Chapter

Effectiveness of Basic Sets of Goncarov and Related Polynomials

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Abstract

The Chapter presents diverse but related results to the theory of the proper and generalized Goncarov polynomials. Couched in the language of basic sets theory, we present effectiveness properties of these polynomials. The results include those relating to simple sets of polynomials whose zeros lie in the closed unit disk $U = \{z : |z| \leq 1\}$. They settle the conjecture of Nassif on the exact value of the Whittaker constant. Results on the proper and generalized Goncarov polynomials which employ the q-analogue of the binomial coefficients and the generalized Goncarov polynomials belonging to the Dq- derivative operator are also given. Effectiveness results of the generalizations of these sets depend on whether $q < 1$ or $q > 1$. The application of these and related sets to the search for the exact value of the Whittaker constant is mentioned.

Keywords: Basic sets, Simple sets, Effectiveness, Whittaker constant, Goncarov polynomials, Dq operator

1. Introduction

The Chapter is on the effectiveness properties of the Goncarov and related polynomials of a single complex variable. It is essentially a compendium of certain results which seem diverse but related to the theory of the proper and generalized Goncarov polynomials.

Our first set of results deals with simple sets of polynomial [1], whose zeros lie in the closed unit disk U. It is a complement of a theorem of Nassif [1] which resolved his conjecture on the value of the Whittaker constant [2]. We provide also the relation between this problem and the theory of the proper Goncarov polynomials.

Next are results on a generalization of the problem where the polynomials are of the form

$$p_0(z) = 1; \quad p_n(z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_n^{n-k} z^k; \quad n \geq 1, \tag{1}$$

and the points $(a_n)_0^\infty$ are given complex numbers with $\begin{bmatrix} n \\ k \end{bmatrix}$ the q-analogue of the binomial coefficient $\binom{n}{k}$. From the results reported, it is shown that the location of the points $(a_k)_0^\infty$ that leads to favorable effectiveness results depends on whether $q < 1$ or $q > 1$. The relation of this problem to the generalized Goncarov polynomials belonging to the Dq-derivative operator is also recorded.

It is shown that applying the results of Buckholtz and Frank [3] on the generalized Goncarov polynomials $\{Q_n(z; z_0, z_1, \ldots, z_{n-1})\}$ belonging to the $\mathrm{D_q}$-derivative operator when $q > 1$, leads to the result that, when the points $(z_k)_0^\infty$ lie in the unit disk U, the resulting polynomials fail to be effective.

Consequently, we provide some results on the polynomials $\{Q_n(z; z_0, z_1, \ldots, z_{n-1})\}$ when

$$|z_k| \leq q^{-k}; \quad k \geq 0, \tag{2}$$

with the obtained results justifying the restriction (2) on the points $(z_k)_0^\infty$.

Finally, we provide other relevant and related results on the properties of the generalized Goncarov polynomials $\{Q_n(z; z_0, z_1, \ldots, z_{n-1})\}$ belonging to the $\mathrm{D_q}$-derivative operator. For a comprehensive and easy reading, background results are provided in the Preliminaries of sections 2.1–2.5.

2. Preliminaries

We record here some background information for easy reading of the contents of the presentation.

2.1 Basic sets and effectiveness

A sequence $\{p_n(z)\}$ of polynomials is said to be basic if any polynomial and, in particular, the polynomials $1, z, z^2, \ldots, z^n, \ldots$, can be represented uniquely by a finite linear combination of the form.

$$z^n = \sum_{k=0} \pi_{n,k} p_k(z); n \geq 0. \tag{3}$$

The polynomials $\{p_n(z)\}$ are linearly independent.

In the representation (3), let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an analytic function about the origin. Substituting (3) into $f(z)$, we have

$$f(z) = \sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty a_n \sum_{k=0} \pi_{n,k} p_k(z).$$

Formally rearranging the terms, we obtain the series

$$\sum_{k=0}^\infty p_k(z) \left[\sum_{n=0}^\infty a_n \pi_{n,k} \right].$$

We write

$$\prod_k (f) = \sum_{n=0}^\infty a_n \pi_{n,k}; k \geq 0.$$

Hence, we obtain the series

$$\sum_{k=0}^\infty \prod_k (f) p_k(z),$$

which is called the basic series associated with the function $f(z)$ and the correspondence is written as

$$f(z) \sim \sum_{k=0}^{\infty} \prod_{k}(f) p_k(z). \tag{4}$$

The coefficients $\{\Pi_k(f)\}$ is the basic coefficients of $f(z)$ relative to the basic set $\{p_k(z)\}$ and is a linear functional in the space of functions $\{f(z)\}$.

If $p_n(z)$ is of degree n then the set is called a simple set and is necessarily a basic set.

The basic series (4) is said to represent $f(z)$ in a disk $|z| \leq r$ where $f(z)$ analytic, if the series is converges uniformly to $f(z)$ in $|z| \leq r$ or that the basic set $\{p_n(z)\}$ represents $f(z)$ in $|z| \leq r$.

When the basic set $\{p_n(z)\}$ represents in $|z| \leq r$ every function analytic in $|z| \leq R, R \geq r$, then the basic set is said to be effective in $|z| \leq r$ for the class $\overline{H}(R)$ of functions analytic in $|z| \leq R$.

When $R = r$, the basic set represents, in $|z| \leq r$, every function which is analytic there and we say that the basic set is effective in $|z| \leq r$.

To obtain conditions for effectiveness, we form the Cannon sum

$$w_n(r) = \sum_{k=0} |\pi_{n,k}| M_k(r), \tag{5}$$

where

$$M_k(r) = \max_{|z|=r} |p_k(z)|. \tag{6}$$

From (3), we have that $w_n(r) \geq r^n$,
so that, if we write

$$\lambda(r) = \lim_{n\to\infty} \sup \{w_n(r)\}^{\frac{1}{n}}, \tag{7}$$

$$\lambda(r) \geq r^n. \tag{8}$$

The function $\lambda(r)$ is called the Cannon function of the set $\{p_n(z)\}$ in $|z| \leq r$.

Theorems about the effectiveness of basic sets are due to Cannon and Whittaker (cf. [2, 4, 5]).

A necessary and sufficient condition for a Cannon set $\{p_n(z)\}$ to be effective, in $|z| \leq r$, is

$$\lambda(r) = r. \tag{9}$$

2.2 Mode of increase of basic sets

The mode of increase of a basic set $\{p_n(z)\}$ is determined by the order and type of the set. If $\{p_n(z)\}$ is a Cannon set, its order is defined, Whittaker [2], by

$$w = \lim_{r\to\infty} \lim_{n\to\infty} \sup \frac{\log w_n(r)}{n \log n}. \tag{10}$$

where $w_n(r)$ is given by (5). The type γ is defined, when $0 < w < \infty$, by

$$\gamma = \lim_{r\to\infty} \frac{e}{w}\left[\lim_{n\to\infty}\sup\{w_n(r)\}^{\frac{1}{n}}\, n^{-w}\right]^{\frac{1}{w}}. \tag{11}$$

The order and type of a set define the class of entire functions represented by the set.

Theorem 2.2.1 (Cannon [6]).

The necessary and sufficient conditions for the Cannon set of polynomials to be effective for all entire functions of increase less than order p type q is

$$\lim_{n\to\infty}\sup\left[\left(\frac{epq}{\mathrm{n}}\right)^{\frac{1}{p}}\{w_n(r)\}^{\frac{1}{n}}\right] \le 1 \text{ for all } r > 0. \tag{12}$$

2.3 Zeros of simple sets of polynomials

The relation between the order of magnitude of the zeros of polynomials belonging to simple sets and the mode of increase of the sets has led to many convergence results, just as that between the order of magnitude of the zeros and the growth of the coefficients has. In the case of the zeros and mode of increase, the approach to achieve effectiveness is to determine the location of the zeros while that between the zeros and the coefficients is to determine appropriate bounds (cf. Boas [7], Nassif [8], Eweida [9]).

