Chapter

Qualitative Analysis for Controllable Dynamical Systems: Stability with Control Lyapunov Functions

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Abstract

The present chapter focuses on some recent work on the qualitatively analysis of dynamical systems, namely stability, a powerful tool with multiple connected appliances. Among them, feedback is a powerful idea which is used extensively in natural and technological systems. In engineering, feedback has been rediscovered and patented many times in many different contexts. Stabilizing a dynamical system could be often easier if we approach *controllable systems*. When the dynamical system is in a controllable form, we can place bounds on its behavior by analyzing the improvement of the linear and nonlinear operators that describe the system. In this chapter it is analyzed how a control in a simple form, could influence the possibility to construct the so-called *Control Lyapunov Function (CLF)* in order to stabilize the dynamical system in study. The main idea is to test multiple cases, in order to get a rich information panel and to make easier the problem of finding a CLF, which is generally a difficult task. As applications, models from excitable media are chosen.

Keywords: dynamical systems in control, Lyapunov stabilities, stabilization by feedback, Lyapunov and storage function, feedback control, computational methods, algebraic methods

1. Introduction: general outlines in feedback and stability for dynamical systems

The term *feedback* is used to refer to a situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled. Feedback is a powerful idea whose principle is based on corrections on the difference between desired and current performance. It was applied, rediscovered and patented in different contexts in engineering. The feedback methods had important improvements in time and, due to its remarkable properties, these improvements had significant importance in all applied sciences models.

A basic feature of feedback is that it changes the dynamics of a system. By modifying the behavior in the sense needed by the application, we can stabilize the model which is initially unstable, or we can obtain responsive systems from sluggish ones.

A survey on control process strategies and applications shows that: (1) a variety of nonlinear controller design techniques are based on input–output linearization; (2) few experimental studies of these techniques have been presented; and (3) many important problems remain unsolved [1].

1.1 Feedback model outlines

There are several types of finite-dimensional, nonlinear process models. The *continuous-time, state-space* model has the form:

$$\dot{x} = f(x) + g(x)u \qquad (1)$$
$$y = h(x).$$

x is the vector of state variables, $x \in R^n$, **u** is the vector of input variables, **y** the vector of controlled output variables, $u, y \in R^m$; **f** and **h** are vectors of nonlinear functions, $f \in R^n, h \in R^m$; finally **g** is a matrix of nonlinear functions.

The single-input, single-output (SISO) case where $m = 1$ is generally easiest and good to facilitate understanding the basic concepts. Consider the Jacobian linearization of the nonlinear model

$$\dot{x} = \left[\frac{\partial f(x_0}{\partial x} + \frac{\partial g(x_0}{\partial x} u_0\right](x - x_0) + g(x_0)(u - u_0) \qquad (2)$$
$$y - y_0 = \frac{\partial h(x_0)}{\partial x}(x - x_0).$$

Using derivation variables, the Jacobian model can be written as a linear state-space system

$$\dot{x} = Ax + Bu \qquad (3)$$
$$y = Cx$$

with obvious definitions for the matrices A, B and C. It is important to note that the Jacobian model is an exact representation of the nonlinear model only at the point (x_0, y_0). As result, a control strategy based on a linearized model may involve unsatisfactory performance and robustness at other operating points.

Roughly speaking, *feedback linearization* is a collection of ways for transforming the original system models into equivalent models of a simpler form. The central idea of feedback linearization is to algebraically transform nonlinear systems dynamics into (fully or partly) linear ones, in order to enable to apply linear control techniques. This approach is essential different from the classic Jacobian linearization, because feedback linearization is realized by an exact state transformation and a feedback law, rather than by linear approximations of the dynamics of the model. More important is the *local* feedback linearization, as it allows avoiding complications associated with the global problem.

After feedback linearization, the input–output model is linear:

$$\dot{\xi} = A\xi + Bv \qquad (4)$$
$$w = C\xi.$$

Here ξ is the vector of transformed variables, $\xi \epsilon R^r$; v is the vector of input variables; w is the vector of output variables, $w, v \epsilon R^m$, and A, B and C are matrices with simple canonical structure. The integer r is the fundamental characteristic of a nonlinear system, named the *relative degree*; if $r < n$, we have to complete the coordinate transformation by adding additional n-r state variables [1].

