Chapter

Study on Approximate Analytical Method with Its Application Arising in Fluid Flow

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Abstract

This chapter is about the, Variational iteration method (VIM); Adomian decomposition method and its modification has been applied to solve nonlinear partial differential equation of imbibition phenomenon in oil recovery process. The important condition of counter-current imbibition phenomenon as $v_i = -v_n$, has been considered here main aim, here is to determine the saturation of injected fluid $S_i(x, t)$ during oil recovery process which is a function of distance ξ and time θ, therefore saturation S_i is chosen as a dependent variable while $x \ \ and t$ are chosen as independent variable. The solution of the phenomenon has been found by VIM, ADM and Laplace Adomian decomposition method (LADM). The effectiveness of our method is illustrated by different numerical.

Keywords: Variational Iteration method (VIM), Adomian decomposition method (ADM), Laplace Adomian decomposition method (LADM), nonlinear partial differential equations

1. Introduction

First, the variational iteration method was proposed by He [1] in 1998 and was successfully applied to autonomous ordinary differential equation, to nonlinear partial differential equations with variable coefficients. In recent times a good deal of attention has been devoted to the study of the method. The reliability of the method and the reduction in the size of the computational domain give this method a wide applicability. The VIM based on the use of restricted variations and correction functional which has found a wide application for the solution of nonlinear ordinary and partial differential equations, e.g., [2–10]. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives and whereas the Adomian decomposition method was before the Nineteen Eighties, it was developed by Adomian [11, 12] for solving linear or nonlinear ordinary, partial and Delay differential equations. A large type of issues in mathematics, physics, engineering, biology, chemistry and other sciences have been solved using the ADM, as reported by many authors [13]. The Adomian decomposition method (ADM) [11–28] is well set systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations

(PDEs), integral equations, integro-differential equations, etc. The ADM is considered as a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It allows us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) [17, 29–46] without unphysical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basic functions, and so forth. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution. More advantages of the ADM over the variational iteration method is mentioned in Wazwaz [22, 28]. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to solve nonlinear operator equations. A key concept of the Adomian decomposition series is that it is computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. The selection behind choice of decomposition is nonunique, which provides a valuable advantage to the analyst, permitting the freedom to design modified recursion schemes for ease of computation in realistic systems.

Same way Laplace Adomian's Decomposition Method (LADM) was first introduced by Khuri [47, 48]. The Laplace Adomian Decomposition Method (LADM) is formed with combination of the Adomian Decomposition Method (ADM) Adomian [29, 49] and Laplace transforms. LADM is a promising method and has been applied in solving various nonlinear systems of differential equations [36, 50–56]. In a variety of applied sciences, systems of partial differential equations have attracted much attention e.g. [50, 57–75]. The general ideas and the essentiality of these systems are of wide applicability. Agadjanov [56] solved Duffing equation with the help of LDM. Elgazery [51, 76] had applied Laplace decomposition method for the solution of Falkner-Skan equation.

In the solution procedure of VIM; many repeated computations and computations of the unneeded forms, which take more time and effort beyond it, so a modification has been shown to reduce these unneeded forms.

On the other hand, few researchers have been discussed imbibition phenomenon in homogenous porous media with different point of view for example, researchers taking different perspectives for this phenomenon; [77, 78] and some others have analyzed it for homogeneous porous medium.

In this Present investigated model, Imbibition takes place over a small part of a large oil formatted region taken as a cylindrical piece of homogeneous porous medium. In this model, we have considered the important condition of counter-current imbibition phenomenon as $v_i = -v_n$, Our purpose is to determine the saturation of injected fluid $S_i(x, t)$ during oil recovery process which is a function of distance ξ and time θ, therefore saturation S_i has been chosen as a dependent variable while x and t are chosen as independent variable.

2. Imbibition phenomenon

It is the process by which a wetting fluid displaces a non-wetting fluid the initially saturates a porous sample, by capillary forces alone. Suppose a sample is completely saturated with a non-wetting fluid, and same wetting fluid is introduced on its surface. There will be spontaneous flow of wetting fluid into the medium, causing displacement of the non-wetting fluid. This is called imbibition phenomenon. The rate of imbibition is greater if the wettability of the porous medium, by the imbibed fluid, is higher.

The mathematical condition for imbibition phenomenon is given by Scheidegger [78]); viz,

$$v_n = -v_i$$

Where v_i & v_n are the seepage velocities of injected & native liquids respectively.
The relation between relative permeability and phase-saturation,

$$k_i = S_i^3$$
$$k_n = 1 - \alpha S_n, \alpha = 1.11$$

Where k_i & k_n denotes fictitious relative permeability. S_i & S_n denotes saturations of injected and native liquids respectively.

