}^{n} h_{i \min}, \text{ where } h_{i \min} = \min_{j}[h_i^{(1)}, \ldots, h_i^{(n)}] \tag{42}$$

The problem of determining the Litvak median can be formulated as the following binary optimization problem (Litvak, 1982; Bury & Wagner, 2000)

$$\min_{y_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i^{(j)} y_{ij} \tag{43}$$

subject to (31)÷(32).

One should note that in the case when in expert judgement tied alternatives can occur the definition of preference vector makes it easy to determine its components and the distance matrix. The preference vector for expert judgement with ties is of the same form for classical as well as for fractional position system. Hence it may be applied for the optimization problem formulated (43).

Example 2.

For a preference order P^2 from Table 1, $P^2 = \{(O_2, O_3), O_4, (O_1, O_5)\}$ one has $\pi^2 = \{3, 0, 0, 2, 3\}$.

5.1.3 The Kemeny median

The distance for the Kemeny median method is defined with the use of pairwise comparison matrices (Kemeny, 1959; Kemeny & Snell, 1960).

For a given preference order of n alternatives presented by the k-th expert (k=1,...,K)

P^k: $O_{i_1}, O_{i_2}, \ldots, O_{i_n}$ the matrix of pairwise comparisons can be constructed as follows (e.g. Litvak, 1982); it is assumed that in expert judgement ties can occur.

$$\mathbf{A}^k = \begin{bmatrix} a_{11}^k, & \ldots, & a_{1n}^k \\ \vdots & \ddots & \vdots \\ a_{n1}^k, & \ldots, & a_{nn}^k \end{bmatrix}, \text{ where } a_{il}^k = \begin{cases} 1 & \text{for } O_i \succ O_l \\ 0 & \text{for } O_i \approx O_l \\ -1 & \text{for } O_i \prec O_l \end{cases}, \ a_{ii}^k \triangleq 0 \qquad (44)$$

i=1, …n, l=1, …, n.

The notation $O_i \succ O_l$ ($O_i \approx O_l$) should be read as follows: the i-th alternative O_i is better than (equivalent to) the l-th alternative O_l with respect to a chosen criterion (a set of criteria). Assume that two preference orders P^{k_1} and P^{k_2} are given. The distance between these two preference orders is defined as follows (e.g. Litvak, 1982; Bury & Wagner, 2000)

$$d(P^{k_1}, P^{k_2}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} \left| a_{il}^{k_1} - a_{il}^{k_2} \right|. \qquad (45)$$

It can be shown that the distance defined in such a way satisfies all the axioms describing the measure of closeness (Litvak, 1982).

The distance between a given preference order P and a set of preference orders given by the experts is defined as

$$d = d(P, P^{(k)}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{K} \left| a_{il}^P - a_{il}^k \right|. \qquad (46)$$

The following equality holds

$$\left| a_{il}^k - a_{il}^P \right| + \left| a_{li}^k - a_{li}^P \right| = 2 \left| a_{il}^k - a_{il}^P \right| = 2 \left| a_{li}^k - a_{li}^P \right| \qquad (47)$$

so the expression (46) can be rewritten as follows

$$d = \sum_{\substack{i \ l \\ (i,l) \in I_P^{(1)}}} \sum_{k=1}^{K} \left| a_{il}^k - a_{il}^P \right| + \sum_{\substack{i \ l \\ (i,l) \in I_P^{(2)}}} \sum_{k=1}^{K} \left| a_{il}^k - a_{il}^P \right| \qquad (48)$$

where

$I_P^{(1)}$ -the set of indices (i,l) for which $O_i \succ O_l$ in the preference order P, i.e. $a_{il}^P = 1$. $\qquad$ (49)

$I_P^{(2)}$ -the set of indices (i,l) for which $O_i \approx O_l$ in the preference order P, i.e. $a_{il}^P = 0$. $\qquad$ (50)

Assume that in the preference order P $O_i \succ O_l$, i.e. $a_{il}^P = 1$. In order to determine the distance of this preference order from a given set $\{P^k\}$, use can be made of coefficients defined as follows

$$r_{il} = \sum_{k=1}^{K} d_{il}\left(P, P^{(k)}\right) = \sum_{k=1}^{K}\left|a_{il}^k - a_{il}^P\right| = \sum_{k=1}^{K}\left|a_{il}^k - 1\right|.$$ $\qquad$ (51)

They are called the loss coefficients and the matrix R=[r_{il}] is called the loss matrix (Litvak,1982). It is assumed that r_{il}=0 for all i=l. It should be noted that elements of the matrix R depend upon the form of preference orders P^k (k=1,...,K) only and are independent of position system assumed (classical as well as fractional - if applicable).