For a dynamical system, the *controllability* problem is to check the existence of a forcing term or control function u(t) such that the corresponding solution of the system will pass through a desired point, $x(t_*) = x_*$. The initial form of the controlled system motivates the form of the control. Thus, a controlled system will have a complex dynamic and therefore, analyzing its stability implies analyzing the nonlinear operators that describe the system's components. In applied sciences and engineering models, the control is aimed to compare the system against the desired behavior and compute corrective actions based on a model of the system's response to external inputs. Therefore, the modern control techniques include the use of algorithms [1].

1.2 Stability outlines

Let us consider the solution of a differential equation representing a physical phenomenon or the evolution of some system. There always is some uncertainty concerning the initial conditions, because, when one attempts to repeat a given experiment, the reproduction of the initial conditions is never entirely identical. It is thus fundamental to be able to recognize the circumstances under which small variations in the initial conditions will only introduce small variations in what follows of the phenomenon.

It is known that stability is a property of the solutions of differential equations in R^n of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ by which, given a "reference" solution $x^*\left(t, t_0^*, x_0^*\right)$, any other solution $x(t, t_0, x_0)$ starting close to $x^*\left(t, t_0^*, x_0^*\right)$ remains close to $x^*\left(t, t_0^*, x_0^*\right)$ for long times. Thus, generally speaking, we can state the question of stability as: "small variations in the initial conditions will imply small variations in what follows for the phenomenon".

The stability concept has the beginnings back in the past, in the analysis of the planets motion. Then Lagrange, Dirichlet had refined the definition, including the boundedness of trajectories. But Lyapunov's work was a corner stone in this area, by analyzing the stability concept, with the help of a positive non-decreasing function which is decreasing along the system trajectories. The Lyapunov functions are a mainstay in the control theory and in the applied sciences modeling.

Within the mathematical context they are implied, the Lyapunov functions provide sufficient conditions for the stability of equilibrium and for analyzing its basin of attraction or more general invariant sets. They characterize the long-time behavior of the solutions depending on their initial solutions. Therefore it appears the natural question how to compute a Lyapunov function for a particular system? Although the existence of Lyapunov functions has been studied in few theorems, it is not provided yet a general method to compute them. The converse theorems which appeared around 1950 were a great help in the issue. A converse theorem generally establishes that, if a system has a certain kind of stability, then there exists a Lyapunov function for the system that characterizes that kind of stability. Still, the converse theorems are not very constructive in practice, since they use the solution trajectory of the system to construct the Lyapunov function and the solution trajectories are usually not known. Krasovski himself noted that [2]:

"One could hope that a method for proving the existence of a Lyapunov function might carry with it a constructive method for obtaining this function. This hope has not been realized".

Numerous computational construction methods have been developed in mathematical community, based on different methods such as linear matrix inequalities, linear programming, series expansion, algebraic methods, theoretic methods and many others.

Very important to notice is the Lyapunov theorem, which enable establishing stability or asymptotic stability of equilibrium points without explicitly computing trajectories [2].

The Lyapunov theorem is of fundamental importance in system theory. It asserts the possibility of establishing stability or asymptotic stability of equilibrium points without explicitly computing trajectories [2].

Theorem 1 (Lyapunov). Let $x_e = 0$ be an equilibrium point for the system (1). Let $V : R^n \to R$ be a positive definite continuously differentiable function.

1. If $\dot{V} : R^n \to R$ is negative semi-definite, then x_e is stable;

2. If $\dot{V}$ is negative definite, then x_e is asymptotically stable.

The theorem assesses the existence of a Lyapunov function but does not provide a method to compute one. In the case of linear systems, this issue arises naturally, but in general computing a Lyapunov function is an open problem giving rise to different ways to construct it.

We recall in what follows the two basic Lyapunov criteria.

a. *The first Lyapunov criterion* is based on the eigenvalues analysis.