2.4 Properties of the Goncarov polynomials

We record in what follows certain properties of the proper and generalized Goncarov polynomials together with the definitions of the q-analogues and the Dq-derivative operator.

The proper Goncarov polynomials $\{G_n(z, z_0, \ldots z_{n-1})\}$ associated with the sequence $\{z_n\}_0^\infty$ of points in the plane are defined through the relations, Buckholtz ([10], p. 194),

$$G_0(z) = 1,$$

$$\frac{z^n}{n!} = \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} G_k(z, z_0, \ldots, z_{k-1});\ \mathrm{n} \ge 1. \tag{13}$$

These polynomials generate any function $f(z)$ analytic at the origin through the Goncarov series

$$f(z) \sim \sum_{k=0}^{\infty} f^k(z_k) G_k(z, z_0, \ldots, z_{k-1}), \tag{14}$$

which represents $f(z)$ in a disk $|z| \le r$, if it uniformly converges to $f(z)$ in $|z| \le r$.

In this case, if $f^{(k)}(z_k) = 0, k \ge 0$, the Goncarov series (14) vanishes and $f \equiv 0$.

A consideration of $g(z) = \sin\frac{\pi}{4}(1 - z)$, for which $g^{(n)}\{(-1)^n\} = 0$ and $\sum_{n=0}^{\infty} g^{(n)}\{(-1)^n\} G_n(z, 1, -1, ..) = 0$ cf. Nassif [8], shows that the Goncarov series does not always represent the associated function and hence certain restrictions have to be imposed on the points $(z_k)_0^\infty$ and on the growth of the function $f(z)$.

Concerning the case where the points $(z_k)_0^\infty$ lie in the unit disk U, the Whittaker constant W (cf. Whittaker, Buckholtz, [2, 10]), is defined as the supremum of the number c with the following property:

If $f(z)$ is an entire function of exponential type less than c and if each of $f, f', f'', \ldots$ has a zero in U then $f(z) \equiv 0$.

Buckholtz [10] obtained an exact determination of the constant W. In fact, if we write

$$H_n = \max |G_k(0; z_0, \ldots, z_{n-1}), \tag{15}$$

where the maximum is taken over all sequences $(z_k)_0^{n-1}$ whose terms lie in U, Buckholtz ([10], Lemma 3) proved that $\lim_{n\to\infty} H_n^{1/n}$ exists and is equal to $\sup_{1\le n<\infty} H_n^{1/n}$.

Moreover, if we put

$$\lim_{n\to\infty} H_n^{1/n} = H = \sup_{1\le n<\infty} H_n^{1/n}, \tag{16}$$

Buckholtz ([10], formula 2) further showed that

$$W = \frac{1}{H}. \tag{17}$$

Employing an equivalent definition of the polynomials $\{G_n(z; z_0, \ldots, z_{n-1})\}$ as originally given by Goncarov [11] in the form

$$G_n(z, z_0, \ldots, z_{n-1}) = \int_{z_0}^{z} ds_1 \int_{z_1}^{s_1} ds_2, \ldots, \int_{z_{n-1}}^{s_{n-1}} ds_n; n \ge 1, \tag{18}$$

and differentiating with respect to z, we can obtain

$$G_n^{(k)}(z, z_0, \ldots, z_{n-1}) = G_{n-k}(z, z_k, \ldots, z_{n-1}); \ 1 \le k \le n-1. \tag{19}$$

Writing

$$G_n(0, z_0, \ldots, z_{n-1}) = F_n(z_0, \ldots, z_{n-1}) \ n \ge 1, \tag{20}$$

then (18) yields, among other results,

$$G_n(z; z_0, \ldots, z_{n-1}) = F_n(z; z_0, \ldots, z_{n-1}) - F_n(z; z_1, \ldots, z_{n-1}); \ n \ge 1, \tag{21}$$

and

$$F_n(0, z_1, \ldots, z_{n-1}) = 0; \ n \ge 1. \tag{22}$$

Applying (21) and (22) to (19) we obtain

$$F_k^{(k)}(z_0, z_1, \ldots, z_{n-1}) = -F_{n-k}(z_0, \ldots, z_{n-1}) \tag{23}$$

for $1 \le k \le n-1$, where the differentiation is with respect to the first argument. Expanding $F_n(z_0, \ldots, z_{n-1})$ in powers of z_0, in the form

$$F_n(z_0, \ldots, z_{n-1}) = \sum_{k=0}^{n} \frac{z_0^k}{k!} F_n^{(k)}(0, z_1, \ldots, z_{n-1}),$$

we arrive through (22) and (23) to the formulae of Levinson [12],

$$F_n(z_0, \dots, z_{n-1}) = -\sum_{k=1}^{n} \frac{z_0^k}{k!} F_{n-k}(z_k, \dots, z_{n-1}). \tag{24}$$

Also, differentiating (18) with respect to z_k, we obtain with Macintyre ([13], p. 243),

$$\frac{\partial}{\partial z_k}\left(G_n^{(k)}(z, z_0, \dots, z_{n-1})\right) = -G_k(z, z_0, \dots, z_{k-1}) G_{n-k-1}(z_k, z_{k-1}, \dots, z_{n-1}) \tag{25}$$

for $0 \le k \le n-1$.

2.5 The q-analogues and D_q derivatives

Let q be a positive number different from 1. The q–analogue of the positive integer n is given by

$$[n] = \frac{q^n - 1}{q - 1}. \tag{26}$$

Also, the q-analogue of $n!$ is

$$[n]! = [n]\,[n-1] \dots [2]\,[1]; n \ge 1; [0]! = 1, \tag{27}$$

and the q-analogue of the binomial coefficient $\binom{n}{k}$ is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \ ; \ 0 \le k \le n. \tag{28}$$

Moreover, the D_q– derivative operator, corresponding to the number q is defined as follows: *If* $f(z)$ is any function of z, then

$$D_q(\, f(z)) = \frac{f(qz) - f(z)}{z(q-1)}, \tag{29}$$

so that when $f(z) = z^n$, then according to (26), we have $D_q(z^n) = [n]z^{n-1}$ and if $f(z) = \sum_{n=0}^{\infty} a_n z^{n-1}$ is any function analytic at the origin then

$$D_q f(z) = \sum_{n=1}^{\infty} [n] a_n z^{n-1}. \tag{30}$$

In [3] we have a generalization of the Goncarov polynomials as in (13) belonging to the operator D such that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$D(\, f(z)) = \sum_{n=1}^{\infty} d_n a_n z^{n-1} \tag{31}$$

associated with the sequence $(z_k)_0^{\infty}$, where $e_n = (d_1 d_2 \dots, d_n)^{-1}, e_0 = 1$ and $(d_n)_1^{\infty}$ is a non-decreasing sequence of numbers to obtain

$$\begin{cases} p_0(z) = 1, \\ e_n z^n = \sum_{k=0}^{n} e_{n-k} z_k^{n-k} P_k(z, ; z_0, \ldots, z_{k-1}); n \geq 1. \end{cases} \tag{32}$$