Making use of the coefficients r_{il} (i,l=1,...,n) the formulae (48) can be rewritten in the following form (Bury & Wagner, 2000; Litvak, 1982):

$$d = \sum_{\substack{i \\ (i,l)\in I_P^{(1)}}} \sum_l r_{il} + \sum_{\substack{i \\ (i,l)\in I_P^{(2)}}} \sum_l \sum_{k=1}^{K}\left|a_{il}^k\right|.$$ $\qquad$ (52)

If in the group judgement ties are not allowed, then $I_P^{(2)} = \{i,i\}$, i=1,...,n. In this case we have (Litvak, 1982)

$$d = \sum_{(i,l)\in I_P^{(1)}} r_{il}.$$ $\qquad$ (53)

This formula can be used to determine the lower bound E of the distance (53). Litvak has shown that in the case under consideration the following theorem is satisfied.
Theorem (Litvak, 1982).

$$E = \sum_{i=1}^{n-1} \sum_{l=i+1}^{n} \min\left(r_{il}, r_{li}\right).$$ $\qquad$ (54)

These results make it possible to propose some heuristic algorithms for determining the Kemeny median (Bury & Wagner, 2000; Litvak, 1982).
The distance (53) can be written in the form

$$d(P, P^{(k)}) = \sum_{i=1}^{n} \sum_{l=1}^{n} r_{il} x_{il},$$ $\qquad$ (55)

where

$$x_{il} = \begin{cases} 1 & \text{for } O_i \succ O_l \text{ in the preference order P} \\ 0 & \text{otherwise} \end{cases}$$ $\qquad$ (56)

The problem of determining a group judgement that minimizes the distance (55) can be formulated as follows

$$\min_{x_{il}} \sum_{i=1}^{n} \sum_{l=1}^{n} r_{il} x_{il} \, , \tag{57}$$

It should be emphasized that this problem (even if ties are not allowed in group judgement) cannot be solved within the framework presented in this subsection. It needs to be formulated within the framework of structures presented for the case of ties.

5.2 The case of ties in group judgement

For this case the fractional notation is to be used i.e.

$$t \in \Psi = \{1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, 3, 3\tfrac{1}{2}, \ldots., n\text{-}1, n\text{-}\tfrac{1}{2}, n\}.$$

5.2.1 The Cook–Seiford median

The problem (30) ÷ (32) is to be modified (double positions T are applied).

The distance matrix $\overline{D}$ is of the form

$$\overline{D} = \begin{array}{c} \\ O_1 \\ O_2 \\ \vdots \\ O_n \end{array} \overset{\displaystyle \begin{array}{cccc} T=2 & T=3 & T=4 & \cdots & T=2n \end{array}}{\begin{bmatrix} \overline{d}_{12} & \overline{d}_{13} & \overline{d}_{14} & & \overline{d}_{12n} \\ \overline{d}_{22} & \overline{d}_{23} & \overline{d}_{12} & & \overline{d}_{12} \\ \vdots & \vdots & \vdots & \ddots & \overline{d}_{12} \\ \overline{d}_{n2} & \overline{d}_{n3} & \overline{d}_{12} & & \overline{d}_{12} \end{bmatrix}} , \tag{58}$$

where

$$\overline{d}_{iT} = \sum_{k=1}^{K} \left| 2q_i^k - T \right| , \tag{59}$$

and q_i^k is the number of fractional position taken by an alternative O_i in the preference order P^k given by the k-th ekspert.

