Chapter

Modified Expression to Evaluate the Correlation Coefficient of Dual Hesitant Fuzzy Sets and Its Application to Multi-Attribute Decision Making

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Abstract

The main objective of this paper is to understand all the existing correlation coefficients (CoCfs) to determine the relation and dependency between two variables of the fuzzy sets and its extensions for solving decision-making (DM) problems. To study the weighted CoCfs between two variables the environment chosen here is dual hesitant fuzzy set (DHFS) which is a generalization of a fuzzy set which considers the hesitant value of both the membership and non-membership elements of a set. Although there exists CoCfs for DHFS but a detailed mathematical analysis suggests that there exists some shortcomings in the existing CoCfs for DHFS. Thus, an attempt has been made to properly understand the root cause of the posed limitation in the weighted CoCfs for DHFS and hence, modified weighted CoCfs for DHFS has been proposed for solving DHFS multi-attribute decision making (MADM) problems i.e., DM problems in which rating value of each alternative over each criterion is represented by a DHFS in the real-life. Also, to validate the proposed expressions of weighted CoCfs for solving DHFS MADM problems, an existing real-life problem is evaluated and a systematic comparison of the solution is presented for clarification.

Keywords: decision-making, dual hesitant fuzzy set, correlation coefficient, multi-attribute decision-making

1. Introduction

Decision-making is a process which has a wide range of real-life applications which requires a great precision for desirable outcomes. Real-world applications like supply chain management, marketing management, healthcare, telecommunication, finance, energy, banking, forestry, pattern recognition, investment, personnel selection etc., has a set of data which includes information with both certainties and uncertainties. The study of uncertainties can be handled well by fuzzy sets [1] and its extensions [1–7], thus measures of decision-making helps in removing and controlling the existing constraints or uncertainties, it increases productivity, helps in better coordination etc.

To rank fuzzy sets and its generalizations there exists various ranking measures like distance measures, similarity measures, score function, accuracy function, certainty function, divergence measure, CoCfs etc. Although in literature there exist expressions to evaluate the CoCf between fuzzy sets and many of its extensions as proposed by several researchers like, the CoCf between two fuzzy sets [8], the CoCf between two intuitionistic fuzzy sets [9–17], the CoCf between two interval-valued intuitionistic fuzzy sets [18], the CoCf between two Pythagorean fuzzy sets [19, 20], the CoCf between two intuitionistic multiplicative sets [21], the CoCf between two hesitant fuzzy sets [22–27], the CoCf between two dual hesitant fuzzy sets [28–30] etc.. Ye [29] proposed an expressions for evaluating the weighted CoCfs between two DHFSs and solved a real-life problem (finding the best investment company) where the uncertainty is represented as a DHFS. However, after a deep study, it is observed that some mathematical incorrect assumptions are considered in the existing weighted CoCf and hence it is scientifically incorrect to apply existing weighted CoCf in real-life MADM problems for DHFSs in its present form. This limitation is a real motivation to modify the CoCfs for DHFSs which would be applicable for the evaluation of the real-life problems. Considering the existing weighted CoCf [29] for solving DHFSs MADM problems as a base, a modified weighted CoCf for DHFSs is proposed and using the modified expressions, the exact results of the real-life problem, considered in the existing paper [29] have been obtained.

The paper is organized as follows. Section 2. Preliminaries. Section 3. A brief review of the existing CoCf of DHFSs is presented here. Section 3.1. Gaps in the existing weighted CoCf for DHFSs. Section 3.2. Mathematical incorrect assumptions. Section 4. It proposes the modified CoCf for DHFSs. Section 5. Origin of the proposed CoCf for the DHFSs is discussed here. Section 6. It presents the exact solution to the existing real-life problem. Section 7. Advantages of modified CoCf for DHFSs. Section 8. Discussion and Concludes the presented paper.

2. Preliminaries

This section states some requisites concerned with the DHFSs and the correlation coefficients while applying in the real-life application during DM process.

Definition 2.1 [31] A set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle | x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$, defined on the universal set XX, is said to be an fuzzy set (FS), where $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element x xin $\tilde{A}$.

Definition 2.2 [1] A set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle | x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1, 0 \le \nu_{\tilde{A}}(x) \le 1, \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \le 1\}$, defined on the universal set X, is said to be an intuitionistic fuzzy set (IFS), where, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represents the degree of membership and degree of non-membership respectively of the element x in $\tilde{A}$. The pair $\langle \mu_{\tilde{A}}, \nu_{\tilde{A}}\rangle$ is called an intuitionistic fuzzy number (IFN) or an intuitionistic fuzzy (IFV), where, $\mu_{\tilde{A}} \in [0,1], \nu_{\tilde{A}} \in [0,1], \mu_{\tilde{A}} + \nu_{\tilde{A}} \le 1$.

Definition 2.3 [29] Let X be an initial universe of objects. A set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ is called a hesitant fuzzy set (HFS), where $\mu_{\tilde{A}}(x^{(s)})$ is a mapping defined by $\mu_{\tilde{A}}(x^{(s)}) : X \to [0,1]$ here, $\mu_{\tilde{A}}(x^{(s)})$ is a set of some different values in [0,1] and $'s'$ represent the number of possible membership degrees of the element $x \in X$ to $\tilde{A}$. For convenience, we call $\mu_{\tilde{A}}(x^{(s)})$ as a hesitant fuzzy element (HFE).