When $d_n = n$, the relations (32) reduce to (6), hence the polynomials $\{p_n(z)\}$ reduce to the proper Goncarov polynomials $\{G_n(z; z_0, \ldots, z_{n-1})\}$. Comparing (30) and (32), Nassif [14] investigated the class of generalized Goncarov polynomials $\{Q_n(z; z_0, \ldots, z_{n-1})\}$ belonging to the Dq- derivative operator when $d_n = [n]$ and $e_n = \frac{1}{[n]!}$ given by,

$$\begin{cases} Q_0(z) = 1 \\ \frac{z^n}{[n]!} = \sum_{k=0}^{n} \frac{z_k^{n-k}}{[n-k]!} Q_k(z\, z_0, \ldots, z_{k-1}); n \geq 1' \end{cases} \tag{33}$$

and the Goncarov series associated with the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is

$$f(z) \sim \sum_{k=0}^{\infty} D_q^k f_k(z_k) Q_k(z; z_0, \ldots, z_{k-1}). \tag{34}$$

Writing

$$R_n(z_0, \ldots, z_{n-1}) = Q_n(0; z_0, \ldots, z_{n-1}) \tag{35}$$

so that

$$R_n(0; z_1, \ldots, z_{n-1}) = 0, \ n \geq 1 \tag{36}$$

then we have from, (32) that

$$R_n(z_0, \ldots, z_{n-1}) = -\sum_{k=0}^{n-1} \frac{z^{n-k}}{[n-k]!} R_k(z_0, \ldots, z_{k-1}). \tag{37}$$

Also, Nassif ([14], Lemma 4.1), proved that

$$Q_n(z; z_0, \ldots, z_{n-1}) = R_n(z_0, \ldots, z_{n-1}) - R_n(z; z_1, \ldots, z_{n-1}). \tag{38}$$

We can verify, with Buckholtz ([10], Lemma 1), from the formulae (33), the following:

$$Q_n(\lambda z, \lambda z_0, \ldots, \lambda z_{n-1}) = \lambda^n Q_n(z, z_0, \ldots, z_{n-1}); \ \ n \geq 1. \tag{39}$$

$$Q_n(z_0, z_0, \ldots, z_{n-1}) = 0; \ \ n \geq 1. \tag{40}$$

$$D_q Q_n(z, z_0, \ldots, z_{n-1}) = Q_{n-1}(z, z_1, \ldots, z_{n-1}); \ \ n \geq 1. \tag{41}$$

And hence, by repeated application of D_q, we obtain

$$D_q^k Q_n(z; z_0, \ldots, z_{n-1}) = Q_{n-k}(z; , z_k, \ldots, z_{n-1}); \ \ 1 \leq k \leq n-1. \tag{42}$$

Expressing $Q_n(z; z_0, \ldots, z_{n-1})$ as a polynomial of degree n in $z_,$ then we have from (27), (29) and (42), that

$$Q_n(z;z_0,\ldots,z_{n-1}) = \sum_{k=0}^{n} \frac{z^k}{[n]!} R_{n-k}(z_k,\ldots,z_{n-1}). \tag{43}$$

The identities (39) and (43) have been obtained, in their general form, in ([3]; formulae (2.5), (2.9)). Also, a combination of (38) and (42) yields

$$D_q^k R_n(0,z_1,\ldots,z_{n-1}) = -R_{n-k}(z_k,\ldots,z_{n-1}), \tag{44}$$

for $1 \le k \le n-1$, where the differentiation is with respect to the first argument. Expanding $R_n(z_0,z_1,\ldots,z_{n-1})$ in powers of z_0, then (36) and (44) imply that

$$R_n(z_0,z_1,\ldots,z_{n-1}) = -\sum_{k=1}^{n} \frac{z_0^k}{[k]!} R_{n-k}(z_k,\ldots,z_{n-1}). \tag{45}$$

Finally, if we put

$$h_n = \max |R_n(z_0,z_1,\ldots,z_{n-1})|, \tag{46}$$

where the maximum is taken over all sequences $(z_k)_0^{n-1}$ and the terms lie in the unit disk U, then Buckholtz and Frank ([3], Corollary 5.2), proved that

$$\lim_{n\to\infty} h_n^{1/n} = h = \sup_{1 \le n < \infty} h_n^{1/n}. \tag{47}$$

Also, in view of the formulae (33), we can verify that, when $q<1$,

$$h \ge h_2^{1/2} = \left(1+\frac{[1]}{[2]}\right)^{\frac{1}{2}} > \left(\frac{3}{2}\right)^{\frac{1}{2}} > 1. \tag{48}$$

3. Results on the zeros of simple sets

3.1 Zeros of simple sets of polynomials and the conjecture of Nassif on the Whittaker constant are discussed here

The following result is known for simple sets of polynomials whose zeros all lie in the unit disk.

Theorem A.([1], Theorem 1).

When the zeros of polynomials belonging to a simple set all lying within or on the unit circle the set will be of increase not exceeding order 1 type 1.378.

Using known contributions in the theory of Goncarov polynomials, we show that the alternative form of the above theorem is as follows:

Theorem 3.1.1 ([Nassif and Adepoju [15], Theorem B)

When the zeros of the polynomials belonging to a simple set all lying in the unit disk, the set will be of increase not exceeding order 1 type $\frac{1}{W}$, where W is the Whittaker constant. It is shown also that the result in this theorem is bes t possible.

Indeed, applying the result of Buckholtz ([10], formula 2), the following theorem which resolved the conjecture of Nassif ([8], p.138), is established.

Theorem 3.1.2 ([15], Theorem B)

Given a positive number ε, a simple set $\{p_n(z)\}$ of polynomials, whose zeros all lie in U can be constructed such that the increase of the set is not less than order 1 type H–ε.

For completeness, we give the proof of Theorem 3.1.1 as a revised version of Theorem A.

Proof of Theorem 3.1.1 (Proof of alternative form of Theorem A)

Let $\{b_n\}_1^\infty$ be a sequence of points lying in the unit disk and consider the set $\{q_n(z)\}$ of polynomials given by

$$q_0(z) = 1;\ q_n(z) = (z + b_n)_1^n \ ;\ n \geq 1. \tag{49}$$

Suppose that z^n admits the representation

$$z^n = \sum_{k=0}^{n} \tilde{w}_{n,k} q_{nk}(z). \tag{50}$$

Then multiplying the matrix of coefficients $\left\{ \binom{n}{k} b_n^{n-k} \right\}$ with its inverse $(\tilde{w}_{n,k})$, we obtain

$$-\tilde{w}_{n,0} = \sum_{k=1}^{n} \binom{n}{k} b_n^k \tilde{w}_{n-k,0}; \quad n \geq 1.$$

Write.

$$u_n = \frac{\tilde{w}_{n,0}}{n!};\ n \geq 0, \tag{51}$$

then the above relation will give

$$u_n = -\sum_{k=1}^{n} \frac{b_n^k}{k!} u_{n-k}; \quad n \geq 1.$$

And to show the dependence of u_n on the points (b_n), this relation can be rewritten as

$$u_n(b_1, b_2, \dots, b_n) = -\sum_{k=1}^{n} \frac{b_n^k}{k!} u_{n-k}(b_1, b_2, \dots, b_{n-k}).$$

Comparing this relation with the identify

$$F_n(z_0, z_1, \dots, z_{n-1}) = -\sum_{k=1}^{n} \frac{z_0^k}{k!} F_n(z_k, \dots, z_{n-1}),$$

of Levinson [12], we infer that

$$u_n(b_1, b_2, \dots, b_n) = F_n(b_n, b_{n-1}, \dots, b_1). \tag{52}$$

Differentiating (50) k times, $k = 1, 2, \dots, n-1$, we obtain that

$$\tilde{w}_{n,k} = \binom{n}{k} \tilde{w}_{n-k,0}(b_{k+1}, b_{k+2}, \dots, b_n). \tag{53}$$

Hence, a combination of (15), (16), (20), (51)-(53) leads to the inequality.

$$|\tilde{w}_{n,k}| \le \frac{n!}{k!} H^{n-k}; \ \ 0 \le k \le n. \tag{54}$$

Observing that $M(q_k; r) \le (1+r)^k$ for any value of $r \ge 0$, then the Cannon sum of the set $\{q_n(z)\}$ for $|z| = r$ will, in view of (54), be

$$w_n(r) = \sum_{k=0}^{n} |\tilde{w}_{n,k}| \mathrm{M}(q_k; r) \le n!\, \mathrm{H}^n \exp\left(\frac{1+r}{\mathrm{H}}\right).$$

It follows from (17) that the set $\{q_n(z)\}$ is of increase not exceeding order 1 type $\frac{1}{W}$. The proof is now completed by applying the results of Walsh and Lucas, cf. Marden ([16], pp. 15,46), with (54) and following exactly the same lines of argument as in ([1], pp.109–110), to arrive at the inequality.

$$|\pi_{n,k}| \le \frac{n!}{k!} \left(H^{n-k}\right). \tag{55}$$

Since $|p_n(z)| \le (1+r)^n$ in $|z| \le r$, it follows that the set $\{p_n(z)\}$ is of increase not exceeding order 1 type H = $\frac{1}{W}$.