The optimization problem is as follows

$$\min_{y_{iT}} \sum_{i=1}^{n} \sum_{T=2}^{2n} \overline{d}_{iT} y_{iT} \, , \tag{60}$$

with the constraints (15)÷(20).

5.2.2 The Litvak median

For the case of ties it is worth to notice that the number of alternatives preceding a given i-th one in the preference order is determined by the level ℓ this alternative takes in the table of structures and equals (ℓ-1). This means that alternatives from the first level are preceded by zero alternatives, alternatives from the second level are preceded by one alternative, etc.

The following example shows this observation.

Example 3.

For a preference order P^2 from Table 1, $P^2 = \{(O_2, O_3), O_4, (O_1, O_5)\}$ one has

$$P^2 = \{4\tfrac{1}{2}, 1\tfrac{1}{2}, 1\tfrac{1}{2}, 3, 4\tfrac{1}{2}\}.$$

Hence

   (O_2, O_3) are placed in the positions 1½, 1½ on the level $\ell =1$; it is evident there are no alternatives preceding those ones, $\pi_2^2 = 0$, $\pi_3^2 = 0$,

   O_4 takes the 3rd position on the level $\ell =3$, it is preceded by two alternatives, $\pi_4^2 = 2$,

   (O_1, O_5) are placed in the positions 4½, 4½ on the level $\ell =4$, they are preceded by three alternatives, $\pi_1^2 = 3$, $\pi_5^2 = 3$.

It may be shown that the components of a preference vector for a respective preference order P may be determined as follows

$$\pi_i^{P(\ell)} = (\ell - 1)\Lambda_\ell, \ \ell =1, \ldots, n, \tag{61}$$

where Λ_ℓ is given by *(14)*.

The preference order $P^2=\{(O_2,O_3), O_4, (O_1, O_5)\}$ presented with the use of the table of structures is as follows

	T	2	3	4	5	6	7	8	9	10
	t	1	1,5	2	2,5	3	3,5	4	4,5	5
$\Lambda_1=1$	$\ell =1$		(O_2, O_3)							
$\Lambda_2=0$	$\ell =2$									
$\Lambda_3=1$	$\ell =3$					O_4				
$\Lambda_4=1$	$\ell =4$								(O_1, O_5)	
$\Lambda_5=0$	$\ell =5$									

Table 7. Positions and levels of alternatives for the preference order P^2.

For a preference order P^2 given above one has

   $\pi_2^{P(1)} = (1-1)\Lambda_1 = 0$, $\pi_3^{P(1)} = 0$,

   $\pi_4^{P(3)} = (3-1)\Lambda_3 = 2$,

   $\pi_1^{P(4)} = (4-1)\Lambda_4 = 3$, $\pi_5^{P(4)} = 3$,

hence $\Lambda = [1,0,1,1,0]$, $\pi^2 = [3, 0, 0, 2, 3]$.

The coefficients $\overline{h}_i^{k(\ell)}$ of distance matrix $\overline{H}$ are of the form:

$$\overline{h}_i^{k(\ell)} = \left| \pi_i^{P(\ell)} - \pi_i^k \right| \ \ i=1,\ldots,n; \ k=1,\ldots K; \ \ell =1,\ldots,n. \tag{62}$$

Summing coefficients $\overline{h}_i^{k(\ell)}$ (62) over k (k=1,…,K) one gets

$$\overline{h}_i^{(\ell)} = \sum_{k=1}^{K} \overline{h}_i^{k(\ell)}. \tag{63}$$

It should be emphasized that the components of preference vectors result from the table of structure, i.e. they depend on position as well as on the level a given alternative is located.