Definition 2.4 [29] Let X be an initial universe of objects. A set $\tilde{A}$ on X then for a given HFE $\mu_{\tilde{A}}(x^{(s)})$, its lower and upper bounds are defined as $\mu_{\tilde{A}}^{-}(x^{(s)}) = \min \mu_{\tilde{A}}(x^{(s)})$ and $\mu_{\tilde{A}}^{+}(x^{(s)}) = \max \mu_{\tilde{A}}(x^{(s)})$, respectively, where $'s'$ represent the number of possible membership degrees of the element $x \in X$ to $\tilde{A}$.

Definition 2.5 [29] Let X be an initial universe of objects. A set $\tilde{A}$ on X then for a given HFE $\mu_{\tilde{A}}(x^{(s)})$, $A_{env}(\mu_{\tilde{A}}(x^{(s)}))$ is called the envelope of $\mu_{\tilde{A}}(x^{(s)})$ which is denoted as $\left(\mu_{\tilde{A}}^{-}, 1-\mu_{\tilde{A}}^{+}\right)$, with the lower bound $\mu_{\tilde{A}}^{-}$ and upper bound $\mu_{\tilde{A}}^{+}$. Also, $A_{env}(\mu_{\tilde{A}}(x^{(s)}))$ establishes the relation between HFS and IFS i.e., $A_{env}(\mu_{\tilde{A}}(x^{(s)})) = \{\langle x, \mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(s)})\rangle\}$, where $\mu_{\tilde{A}}(x^{(s)}) = \mu_{\tilde{A}}^{-}$ and $\nu_{\tilde{A}}(x^{(s)}) = 1-\mu_{\tilde{A}}^{+}$.

Definition 2.6 [29] For a HFE $\mu_{\tilde{A}}$, $s(\mu_{\tilde{A}}) = \left(\frac{1}{l(\mu_{\tilde{A}})}\right)\sum_{\gamma \in \mu_{\tilde{A}}} \gamma$ is called the score function of $\mu_{\tilde{A}}$, where $l(\mu_{\tilde{A}})$ is the number of the values in $\mu_{\tilde{A}}$. For any two HFEs $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$, the comparison between two HFEs is done as follows:

i. If $s\left(\mu_{\tilde{A}_1}\right) > s\left(\mu_{\tilde{A}_2}\right)$, then $\mu_{\tilde{A}_1} > \mu_{\tilde{A}_2}$.

ii. If $s\left(\mu_{\tilde{A}_1}\right) = s\left(\mu_{\tilde{A}_2}\right)$, then $\mu_{\tilde{A}_1} = \mu_{\tilde{A}_2}$.

Let $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$ be two HFEs such that $l\left(\mu_{\tilde{A}_1}\right) \neq l\left(\mu_{\tilde{A}_2}\right)$. For convenience, let $l = \max\left\{l\left(\mu_{\tilde{A}_1}\right), l\left(\mu_{\tilde{A}_2}\right)\right\}$, then while comparing them, the shorter one is extended by adding the same value till both are of same length. The selection of the value to be added is dependent on the decision makers risk preferences. For example (adopted from 28), let $\mu_{\tilde{A}_1} = \{0.1, 0.2, 0.3\}$, $\mu_{\tilde{A}_2} = \{0.4, 0.5\}$ and $l\left(\mu_{\tilde{A}_1}\right) > l\left(\mu_{\tilde{A}_2}\right)$, then for the correct arithmetic operations $\mu_{\tilde{A}_2}$ must be extended to $\mu_{\tilde{A}_2}{}'$, i.e. either $\mu_{\tilde{A}_2}{}' = \{0.4, 0.5, 0.5\}$ as an optimist or $\mu_{\tilde{A}_2}{}' = \{0.4, 0.4, 0.5\}$ as a pessimist depending on the risk taking factor of the decision-maker though their results would vary definitely.

Definition 2.7 [29] A set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})\rangle | x \in X\}$ is called a DHFS, where, $\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})$ is a mapping defined by $\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)}) : X \to [0,1]$, here $\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})$ is a set of some different values in [0,1], $'s'$ represent the number of possible membership degrees and $'t'$ represent the number of possible non membership degrees of the element $x \in X$ to $\tilde{A}$. For convenience, we call $d = \{\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})\}$ as a dual hesitant fuzzy element (DHFE).

Definition 2.8 [29] Let $d_1 = \left\{\mu_{\tilde{A}_1}, \nu_{\tilde{A}_1}\right\}$ and $d_2 = \left\{\mu_{\tilde{A}_2}, \nu_{\tilde{A}_2}\right\}$ be any two DHFEs, then the score function for DHFSs $d_i (i = 1, 2)$ is defined as $s(d_i) = \left(\frac{1}{l\left(\mu_{\tilde{A}_i}\right)}\right)\sum_{\gamma \in \mu_{\tilde{A}_i}} \gamma - \left(\frac{1}{m\left(\nu_{\tilde{A}_i}\right)}\right)\sum_{\eta \in \nu_{\tilde{A}_i}} \eta \ (i = 1, 2)$ and the accuracy function for DHFSs $d_i (i = 1, 2)$ is defined as $p(d_i) = \left(\frac{1}{l\left(\mu_{\tilde{A}_i}\right)}\right)\sum_{\gamma \in \mu_{\tilde{A}_i}} \gamma + \left(\frac{1}{m\left(\nu_{\tilde{A}_i}\right)}\right)\sum_{\eta \in \nu_{\tilde{A}_i}} \eta \ (i = 1, 2)$ where $l\left(\mu_{\tilde{A}_i}\right)$ and $m\left(\nu_{\tilde{A}_i}\right)$ are the number of the values in $\mu_{\tilde{A}_i}$ and $\nu_{\tilde{A}_i}$ respectively. For any two DHFEs d_1 and d_2, the comparison between two DHFEs is done as follows:

i. If $s(d_1) > s(d_2)$, then $d_1 > d_2$.

ii. If $s(d_1) = s(d_2)$, then check the accuracy function of DHFSs

1. If $s(d_1) > s(d_2)$, then $d_1 > d_2$.

2. If $s(d_1) = s(d_2)$, then $d_1 = d_2$.

Definition 2.9 [29] Correlation coefficient of HFSs.