3. Mathematical structure of the model

According to the Darcy's law, the basic equations of the phenomenon as; [78]

$$v_i = -\left(\frac{k_i}{\delta_i}\right) K \frac{\partial p_i}{\partial x} \tag{1}$$

$$v_n = -\left(\frac{k_n}{\delta_n}\right) K \frac{\partial p_n}{\partial x} \tag{2}$$

$$v_i = -v_n \tag{3}$$

$$p_c = p_n - p_i \tag{4}$$

$$\varphi\left(\frac{\partial S_i}{\partial t}\right) + \frac{\partial v_i}{\partial x} = 0 \tag{5}$$

$$\varphi\left(\frac{\partial S_n}{\partial t}\right) + \frac{\partial v_n}{\partial x} = 0 \tag{6}$$

Where v_i and v_n are the seepage velocities, k_i and kare the relative permeabilities δ_i and δ_n are the kinematic viscosities (which are constants), p_i and p_n are pressure of the injected and native liquid respectively, φ and K are the porosity and the permeability of the homogeneous porous medium; S_i is the saturation of the injected liquid; p_c is the capillary pressure and t is the time. The co-ordinate x is measured along the axis of the cylindrical medium, the origin being located at the imbibition face x=0.

Combing equations (1)-(5) and using the relation for capillary pressure as, $p_c=\beta S_i$ [70], we get,

$$\varphi\left(\frac{\partial S_i}{\partial t}\right) + \frac{\partial}{\partial x}\left[KD(S_i)\beta\left(\frac{\partial S_i}{\partial x}\right)\right] = 0 \tag{7}$$

Where $D(S_i) = \frac{k_i k_n}{\delta_n k_i + \delta_i k_n}$ and β being small capillary pressure coefficient.
It is assumed is that an average value of $D(S_i) = \overline{D}\ (S_i)$
Using the transformation,

$$\xi = \frac{x}{L} \quad, \theta = \frac{Lt}{\varphi L^2}, \quad 0 \le x \le \frac{LS_{io}}{B}. \tag{8}$$

Eq. (7), becomes;

$$\left(\frac{\partial S_i}{\partial \theta}\right) + \beta \overline{D}(S_i)\frac{\partial^2 S_i}{\partial \xi^2} = 0$$

$\frac{\partial S_i}{\partial \theta} = -\beta \overline{D}(S_i)\frac{\partial {S_i}^2}{\partial \xi^2}$

$$\frac{\partial S_i}{\partial \theta} = \varepsilon \frac{\partial {S_i}^2}{\partial \xi^2} \text{ Where } \varepsilon = -\beta \overline{D}(S_i) \tag{9}$$

By the Hopf-Cole transformation [79, 80] equation (9) reduces to the Burger's equation.

$$S^*_{i_\theta} + S^*_i S^*_{i_\xi} = \varepsilon S^*_{i\ \xi\xi} \tag{10}$$

With the condition
$S^*_i(\xi, 0) = S^*_{i_0} e^{\xi}$ at time $\theta = 0$ and $\xi > 0$

3.1 Solution of the Burger's equation by variational iteration method

To add the basic concepts of VIM, considering the below mentioned nonlinear partial differential equations:

$$\begin{aligned} &Lu(x,t) + Ru(x,t) + Nu(,t) = g(x,t), \\ &u(x,0) = e^x \end{aligned} \tag{11}$$

Where $L = \left(\frac{\partial}{\partial t}\right), R$ is a linear operator which has partial derivatives with respect to x, Nu(x,t) is a nonlinear term and g(x,t) is an inhomogeneous term.
As per the VIM [6, 7];

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda\{LU_n + \overline{RU_n} + \overline{NU_n} - g\}d\tau \tag{12}$$

Where λ is called a general Lagrange multiplier [81, 82] which can be identified optimally via vatiational theory, $\overline{RU_n}$ and $\overline{NU_n}$ are considered as restricted variations,
i.e. $\delta\overline{RU_n} = 0, \delta\overline{NU_n} = 0$ calculating variation with respect to U_n;

$$\begin{aligned} &\lambda'(\tau) = 0 \\ &1 + \lambda(\tau)_{\tau=t} = 0 \end{aligned} \tag{13}$$

The Lagrange multiplier, therefore, can be considered as λ=-1.
Now, substituting the multiplier in (12), then

$$U_{n+1}(x,t) = U_n - \int_0^t \{L(U_n) + R(U_n) + N(U_n) - g\}d\tau \tag{14}$$

$$S^*_{i_\theta} + S^*_i S^*_{i_\xi} = \varepsilon S^*_{i\ \xi\xi} \tag{15}$$

With the constrain
$S^*_i(\xi, 0) = S^*_{i_0} e^{\xi}$ at time $\theta = 0$ and $\xi > 0$