As components of preference vectors π^k, k=1, ..., K are the same regardless of the notation of positions applied, it can be seen that $H = \overline{H}$. The bar over H indicates another way of defining the components of preference vector only. One has

$$
\overline{H} = \begin{array}{c} \\ O_1 \\ O_2 \\ \vdots \\ O_n \end{array}
\begin{array}{cccc}
\ell = 1 & \ell = 2 & \cdots & \ell = n \\
\left[\begin{array}{cccc}
\overline{h}_1^{(1)} & \overline{h}_1^{(2)} & \cdots & \overline{h}_1^{(n)} \\
\overline{h}_2^{(1)} & \overline{h}_1^{(2)} & \cdots & \overline{h}_2^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{h}_n^{(1)} & \overline{h}_1^{(2)} & \cdots & \overline{h}_n^{(n)}
\end{array}\right]
\end{array} .
\tag{64}
$$

The optimization problem for Litvak median is as follows:

$$
\min_{y_{iT}^\ell} \sum_{i=1}^{n} \sum_{T=2}^{2n} \sum_{\ell=\ell_T}^{\overline{\ell_T}} \overline{h}_i^{(\ell)} y_{iT}^\ell
\tag{65}
$$

subject to (15)÷(20).

5.2.3 The Kemeny median

If ties can occur in the group judgement, an optimization problem to be solved is more difficult. As it was mentioned before, the Kemeny median is defined with the use of pairwise comparisons. To solve the problem of determining group judgement one has to introduce some additional constraints.

The problem of finding a preference order $\hat{P}$ such that the distance (52) is minimized can be formulated as follows:

$$
\min_{x_{il}, z_{il}} \left(\sum_{i=1}^{n} \sum_{l=1}^{n} r_{il} x_{il} + \sum_{i=1}^{n-1} \sum_{l=i+1}^{n} b_{il} z_{il} \right),
\tag{66}
$$

where
x_{il} is given by (56),

$$
z_{il} = \begin{cases} 1 & \text{for } O_i \approx O_l \text{ in the preference order P} \\ 0 & \text{otherwise} \end{cases}
\tag{67}
$$

$$
b_{il} = \left(\sum_{k=1}^{K} \left| a_{il}^k \right| \right)
\tag{68}
$$

r_{il} is given by (51),
subject to a specially defined set of constraints. This set can be divided into two groups. First one contains those resulting from the table of structure. The second one consists of constraints resulting from the definition of the Kemeny median.
The following binary variables are introduced:

$$
x_{il}^\ell = \begin{cases} 1 & \text{if } O_i \text{ located on the level } \ell \text{ is such that } O_i \succ O_l \text{ in the preference order P,} \\ & \text{for } O_l \text{ from an arbitrary level } \lambda > \ell \\ 0 & \text{otherwise} \end{cases}
\tag{69}
$$

$$z_{il}^{\ell} = \begin{cases} 1 & \text{if } O_i \approx O_1 \text{ on the level } \ell \text{ in the preference order P} \\ 0 & \text{otherwise} \end{cases} . \tag{70}$$

The constraints are as follows
for $\ell = 1, \ldots, n-1, i = 1, \ldots, n, l = 1, \ldots, n$:

$$x_{il} = \sum_{\ell=1}^{n-1} x_{il}^{\ell} \tag{71}$$

$$z_{il} = \sum_{\ell=1}^{n-1} z_{il}^{\ell} . \tag{72}$$

It should be noticed that

$$x_{il}^{\ell} = 1 \quad \text{iff} \quad \left(\underbrace{\sum_{T=2\ell}^{n+\ell-1} y_{iT}^{\ell} = 1}_{\substack{\text{preceding} \\ \text{alternative}}} \right) \wedge \left(\underbrace{\sum_{\lambda=\ell+1}^{n} \sum_{T=2\lambda}^{n+\lambda} y_{lT}^{\lambda} = 1}_{\substack{\text{preceded} \\ \text{alternative}}} \right) \tag{73}$$

and

$$z_{il}^{\ell} = 1 \quad \text{iff} \quad \left(\sum_{T=2\ell+1}^{n+\ell} y_{iT}^{\ell} = 1 \right) \wedge \left(\sum_{T=2\ell+1}^{n+\ell} y_{lT}^{\ell} = 1 \right) \tag{74}$$