The values in HFEs are generally not in order, so they are arranged in descending order i.e., for HFE $\mu_{\tilde{A}}$, let $\sigma : (1, 2, \ldots, n) \to (1, 2, \ldots, n)$ be such that $\mu_{\tilde{A}\sigma(j)} \geq \mu_{\tilde{A}\sigma(j+1)}$ for $j = 1, 2, \ldots, n-1$ and $\mu_{\tilde{A}\sigma(j)}$ be the jth largest value in $\mu_{\tilde{A}}$.

Definition 2.9.1 Let $X = \{x_1, x_2, \ldots, x_n\}$ be an initial universe of objects and a set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ be a HFS, then the information energy of $\tilde{A}$ is defined as $E_{HFS}(\tilde{A}) = \sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu^2_{\tilde{A}\sigma(j)}(x_i)\right)$, where $l_i = l(\mu_{\tilde{A}}(x_i))$ denotes the total number of membership values in $\mu_{\tilde{A}}(x_i)$, $x_i \in X$.

Definition 2.9.2 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universal set and a set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x^{(s)})\rangle | x \in X\}$ be any two HFSs on X, then the correlation between $\tilde{A}$ and $\tilde{B}$ is defined by $C_{HFS}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu_{\tilde{A}\sigma(j)}(x_i)\ \mu_{\tilde{B}\sigma(j)}(x_i)\right)$ where $l_i = \max\{l(\mu_{\tilde{A}}(x_i)), l(\mu_{\tilde{B}}(x_i))\}$ for each $x_i \in X$. Also, when $l(\mu_{\tilde{A}}(x_i)) \neq l(\mu_{\tilde{B}}(x_i))$, then they can be made equal by adding number of membership values in HFE which has least number of membership values in it. This can be done by adding the smallest membership values to make the lengths of both HFE $\tilde{A}$ and $\tilde{B}$ equal i.e. $l(\mu_{\tilde{A}}(x_i)) = l(\mu_{\tilde{B}}(x_i))$. For example, $\tilde{A} = \{\langle 0.3, 0.6, 0.8\rangle\}$ and $\tilde{B} = \{\langle 0.5, 0.4\rangle\}$, be any two HFSs and their lengths are not equal therefore it can be made equal as $\tilde{A} = \{\langle 0.3, 0.6, 0.8\rangle\}$ and $\tilde{B} = \{\langle 0.5, 0.4, 0.4\rangle\}$ respectively.

Definition 2.9.3 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universal set and a set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x^{(s)})\rangle | x \in X\}$ be any two HFSs on X, then the correlation coefficient between $\tilde{A}$ and $\tilde{B}$ is defined by

$$\rho_{HFS}(\tilde{A}, \tilde{B}) = \frac{C_{HFS}(\tilde{A}, \tilde{B})}{\sqrt{E_{HFS}(\tilde{A})}\sqrt{E_{HFS}(\tilde{B})}} = \frac{\sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu_{\tilde{A}\sigma(j)}(x_i)\ \mu_{\tilde{B}\sigma(j)}(x_i)\right)}{\sqrt{\sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu^2_{\tilde{A}\sigma(j)}(x_i)\right)}\sqrt{\sum_{i=1}^{n}\left(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu^2_{\tilde{B}\sigma(j)}(x_i)\right)}}.$$

Definition 2.10 [29] Correlation coefficient of DHFSs.

The values in DHFEs are generally not in order, so they are arranged in descending order i.e., for DHFE $d = \{\mu_{\tilde{A}}, \nu_{\tilde{A}}\}$, let $\sigma : (1, 2, \ldots, n) \to (1, 2, \ldots, n)$ be such that $\mu_{\tilde{A}\sigma(s)} \geq \mu_{\tilde{A}\sigma(s+1)}$ for $s = 1, 2, \ldots, n-1$, and $\mu_{\tilde{A}\sigma(s)}$ be the s^{th} largest value in $\mu_{\tilde{A}}$; let $\delta : (1, 2, \ldots, m) \to (1, 2, \ldots, m)$ be such that $\nu_{\tilde{A}\delta(t)} \geq \nu_{\tilde{A}\delta(t+1)}$ for $t = 1, 2, \ldots, m-1$, and $\nu_{\tilde{A}\delta(t)}$ be the t^{th} largest value in $\nu_{\tilde{A}}$.

Definition 2.10.1 Let $X = \{x_1, x_2, \ldots, x_n\}$ be an initial universe of objects and a set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)}, \nu_{\tilde{A}}(x^{(t)},))\rangle | x \in X\}$ be a DHFS, then the information energy of $\tilde{A}$ is defined as $E_{DHFS}(\tilde{A}) = \sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu^2_{\tilde{A}\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i}\nu^2_{\tilde{A}\sigma(t)}(x_i)\right)$, where $k_i = k(\mu_{\tilde{A}}(x_i))$ denotes the total number of membership values in $\mu_{\tilde{A}}(x_i)$ and $l_i = l(\nu_{\tilde{A}}(x_i))$ denotes the total number of non-membership values in $\nu_{\tilde{A}}(x_i)$ respectively.