To solve equation (10) by VIM, substituting in equation (14) by

$$RU_n = -U_{n_x}^2$$
$$NU_n = U_n(U_n)_{xx}$$

& g(x,t) = 0
And can obtain the following variational iteration formula:

$$S_{i\ n+1}^* = S_{i_n}^* - \int_0^\theta \left\{ S_{i_{n_\tau}}^* + S_{i_n}^* \left(S_{i_n}^* \right)_\xi - \varepsilon S_{i_{n\xi\xi}}^* \right\} d\tau \tag{16}$$

Using (14), the approximate solutions $U_n(x,t)$are obtained by substituting;

$$S_i^*(\xi, 0) = S_{i_0}^* e^\xi \tag{17}$$

Approximate solutions are given below;

$$S_{i_1}^* = S_{i_0}^* e^\xi - \beta_1^0 \theta; \quad where\ \beta_1^0 = \left(S_{i_0}^{*\,2} e^{2\xi} - \varepsilon S_{i_0}^* e^\xi \right)$$

$$S_{i_2}^* = S_{i_0}^* e^\xi - \beta_1^0 \theta + \beta_1^1 \frac{\theta^2}{2} \quad where\ \beta_1^1 = \beta_1^0 S_{i_0}^* e^\xi$$

Similarly,

$$S_{i_3}^* = S_{i_0}^* e^\xi - \beta_2^1 \frac{\theta^2}{2!} + \beta_2^2 \frac{\theta^3}{3!}$$

And so on … ..
Notes on VIM
From the analysis we can observed is this:

1. VIM can contain a series solution not exactly like ADM.

2. VIM needs many modifications to overcome the wasted time in the repeated calculations and unneeded terms.

To overcome these problems, following ADM and LADM is suggested.
Now applying ADM to equation (10); we get

$$S_i^*(\xi, \theta) = L_\theta^{-1} \left[S_i^* S_{i_\xi}^* - \varepsilon S_{i_{\xi\xi}}^* \right] \tag{18}$$

And recursive relation is:

$$S_i(\xi, 0) = e^\xi$$

Then:

$$S_{i_1}^*(\xi, \theta) = \beta_1^0 \theta$$

$$S_{i_2}^*(\xi, \theta) = \beta_{12}^0 \frac{\theta^3}{3} - \varepsilon \beta_2^1 \frac{\theta^2}{2}$$

$$S^*_{i_3}(\xi,\theta) = \beta^0_{123}\frac{\theta^4}{4} - \varepsilon\beta^1_3\frac{\theta^3}{3}$$

and so on …

Now, applying (LADM) Laplace transform with respect to t on both sides of (10);

$$S^*_i(x,t) = L^{-1}\left[\frac{1}{s}L\left[S^*_{i_0}S^*_{i_{0_\xi}} - \varepsilon S^*_{i_{0_{\xi\xi}}}\right]\right]$$

$$S^*_{i_1} = \beta^0_1 e^{\xi}\theta$$

$$S^*_{i_2} = \left(\beta^{0^2}_1 e^{2\xi} - \varepsilon\beta^0_1 e^{\xi}\right)\frac{\theta^2}{2!}$$

$$S^*_{i_3} = \left(\beta^3_0 - \varepsilon\beta^3_1\right)\frac{\theta^3}{3!}$$

And so on …

4. Interpretation

It is concluded that for the non linear partial differential equation of imbibitions phenomenon in oil recovery process, through graphs, it has been observed that the

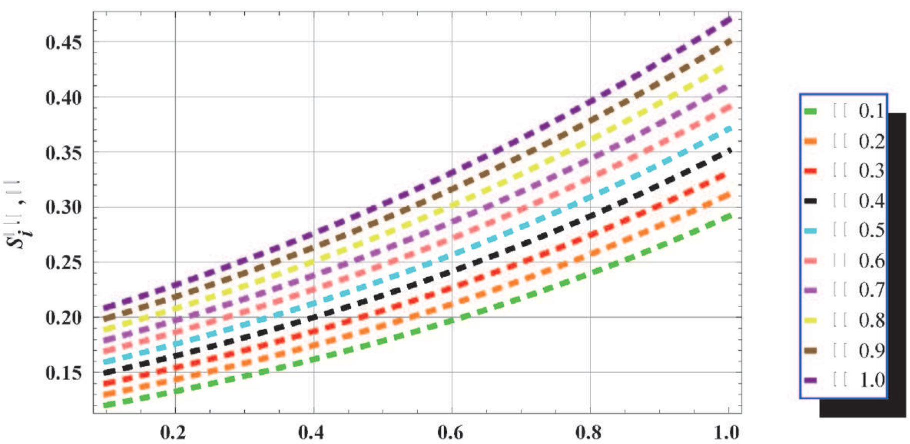

Figure 1.
Plot of Saturation $S^*_i(\xi,\theta)$ versus ξ for VIM Solution.