This completes the proof of the theorem.

3.2 Background and the proof of the conjecture

Before the proof of Theorem 3.2.1, we note that we can take, $\varepsilon < H - 1$.(In fact, according to Macintyre ([13]; p. 241), we have H > $\frac{1}{0.7378}$). Hence it follows from (16) that corresponding to ε, there exists an integer m such that

$$m > (\log \mathrm{H}) / \log\left(1 + \frac{\in}{2\,\mathrm{H}}\right) \tag{56}$$

such that

$$H^{\frac{1}{m}}_m > H - \frac{\in}{2}. \tag{57}$$

Moreover, from (20), the definition (15) ensures the existence of the points $(a_k)_1^m$ lying in $|z| \le 1$ such that

$$H_m = |F_m(a_m, a_{m-1}, \dots, a_1)|. \tag{58}$$

Having fixed the integer m and the sequence $(a_k)_1^m$, the following Lemma is to be first established.

Lemma 3.2.1 ([15], Lemma 3.2).

For any integer $j \ge 1$, write

$$f_j(z_1, z_2, \dots, z_j) = F(\,j+1)_{m+j}(a_m, \dots, a_i; z_j; a_m, \dots, a_i; z_{j-1}; \dots, a_m, \dots, a_i; z_1; a_m, \dots, a_1) \tag{59}$$

Then, the complex numbers $(\xi_k)_1^\infty$ can be chosen so that

$$|\xi_k| = 1; \quad k \geq 1, \tag{60}$$

and

$$\left| f_j\left(\xi_1, \xi_2, \dots, \xi_j\right) \right| = H_m^{j+1}; \quad j \geq 1. \tag{61}$$

Proof.

The proof is by induction.

When $j = 1$, we have from (59) that

$$f_1(z_1) = F_{2m+1}(a_m, \dots, a_1; z_1; a_m, \dots, a_1).$$

Then the value ξ_1 will be chosen so that

$$|\xi_1| = 1; \left| f_1(\xi_1) \right| = \sup_{|z_1| \leq 1} \left| f_1(z_1) \right|. \tag{62}$$

Applying the identify (25) of Macintyre to $F_{2m+1}(a_m, \dots, a_1; z_1; a_m, \dots, a_1)$, we obtain

$$\frac{d}{dz_1}\left(f_1(z_1)\right) = -F_m(a_m, \dots, a_1)\, G_m(z_1; a_m, \dots, a_1),$$

so that (20) and (58) imply that

$$\left| f_1'{}'(0) \right| = H_m^2,$$

where the prime denotes differentiation with respect to z_1.

Hence, in view of (62), Cauchy's inequality yields

$$\left| f_1(\xi_1) \right| \geq H_m^2,$$

and the inequality (61) is satisfied for $j = 1$. Suppose then that, for some value $j = k$, the complex numbers $\xi_1, \xi_2, \dots, \xi_k$ have been chosen satisfying (60) and (61).

The numbers ξ_{k+1} will be fixed so that

$$\begin{cases} |\xi_{k+1}| = 1 \\ |F_{k+1}(\xi_1, \xi_2, \dots, \xi_{k+1})| = \sup\limits_{|z_{k+1}| \leq 1} |F_{k+1}(\xi_1, \xi_2, \dots, \xi_k, z_{k+1})| \cdot \end{cases} \tag{63}$$

Proceeding in a similar manner as for the Case $j = 1$ and applying the identity (25) of Macintyre with (58), (59) and (61),we can obtain the inequality.

$$\left| f_{k+1}^1(\xi_1, \xi_2, \dots, \xi_k, 0) \right| \geq H_m^{k+2}, \tag{64}$$

where the prime denotes differentiation with respect to z_{k+1}..

Applying Cauchy's inequality to the polynomial $F_{k+1}(\xi_1, \xi_2, \dots, \xi_k, z_{k+1})$,we can deduce, using (63) and (64), that

$$|F_{k+1}(\xi_1, \xi_2, \dots, \xi_k, z_{k+1})| \geq H_m^{k+2}.$$

Hence, by induction, the inequality (61) of the Lemma is established.

We now prove theorem 3.1.2.

The required simple set $\{p_n(z)\}$ of polynomials is constructed as follows:

$$\begin{cases} P_0(z) = 1, \\ p_{j(m+1)}(z) = \left(z + \xi_j\right)^{j(m+1)}; \; j \geq 1, \\ p_{j(m+1)+i}(z) = \left(z + a_j\right)^{j(m+1)+i}; \; 1 \leq i \leq m; j \geq 0, \end{cases} \tag{65}$$

where the points $(a_k)_1^m$ are chosen to satisfy (63) and the numbers $(\xi_k)_1^\infty$ are fixed as in the Lemma.

It follows that the zeros of the polynomials $\{p_n(z)\}$ all lie in the unit disk U.

Also, if z^n admits the unique linear representation.

$$z^n = \sum_{k=0}^{n} \pi_{n,k} p_k(z), \tag{66}$$

and if we write

$$u_n = \frac{\pi_{n,0}}{n!} \;\; ; \;\; n \geq 0, \tag{67}$$

then from the relation (52), we deduce from (59) and (65), that

$$u_{(j+1)m+j} = f_j\left(\xi_1, \xi_2, \dots, \xi_j\right); \; j \geq 1. \tag{68}$$

Now, in view of (66), the Cannon sum of the set $\{p_n(z)\}$ for $|z| = r$, is $w_n(r) > |\pi_{n,0}|$.

Hence, combining (57), (61), (67) and (68) yields

$$w_{(j+1)m+j}(r) \geq \{(j+1)m + j\}! \left(H - \frac{\in}{2}\right)^{(j+1)m}; \; j \geq 1.$$

vIt follows from this inequality and Theorem 3.1 that the order of the set $\{P_n(z)\}$ is exactly 1 and since $H - \frac{\in}{2} > 1$, the type of the set will be

$$\gamma \geq \left(H - \frac{\in}{2}\right)^{\frac{m}{m+1}} \geq \left(H - \frac{\in}{2}\right)^{\frac{m=1}{m}}. \tag{69}$$

In view of the inequality (56), we deduce from (69) that

$$\gamma > H - \in$$

and Theorem 3.1.2 is established.

This settles the conjecture.