Let us consider the following continuous-time nonlinear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), u(t)). \tag{5}$$

In the vicinity of the equilibrium point $(x_{0,}u_0)$, let us consider the corresponding linearized system:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t). \tag{6}$$

This criterion has three distinct cases for the eigenvalues λ_i of the matrix A [3]:

i. If $Re\ \lambda_i < 0$ for all i, then (x_0, u_0) is asymptotically stable;

ii. If there exits at least one i such as $Re\ \lambda_i > 0$ then (x_0, u_0) is unstable;

iii. If there exits at least one i such as $Re\ \lambda_i = 0$ and for all other λ_j, $j \neq i$, $Re\ \lambda_j < 0$, then we cannot conclude anything about the stability of (x_0, u_0). In this case we say that the criterion is not effective

b. *The second Lyapunov criterion*

Theorem 2. Consider the dynamical system in R^n:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{7}$$

and let $x = 0$ be its unique equilibrium point. If there exists a continuously differentiable function $V : R^n \to R$ such that:

$$V(0) = 0; \tag{8}$$

$$V(\boldsymbol{x}) > 0, \forall x \neq 0; \tag{9}$$

$$\|\boldsymbol{x}\| \to \infty \Rightarrow V(\boldsymbol{x}) \to \infty; \tag{10}$$

$$\dot{V}(\boldsymbol{x}) < 0 \forall x \neq 0. \tag{11}$$

Then $\boldsymbol{x} = 0$ is global asymptotically stable.

The condition (9) refers to the *monotonicity* of the Lyapunov function. We say that V is decreasing along trajectories, using the *orbital derivative* given by:

$$\dot{V}(\boldsymbol{x}) = \left\langle \frac{\partial V(\boldsymbol{x})}{\partial x}, f(\boldsymbol{x}) \right\rangle \tag{12}$$

where $\langle , \rangle$ is the inner product in R^n and $\frac{\partial V}{\partial x}$ is the gradient of V. Also, the condition (10) refers to the requirement for V to be *radially unbounded*.

We could ask if Lyapunov functions always exist, and if so, how could we find such a function? For the first part of the question the answer is generally positive but, finding a Lyapunov function is not immediate, since the converse theorems assume the knowledge of the solutions of the system (7) [2, 3]. Therefore refining the definition of Lyapunov function and establishing a more specific context was very necessary.

An important aim in qualitative analysis of the stability is to search if the solutions remain close to the equilibrium and moreover, if they converge towards it. Therefore the search for the Lyapunov function must be more accurate. The *strict Lyapunov functions* can achieve this goal. Designed as generalization of the energy in a physical dissipative system, they preserve the property of decreasing energy along trajectories and thus, the solutions of the system converge to a (local) minimum of energy.

Definition 1. A *strict Lyapunov* function for the equilibrium x_0 of (7) is a real-valued, continuously differentiable function $V : U \subset R^n \rightarrow R$ defined on a neighborhood U of x_0 which satisfies:

a. *Minimum*. V has a minimum at x_0, i.e. $V(x) \geq 0$ *for all* $x \in U$ *and* $V(x) = 0$ *iff* $x = x_0$

b. *Decrease*. V is strictly decreasing along solution trajectories of (7) in U except for the equilibrium. A sufficient condition is $\dot{V} < 0$ *for all* $x \in U \setminus x_0$.

Thus, two important properties are deduced for a strict Lyapunov function [4]:

- If we have a strict Lyapunov function, then the equilibrium is asymptotically stable;
- Compact sublevel sets of a strict Lyapunov function are subsets of the basin of attraction of the equilibrium.

This chapter is organized as follows. The section 2 is dedicated to Lyapunov functions computational analysis. There are exposed the basic outlines of CLF concept and also the related outlines: LMI approach and SOS Lyapunov functions. In the section 3, a computational Lyapunov function is searched for the mixing flow dynamical system in a slightly perturbed form. After presenting the mathematical context of the 2d mixing flow dynamical system, together with recent results in the field, the results of searching a CLF for the mixing flow are presented. The section 4 is dedicated for conclusions and further aims in the topic. The chapter ends with references.

2. Computational Lyapunov stability analysis

2.1 Control Lyapunov functions

The concept of control Lyapunov function (CLF) is a very useful appliance in solving stability tasks. We search to stabilize a nonlinear system by selecting a

Lyapunov function *V(x)* and then try to find a feedback control *u(x)* that gives $\dot{V}(x, u(x))$ negative definite. If with an arbitrary choice of V this attempt may fail, when V(x) is a CLF, to find a stabilizing control law *u(x)* is easier.

Similar with the system (5), we can define a control system like follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{13}$$

where $\boldsymbol{u} \in U \subset R^m$ is the control. We speak about an *open-loop control* if u is function of time, $\boldsymbol{u} = \boldsymbol{u}(t)$ and *closed-loop* if $\mathbf{u} = \mathbf{k}(\mathbf{x})$. The closed-loop control is in fact the *feedback control*. We speak also about *feedback stabilized system* if the feedback has been fixed, $\mathbf{u} = \mathrm{k}(\mathbf{x})$ and the equilibrium in the origin has a desired stability property.