For the case of binary minimization problem (66) with positive coefficients of the objective function, in order to satisfy this condition the following inequalities should hold true

$$x_{il}^{\ell} \geq \underbrace{\sum_{T=2\ell}^{n+\ell-1} y_{iT}^{\ell}}_{\substack{\text{preceding} \\ \text{alternative}}} + \underbrace{\sum_{\lambda=\ell+1}^{n} \sum_{T=2\lambda}^{n+\lambda} y_{lT}^{\lambda}}_{\substack{\text{preceded} \\ \text{alternative}}} - 1 \tag{75}$$

and

$$z_{il}^{\ell} \geq \sum_{T=2\ell+1}^{n+\ell} \left(y_{iT}^{\ell} + y_{lT}^{\ell} \right) - 1 . \tag{76}$$

Hence the optimization problem becomes:
solve (66) subject to the constraints (67) ÷ (76) and (15)÷(20).

6. Numerical examples

6.1 The Cook-Seiford median
Example 4.
Given the set of five alternatives and eleven experts' judgements.
There are no ties in experts' judgements.

P^1 = $\{O_4, O_1, O_3, O_2, O_5\}$
P^2 = $\{O_3, O_1, O_5, O_2, O_4\}$
P^3 = $\{O_3, O_1, O_5, O_2, O_4\}$
P^4 = $\{O_2, O_1, O_5, O_4, O_3\}$
P^5 = $\{O_1, O_5, O_4, O_3, O_2\}$
P^6 = $\{O_3, O_5, O_2, O_4, O_1\}$
P^7 = $\{O_4, O_1, O_3, O_5, O_2\}$
P^8 = $\{O_1, O_3, O_5, O_2, O_4\}$
P^9 = $\{O_5, O_1, O_2, O_4, O_3\}$
P^{10} = $\{O_1, O_2, O_3, O_5, O_4\}$
P^{11} = $\{O_1, O_5, O_2, O_3, O_4\}$

The lower bound C (29) of the distance is equal to 51.
If ties are not allowed in group judgement the solutions obtained are

$$\{O_1, O_3, O_5, O_2, O_4\}$$
$$\{O_1, O_5, O_3, O_2, O_4\}$$

and the distance (30) from the set of preference orders given by experts is equal to 58.
The group judgement with ties is $\{O_1, (O_2, O_3, O_5), O_4\}$.
The distance (60) from the set of preference orders given by experts is equal to 56.
Example 5.
Given the set of five alternatives and eleven experts' judgements.
Ties are allowed in experts' judgements.

P^1 = $\{O_3, (O_4, O_5), (O_1, O_2)\}$
P^2 = $\{(O_1, O_2, O_3), (O_4, O_5)\}$
P^3 = $\{O_3, O_2, (O_1, O_4, O_5)\}$
P^4 = $\{O_2, (O_1, O_5), (O_3, O_4)\}$
P^5 = $\{(O_1, O_4), O_3, O_5, O_2\}$
P^6 = $\{(O_2, O_4), (O_1, O_5), O_3\}$
P^7 = $\{(O_1, O_2, O_3, O_5), O_4\}$
P^8 = $\{O_5, (O_1, O_3), O_4, O_2\}$
P^9 = $\{O_2, (O_1, O_3, O_4), O_5\}$
P^{10} = $\{O_1, O_2, (O_3, O_4, O_5)\}$
P^{11} = $\{O_1, (O_2, O_3), (O_4, O_5)\}$

The lower bound C (29) of the distance is equal to 52.
If ties are allowed in experts' judgements only then the solutions obtained are as follows

$$\{O_1, O_2, O_3, O_5, O_4\}$$
$$\{O_2, O_1, O_3, O_5, O_4\}$$

and the distance (30) from the preference orders given by experts is equal to 66.
The group judgement with ties is $\{(O_1, O_2, O_3), (O_4, O_5)\}$.
The distance (60) from the set of preference orders given by experts is equal to 60.

6.2 The Litvak median

Example 6.

Given the set of six alternatives and eleven experts' judgements.

There are no ties in experts' judgements.