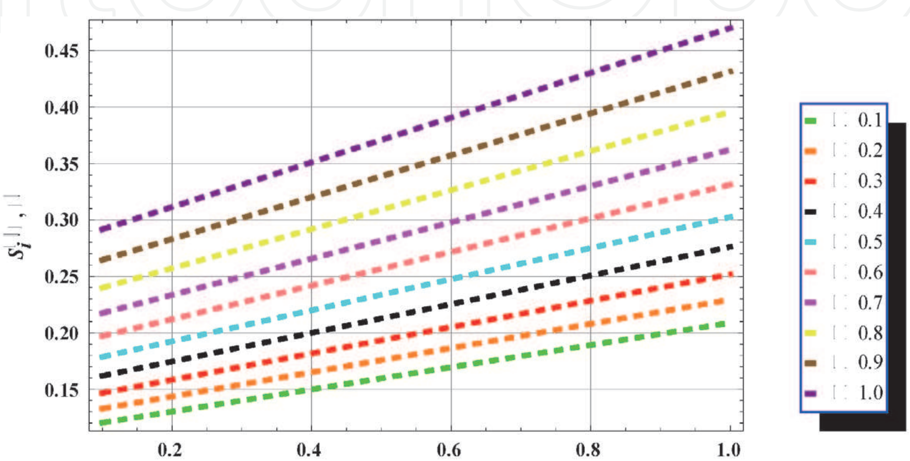

Figure 2.
Plot of Saturation $S^*_i(\xi,\theta)$ versus θ for VIM Solution.

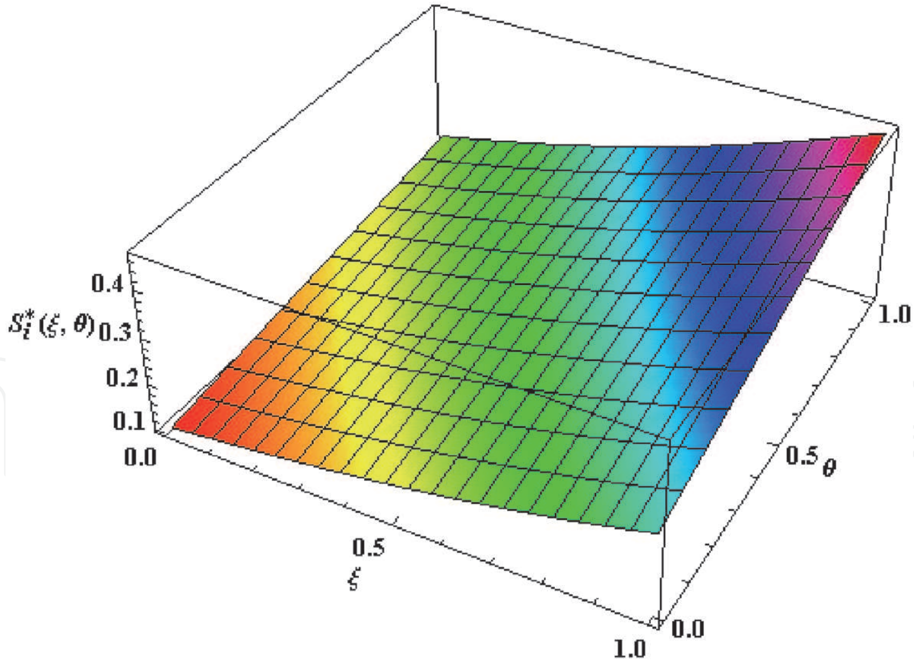

Figure 3.
3-Dimensional VIM Solution.

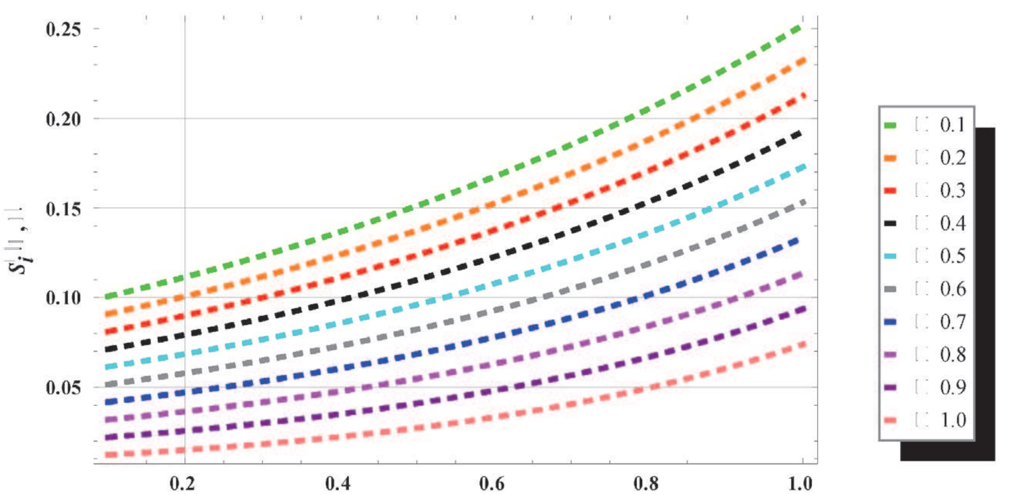

Figure 4.
Plot of Saturation $S_i^*(\xi,\theta)$ versus ξ for ADM Solution.