4. Generalization

4.1 As a generalization of the above problem, we consider the simple set $\{p_n(z_n)\}$ given by

$$p_0(z) = 1; p_n(z) = p_n(z; a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_n^{n-k} z^k; n \geq 1, \tag{70}$$

where $\begin{pmatrix} n \\ k \end{pmatrix}$ is the q-analogue of the binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$ and $(a_k)_1^\infty$ is a sequence of given complex numbers. The set $\{p_n(z)\}$ is in fact, the q-analogue of the set $\{q_n(z)\}$ in (49). This study is motivated by the fact that this set is related to the generalized Goncarov polynomials belonging to the Dq-derivative operator. Our results show that effectiveness properties of the set.

$\{p_n(z)\}$ depend on whether $q < 1$ or $q > 1$.

We establish the following:

Theorem 4.1.1 ([17], Theorem 1.1)

When the points $(a_k)_1^\infty$ all lie in the unit disk U, the corresponding set $\{p_n(z)\}$ for $q < 1$,will be effective in $|z| \leq r$ for $r \geq \frac{h}{1-q}$, where h is as in (47).

Theorem 4.1.2. ([17], Theorem 3.1)

Given $\in > 0$, the points $(a_k)_1^\infty$ lying in $|z| \leq 1$ can the chosen so that the correspondence set $\{p_n(z)\}$ of (70) with $q < 1$ will not be effective in $|z| < r$ for $r < \frac{h-\in}{1-q}$ ..

Theorem 4.1.3 ([17], Theorem 1.2)

When $q > 1$ and

$$|a_k| \leq q^{-k} \;\; ; q \geq 1 \tag{71}$$

the corresponding set $\{p_n(z)\}$ of (70) will be effective in $|z| \leq r$ for $r > \frac{q\gamma}{q-1}$, where $\frac{1}{\gamma}$ is the least root of the equation.

$$\sum_{n=0}^{\infty} q^{-n^2} x^n = 2. \tag{72}$$

Theorem 4.1.2 shows that the result in Theorem 4.1.1 is best possible. Also, the restriction (71) on the sequence $(a_k)_1^\infty$ when $q > 1$, is shown to be justified in the sense that if the restriction is not satisfied, the corresponding set $\{p_n(z)\}$ may be of infinite order and not effective.

Proof.

Proof of Theorem 4.1.1 is similar to the first part of Theorem 3.1.1.

Let z^n admits the representation

$$z^n = \sum_{k=0}^{n} \pi_{n,k}(a_1, a_2, \ldots, a_n) p_k(z), \tag{73}$$

then multiplying the matrix of coefficients $\left\{ \begin{bmatrix} n \\ k \end{bmatrix} a^n{}_{n-k} \right\}$ of the set $\{p_n(z)\}$ with the inverse matrix $(\pi_{n,k})$ we obtain

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^n{}_{n-k} \pi_{k,0}(a_1, a_2, \ldots, a_k) = 0; \;\; n \geq 1.$$

Putting

$$v_0 = 1, v_k = v_k(a_1, a_2, \ldots \ldots, a_k) = \frac{1}{[k]!} \pi_{k,o}(a_1, \ldots, a_{n-k'}), \tag{74}$$

the above relation yields

$$v_n(a_1, \ldots \ldots, a_n) = -\sum_{k=1}^{n} \frac{a^k}{[k]!} v_{n-k}(a_1, \ldots, a_{n-k'}) \tag{75}$$

Comparing the formulae (45) and (75) we infer that

$$v_k(a_1, \ldots \ldots, a_k) = R_k(a_k, \ldots \ldots, a_1). \tag{76}$$

Moreover, operating Dq on the polynomials $\{p_n(z)\}$, we can deduce, from (28) and (29), that

$$D_q(p_k(z; a_k)) = [K]p_{k-1}(z, a_k); \; k \geq 1. \tag{77}$$

Hence, when the operator Dq acts on the representation (73), then (77) leads to the equality

$$\pi_{n,k}(a_1, \ldots \ldots, a_n) = \frac{[n]}{[k]} \pi_{n-1,k-1}(a_1, \ldots, a_n),$$

which, on reduction, yields

$$\pi_{n,k}(a_1, \ldots \ldots, a_n) = \begin{bmatrix} n \\ k \end{bmatrix} \pi_{n-k,0}(a_{K+1}, \ldots, a_n); 0 \leq k \leq n. \tag{78}$$

Applying (74), (76) and (78), we obtain

$$\pi_{n,k}(a_1, \ldots \ldots, a_n) = \frac{[n]!}{[k]!} R_{n-k}(a_n, \ldots, a_{K+1}); 0 \leq k \leq n. \tag{79}$$

Identify (79) is the bridge relation between the set $\{p_n(z)\}$ and the Goncarov polynomials mentioned earlier.

Suppose $q < 1$ and assume that

$$r \geq \frac{h}{1-q}. \tag{80}$$

Since $h > 1$ as in (47), and restricting the points $(a_k)_1^\infty$ to lie in the unit disk U as in the theorem, it follows from (28) and (80) that

$$\mathrm{M}(p_k; r) \leq (k+1)r^k \; ; k \geq 0. \tag{81}$$

The Cannon sum of the set $\{p_n(z)\}$ for $|z| = r,$ is evaluated from (46), (47), (79), (80) and (81) to obtain

$$w_n(r) = \sum_{k=0}^{n} |\pi_{n,k}| \mathrm{M}(p_k; r) \leq (n+1)^2 r^n, \tag{82}$$

from which it follows that the set $\{p_n(z)\}$ is effective in $|z| \leq r$ for $r \geq \frac{h}{1-q}$ and the theorem is established.

5. Proof

5.1 Proof of Theorem 4.1.2

We argue as in the Proof of Theorem 3.1.2. We first obtain an identity similar to (25) of Macintyre using the following Lemma:

Lemma 5.1.1.

For $n \geq 1$ and $k \geq 0$, the following identity holds.

$$\begin{cases} D_{q,z_k} Q_{k+n}(z; z_0, \ldots, z_{k+n-1}) \\ = -Q_k(z; z_o, \ldots, z_{k-1}) Q_{n-1}(z_k, z_{k+1}, \ldots, z_{k+n-1}), \end{cases} \tag{83}$$

where D_{q,z_k} denote the Dq-derivative with respect to z_k.

Proof of Lemma

The proof is by induction.

For $n = 1, k \geq 0$, we have from the construction formulae (33),

$$Q_{K+1}(z; z_0, \ldots, z_k) = \frac{z^{k+1}}{[k+1]!}$$
$$- \sum_{j=0}^{k-1} \frac{z_j^{k+1-j}}{[k+1-j]!} Q_j\left(z; z_0, \ldots, z_{j-1}\right) - z_k Q_k\left(z; z_0, \ldots, z_{j-1}\right).$$

Hence, operating D_{q,z_k} on this equality, we have that

$$D_{q,z_k} Q_{k+1}(z; z_0, \ldots, z_{k-1}) = -Q_k(z; z_0, \ldots, z_{k-1}),$$

so that the identity (83) is satisfied for $n = 1, k \geq 0$. Suppose that (83) is satisfied for$n = 1, 2, \ldots, m; k \geq 0$. The formulae (33) can be written for $k + m + 1$ in the form,

$$Q_{k+m+1}(z; z_0, \ldots, z_{k+m}) = \frac{z^{k+m+1}}{[k+m+1]!} - \sum_{j=0}^{k-1} \frac{z_j{}^{k+m+1-j}}{[k+m+1-j]!} Q_j\left(z; z_{0, \ldots, z_{j-1}}\right)$$

$$- \frac{z^{m+1}{}_k}{[m+1]!} \left(Q_k(z; z_0, \ldots, z_{k-1})\right) - \sum_{j=1}^{m} \frac{z_{k+j}{}^{m+1-j}}{[m+1-j]!} Q_{k+j}\left(z; z_0, \ldots, z_{k+j-1}\right).$$