The system (13) is called *locally, asymptotically null-controllable*, [4] if for every ρ in a neighborhood of the origin there is an open-loop control u such that the solution of the system with initial value ρ tends asymptotically towards the origin, i.e. if it is possible to steer the system state asymptotically to the origin. A *control Lyapunov function* (CLF) for such a system, introduced by Sontag 1983 [5], is a positive definite function V such that

$$inf_{\ u \in U} \nabla V(\boldsymbol{x}) \bullet \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\gamma(\|\boldsymbol{x}\|) \tag{14}$$

where γ is a comparison function [4]. Asymptotic null-controllability cannot be characterized by smooth control Lyapunov functions and one must resort to more general definitions of differentiability like the Dini- or the proximal sub-differential [6].

For asymptotically null-controllable systems the equilibrium at the origin is sometimes referred to as weakly asymptotically stable, in contrast to strongly asymptotically stable equilibrium, where every choice of **u** leads to states being attracted asymptotically to the equilibrium.

As mentioned in the previous section, the stability concept has a lot of approaches: we have the classic Lagrange, Dirichlet and Lyapunov stability but, depending on the context, we also have input–output stability, hyperstability, input-to-state stability [3]. The last one, input to state stability (ISS) was introduced by Sontag [7] and it is interesting by the idea of characterizing a certain kind of stability at the origin imposing for the Lyapunov function V the condition:

$$\nabla V(\boldsymbol{x}) \bullet \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \leq -\gamma(\|\boldsymbol{x}\|) + \alpha(\|\boldsymbol{u}\|) \tag{15}$$

where α and γ are comparison functions [4]. The origin is thus an asymptotically stable equilibrium of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, 0)$ and a practically stable equilibrium of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ for $\|\boldsymbol{u}\| \leq u_{max}$, with $u_{max} > 0$ a (not too large) constant. Moreover, the smaller u_{max} is, usually interpreted as a bound on the perturbation u, the closer solutions of the system will be to zero in the long run.

2.2 Stability analysis using sum of squares Lyapunov functions

The stability of dynamical systems is basically carried out by Lyapunov theory. For linear systems, the construction of an "energy-like function" – the Lyapunov function, fulfilling certain positivity conditions, is not difficult. For a system $\dot{x} = Ax$ this implies finding a matrix P such that $A^T P + PA$ is negative definite [8]. Then the associated Lyapunov function is given by $V(x) = x^T Px$. But although obviously, it was seen only recently that, in this context, both $V(x)$, $\dot{V}(x)$ are *sum of squares functions*!

Sum of squares (SOS) optimization is a quite new technique at the interface between convex optimization and computational algebra. Recently it had significant impact not only in optimization, but over several disciplines as well, especially in control theory. Besides the stability analysis of nonlinear systems, SOS has opened a new direction in approaching different types of systems and answering different analysis questions. The SOS technique generalizes a well-known computational appliance in linear robust control theory, "Linear Matrix Inequalities" – LMI. Parrilo and Ahmadi had important contributions in this field. [9] Using LMI in different analysis problems is advantageous, since there are efficient algorithms developed in the framework of semi-definite programming (SDP) [8, 9]. The SOS uses these types of algorithms, but all questions are formulated at polynomial level, or in polynomial-matrix terms.

Related to the Lyapunov functions, to construct them "by test" requires analytic skills of the researchers and moreover, depend on the small-state dimensions. When the vector field f and the Lyapunov function candidate V are both polynomial, the Lyapunov conditions are polynomial non-negativity conditions, which are quite hard to test. This could be one of the reasons for lack of efficiency in the algorithmic construction of a Lyapunnov function. But, if the non-negativity conditions are replaced by SOS conditions, then constructing the Lyapunov function can be done efficiently using semi-definite programming.

There are a lot of the control problems which follow the same two steps: i) recasting the primal problem as a Lyapunov-type problem and then, ii) constructing a sum-of-squares relaxation to the problem. We present in what follows a theoretical tool which is basic in the sum of squares computational approach [8].