Definition 2.10.2 Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set and a set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)}, \nu_{\tilde{A}}(x^{(t)},))\rangle | x \in X\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x^{(s)}, \nu_{\tilde{B}}(x^{(t)},))\rangle | x \in X\}$ be any two DHFSs on X, then the correlation between $\tilde{A}$ and $\tilde{B}$ is defined by $C_{DHFS}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{A}\sigma(s)}(x_i)\ \mu_{\tilde{B}\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{A}\delta(t)}(x_i)\ \nu_{\tilde{B}\delta(t)}(x_i)\right)$ where $k_i = \max\{k(\mu_{\tilde{A}}(x_i)), k(\mu_{\tilde{B}}(x_i))\}$ $l_i = \max\{l(\nu_{\tilde{A}}(x_i)), l(\nu_{\tilde{B}}(x_i))\}$ for each $x_i \in X$. Also, when $k(\mu_{\tilde{A}}(x_i)) \neq k(\mu_{\tilde{B}}(x_i))$ or $l(\nu_{\tilde{A}}(x_i)) \neq l(\nu_{\tilde{B}}(x_i))$, then they can be made equal by adding some elements in DHFE which has least number of elements in it. This can be done by adding the smallest membership values or smallest non-membership values to make the lengths of both DHFE $\tilde{A}$ and $\tilde{B}$ equal i.e. $k(\mu_{\tilde{A}}(x_i)) = k(\mu_{\tilde{B}}(x_i))$ or $l(\nu_{\tilde{A}}(x_i)) = l(\nu_{\tilde{B}}(x_i))$. For example, $\tilde{A} = \{\langle\{0.3, 0.8\}, \{0.2, 0.5\}\rangle\}$ and $\tilde{B} = \{\langle\{0.1, 0.7\}, \{0.8, 0.9, 0.4\}\rangle\}$, be any two DHFSs and their lengths are not equal therefore it can be made equal as $\tilde{A} = \{\langle\{0.3, 0.8\}, \{0.2, 0, 5, 0.2\}\rangle\}$ and $\tilde{B} = \{\langle\{0.1, 0.7\}, \{0.8, 0.9, 0.4\}\rangle\}$ respectively.

Definition 2.10.3 Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set and a set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x^{(s)}, \nu_{\tilde{A}}(x^{(t)},))\rangle | x \in X\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x^{(s)}, \nu_{\tilde{B}}(x^{(t)},))\rangle | x \in X\}$ be any two DHFSs on X, then the correlation coefficient between $\tilde{A}$ and $\tilde{B}$ is defined by $\rho_{DHFS}(\tilde{A}, \tilde{B}) = \frac{C_{DHFS}(\tilde{A}, \tilde{B})}{\sqrt{E_{DHFS}(\tilde{A})}\sqrt{E_{DHFS}(\tilde{B})}}$

$$= \frac{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{A}\sigma(s)}(x_i)\ \mu_{\tilde{B}\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{A}\delta(t)}(x_i)\ \nu_{\tilde{B}\delta(t)}(x_i)\right)}{\sqrt{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu^2_{\tilde{A}\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i}\nu^2_{\tilde{A}\sigma(t)}(x_i)\right)}\sqrt{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu^2_{\tilde{B}\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i}\nu^2_{\tilde{B}\sigma(t)}(x_i)\right)}}.$$

3. Brief review of the existing CoCf between two DHFSs

In the existing literature [29] it is claimed that, there does not exist any expression to evaluate the CoCf between two DHFSs, so to fill this gap, the expression (1) is proposed to evaluate the weighted CoCf between two DHFSs $A = \left\{\left\langle h_{A\sigma(s)}(x_i), g_{A\sigma(t)}(x_i)\right\rangle\right\}$ and $B = \left\{\left\langle h_{B\sigma(s)}(x_i), g_{B\sigma(t)}(x_i)\right\rangle\right\}$, where $i = 1, 2, \dots, n$, and s, t represents the number of values in $h_{A\sigma(s)}$ and $g_{A\sigma(t)}$ respectively.

$$\rho_{WDHFS}(A,B) = \frac{\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h_{A\sigma(s)}(x_i)\ h_{B\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g_{A\sigma(t)}(x_i)\ g_{B\sigma(t)}(x_i)\right)\right)}{\sqrt{\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h^2_{A\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g^2_{A\sigma(t)}(x_i)\right)\right)}\sqrt{\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h^2_{B\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g^2_{B\sigma(t)}(x_i)\right)\right)}} \tag{1}$$

where,

i. w_i represents the normalized weight $\left(w_i \geq 0, \text{and } \sum_{i=1}^{n} w_i = 1\right)$ of the i^{th} element.

ii. n represents the number of elements.

iii. $h_{A\sigma(s)}(x_i)$ and $g_{A\sigma(t)}(x_i)$ are two sets of some values in $[0, 1]$. Out of these two, $h_{A\sigma(s)}(x_i)$ represents the set of all the possible membership degree and $g_{A\sigma(t)}(x_i)$ represents the set of all the possible non-membership degree.

iv. k_i represents the number of values in $h_{A\sigma(s)}(x_i)$.

v. l_i represents the number of values in $g_{A\sigma(t)}(x_i)$.