$P^1 =$ $\{O_2, O_1, O_3, O_5, O_6, O_4\}$

$P^2 =$ $\{O_1, O_3, O_2, O_5, O_6, O_4\}$

$P^3 =$ $\{O_4, O_5, O_6, O_3, O_1, O_2\}$

$P^4 =$ $\{O_5, O_1, O_4, O_6, O_2, O_3\}$

$P^5 =$ $\{O_5, O_4, O_2, O_6, O_1, O_3\}$

$P^6 =$ $\{O_2, O_5, O_4, O_6, O_3, O_1\}$

$P^7 =$ $\{O_1, O_5, O_2, O_4, O_3, O_6\}$

$P^8 =$ $\{O_6, O_1, O_2, O_5, O_3, O_4\}$

$P^9 =$ $\{O_2, O_5, O_6, O_4, O_3, O_1\}$

$P^{10} =$ $\{O_4, O_2, O_6, O_1, O_3, O_5\}$

$P^{11} =$ $\{O_4, O_5, O_6, O_1, O_2, O_3\}$

The lower bound G (40) of the distance is equal to 82.

If ties are not allowed in group judgement the solutions obtained are

$$\{O_4, O_5, O_2, O_6, O_1, O_3\}$$
$$\{O_2, O_5, O_4, O_6, O_1, O_3\}$$

The distance (43) from the set of preference orders given by experts is equal to 98.

The group judgement with ties is $\{(O_2, O_4, O_5), O_6, O_1, O_3\}$

The distance (65) from the set of preference orders given by experts is equal to 96.

Example 7.

Given the set of six alternatives and eleven experts' judgements.

Ties are allowed in experts' judgements.

$P^1 =$ $\{O_1, O_6, O_4, (O_2, O_3), O_5\}$

$P^2 =$ $\{(O_1, O_2), O_4, O_6, (O_3, O_5)\}$

$P^3 =$ $\{O_4, (O_1, O_2), O_5, (O_3, O_6)\}$

$P^4 =$ $\{(O_2, O_3, O_6), O_4, O_1, O_5\}$

$P^5 =$ $\{(O_3, O_5), (O_1, O_2, O_6), O_4\}$

$P^6 =$ $\{(O_2, O_4, O_5), (O_3, O_6), O_1\}$

$P^7 =$ $\{O_3, (O_4, O_5, O_6), (O_1, O_2)\}$

$P^8 =$ $\{O_1, (O_3, O_6), (O_2, O_4, O_5)\}$

$P^9 =$ $\{(O_1, O_5), O_3, (O_2, O_6), O_4\}$

$P^{10} =$ $\{(O_3, O_6), O_2, O_4, O_5, O_1\}$

$P^{11} =$ $\{O_1, O_2, (O_3, O_4, O_5), O_6\}$

The lower bound G (42) of the distance is equal to 95.

The solutions obtained for the case of no ties in group judgement are

$$\{O_1, O_3, O_2, O_6, O_4, O_5\}, \qquad \{O_1, O_3, O_2, O_6, O_5, O_4\}$$
$$\{O_1, O_3, O_2, O_4, O_6, O_5\}, \qquad \{O_1, O_2, O_3, O_6, O_4, O_5\}$$
$$\{O_1, O_2, O_3, O_6, O_5, O_4\}, \qquad \{O_1, O_2, O_3, O_4, O_6, O_5\}$$

The distance (43) from the set of preference orders given by experts is equal to 116.

The group judgement with ties is $\{(O_1, O_2, O_3, O_4, O_5, O_6)\}$.

The distance (65) from the set of preference orders given by experts is equal to 97. It is worth to note that in this example making all the alternatives equivalent significantly improved the distance.

6.3 The Kemeny median

Example 8.

Given the set of seven alternatives and eleven experts' judgements.

There are no ties in experts' judgements.