Given $x \in R^n$ we denote the ring of multivariable polynomials with real coefficients by *R[x]* and the subset of sum-of-squares polynomials in the variable x by $\sum[x]$. Sometimes it may be necessary to indicate the maximum degree of a polynomial or sum-of-squares polynomial in which case we use the subscript notation $R_d[x]$ or $\sum_d [x]$ where d is a positive integer.

Theorem 3. Let $x^* = 0 \in D \subset R^n$ an equilibrium point of (7). If there exists a function $V : D \to R$ continuously differentiable such that the following hold:

$$V(0) = 0; \tag{16}$$

$$V(\boldsymbol{x}) > 0, \forall x \in D \setminus \{0\} \tag{17}$$

$$\dot{V}(\boldsymbol{x}) \le 0 \forall x \in D. \tag{18}$$

Then x^* is stable. Moreover, x^* is asymptotically stable if (11) holds.

Theorem 4. Let $x^* = 0$ be equilibrium for (7) and assume that $D \subset R^n$ is a given domain which includes x^*. Assume there exists a continuously differentiable function $V : D \to R$ and positive constants k_1, k_2, k_3 such that

$$k_1 \|x\|^p \le V(x) \le k_2 \|x\|^p \tag{19}$$

$$\dot{V}(x) \le -k_3 \|x\|^p, p \in Z \tag{20}$$

for all $t \ge 0, x \in D$. Then x^* is exponentially stable. Furthermore, if the assumptions hold when $D = R^n$, then x^* is globally exponentially stable.

The following theorem illustrates how sum-of-squares programming can be used to construct a polynomial stability certificate, in this case a Lyapunov function, for an equilibrium point of (7). The domain D must be defined. In particular D must be representable as a semi-algebraic set. If we consider that the domain of interest D is represented by all points that satisfy $\beta(x) \le 0, \beta \in R[x]$, then we have the following result [8].

Theorem 5. If there exists a polynomial function V, sum-of-squares polynomials r_1, r_2 and positive definite polynomial functions φ_1, φ_2 all of bounded degree, such that

$$V(x) + r_1(x)\beta(x) - \varphi_1(x)\ \epsilon\Sigma[x] \tag{21}$$

$$-\dot{V}(x) + r_2(x)\beta(x) - \varphi_2(x)\ \epsilon\Sigma[x] \tag{22}$$

then the equilibrium point x^* of (7) is asymptotically stable.

A few questions immediately come to mind: Do stable polynomial systems always admit a sum-of-squares Lyapunov function? How conservative are we being by limiting ourselves to positive polynomials that admit a sum-of-squares decomposition? Can we determine a priori the degree of the Lyapunov function required? The answer to the first question is simply, no.

There is a vast literature on the SOS method to compute Lyapunov functions in various settings and for different kinds of systems [4]. Between them, Parillo given important result on SOS and SDP (Semi Definite Problems) programming in finding Lyapunov functions [9]. As specified above, he introduced an efficient LMI program for finding Lyapunov functions as sum of squares for polynomial systems. The above setting is simplified for finding global Lyapunov functions, but if we are interested to find SOS Lyapunov functions on a compact domain, is important to mention the "Positivstellensatz" as useful appliance [9].

In the case of a dynamical system of a simple polynomial form, the stability study and Lyapunov function search can begin with a function test which facilitates the further analysis, as we see in the section 3.

3. Existence of control Lyapunov function for dynamical systems from excitable media

3.1 Recent results

The concerning for the dynamical models arising from excitable media is not new. The dynamical systems modeling the mixing flow have an important place, because of the complexity of the model. It is in fact about far from equilibrium class of models, with a very sensitive behavior to initial conditions.

Let us recall the statistical idea of a flow, generally represented by the application

$$\boldsymbol{x} = \Phi_t(\boldsymbol{X}), \boldsymbol{X} = \Phi_{t=0}(\boldsymbol{X}). \tag{23}$$

That means, X is mapped in x after a time t. In continuum mechanics, the relation (23) is named *flow*, it is a diffeomorphism of class C^k and must satisfy the relation

$$0 < J < \infty, J = \det\left(\frac{\partial x_i}{\partial X_j}\right), J = \det(D\Phi_t(x)) \tag{24}$$

where D denotes the derivation operation with respect to the reference configuration, in this case with $\mathbf{X}$. If the Jacobean J is unitary, it is said we have an isochoric flow. The relation (24) implies two particles, $\mathbf{X}_1$ and $\mathbf{X}_2$ which occupy the same position $\mathbf{x}$ at a given moment, or a particle which splits in two parts. That means, non-topological motions like break up or disintegration *are not allowed.*