Its claimed that if $w_i = \frac{1}{n}$ for all i then the expression (1) will be transformed into expression (2).

$$\rho_{DHFS}(A,B) = \frac{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h_{A\sigma(s)}(x_i)\, h_{B\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g_{A\sigma(t)}(x_i)\, g_{B\sigma(t)}(x_i)\right)\right)}{\sqrt{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h^2_{A\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g^2_{A\sigma(t)}(x_i)\right)\right)}\sqrt{\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(h^2_{B\sigma(s)}(x_i)\right) + \frac{1}{l_i}\sum_{t=1}^{l_i}\left(g^2_{B\sigma(t)}(x_i)\right)\right)}}. \tag{2}$$

3.1 Gaps in the existing weighted CoCf for DHFSs

In this paper, it is claimed that the existing CoCf (1) [29] is not valid in its present form. To prove that this claim is valid, there is a need to discuss the origin of the expressions (1). Therefore, the same is discussed in this section.

It can be easily verified that the expression (1) can be obtained mathematically in the following manner:

$$\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i} h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i} g_{A\sigma(t)}(x_i)\, g_{B\sigma(t)}(x_i)\right)$$
$$= \sum_{i=1}^{n}\left(\sum_{s=1}^{k_i} w_i \frac{1}{k_i}\left(h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)\right)\right) + \sum_{i=1}^{n}\left(\sum_{t=1}^{l_i} w_i \frac{1}{l_i}\left(g_{A\sigma(t)}(x_i)\, g_{B\sigma(t)}(x_i)\right)\right)$$
$$= \left(\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{A\sigma(s)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{B\sigma(s)}(x_i)\right)\right) + \left(\left(\sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{A\sigma(t)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{B\sigma(t)}(x_i)\right)\right)$$

Assuming, $X_1 = \sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{A\sigma(s)}(x_i)$, $Y_1 = \sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{B\sigma(s)}(x_i)$, $X_2 = \sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{A\sigma(t)}(x_i)$ and $Y_2 = \sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{B\sigma(t)}(x_i)$.

$$\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i} h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i} g_{A\sigma(t)}(x_i)\, g_{B\sigma(t)}(x_i)\right) = (X_1Y_1 + X_2Y_2) \le \sqrt{X_1^2 + X_2^2}\sqrt{Y_1^2 + Y_2^2}$$

$$\le \left(\sqrt{\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{A\sigma(s)}(x_i)\right)^2 + \left(\sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{A\sigma(t)}(x_i)\right)^2}\right) \times$$
$$\left(\sqrt{\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}} h_{B\sigma(s)}(x_i)\right)^2 + \left(\sum_{i=1}^{n}\sum_{t=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}} g_{B\sigma(t)}(x_i)\right)^{2.}}\right)$$

$$\leq \left(\left(\sqrt{\sum_{i=1}^{n}\left(\sqrt{w_i}\right)^2\left(\left(\sum_{s=1}^{k_i}\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\right)}\right)\right)\times$$

$$\left(\sqrt{\sum_{i=1}^{n}\left(\sqrt{w_i}\right)^2\left(\left(\sum_{s=1}^{k_i}\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\right)}\right)$$

$$\leq \left(\left(\sqrt{\sum_{i=1}^{n}w_i\left(\left(\sum_{s=1}^{k_i}\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\right)}\right)\right)\times$$

$$\left(\sqrt{\sum_{i=1}^{n}w_i\left(\left(\sum_{s=1}^{k_i}\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\right)}\right)$$

$$\leq \left(\left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{A\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{A\sigma(t)}(x_i)\right)}\right)\right)\times$$

$$\left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{B\sigma(t)}(x_i)\right)}\right)$$

$$\Rightarrow \sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)$$

$$\leq \left\{\begin{array}{l}\left(\left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{A\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{A\sigma(t)}(x_i)\right)}\right)\right)\times\\ \left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{B\sigma(t)}(x_i)\right)}\right)\end{array}\right\}$$

$$\Rightarrow \frac{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)}{\left\{\left(\left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{A\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{A\sigma(t)}(x_i)\right)}\right)\right)\times\left(\sqrt{\sum_{i=1}^{n}w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h^2_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g^2_{B\sigma(t)}(x_i)\right)}\right)\right\}}$$

$$\leq 1.$$

3.2 Mathematical incorrect assumptions

In this section, the mathematical incorrect assumptions, considered in existing literature [29] to obtain the expressions (1) have been discussed.

It can be easily verified from Section 3.1 that to obtain the expressions (1) it have been assumed that,

i. $\sum_{i=1}^{n}\left(\sum_{s=1}^{k_i}\frac{w_i}{k_i}h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)\right)=\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}}h_{A\sigma(s)}(x_i)\right)\times\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i}\frac{\sqrt{w_i}}{\sqrt{k_i}}h_{B\sigma(s)}(x_i)\right)$,

ii. $\sum_{i=1}^{n}\left(\sum_{t=1}^{l_i}\frac{w_i}{l_i}g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)=\left(\sum_{i=1}^{n}\sum_{s=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}}g_{A\sigma(t)}(x_i)\right)\times\left(\sum_{i=1}^{n}\sum_{s=1}^{l_i}\frac{\sqrt{w_i}}{\sqrt{l_i}}g_{B\sigma(t)}(x_i)\right)$

iii. $\left(\sum_{s=1}^{k}\frac{1}{\sqrt{k}}h_{A\sigma(s)}(x_1)\right)^2=\frac{1}{k}\sum_{s=1}^{k}h^2_{A\sigma(s)}(x_1)$

iv. $\left(\sum_{t=1}^{l}\frac{1}{\sqrt{l}}g_{A\sigma(t)}(x_1)\right)^2=\frac{1}{l}\sum_{t=1}^{l}g^2_{A\sigma(t)}(x_1)$

v. $\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} h_{B\sigma(s)}\ (x_1)\right)^2 = \frac{1}{k}\sum_{s=1}^{k} h^2_{B\sigma(s)}(x_1)$

vi. $\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} g_{B\sigma(t)}\ (x_1)\right)^2 = \frac{1}{l}\sum_{t=1}^{l} g^2_{B\sigma(t)}(x_1).$