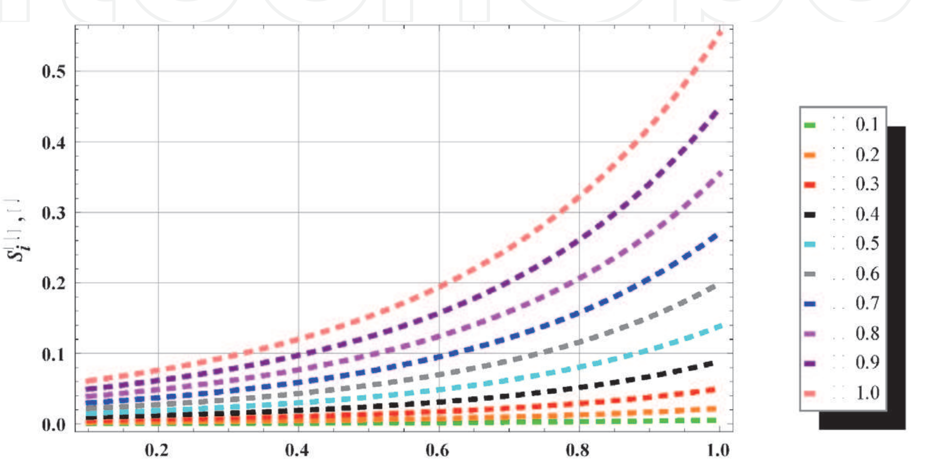

Figure 5.
Plot of Saturation $S_i^*(\xi,\theta)$ versus ξ for LADM Solution.

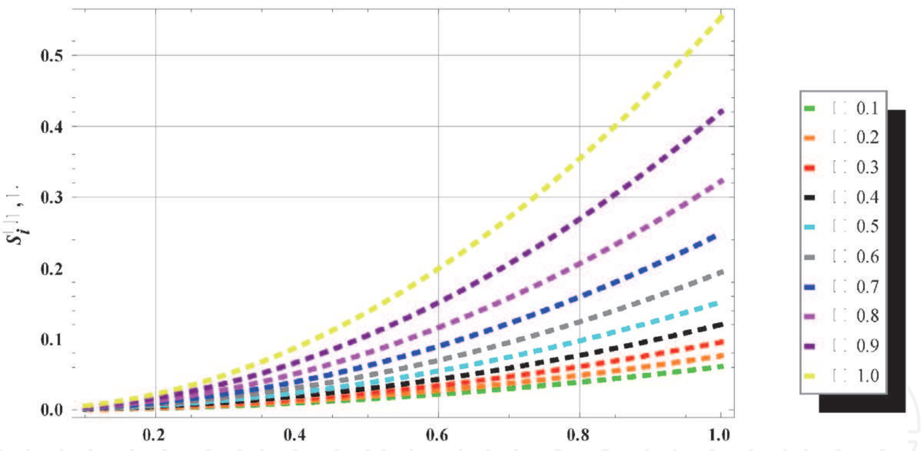

Figure 6.
Plot of Saturation $S_i^*(\xi, \theta)$ versus θ for LADM Solution.

