Chapter

Basics of Fluid Dynamics

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Abstract

In this chapter, studies on basic properties of fluids are conducted. Mathematical and scientific backgrounds that helps sprint well into studies on fluid mechanics is provided. The Reynolds Transport theorem and its derivation is presented. The well-known Conservation laws, Conservation of Mass, Conservation of Momentum and Conservation of Energy, which are the foundation of almost all Engineering mechanics simulation are derived from Reynolds transport theorem and through intuition. The Navier–Stokes equation for incompressible flows are fully derived consequently. To help with the solution of the Navier–Stokes equation, the velocity and pressure terms Navier–Stokes equation are reduced into a vorticity stream function. Classification of basic types of Partial differential equations and their corresponding properties is discussed. Finally, classification of different types of flows and their corresponding characteristics in relation to their corresponding type of PDEs are discussed.

Keywords: Navier–Stokes equation, engineering simulation, conservation laws, Reynolds transport equation, modeling, simulation, fluid dynamics, fluid flow, nonlinear differential equations

1. Introduction

1.1 Mathematical background

In this subsection, important mathematical formulations and ideas that help understand fluids are discussed.

All mathematical formulations presented in this chapter will be performed using vectors. Therefore, it is advised to have a good knowledge of the fundamental theorems of vector calculus.

1.1.1 Green's theorem

Assuming a closed curve **C**, the Green's theorem expresses contour integral of **C** in terms of a two dimensional region **R,** bounded by **C** [1–3].

The Green's theorem is given as

$$\int_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \oint_c P dx + Q dy \qquad (1)$$

1.1.2 Stokes theorem

Let Q be a vector field, s be an oriented surface be a closed surface oriented by the Right hand rule, Stokes theorem states that

$$\int_s [\nabla \times \mathbf{Q}]\boldsymbol{ds} = \oint \boldsymbol{Qdr} \tag{2}$$

where r is such that dr/ds is the unit tangent vector and s the arc length of C. The curve of the line integral, C, must have positive orientation, meaning that dr points counterclockwise when the surface normal, dS, points toward the viewer, as per the right-hand rule [1–3].

1.1.3 Divergence theorem

Divergence theorem is a relation to convert volume integral into areal integral.

Let v be a volume in a three dimensional space, and $\boldsymbol{\Omega}$ be the surface boundary. Let $\boldsymbol{\eta}$ be a unit normal pointing outward from the surface. $\mathbf{Q}$ being any vector field, the Divergence theorem is given as

$$\int_v Q.vdv = \int_\Omega Q.\eta dA \tag{3}$$

The Divergence theorem is a very important concept, as shall be discussed in later sections, in the area of fluid dynamics, especially in studying the Flux terms [1–6].

The Divergence Theorem is equally applicable to tensors.

1.1.4 Leibniz integral rule

Leibniz integral rule gives a formula for differentiating a definite integral whose limits are a function of definite variable [1–6]. If Q be a field that is a function of time t and space X, the Leibniz Rule is given as

$$\frac{d}{dx}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial x}dx + b'(t)f(b(t),\mathrm{t}) - \mathrm{a'(t)f(a(t),t)} \tag{4}$$

If a and b are constant, the second and third terms go to zero.

1.2 Background on fluids

1.2.1 What are fluids?

Air is all around us. We drink water every day. We clean ourselves and our environment using water. Almost 71% of the earth is covered with water. Our lives are highly interrelated with fluids. This highly necessitates the study of fluids.

Fluids can be found in liquid or Gaseous state.

Fluids, in its Engineering sense, can be defined as a material that shear constantly in the presence of a very small disturbance (Force and/or Gradient). Assume we pour water over a horizontal plate. The water will flow horizontally in all directions even if there is no gradient applied until it reaches stable position of a very minimal depth.

The study of fluid dynamics is a very important area and is very useful in the area of modeling and simulation of fluid flow.

1.2.2 Scale

Scale is a very important concept when studying a natural phenomenon.

It helps understand where to position oneself to look at his/her study. There are two major categories of scales, namely, microscopic and macroscopic scales.

If we see water with our naked eye, it is continuous and smooth, i.e. macro scale. But when looked under microscope, we see small discontinuity. Zooming it a bit more, it becomes more discontinuous. It somehow looks like a dense crowd in a subway. Again zooming it more, we see groups (large chunks) of circles grouped together and moving along with each other. Finally, if we zoom it enough, we can see groups with three circles joined and moving together. The three circles joined together are water molecules (H2O), with two hydrogen atoms and one oxygen atom (micro scale).

The scale below molecular level, i.e., molecules, atoms, subatomic particles are microscopic scale. The scale above which can be seen with the naked eye is commonly called macroscopic scale. In fluid mechanics, and also in solid mechanics, macroscopic level of study is performed. In fluid mechanics, as in the case of solid mechanics, materials are continuous and are thought of being composed of macroscopic elements (chunks). A chunk of fluid and solid, called a control volume, is used to study the overall property of fluids and solids respectively.