Let us consider an example,

Example 1: Let

$$A = \left\{ \begin{array}{c} \langle x_1, \{0.1, 0.2, 0.5\}, \{0.3\}\rangle, \langle x_2, \{0.2, 0.4, 0.6\}, \{0.4, 0.5, 0.8\}\rangle, \\ \langle x_3, \{0.1, 0.2, 0.4\}, \{0.6, 0.8, 0.9\}\rangle, \langle x_4, \{\{0.2, 0.4, 0.1\}, \{0.8, 0.9, 0.6\}\}\rangle \end{array} \right\} \text{ and} \tag{3}$$

$$B = \left\{ \begin{array}{c} \langle x_1, \{\{0.2, 0.3, 0.5\}, \{0.3, 0.6, 0.9\}\}\rangle, \langle x_2, \{\{0.2, 0.3, 0.7\}, \{0.1, 0.9\}\}\rangle, \\ \langle x_3, \{\{0.6, 0.3, 0.5\}, \{0.9, 0.2, 0.3\}\}\rangle, \langle x_4, \{\{0.5\}, \{0.9\}\}\rangle \end{array} \right\}$$

be two DHFS and let $w = (0.3, 0.2, 0.1, 0.4)^T$ be the weight vector of x_i. Then, it can be easily verified that

$$\sum_{i=1}^{n}\left(\sum_{s=1}^{k_i} \frac{w_i}{k_i}\left(h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)\right)\right) = 0.1289,$$

$$\left(\sum_{i=1}^{n}\sum_{s=1}^{k_i} \frac{\sqrt{w_i}}{\sqrt{k_i}} h_{A\sigma(s)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{s=1}^{k_i} \frac{\sqrt{w_i}}{\sqrt{k_i}} h_{B\sigma(s)}(x_i)\right) = 1.1335$$

It is obvious that

$$\sum_{i=1}^{n}\left(\sum_{s=1}^{k_i} \frac{w_i}{k_i} h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)\right) \neq \left(\sum_{i=1}^{n}\sum_{s=1}^{k_i} \frac{\sqrt{w_i}}{\sqrt{k_i}} h_{A\sigma(s)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{s=1}^{k_i} \frac{\sqrt{w_i}}{\sqrt{k_i}} h_{B\sigma(s)}(x_i)\right).$$

Also, it can be easily verified that

$$\sum_{i=1}^{n}\left(\sum_{t=1}^{l_i} \frac{w_i}{l_i}\left(g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)\right) = 0.4003,$$

$$\left(\sum_{i=1}^{n}\sum_{s=1}^{l_i} \frac{\sqrt{w_i}}{\sqrt{l_i}} g_{A\sigma(t)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{s=1}^{l_i} \frac{\sqrt{w_i}}{\sqrt{l_i}} g_{B\sigma(t)}(x_i)\right) = 3.1862.$$

It is obvious that,

$$\sum_{i=1}^{n}\left(\sum_{t=1}^{l_i} \frac{w_i}{l_i} g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right) \neq \left(\sum_{i=1}^{n}\sum_{t=1}^{l_i} \frac{\sqrt{w_i}}{\sqrt{l_i}} g_{A\sigma(t)}(x_i)\right) \times \left(\sum_{i=1}^{n}\sum_{t=1}^{l_i} \frac{\sqrt{w_i}}{\sqrt{l_i}} g_{B\sigma(t)}(x_i)\right).$$

Furthermore, it can be easily verified that

$$\left(\sum_{s=1}^{k_i} \frac{1}{\sqrt{k_i}} h_{A\sigma(s)}\ (x_1)\right)^2 = 1.02, \quad \frac{1}{k_i}\sum_{s=1}^{k_i} h^2_{A\sigma(s)}(x_1) = 0.4267,$$

$$\left(\sum_{t=1}^{l_i} \frac{1}{\sqrt{l_i}} g_{A\sigma(t)}\ (x_1)\right)^2 = 4.5800, \quad \frac{1}{l_i}\sum_{t=1}^{l_i} g^2_{A\sigma(t)}(x_1) = 1.6467.$$

$$\left(\sum_{s=1}^{k_1} \frac{1}{\sqrt{k_1}} h_{B\sigma(s)}(x_1)\right)^2 = 2.8232, \quad \frac{1}{k_1}\sum_{s=1}^{k_1} h_{B\sigma(s)}^2(x_1) = 0.8166.$$

$$\left(\sum_{t=1}^{l_1} \frac{1}{\sqrt{l_1}} g_{B\sigma(t)}(x_1)\right)^2 = 4.1960, \quad \frac{1}{l_1}\sum_{t=1}^{l_1} g_{B\sigma(t)}^2(x_1) = 1.4133.$$

It is obvious that

i. $\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} h_{A\sigma(s)}(x_1)\right)^2 \neq \frac{1}{k}\sum_{s=1}^{k} h_{A\sigma(s)}^2(x_1)$

ii. $\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} g_{A\sigma(t)}(x_1)\right)^2 \neq \frac{1}{l}\sum_{t=1}^{l} g_{A\sigma(t)}^2(x_1)$

iii. $\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} h_{B\sigma(s)}(x_1)\right)^2 \neq \frac{1}{k}\sum_{s=1}^{k} h_{B\sigma(s)}^2(x_1)$

iv. $\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} g_{B\sigma(t)}(x_1)\right)^2 \neq \frac{1}{l}\sum_{t=1}^{l} g_{B\sigma(t)}^2(x_1).$

Thus, Example 1 verifies that the considered mathematical assumptions in the existing literature [29] to obtain the weighted correlation coefficient expressions (1) for DHFSs are not valid.