The mixing flow is a special type of flow, implying a basic fluid (water) in which a biological material is moving (mixing) in different conditions and with different velocities. Therefore, the *stretching and folding* are strongly related phenomena. With respect to X there is defined the basic measure of deformation, the *deformation gradient* $\boldsymbol{F}$ [10]:

$$\mathbf{F} = (\nabla_X \Phi_t(\mathbf{X}))^T, F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right). \tag{25}$$

For a material filament and correspondingly for a material surface, in a mixing flow, there are defined another two basic deformation measures, the *length deformation* λ and *surface deformation* η. In this context, a specific analysis for deformations of infinitesimal elements is the so-called "*good mixing concept*", related to the boundaries of the quantities λ and η. The class of flows with a special form of F is of very large interest in the literature, as it contains the so-called "constant stretch history motion" (CSHM flows). Details can be found in [10].

When studying the mixing flow phenomena, one starts from the widespread kinematic 2d mixing flow

$$\begin{cases} \dot{x_1} = Gx_2 \\ \dot{x_2} = KGx_1, \end{cases} -1 < K < 1, G \in R. \tag{26}$$

Although this is a linear model, when associating the corresponding initial condition.

$$x_1(0) = x_1(t=0) = X_1; x_2(0) = x_2(t=0) = X_2 \tag{27}$$

it is obtained a complex solution for the Cauchy problem (26)-(27) [11]. From geometric standpoint, the streamlines of the above model satisfy the relation $x_2^2 - K \bullet x_1^2 = const.$ and this is corresponding to some ellipses with the axes rate $\left(\frac{1}{|K|}\right)^{1/2}$ if K is negative, and to some hyperbolas with the angle $\beta = \arctan\left(\frac{1}{|K|}\right)^{1/2}$ between the extension axis and x_2, if K is positive [10].

To this broad isochoric flow, we can associate easily the corresponding 3d dynamical system [11]:

$$\begin{cases} \dot{x}_1 = G \bullet x_2 \\ \dot{x}_2 = K \bullet G \bullet x_1, \ -1 < K < 1, c = const. \\ \dot{x}_3 = c \end{cases} \tag{28}$$

with the third component for the moving velocity of the system.

In the 3d case, the non-periodic model exhibits a complicate behavior. A lot of comparative computational analysis proved the great influence of the parameters on the model behavior, leading to far from equilibrium models [11]. The perturbed model was also taken into account, and it was found out that its sensitivity with respect to the parameters is significant, both in 2d and 3d case [11, 12].

3.2 Existence of a CLF for the mixing flow dynamical system in a slightly perturbed form

The central aim in the study of the mixing flow dynamical system was associated rather with the fluid mechanics standpoint, namely analyzing the *efficiency of mixing* [10, 11].This is a concept which implies the analysis of deformation efficiencies in length and surface for the material mixed in the basic fluid. The physical phenomena associated are the *multiphase flow* phenomena, and the analysis and numeric simulation for these complex flows is in study. Briefly, in order to obtain a good behavior for the deformation efficiencies, the mathematical context must take into account the following three stages:

- modeling the global swirling streamlines;
- local modeling of the concentrated vorticity structure;
- introducing the elements of chaotic turbulence.

Because of its complexity, any perturbation on the mixing flow model changes the behavior of the model in a significant way. Therefore its stability is an important and challenging task. If in [13] it was found a SOS Lyapunov function working both for the initial and the feedback linearized form of the mixing flow dynamical system, this chapter brings the novelty of approaching the mixing flow dynamical system as a *polynomial differential system*, in order to get an optimal search for a Lyapunov function.

When taking into account the feedback control, the transformed model has a significant different repartition of the parameters in the model [12]. The stability analysis was started with the 2d case, with slight perturbations. In what follows, we consider the same slightly perturbed form of the 2d dynamical system as in [13], namely

$$\begin{cases} \dot{x_1} = Gx_2 + x_1 \\ \dot{x_2} = KGx_1 - x_2 \end{cases} - 1 < K < 1, G \in R. \tag{29}$$

For this model, it was found a strong result. A Lyapunov function V was found *both* for the initial model (29) and for its feedback linearized form. Namely, it was constructed the sum-of-squares function

$$V : R^2 \to R, V(\boldsymbol{x}) = x_1^2 + \frac{1}{|K|}x_2^2, \forall \boldsymbol{x} = (x_1, x_2) \in R^2. \tag{30}$$

The conditions (8)–(11) are fulfilled for suitable conditions on the parameters, for both models, so V is a Lyapunov function, and the origin is an asymptotically stable equilibrium point.