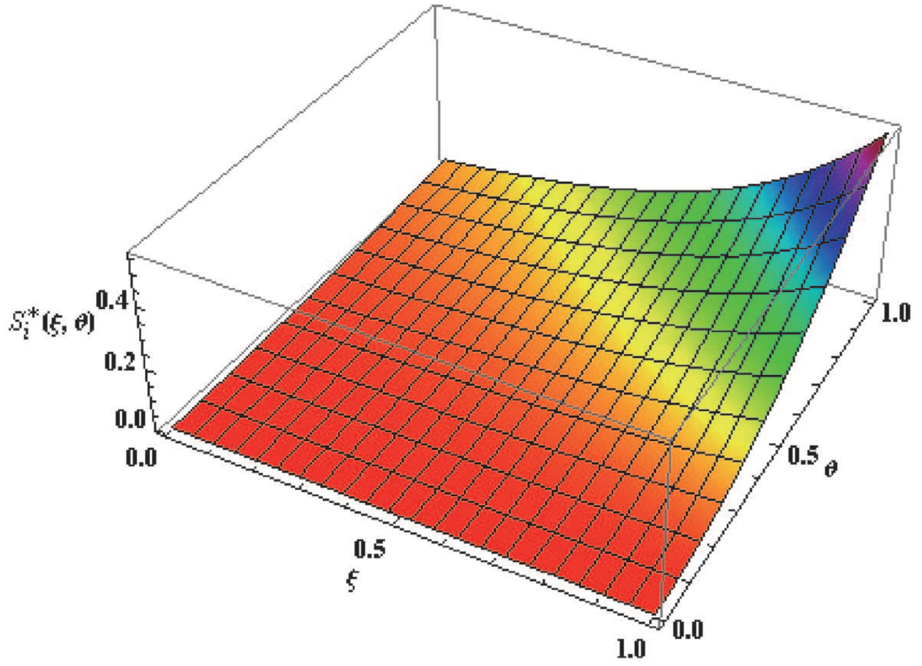

Figure 7.
3-Dimensional LADM Solution

saturation of injected water during imbibition, increases and it is noted that LADM gives faster accuracy compare to VIM and ADM (**Figures 1**–7).

5. Conclusions

The VIM, the ADM and the LADM are successfully applied to Burger's equation. The results which are obtained by ADM are a powerful mathematical tool to solve nonlinear partial differential equation. It has been noted that this method is reliable and requires fewer computations; and scheme LADM gives better and very faster accuracy in comparison with VIM.

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References

[1] He, J.H. (1998). Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg. 167, 69–73.

[2] Al-Hayani, W.(2011). Adomian decomposition method with Green's function for sixth- Orderboundary value problems, Comp. Math. Appl. 61, 1567–1575.

[3] Al-Hayani, W.and. Casasús, L (2005). Approximate analytical solution of fourth order Boundaryvalue problems, Numer. Algorithms 40, 67–78

[4] Abdou, M.A. and Soliman, A.A. (2005). New applications of variational iteration method, Physica D 211 (1-2), 1-8.

[5] Abdou, M.A. and Soliman, A.A. (2005). Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (20), 245-251.

[6] Momani, S.,Abuasad, S. (2005). Application of He's varitional iteration method to Helmholtz equation, Chaos Solitons & Fractals 27, 1119-1123.

[7] Odibat, Z.M. , Momani,S.(2006). Application of variational iteration method to nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 7 (1), 27- 34.

[8] Soliman, A. A. (2006). Numerical simulation and explicit solutions of KdV–Burgers' and Lax's seventh-order KdV equations, Chaos Solitons & Fractals; 29 (2), 294-302.

[9] Scott, M. R. (1973). Invariant Imbedding and its Applications to Ordinary Differential Equations, Addison-vesley.

[10] Sweilam, N.Hand Khader, M.M. (2007). Variational iteration method for one dimensional nonlinear thermoelasticity, Chaos Solitons & Fractals, 32 (1),145-149.

[11] Adomian,G.91989). Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer Academic, Dordrecht.

[12] Adomian, G. (1994). Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic, Dordrecht.

[13] Serrano, S.E. (2010). Hydrology for Engineers, Geologists, and Environmental rofessionals: An Integrated Treatment of Surface, Subsurface, and Contaminant Hydrology, Second Revised Edition, HydroScience, Ambler, PA.

[14] Adomian, G. (1983) R. Rach, Inversion of nonlinear stochastic operators, J. Math. Anal. Appl.91, 39–46.

[15] Adomian,G. (1983). Stochastic Systems, Academic, New York.

[16] Adomian, G.(1986). Nonlinear Stochastic Operator Equations, Academic, Orlando, FL.

[17] Adomian, G, Rach, R. (1993). A new algorithm for matching boundary conditions in decomposition solutions, Appl. Math. Comput. 58, 61–68.

[18] Bellman, R. E. and Adomian, G. (1985). Partial Differential Equations: New Methods for their Treatment and Solution, D.Reidel, Dordrecht.

[19] Bellomo, N.and Riganti, R. (1987). Nonlinear Stochastic System Analysis in Physics and Mechanics, World Scientific, Singapore and River Edge, NJ,

[20] Cherruault, Y. (1998). Modèles et méthodes mathématiques pour les

sciences du vivant,Presses Universitaires de France, Paris.

[21] Wazwaz, A. M. (1997). A First Course in Integral Equations, World Scientific, Singapore and River Edge, NJ.

[22] Wazwaz, A. M. (2002). Partial Differential Equations: Methods and Applications, A. A. Balkema,Lisse, The Netherlands.

[23] Wazwaz, A. M. (2009). Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Beijing, and Springer-Verlag, Berlin.

[24] Wazwaz, A.M. (2011). Linear and Nonlinear Integral Equations: Methods and Applications,Higher EducationPress, Beijing, and Springer-Verlag, Berlin.

[25] Wazwaz, A.M. (2000). Approximate solutions to boundary value problems of higher order by themodified decomposition method, Comput. Math. Appl. 40, 679–691.

[26] Wazwaz, A.M. (2000). The modified Adomian decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order, Int. J. Nonlinear Sci. Numer. Simul. 1, 17–24.