4. Proposed CoCf for the DHFSs

Considering the above mentioned limitation in Section 3 as a motivation, an attempt has been made to modify the existing expression (1) [29], and hence the weighted CoCf for DHFSs is proposed which is represented in expression (3).

$$\rho_{WDHFS}(A,B) = \frac{\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i} h_{A\sigma(s)}(x_i) h_{B\sigma(s)}(x_i) + \frac{1}{l_i}\sum_{t=1}^{l_i} g_{A\sigma(t)}(x_i) g_{B\sigma(t)}(x_i)\right)}{\sum_{i=1}^{n} w_i\left(\left(\sqrt{\sum_{s=1}^{k_i}\left(\frac{h_{A\sigma(s)}}{\sqrt{k_i}}\right)^2 \sum_{s=1}^{k_i}\left(\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2}\right) + \left(\sqrt{\sum_{t=1}^{l_i}\left(\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2 \sum_{t=1}^{l_i}\left(\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2}\right)\right)} \tag{4}$$

where,

i. w_i represents the normalized weight $\left(w_i \geq 0, \text{and } \sum_{i=1}^{n} w_i = 1\right)$ of the i^{th} element.

ii. n represents the number of elements.

iii. $h_{A\sigma(s)}(x_i)$ and $g_{A\sigma(t)}(x_i)$ are two sets of some values in $[0,1]$. Out of these two, $h_{A\sigma(s)}(x_i)$ represents the set of all the possible membership degree and $g_{A\sigma(t)}(x_i)$ represents the set of all the possible non-membership degree.

iv. k_i represents the number of values in $h_{A\sigma(s)}(x_i)$.

v. l_i represents the number of values in $g_{A\sigma(t)}(x_i)$.

5. Origin of the proposed CoCf for the DHFSs

The modified expression (3) has been obtained mathematically as follows:

$$\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i} h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i} g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)$$

$$=\sum_{i=1}^{n} w_i\left(\left(\sum_{s=1}^{k_i}\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}\,\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)+\left(\sum_{t=1}^{l_i}\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\,\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)\right)$$

Assuming,
$X^{(s)}=\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}$, $Y^{(s)}=\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}$,
$X^{(t)}=\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}$ and $Y^{(t)}=\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}$

$$\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i} h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i} g_{A\sigma(t)}(x_i)g_{B\sigma(t)}(x_i)\right)=\sum_{i=1}^{n} w_i\left(\sum_{s=1}^{k_i}X^{(s)}\,Y^{(s)}+\sum_{t=1}^{l_i}X^{(t)}Y^{(t)}\right)$$

$$\le\sum\nolimits_{i=1}^{n} w_i\left(\sqrt{\left(\sum\nolimits_{s=1}^{k_i}\left(X^{(s)}\right)^2\times\sum\nolimits_{s=1}^{k_i}\left(Y^{(s)}\right)^2\right)}+\sqrt{\sum\nolimits_{t=1}^{l_i}\left(X^{(t)}\right)^2\times\sum\nolimits_{t=1}^{l_i}\left(Y^{(t)}\right)^2}\right)$$

$$\le\sum\nolimits_{i=1}^{n} w_i\left(\left(\sqrt{\sum\nolimits_{s=1}^{k_i}\left(\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2\times\sum\nolimits_{s=1}^{k_i}\left(\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2}\right)+\left(\sqrt{\sum\nolimits_{t=1}^{l_i}\left(\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\times\sum\nolimits_{t=1}^{l_i}\left(\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2}\right)\right)$$

$$\Rightarrow\frac{\sum_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum_{s=1}^{k_i}h_{A\sigma(s)}(x_i)h_{B\sigma(s)}(x_i)+\frac{1}{l_i}\sum_{t=1}^{l_i}g_{A\sigma(t)}(x_i)\,g_{B\sigma(t)}(x_i)\right)}{\sum_{i=1}^{n} w_i\left(\left(\sqrt{\sum_{s=1}^{k_i}\left(\frac{h_{A\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2\times\sum_{s=1}^{k_i}\left(\frac{h_{B\sigma(s)}(x_i)}{\sqrt{k_i}}\right)^2}\right)+\left(\sqrt{\sum_{t=1}^{l_i}\left(\frac{g_{A\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2\times\sum_{t=1}^{l_i}\left(\frac{g_{B\sigma(t)}(x_i)}{\sqrt{l_i}}\right)^2}\right)\right)}\le 1.$$

6. Exact results of the existing real life problem

There is an investment company, which intends to invest a sum of money in the best alternative [29]. There are four available alternatives, A_1: a car company, A_2: a food company, A_3: a computer company, and A_4: an arms company. The investment company considers three attributes, C_1: the risk analysis, C_2: the growth analysis, and C_3: the environment impact analysis to consider the best alternatives. Since, there is a need to identify the best investment company among A_1, A_2, A_3 and A_4, with respect to an ideal alternative A^* on the basis of three different attributes C_1, C_2, and C_3, it is assumed that:

i. The weights assigned to the attributes C_j ($j = 1, 2$ and 3) are 0.35, 0.25 and 0.40 respectively.

ii. The DHFS $A^* = \{h^*, g^*\} = \{\{1\}, \{0\}\}$ ($j = 1, 2$ and 3) represents the ideal alternative.

iii. The $(i.j)^{th}$ element of **Table 1**, represented by a DHFS, represents the rating value of the i^{th} alternative over the j^{th} attribute i.e. D is a dual hesitant fuzzy decision matrix.