In what follows we consider the problem of finding a *control Lyapunov function* (CLF) for the model (29). We use a simplified form of the control system (5), namely we consider a control system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})u \tag{31}$$

where f and g are smooth vector fields, $x(t) \in R^n, u(t) \in R, f(0) = 0$.

Definition 2. A function V is a control Lyapunov function (CLF) for this system if $V : R^n \to R$ is a smooth, radially bounded, and positive definite function such that

$$inf_{\ u \in R} \left\{ \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) + \frac{\partial V}{\partial \boldsymbol{x}} g(\boldsymbol{x})u \right\} < 0, \forall x \neq 0. \tag{32}$$

Existence of such a V implies that (29) is globally asymptotically stabilizable at origin.

We have to notice that the condition (32) for the CLF is in fact equivalent with that in (14) introduced by Sontag [6]. If such a CLF is given, it is shown [6] that a feedback low u = k(x) with k(0) = 0, can be constructed from the CLF producing a closed loop globally asymptotically stable. Hence, the problem of globally asymptotically stabilizing (31) is reduced to finding a CLF for the system.

Finding a CLF is a not trivial problem. It has been approached by multiple techniques, and Linear Programming, Positivstellensatz (P-sat) [14] and Linear Matrix Inequalities are only few examples. For the present aim the LMI approach [15, 16] helped to get better track of the model behavior.

A CLF must satisfy the inequality (32) which can be further evaluated as:

$$inf_{\ u \in R}\left\{\frac{\partial V}{\partial \boldsymbol{x}}f(\boldsymbol{x})+\frac{\partial V}{\partial \boldsymbol{x}}g(\boldsymbol{x})u\right\}=\begin{cases}-\infty & when\ \dfrac{\partial V}{\partial \boldsymbol{x}}g(\boldsymbol{x})\neq 0\\ \dfrac{\partial V}{\partial \boldsymbol{x}}f(\boldsymbol{x}) & when\ \dfrac{\partial V}{\partial \boldsymbol{x}}g(\boldsymbol{x})=0\end{cases}. \tag{33}$$

For a fixed x such that $\frac{\partial V}{\partial x}g(x)\neq 0$, we can make the inequality (32) hold by choosing a large value of u of the correct sign. Therefore, is essential to establish the set of x such that $\frac{\partial V}{\partial x}g(x)=0$. Thus, we want

$$\frac{\partial V}{\partial \boldsymbol{x}}f(\boldsymbol{x})<0\forall x\in R^n\ such\ that\ \frac{\partial V}{\partial \boldsymbol{x}}g(\boldsymbol{x})=0, x\neq 0. \tag{34}$$

Let us consider the 2d mixing flow dynamical model in its perturbed form (29). We have a polynomial system of differential equations, with monomials in the right hand sides of the system. In order to put it in a controlled form like (31), we take into account a simple control u at each of the equations of the model (29), namely

$$\begin{cases}\dot{x_1}=Gx_2+x_1-u\\ \dot{x_2}=KGx_1-x_2+u\end{cases} -1<K<1, G\in R. \tag{35}$$

The vector fields **f** and **g** are $f=\begin{pmatrix}Gx_2+x_1\\ KGx_1-x_2\end{pmatrix}, g=\begin{pmatrix}-1\\ 1.\end{pmatrix}$

We are looking for a positive definite symmetric matrix P, $P=\begin{pmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{pmatrix}$ and define a CLF function like

$$V=\frac{1}{2}\boldsymbol{x}^T P\boldsymbol{x}, \tag{36}$$

that means

$$V=\frac{1}{2}\left(x_1^2P_{11}+2x_1x_2P_{12}+x_2^2P_{22}\right)$$

such that (34) holds.

So, we search P by setting the condition $\frac{\partial V}{\partial x}g(\boldsymbol{x})=0$ and studying when the condition $\frac{\partial V}{\partial x}f(\boldsymbol{x})<0$ is fulfilled.