[27] Wazwaz, A.M. (2000). A note on using Adomian decomposition method for solving boundary value problems, Found. Phys. Lett. 13, 493–498.

[28] Wazwaz, A.M. (2000). The numerical solution of special eighth-order boundary value problems by the modified decomposition method, Neural Parallel Sci. Comput. 8,133–146.

[29] Adomian, G.(1994). Solving Frontier Problems of Physics: Decomposition method.Kluwer, Boston, MA.

[30] Al-Sawalha, M.M., Noorani, M.S.M. and. Hashim, I. (2008). Numerical experiments on the hyperchaotic Chen system by the Adomian decomposition method, Int. J. Comput. Methods 5, 403–412.

[31] Bigi, D. and Riganti, R. (1986). Solution of nonlinear boundary value problems by the decomposition method, Appl. Math. Modelling 10, 49–52.

[32] Casasús L.and Al-Hayani, W. (2000). The method of Adomian for a nonlinear boundary value problem, Rev. Acad. Canar. Cienc. 12, 97–105.

[33] Duan, J.S. and Rach,R. (2011). New higher-order numerical one-step methods based on the Adomian and the modified decomposition methods, Appl. Math. Comput. 218, 2810–2828.

[34] Ebadi G. and Rashedi, S. (2010). The extended Adomian decomposition method for fourth order boundary value problems,Acta Univ. Apulensis 22, 65–78.

[35] He,J.H. (1997). A new approach to nonlinear partial differential equations, Comm. Nonlinear Sci. Numer. Simul. 2 (1997) 230–235.

[36] Hussain, M. and Khan, M. (2010). Modied Laplace decomposition method, Appl. Math. Sci, 36, 1769-1783.

[37] Jun-Sheng Duan a, Randolph Rach , Dumitru B˘aleanu , Abdul-Majid Wazwaz. (2012). A review of the Adomian decomposition method and its applications to fractional differential equations. Commun. Frac. Calc. 3 (2), 73 – 99.

[38] Jasem Fadaei. (2011). Application of Laplace − Adomian Decomposition Method on Linear and Nonlinear System of PDEs. Applied Mathematical Sciences, Vol. 5, 27, 1307 –1315.

[39] Jang, B. (2008). Two-point boundary value problems by the extended Adomian decomposition

method, J. Comput. Appl.Math. 219, 253–262.

[40] Khan, Y. (2009). An effective modification of the Laplace decomposition method for Nonlinear equations, International Journal of Nonlinear Sciences and Numerical Simulation, 10, 1373-1376.

[41] Lesnic, D. (2008). The decomposition method for nonlinear, second-order parabolic partial differential equations, Int. J. Comput. Math. Numeric. Simulat. 1, 207–233.

[42] Rach R.and. Baghdasarian, A . (1990). On approximate solution of a nonlinear differential equation, Appl. Math. Lett. 3, 101–102.

[43] Tatari, M.and Dehghan, M.(2006). The use of the Adomian decomposition method for solving multipoint boundary value problems, Physica Scripta 73, 672–676.

[44] Tsai, P.Y. and Chen, C.K. (2010). An approximate analytic solution of the nonlinear Riccati differential equation, J. Frank. Inst. 347, 1850–1862.

[45] Wazwaz A.M. and Rach, R. (2011). Comparison of the Adomian decomposition methodand the variational iteration method for solving the Lane-Emden equations of the first and second kinds, Kybernetes 40, 1305–1318.

[46] Wazwaz, A.M. (2012). A reliable study for extensions of the Bratu problem with boundary conditions, Math. Methods Appl. Sci. 35 (2012) 845–856.

[47] Khuri, S. A. (2001). A Laplace decomposition algorithm applied to a class ofNonlineardifferential equations," J. Math. Annl. Appl., 4, 141-155.

[48] Khuri, S. A. (2004). A new approach to Bratus problem," Appl. Math. Comp., 147 (2004) 31-136.

[49] Adomian, G.(1988). A review of the decomposition method in applied mathematics," Journal of mathematical analysis and applications, vol. 135, 501-544.

[50] Duan, J.S. and Rach, R. (2011). A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, Appl.Math. Comput. 218, 4090–4118.

[51] Elgazery, S. Nasser. (2008). Numerical solution for the Falkner-Skan equation, Chaos, Solitons and Fractals, 35, 738 - 746.

[52] Fadaei, J. and Moghadam, M.M. (2012). Numerical Solution of Systems of Integral Differential equations by Using Modied Laplace Adomian Decomposition Method, World Applied Sciences Journal 19, 1818-1822.

[53] Ongun, M.Y. (2011). The Laplace Adomian Decomposition Method for solving a model for HIV infection of CD4+T cells, Math. Comp.Modell, 53, 597-603.