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} \{\{0.5, 0.4, 0.3\}, \{0.4, 0.3\}\} & \{\{0.6, 0.4\}, \{0.4, 0.2\}\} & \{\{0.3, 0.2, 0.1\}, \{0.6, 0.5\}\} \\ \{\{0.7, 0.6, 0.4\}, \{0.3, 0.2\}\} & \{\{0.7, 0.6\}, \{0.3, 0.2\}\} & \{\{0.7, 0.6, 0.4\}, \{0.2, 0.1\}\} \\ \{\{0.6, 0.4, 0.3\}, \{0.3\}\} & \{\{0.6, 0.5\}, \{0.3\}\} & \{\{0.6, 0.5\}, \{0.3, 0.1\}\} \\ \{\{0.8, 0.7, 0.6\}, \{0.2, 0.1\}\} & \{\{0.7, 0.6\}, \{0.2\}\} & \{\{0.4, 0.3\}, \{0.2, 0.1\}\} \end{bmatrix} \end{array}.$$

Table 1.
Rating values of the alternatives over the attributes.

Existing real-life problem [29]	Existing expressions (1) [29]	Proposed expressions (3)
Best investment company among A_1, A_2, A_3, and A_4	$\rho_1((A^*, A_1)) = 0.5981$ $\rho_1((A^*, A_2)) = 0.9200$ $\rho_1((A^*, A_3)) = 0.8668$ $\rho_1((A^*, A_4)) = 0.9088$ $A_2 > A_4 > A_3 > A_1$ i.e. A_2 is the best alternative.	$\rho_1((A^*, A_1)) = 0.9670$ $\rho_1((A^*, A_2)) = 0.9822$ $\rho_1((A^*, A_3)) = 0.9852$ $\rho_1((A^*, A_4)) = 0.9935$ $A_4 > A_3 > A_2 > A_1$ i.e. A_4 is the best alternative.

Table 2.
Results of the considered real-life problem.

Then, by applying the existing expression (1) [29] the obtained preferred company is A_2 i.e. the food company is the best alternative for the investment. However it is discussed in Section 3 that the expression (1) [29] is not valid in its present form since it is scientifically incorrect. Therefore, the result of the considered real-life problem, obtained in existing literature [29], is also not exact. Thus, to obtain the exact results of the existing problem [29], the proposed CoCf represented by expression (3) is utilized and the solution is obtained successfully. Furthermore, comparison of the results of the considered real-life problem is obtained by the existing expression (1) [29] as well as by the modified expression (3), and the results are shown below in **Table 2**.

From the above obtained results as shown in **Table 2**, it is obvious that according to existing expression (1), A_2 i.e. the food company is the most preferred company to invest the money, while, according to the proposed expression (3), A_4 i.e. arms company is the most preferred company to invest the sum of the money by the investment company.

7. Advantages of the proposed measure

The proposed correlation coefficient measure is an efficient tool which has the following advantages for solving the decision-making problems under the dual hesitant fuzzy environment.

i. Dual hesitant fuzzy set is an extension of hesitant fuzzy set (HFS), and intuitionistic fuzzy set (IFS) which contains more information i.e., it has wider range of hesitancy included both in membership and non-membership of an object than the others fuzzy sets (HFSs, deals with only membership hesitant degrees and IFSs deals with both membership degree and non-membership degree).

ii. It is observed in the suggested modified approach that the correlation coefficients of HFS [22–24], IFS [9–17] are the special cases of the proposed correlation coefficients of DHFSs. Thus, it can be comprehended that the proposed correlation coefficients for DHFSs is quite efficient in solving the decision-making problems under HFS, IFS, environment, whereas the existing methods [9–17, 22–24] poses some limitations.

iii. Since DHFSs contains more information in the data in relation to the uncertainties involved in comparison to the IFS, HFS environment hence the proposed tool is efficient in giving an appropriate solution in real-life applications in decision-making problems.

8. Conclusions

This paper is an outcome of the deep analysis made in understanding the ranking measures of DHFS using CoCf. In the present paper, a deep mathematical analysis is made to study the CoCf of a DHFS and it's concluded that there exist certain limitations in the existing CoCf [29] for DHFS. These shortcomings are pointed out with a detailed mathematical derivation which suggests that there are some mathematical incorrect assumptions involved hence, it is not appropriate to apply the existing CoCf of a DHFS in its present form. This limitation encouraged to propose a valid mathematical expression for ranking DHFSs in terms of CoCfs. Therefore, a new CoCf given by expression (3) is proposed for DHFSS which is a modified form of expression (1) [29]. To validate the claim of the modified expressions of the CoCf for DHFSs the detailed mathematical derivation is stated and the results of the real-life problems considered in existing paper [29] are obtained and to validate the obtained results a systematic comparison between the results are made.

Conflict of interest

The authors declare that they do not have any financial or associative interest indicating a conflict of interest in about submitted work.

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