Hence, the derivative D_{q,z_k} operating on this equation gives, in view of (83),

$$D_{q,z_k} Q_{k+m+1}(z; z_0, \ldots, z_{k+m}) = -\frac{z^m}{[m]!} Q_k(z; z_0, \ldots, z_{k-1})$$
$$+ \sum_{j=1}^{m} \frac{z_{k+j}^{m+1-j}}{[m+1-j]!} Q_K(z; z_0, \ldots, z_{K-1}) \times Q_{j-1}\left(z_k; z_{k+1}, \ldots, z_{K+j-1}\right).$$

Or equivalently,

$$D_{q,zk} Q_{k+m+1}(z; z_0, \ldots, z_{k+m}) = -Q_k(z; z_0, \ldots, z_{k-1})$$
$$\times \left[\frac{z_k^m}{[m]!} - \sum_{j=0}^{m-1} \frac{z_{k+j+1}^{m+j}}{[m-j]!} Q_j\left(z_k; z_{k+1}, \ldots, z_{K+j}\right)\right].$$

Hence, formulae (33) imply that

$$D_{q,z_k} Q_{k+m+1}(z; z_0, \dots, z_{k+m}) = -Q_k(z; z_0, \dots, z_{k-1}) Q_m(z_k; z_{k+1}, \dots, z_{K+m}),$$

and the relation (83) is also valid for $n = m + 1$

The Lemma is thus proved by induction. Now, following similar lines paralleling those of the proof of Theorem 3.1.2, we need to establish a Lemma similar to that used for Theorem 3.1.2.

Indeed, observing that $h > 1$ as in (39), the $\in > 0$ of Theorem 4.1.2 can always be picked less than $h - 1$. Also, from (39) it follows that, corresponding to the number $\in$, there exists an integer m for which

$$m > (\log h) / \log \left(1 + \frac{\in}{2h}\right), \tag{84}$$

such that

$$h_m^{\frac{1}{m}} > h - \frac{\in}{2}. \tag{85}$$

Also, from the definition (46) of h_m, the points $(\alpha_i)_1^m$ lying in U can be chosen so that

$$h_m = |R_m(\alpha_m, . \dots, \alpha_1)|. \tag{86}$$

With this choice of the integer m and the points $(\alpha_i)_1^m$, the Lemma to be established is the following:

Lemma 5.1.2.

With the notation

$$u_j\left(z_1, z_{2,\dots,z_j}\right) = R_{(\ j+1)m+j}\left(\alpha_m, \dots, \alpha_1; z_j; \alpha_m, \dots, \alpha_1; z_{\ j-1}; \dots; \alpha_m, \dots, \alpha_1; \alpha_m, \dots, \alpha_1\right), \tag{87}$$

we can choose a sequence $\left(\xi_j\right)_1^m$ of points on $|z| = 1$ such that

$$\left|u_j\left(\xi_1, \xi_2, \dots, \xi_j\right)\right| \geq m^{\ j+1}; j \geq 1. \tag{88}$$

Proof.

We first observe, from a repeated application of (30), that an analytic function $f(z)$ regular at the origin, can be expanded in a certain disk $|z| \leq 1$ in a series of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} D_q^n f(0).$$

Hence, by Cauchy's inequality, we have

$$\mathrm{M}(f, r) \geq r \left|D_q f(0)\right|. \tag{89}$$

Applying the usual induction process, we obtain, from (87) for the case $j = 1$, that

$$u_1(z_1) = R_{2m+1}(\alpha_m, \dots \dots, \alpha_1; z_i; \alpha_m, \dots \dots, \alpha_1)$$

Hence the identity (83) yields

$$\begin{aligned} D_q u_1(z_1) &= D_{q,z_1} Q_{2m+1}(0; \alpha_m, \dots \dots, \alpha_1; z_i; \alpha_m, \dots \dots, \alpha_1) \\ &= -R_m(\alpha_m, \dots \dots, \alpha_1) Q_m(z_i, \alpha_m, \dots \dots, \alpha_1). \end{aligned}$$

Therefore, we obtain

$$D_q u_1(0) = R_m^2(\alpha_m, \dots \dots, \alpha_1), \tag{90}$$

where the D_q is operating with respect to z_1.

Pick the number ξ_1, with $|\xi_1| = 1$, such that

$$|u_1(\xi_1)| = \sup\{|u_1(z_1)| : |z_1| = 1\};$$

hence, a combination of (86), (89) and (90) yields

$$|u_1(\xi_1)| \geq h_m^2,$$

and the inequality (88) is satisfied for $j = 1$. The similarity with the proof of Lemma 3.2.1 shows that the proof of this Lemma can be completed in the same manner as that for ealier Lemma.

We can now prove Theorem 5.1.4.

We note that the points $(a_k)_1^\infty$ lying in U which define the required set $\{p_n(z)\}$ of polynomials (70), are chosen as follows:

$$\begin{cases} a_{\ j(m+1)} = \xi_j \\ a_{\ j(m+1)+i} = \alpha_i \ ; 1 \leq i \leq m; j \geq 0 \end{cases}, \tag{91}$$

where the points $(\alpha_i)_1^m$ are fixed as in (86) and the sequence $\left(\xi_j\right)_0^\infty$ of points is determined as in Lemma 5.1.2; and the integer m is chosen as in (84) and (85).

If z^n admits the representation (86), then applying (79), (87) and (91) we have that

$$\pi_{(\ j+1)m+j,o} = [(\ j+1)m+i]! u_j\left(\xi_1, \xi_2, \dots, \xi_j\right), j \geq 1, \tag{92}$$

so that, for the Cannon sum of the set $\{p_n(z)\}$ for $|z| = r,$ we obtain, from (85), (88) and (92),

$$w_{(\ j+1)m+j}(r) > [(\ j+1)m + j]! \left(h - \frac{\in}{2}\right)^{(\ j+1)m}; \ r > 0.. \tag{93}$$

Since $q < 1$, we have that

$$\lim_{n\to\infty} ([n]!)^{1/n} = \frac{1}{1-q}. \tag{94}$$

Hence, (93) and (94) yield, for the Cannon function,

$$\begin{aligned}\lambda(r) &= \limsup_{n\to\infty}\{w_n(r)\}^{1/n}\\ &\geq \limsup_{j\to\infty}\left\{w_{(j+1)m+j}(r)\right\}^{1/(j+1)m+j}\\ &\geq \frac{1}{1-q}\left(h-\frac{\in}{2}\right)^{\frac{m}{m+1}};\ r>0.\end{aligned}$$

Noting that $h-\frac{\in}{2}>1$, we conclude, from (84), as in the proof of Theorem (50), that

$$\lambda(r)\geq\frac{h-\in}{1-q}\ ;\ r>0,$$

and $\{p_n(z)\}$ will not be effective in $|z|\leq r$ for $r<\frac{h-\in}{1-q}$. This completes the proof.

5.2 Proof of Theorem 4.1.3

Let $\{p_n(z)\}$ be the basic set in (70) with $q>1$. We first justify the statement that if the restriction (71) is not satisfied the corresponding set $\{p_n(z)\}$ may be of infinite order.

For this, we put

$$a_k=t^k\ \ ;\ k\geq 1, \tag{95}$$

and let t be such that

$$|t|=\beta,\frac{1}{q}<\beta<q \tag{96}$$

We claim that, in this case, the corresponding set $\{p_n(z)\}$ will be of infinite order and hence the effectiveness properties of the set will be violated.