[54] Saei, F., Dastmalchi, F.,Misagh, D., Zahiri, Y.,Mahmoudi, M., Salehian, V. and Rafati Maleki, N. (2013). New Application of Laplace Decomposition Algorithm For Quadrtic Riccati Differential Equation by Using Adomian's Polynomials, Life Science Journal,10, 3s.

[55] Yusufoglu, E. (2006). Numerical solution of Dung equation by the Laplace decomposition algorithm, Appl. Math. Comput, 177, 572-580.

[56] Yusufoglu, Elcin,(Agadjanov), (2006). Numerical solution of Duffing equation by the Laplace decomposition algorithm, Appl. Math. Comput. 177, 572 - 580.

[57] Abbasbandy, S.(2006). Iterated He's homotopy Perturbation method for

quadratic Riccati differential equation, Applied Mathematics and Computation 175, 581-589

[58] Adomian G and Rach, R. (1993). Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition, J. Math. Anal. Appl. 174,118–137.

[59] Al-Mazmumy, M.and Al-Malki,H. (2015). The modified Adomian Decomposition Method for solving Nonlinear Coupled Burger's Euations, Nonlinear Analysis and Differential Equations, Vol. 3, No.3, 111-122.

[60] Dehghan, M.and Tatari, M. (2010). Finding approximate solutions for a class of third-order non-linear boundary value problems via the decomposition method of Adomian, Int. J. Comput. Math. 87,1256–1263.

[61] Ebaid, A.E. (2010). Exact solutions for a class of nonlinear singular two-point boundary value problems: The decomposition method, Z. Naturforsch. 65a (2010) 1–6.

[62] Ebaid, A.E. (2011). A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method, J. Comput. Appl. Math. 235, 1914–1924.

[63] Geny, F.Z., Lin, I.Z., Cui, M.G. (2009). A piecewise variational iteration method for Riccati differential equations, Computers and mathematics with Applications 58, 2518-2522.

[64] Hashim, I., Adomian, G. (2006). Decomposition method for solving BVPs for fourth- order integro-differential equations, J. Comput. Appl. Math. 193, 658–664.

[65] He, J.H. (2004). Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos Solitons Fractals 19, 847–851

[66] He, J.H. (2000). Variational iteration Method for autonomons ordinary differential systm, Applied Mathematics and Computation 114, 115-123.

[67] He, J.H.(1999). Variational iteration Method – a kind of non-linear analytical technique; some examples, International journal of Non-linear Mechanics 34, 699-708.

[68] He, J.H. (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167, 57–68.

[69] Onur Kıymaz, Ayşegül Çetinkaya. (2010). Variational Iteration Method for a Class of Nonlinear Differential Equations. Int. J. Contemp. Math. Sciences, Vol. 5, 37:1819 – 1826

[70] Oroveanu,T.(1963). Scurgerea fluiidelor prin medii poroase neomogene, Editura Academiei Republicii Populare Romine,92, 328.

[71] Rach, R and Adomian, G. (1990). Multiple decompositions for computational convenience, Appl.Math. Lett. 3, 97–99.

[72] Sambath, M. and Balachandran, K. (2016). Laplace Adomian decomposition method for solving a fish farm model. Nonauton. Dyn. Syst. 3,104–111.

[73] Serrano, S.E. (2011). Engineering Uncertainty and Risk Analysis: A Balanced Approach to Probability, Statistics, Stochastic Modeling, and Stochastic Differential Equations, second Revised Edition, HydroScience, Ambler, PA.

[74] Taiwo, O. and Odetunde, O. (2010). On the numerical approximation of delay differential equations by a decomposition method," Asian Journal

of Mathematics and Statistics,vol. 3, pp. 237-243.

[75] Verma, A.P. (1970): Perturbation solution in imbibition in a cracked porous medium of small inclination, IASH, 13, 1, 45-51.

[76] Elgazery, S.N. (2008). Numerical solution for the Falkner-Skan equation, Chaos, Solitons and Fractals, 35,738-746.

[77] Graham, R.A. and Richardson, J.G. (1959): Theory and applications of imbibition phenomena in recovery of oil, Trans, Aime, 216.

[78] Scheidegger, A.E. (1960). The Physics of Flow through Porous Media, University of Toronto press.

[79] Cole, J. D. (1951): On a quasilinear parabolic equation occurring in aerodynamics. Q. Appl. Math. g,225-236.

[80] Hopf, E. (1950): The partial differential equation ut + uux = ε uxx comm. Pure Appl. Math. 3, pp – 201-230.

[81] Tamer A. Abassy, Magdy A. El-Tawil, H. Fl. Zoheiry, Towards a modified Variational iteration method, Journal of Computational and Applied mathematics 207 (2007) 137-147.

[82] Tamer A. Abassy, Improved Adomian decomposition method; Computers and Mathematics with Applications 59 (2010) 42-54.