We have scalar inners in the condition (34). So, $\frac{\partial V}{\partial x}g(\boldsymbol{x})=0$ implies

$$(P_{12}-P_{11})x_1+(P_{22}-P_{12})x_2=0.$$

From here we take x_2 function of x_1 and replace in $\frac{\partial V}{\partial x}f(\boldsymbol{x})$. Thus, we obtain a quadratic form like follows:

$$\frac{\partial V}{\partial \boldsymbol{x}}f(\boldsymbol{x})=(P_{11}+KGP_{12})x^2+(P_0GP_{11}+P_0KGP_{22})x+(P_{12}-P_{22})P_0^2, \tag{37}$$

with the notation

$$P_0 = \frac{P_{12} - P_{11}}{P_{12} - P_{22}}.$$

The sign of the form (37) is of course dominated by the coefficient of x^2. Thus, we have

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) < 0$$

when

$$P_{11} + KGP_{12} < 0. \tag{38}$$

Taking into account that K,G are also real parameters, the inequality (38) implies few tests. Therefore for the moment we choose the matrix P like

$$P = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}. \tag{39}$$

With this choice we find the condition

$$KG < -2, \tag{40}$$

which is a feasible condition since from the model we have $-1 < K < 1, G \in R$.

Now we can verify if the test value (39) of P can produce a feasible CLF function V. LMI approach has very reliable numeric tools [14, 15]. We choose for the present aim the basic matrix inequality, namely the Lyapunov inequality:

$$A^T P + PA \leq 0 \tag{41}$$

where A is the system matrix and P is the matrix in the definition (36) of CLF.

In our model (35), the matrix A is given by:

$$A = \begin{pmatrix} 1 & KG \\ G & -1 \end{pmatrix}, \tag{42}$$

and thus we obtain

$$A^T P + PA = \begin{pmatrix} 2P_{11} + 2P_{12}KG & P_{11}G + KGP_{22} \\ P_{11} + P_{22}KG & 2P_{12}G - 2P_{22} \end{pmatrix}. \tag{43}$$

Replacing with the test values from (39) for P, the condition "$A^T P + PA$ negative semidefinite" implies the following conditions for the parameters

$$KG < -2, \tag{44}$$
$$G < -\frac{1}{2} \bullet \frac{1}{1+K}.$$

These conditions are feasible, for a negative G.

Thus we can conclude that, in feasible conditions for the parameters, the model (35) can admit a CLF V and thus the model can be globally stabilizable at the origin.

4. Conclusions

In the present chapter, the aim of stabilizing the two-dimensional dynamical systems arising from excitable media is taken into account.

Stabilizing a dynamical system from excitable media is not an easy task, especially because of the sensitive dependence of these models on initial conditions. The mixing flow models are not an exception, since the repartition of their parameters has a great influence on the model trajectory behavior.

The Lyapunov function is a strong appliance in studying stability. Still, constructing it is not easy. This is why approaching this for controllable systems make easier the task of searching the stability.

Searching for a Lyapunov function becomes easier if we take into account the iterative algorithm in the "sum of squares" programming. For the mixing flow example of this chapter, the test values (39) for the matrix P shown that in feasible conditions for the parameters, we can construct a CLF for the model (35) and thus the model is globally stabilizable in the origin.

Constructing the CLF functions for controllable systems provide interesting results, and this is shown by the above 2d mixing flow model. The 2d mixing flow model is controllable in some slightly perturbed forms [12, 13]. Therefore, in this chapter the standpoint is that we started with the simple control u taken on both equations of the dynamical system. The control is not implied in the further calculus on finding the CLF V, which made easier the above analysis. A next aim is to find CLF functions for other perturbations cases of the mixing flow model, on one hand, and also to approach iterative LMI algorithms for finding the matrix P. The test values (39) for the matrix P give rise to an easier further statement of a SDP problem for the mixing flow dynamical system.

Thus, this chapter adds a new standpoint in the mathematical context of the mixing flow dynamical system, by considering it as a polynomial system of differential equations. For a model whose mathematical apparatus is rather associated with mechanics and fluid mechanics issues, approaching the stability by searching Lyapunov functions with computational algebraic appliances is new. This approach could open a new way in the study of stabilizing of the mixing flow dynamical model - which is a far from equilibrium model, and also in the qualitative analysis of the differential systems associated to transport phenomena.

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