Now, in the identity (37), we let

$$z_k=a_{n-k}=t^{n-k};\ 0\leq k\leq n-1,$$

to obtain

$$\sum_{k=0}^{n}\frac{t^{nk}}{[k]!}R_{n-k}\left(t^{n-k},\ \dots,t\right)=0;n>0. \tag{97}$$

Put

$$R_j\left(t^j,\ \dots,t\right)=t^{\frac{1}{2}j(j-1)}u_j;j\geq 1, \tag{98}$$

so that (97) yields

$$\sum_{k=0}^{n}\frac{t^{\frac{1}{2}k(k+1)}}{[k]!}u_{n-k}=0\ \ ;\ n>0. \tag{99}$$

Hence, if we put

$$u(z) = \sum_{n=0}^{\infty} u_n z^n, \tag{100}$$

then (97) implies that

$$u(z) = \frac{1}{\varphi(z)}, \tag{101}$$

where

$$\phi(z,t) = \sum_{n=0}^{\infty} \frac{t^{-\frac{1}{2}n(n-1)}}{[n]!} z^n.$$

Since $|t| = \beta < q$, the function $\phi(z,t)$ is entire of zero order and hence it will have zeros in the finite part of the plane.

Let

$$\sigma = \inf\{|z|; \varphi(z) = 0\} < \infty, \tag{102}$$

then from (100) and (101), we have $\limsup_{n\to\infty} |u_n|^{\frac{1}{n}} = \frac{1}{\sigma} > 0$.

Thus, for the Cannon sum of the set $\{p_n(z)\}$, we have, from (79), (96) and (98), that

$$w_n(r) > |\pi_{n,0}| = [n]! \beta^{\frac{1}{2}n(n-1)} |u_n|. \tag{103}$$

Since $q > 1$ and $\beta > \frac{1}{q}$ then, in view of (102), we deduce from (103) that the set $\{p_n(z)\}$ is of infinite order; as claimed.

To prove Theorem 4.1.3 we first note, from (72), that if we put

$$c = \frac{q\gamma}{q-1}, \tag{104}$$

then

$$c > \frac{1}{q-1}. \tag{105}$$

We then multiply the matrix $\left\{ \begin{bmatrix} n \\ k \end{bmatrix} a_n^{n-k} \right\}$ with the inverse$(\pi_{n,k})$ to get

$$\pi_{n,k} = -\sum_{j=k}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} a_n^{n-j} \pi_{j,k} : \ n > k; \pi_{k,k} = 1. \tag{106}$$

Now, imposing the restriction (71) on the points $(a_k)_1^{\infty}$, we have from (105) and (106) that

$$|\pi_{k+1,k}| \leq c.$$

Thus, the inequality

$$|\pi_{m\,k}| \le c^{m-k} \;\; ; m \ge k, \tag{107}$$

is true for $m = k, \; k+1$.

To prove (107), in general, we observe that, since $q > 1$,

$$\begin{bmatrix} n \\ j \end{bmatrix} \le q^{\,j(n-j)} \left\{ \frac{q}{q-1} \right\}^{n-j} \;\; ; \;\; 1 \le j \le n. \tag{108}$$

Assume that (107) is satisfied for $m = k, k+1, \dots, n-1$; then a combination of (71), (72), (104), (106), (107) and (108) leads to the inequality.

$$|\pi_{n,k}| \le c^{n-k} \sum_{j=1}^{\infty} \left(\frac{q}{c(q-1)} \right)^{j} q^{-j^2} = c^{n-k}.$$

Hence, it follows by induction, that the inequality (107) is true for $m \ge k$. Noting that

$$\begin{bmatrix} k \\ j \end{bmatrix} = q^{\,j(k-j)} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \quad q > 1,$$

where $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ is the q–analogue of $\begin{pmatrix} k \\ j \end{pmatrix}$, $q^1 = \frac{1}{q} < 1$, we then deduce from (70) and (71), that

$$\begin{aligned} \mathrm{M}(p_k; r) &\le r^k \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} q^{-j^2} r^{-j} \\ &\le r^k \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} q^{-\frac{1}{2}j(\,j-1)} (qr)^{-j} \;\; ; \;\; q > 1. \end{aligned}$$

Appealing to a result of Al-Salam ([18]; formula 2.5), we deduce that

$$\mathrm{M}(p_k; r) \le r^k \prod_{j=1}^{k} \left(1 + \frac{1}{q^j r} \right) \; ; k \ge 1, r > 0. \tag{109}$$

The Cannon sum of the set $\{p_n(z)\}$ for $|z| = r$ can be evaluated from (107) and (109) in the form

$$w_n(r) \le \left\{ \prod_{j=1}^{n} \left(1 + \frac{1}{q^j r} \right) \right\} \sum_{k=0}^{n} c^{n-k} r^k. \tag{110}$$

Hence, when $r \ge c$ we should have

$$w_n(r) \le (n+1) \left\{ \prod_{j=1}^{n} \left(1 + \frac{1}{q^j r} \right) \right\} r^n,$$

from which it follows that the set $\{p_n(z)\}$ is effective in $|z| \le r$ and Theorem 4.1.3 is proved.

6. Other related results

The Goncarov polynomials belonging to the Dq–derivative operator have other properties of interest and worth recording. Hence, we present, in this section, more results regarding the Goncarov polynomials $\{Q_n(z); z_0, \ldots, z_{n-1}\}$ as defined in (84) which belong to the derivative operator Dq and whose points $(z_n)_0^\infty$ lie in the unit disk U for which $q < 1$ or $q > 1$.

When $q < 1$, the result of Buckoltz and Frank ([3]; Theorem 1.2) applied to the derivative operator Dq leads, in the language of basic sets, to the following theorem:

Theorem 6.1 ([19], Theorem 1).

The set of Gancarov polynomials $\{Q_n(z); z_0, \ldots, z_{n-1}\}$ belonging to the Dq operator, with $q < 1$ and associated with the sequence of points $(z_n)_0^\infty$ in U, is effective in $|z| \leq r$ for $r \geq \frac{h}{1-q}$.

Theorem 1.5 of Buckholtz and Frank [3] shows that the result of Theorem 6.1 above is best possible. They also showed that when $q > 1$ the Goncarov polynomials fail to be effective and also, that if $|z| \leq q^{-n}$, no favorable effectiveness results will occur, thus justifying the restriction $|z| \leq q^{-n}$ on the points $(z_n)_0^\infty$.

We also state and prove the following theorem.

Theorem 6.2 ([19], Theorem 2).

Suppose that $q > 1$ and that the points $(z_n)_0^\infty$ satisfy the restriction (111). Then the Goncarov set $\{Q_n(z; z_0, \ldots, z_{n-1})\}$ belonging to the Dq–derivative operator, will be effective in $|z| \leq r$ for $r \geq \frac{hq}{q-1}$ and this result is best possible.

To prove this theorem we put, as in the proof of Theorem (72),

$$q^1 = \frac{1}{q}, \tag{111}$$

so that $q^1 < 1$ and we differentiate between the Goncarov polynomials belonging to the operations Dq and Dq^1 by adopting the notation.

$$Q_n(z; z_0, \ldots, z_{n-1}) \text{ and } p_n(z; z_0, \ldots, z_{n-1}),$$

for these respective polynomials. Thus, the constructive formulae (33) for these polynomials will be

$$Q_n(z; z_0, \ldots, z_{n-1}) = \frac{z^n}{[n]!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{[n-k]!} Q_k(z; z_0, \ldots, z_{k-1}), \tag{112}$$

and

$$P_n(z; z_0, \ldots, z_{n-1}) = \frac{z^n}{\{n\}!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{\{n-k\}!} P_k(z; z_0, \ldots, z_{k-1}), \tag{113}$$

where $[k]!$ and $\{k\}!$ are the respective q and q^1 analogues of the factorial k. With this notation, the following Lemma is to be proved.

Lemma 6.1.