$P^1 = \quad \{O_1, O_2, O_6, O_7, O_5, O_3, O_4\}$

$P^2 = \quad \{O_2, O_6, O_4, O_5, O_1, O_7, O_3\}$

$P^3 = \quad \{O_3, O_2, O_6, O_1, O_7, O_4, O_5\}$

$P^4 = \quad \{O_4, O_1, O_6, O_5, O_3, O_2, O_7\}$

$P^5 = \quad \{O_4, O_6, O_1, O_3, O_7, O_5, O_2\}$

$P^6 = \quad \{O_4, O_2, O_7, O_6, O_1, O_3, O_5\}$

$P^7 = \quad \{O_4, O_7, O_6, O_1, O_3, O_2, O_5\}$

$P^8 = \quad \{O_6, O_3, O_7, O_1, O_2, O_4, O_5\}$

$P^9 = \quad \{O_2, O_5, O_7, O_1, O_3, O_4, O_6\}$

$P^{10} = \quad \{O_7, O_6, O_2, O_5, O_3, O_4, O_1\}$

$P^{11} = \quad \{O_6, O_3, O_1, O_2, O_7, O_4, O_5\}$

The lower bound E (54) of the distance is equal to 166.

If ties are not allowed in group judgement the solutions are

$$\{O_6, O_1, O_2, O_7, O_3, O_4, O_5\}, \qquad \{O_6, O_1, O_3, O_2, O_7, O_4, O_5\}$$

and the distance (55) from the set of preference orders given by experts is equal to 170.

The group judgement derived for the problem (66) ÷ (76) (with ties assumed) is $\{O_6, O_1, O_2, O_7, O_3, O_4, O_5\}$. It is worth to note that it doesn't contain tied alternatives even though ties were allowed.

The distance (66) from the set of preference orders given by experts is equal to 170.

Example 9.

Given the set of seven alternatives and eleven experts' judgements.

Ties are allowed in experts' judgements.

$P^1 = \quad \{O_1, O_2, O_4, O_7, (O_3, O_5, O_6)\}$

$P^2 = \quad \{(O_4, O_6), O_7, O_5, (O_2, O_3), O_1\}$

$P^3 = \quad \{O_4, O_7, (O_1, O_2, O_6), (O_3, O_5)\}$

$P^4 = \quad \{O_2, O_3, (O_4, O_5, O_6), O_7, O_1\}$

$P^5 = \quad \{(O_3, O_5, O_6, O_7), (O_1, O_4), O_2\}$

$P^6 = \quad \{O_4, O_6, (O_1, O_2), (O_3, O_5), O_7\}$

$P^7 = \quad \{O_1, (O_3, O_4, O_5, O_7), (O_2, O_6)\}$

$P^8 = \quad \{O_4, (O_3, O_6), O_7, (O_1, O_2), O_5\}$

$P^9 = \quad \{O_5, O_6, O_4, O_2, O_3, (O_1, O_7)\}$

$P^{10} =$ $\{O_5, O_1, O_3, (O_2, O_4, O_6), O_7\}$

$P^{11} =$ $\{(O_1, O_2), (O_3, O_4), (O_5, O_6), O_7\}$

The lower bound E (52) of the distance is equal to 187.

If ties are not allowed in group judgement the solutions obtained are

$$\{O_4, O_6, O_1, O_2, O_3, O_5, O_7\}, \qquad \{O_4, O_6, O_1, O_2, O_5, O_3, O_7\}$$

and the distance (55) from the set of preference orders given by experts is equal to 187.

The group judgement with ties is $\{O_4, (O_1, O_2, O_3, O_5, O_6), O_7\}$

and the distance (66) from the set of preference orders given by experts is equal to 177.

7. Concluding remarks

In the paper it is shown that the fractional notation and the structure table can be successfully applied to find the group judgement with ties for the distance defined as the Cook-Seiford, Litvak and Kemeny median. On the basis of results obtained it seems reasonable to make an attempt at applying the approach discussed to the other methods of determining group judgement with ties or in the case when some alternatives are not comparable.

8. References

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Edited by Harald Aschemann

The book New Approaches in Automation and Robotics offers in 22 chapters a collection of recent developments in automation, robotics as well as control theory. It is dedicated to researchers in science and industry, students, and practicing engineers, who wish to update and enhance their knowledge on modern methods and innovative applications. The authors and editor of this book wish to motivate people, especially under-graduate students, to get involved with the interesting field of robotics and mechatronics. We hope that the ideas and concepts presented in this book are useful for your own work and could contribute to problem solving in similar applications as well. It is clear, however, that the wide area of automation and robotics can only be highlighted at several spots but not completely covered by a single book.

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