The following identity is true for $n \geq 1$ and $q > 1$:

$$q^{-\frac{1}{2}n(n+1)} Q_n\left(q^n z; q^n z_0, \ldots\ldots, q z_{n-1}\right) = p_n(z; z_0, \ldots, z_{n-1}). \tag{114}$$

Proof.

We finish note, from the definition of the analogue $[k]!$ and $\{k\}!$, that

$$\frac{q^{n^2}}{\{n\}!} = \frac{q^{\frac{1}{2}n(n+1)}}{\{n\}!}; \;\; n \geq 1, \tag{115}$$

and

$$\frac{q^{(n-k)^2+(n-k)k}}{\{n-k\}!} = \frac{1}{\{n-k\}!} q^{\frac{1}{2}n(n+1)-\frac{1}{2}k(k+1)} \; ; \;\; 0 \leq k \leq n. \tag{116}$$

Hence, applying the relations (37) and (112) to $\{Q_N(q^n z; q^n z_0, \ldots, q z_n)\}$, we get

$$Q_n(qz, q^n z_0, \ldots, qz_{n-1}) = \frac{q^{n^2}}{\{n\}!} z^n - \sum_{k=0}^{n-1} \frac{q^{(n-k)^2+(n-k)k}}{\{n-k\}!} z_k^{n-1} Q_k(q^k z, q_k z_0, \ldots, qz_{k-1}).$$

Hence, the relations (115) and (116) can be introduced to yield

$$\begin{aligned} q^{-\frac{1}{2}n(n+1)} Q_n(q^n z, q^n z_0, \ldots, qz_{n-1}) = {} & \frac{z^n}{\{n\}!} \\ & - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{\{n-k\}!} q^{-\frac{1}{2}k(k+1)} Q_k(q^k z, q^k z_0, \ldots, qz_{k-1}). \end{aligned} \tag{117}$$

Now, since

$$q^{-1} Q_1(qz; qz_0) = z - z_0 = p_1(z; z_0),$$

the identity (114) is satisfied for $n = 1$.

Moreover, if (114) is valid for $k = 1, 2, \ldots, n-1$, the relations (113) and (117) will give

$$\begin{aligned} q^{-\frac{1}{2}n(n+1)} Q_n(q^n z, q^n z_0, \ldots, qz_{n-1}) &= \frac{z^n}{\{n\}!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{\{n-k\}!} P_k(z; z_0, \ldots, z_{k-1}) \\ &= P_n(z; z_0, \ldots, z_{n-1}), \end{aligned}$$

and hence the Lemma is established.

Proof of Theorem 6.2.

Write

$$z_k = q^{-k} a_k; \;\; k \geq 0, \tag{118}$$

so that the restriction (111) implies that

$$|a_k| \leq 1 \; ; \;\; k \geq 0 \tag{119}$$

Therefore, a combination of (37), (114), (118) yields

$$Q_n(z; z_0, \ldots, z_{k-1}) = q^{-\frac{1}{2}k(k+1)} P_k(z, z_0, \ldots, z_{k-1}). \tag{120}$$

Also, by actual calculation we have that

$$\frac{[n]!}{[n-k]!} q^{-k(n-k)-\frac{1}{2}k(k+1)} = \frac{\{n\}!}{\{n-k\}!} \quad ; \quad 0 \le k \le n \tag{121}$$

Inserting (118), (120) and (121) into (33), we obtain

$$z^n = \sum_{k=0}^{n} \frac{[n]!}{[n-k]!} z_k^{n-k} Q_k(z; z_0, \dots, z_{k-1})$$

$$= \sum_{k=0}^{n} \frac{\{n\}!}{\{n-k\}!} a_k^{n-k} P_k(z, a_0, \dots, a_{k-1}),$$

in the sense that each term in the sum on the left hand side of this relation is equal to the corresponding term in the sum on the right hand side.

Hence, if

$$\mathrm{M}_k(r) = \sup_{|z|=r} |Q_k(z;{}_0, \dots, {}_{k-1})|$$

$$m_k(r) = \sup_{|z|=r} |P_k(z; a_0, \dots, a_{k-1})|$$

and $\Omega_n(r)$ and $w_n(r)$ are the respective Cannon sums of the sets $\{Q_n(z; z_0, \dots, z_{n-1})\}$ and $\{P_n(z; a_0, \dots, a_{n-1})\}$, it follows that

$$\Omega_n(r) = \sum_{k=0}^{n} \frac{[n]!}{[n-k]!} |z_k|^{n-k} \mathrm{M}_k(r) \tag{122}$$

$$= \sum_{k=0}^{n} \frac{\{n\}!}{\{n-k\}!} |a_k|^{n-k} m_k(r) = w_n(r).$$

Since the points $(a_k)_0^\infty$ lie in U, from (119), then applying Theorem 6.1 we deduce from (122) that the set $\{Q_n(z; z_0, \dots, z_n)\}$ will be effective in $|z| \le r$ for $r \ge \frac{h}{1-q} = \frac{qh}{q-1}$ as to be proved.

To show that the result of the Theorem is best possible we appeal to Theorem 1.5 of Buckholtz and Frank [3] to deduce that the set $\{P_n(z; a_0, \dots, a_{n-1})\}$ may not be effective in $|z| \le r$ for $r < \frac{qh}{q-1}$.

In view of the relation (122), we may conclude that the set $\{Q_n(z; z_0, \dots, z_n)\}$ will not be effective in $|z| \le r$ for $r < \frac{qh}{q-1}$ and Theorem 6.2 is fully established.

6.1 The case of Goncarov polynomials with $Z_k = at^k, k \ge 0$

Nassif [14] studied the convergence properties of the class of Goncarov polynomials $\{Q_n(z; z_0, \dots, z_{n-1})\}$ generated through the q^{th} derivative described in (33) where now, $z_k = at^k, k \ge 0$ and a and t are any complex numbers. By considering possible variations of t and q, it was shown that except for the cases $|t| \ge 1, q < 1$ and $|t| > \frac{1}{q}; q > 1$, all other cases lead to the effectiveness of the set $Q_n(z; a, at, \dots at^{n-1})$ in finite circles ([14]; Theorems 1.1, 1.2, 1.3, 3.2, 3.3).

6.2 Quasipower basis (QP-basis)

Kazmin [20] announced results on some systems of polynomials that form a quasipower basis, (QP-basis), in specified spaces. These include the systems of Goncarov polynomials and of polynomials of the form:

$$\{(z+\alpha_n)^n\}, n=0,i,2\ldots;\alpha_n\in[-1,1]. \tag{123}$$

For full details of QP-basis and some of the results announced, cf. ([20]; Corollaries 3, 4).

Of interest is his results that the system in (123), for arbitrary sequence $\{a\}_0^\infty$ of complex numbers with $|a_n|\leq 1$, forms a QP- basis in the space $[1,\sigma]$, for $0<\sigma<W$ and in the space $[1,\sigma)$, for $0<\sigma\leq W$, where W = 0.7377 is the Whittaker constant. This value of W = 0.7377 is attributed to Varga [21]. He also added that Corollaries 3 and 4 contain known results in [5, 9, 15, 22, 23].

7. Conclusions

The chapter presents a compendium of diverse but related results on the convergence properties of the Goncarov and Related polynomials of a single complex variable. Most of the results of the author (or joint), have appeared in print but are here presented in considerable details in the proofs and in their development, for easy reading and assimilation. The results of other authors are summarized with related and relevant ones mentioned to complement the thesis of the chapter. Some recent works related to the Goncarov and related polynomials, cf. [24–29], which provides further applications are included in the references.

The comprehensiveness of the presentation is for the needs of those who may be interested in the subject of the Goncarov polynomials in general and also in their application to the problem of the determination of the exact value of the Whittaker constant, a problem that is still topical and challenging.

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