1.2.3 Frame of reference

In engineering mechanics, there are three types of frame of reference. The lagrangian, Eulerian and the Arbitrary-Lagrangian–Eulerian frame of reference [7].

- **Lagrangian Frame of Reference (L.F.R):-** In the lagrangian frame of reference, properties of material points are studied by tracing individual material elements. Let us assume there is a hypothetical grid of reference aligned with the material element. In the L.F.R., the reference grid is not stationary and deforms together with the domain.

  In fluids, the L.F.R. study can be implemented by using streak lines (dyes). The movement of the dye in the fluid is assumed to be a particle of fluid to be studied. There are some cases that the lagrangian approach can be used. In cases of solid mechanics, since the particles undergo very small deformations, we can allow the reference grid to deform along with the body. Hence, the Lagrangian frame of reference is preferred.

- **Eulerian Frame of Reference(EFR):-** In the EFR case, we assume a stationary reference grid to monitor properties at a specific point and time. Normally, in fluids, it is very difficult and can be unnecessary to track infinitely many fluid particles. Hence, the Lagrangian frame of reference cannot be used.

  Instead, the Eulerian frame of reference studies properties of fluids at a specific space and time, which makes it convenient to study fluids. It is performed by tracing properties of the fluid at each stationary grid point.

- **Arbitrary Eulerian Lagrangian Frame of Reference (ALE)**:

  We have seen that in case of the Lagrangian Frame of Reference, the reference grid moves independently and in the case of the Eulerian Frame of Reference, the grid is stationary.

In the case of the Arbitrary Eulerian Lagrangian Frame of Reference, the reference grid moves independently with the material element. This type of frame of reference is called the ALE. The ALE frame of reference is widely used in the study of fluid–structure Interaction problems.

1.2.4 Types of flow

1.2.4.1 Laminar and Turbulent Flow

Flows can be broadly classified as Laminar and Turbulent flows. At a certain velocity and viscosity, flows are stable and have a defined property. The viscosity tend to dissipate the velocity and pressure terms and take the responsibility of calming the flow that it has a defined property, which is termed as **Laminar flow** [1, 8].

But when the viscosity is small as compared to that of velocity of the flow, the flow shall have unpredictable and chaotic property, thereby called **Turbulent flow** [1, 8].

The relation between the velocity and viscosity of a flow can be described by a dimensionless number called Reynold's Number, which is given as the ratio of Velocity times a length scale to that of viscosity.

$$\mathrm{Re} = \frac{\rho U l}{\mu} \tag{5}$$

Where Re is the Reynold's Number, ρ is the density of the fluid, U is the velocity, l is the Length scale, and μ is the dynamic Viscosity.

As the Re is below a certain value for a specific value for a certain flow, the flow is classified as a Laminar Flow, and is classified as Turbulent when Re is higher than the stipulated value for the specific type of flow.

1.2.4.2 Viscous and Inviscid Flow

A fluid is formally termed viscous if the shear stress is directly proportional to the shear strain rate. In solids, materials with stress directly proportional to strain are called Elastic material, or said to obey Hook's Law.

A fluid is inviscid if shear stress is not directly proportional to shear strain rate. By the same token, solids are termed as Inelastic if stress and strain are not directly related.

Viscous fluids exhibit nonlinear behavior. This can intuitively be demonstrated by the feel someone will have if s/he spills a ketchup or oil and water. Water is relatively in viscid while oil or ketchup is viscous.

Viscosity dissipates energy thereby, stabilizes a fast moving Laminar flow [1, 2, 8–12]. But, it can some destabilize a flow more in some Turbulent flow.

While viscosity dissipates energy, elastic materials store energy.

2. Conservation laws

In fluid mechanics, for convenience, individual fluid particles are called fluid particles. And a set of fluid particles comprise a fluid element or a fluid system. Therefore, one fluid element can comprise of many fluid particles, as can be seen in **Figure 1**.

To find answers to different physical problems that arise in fluids, we can simply apply the fundamental laws of physics. But, the problem is that, physical laws are obtained in the Lagrangian frame of reference form. Therefore, it is necessary to customize it to the Eulerian frame of reference. To do that, we use the famous the Reynolds Transport Theorem.

2.1 Reynold's transport equation

Many of Fluid Dynamics problems are of interest in understanding and solving what is currently going on at a specific point and time, rather than tracking particles (Eulerian rather than Lagrangian Study).

But unfortunately, Physical laws like Newton's laws can be applied in the Lagrangian Frame of Reference (**Figure 2**).

The Reynolds transport theorem converts Eulerian Study into the Lagrangian one so that the physical laws can be customized to the Eulerian frame of reference.

The above figure shows a material element (fluid element) in a motion. At time t, the fluid element was at position 1. And after a time increase of Δt,i.e. at time $t + \Delta t$, it moves to position 3. In the moving process, the element t and $t + \Delta t$ intersected at position 2.

Now, let N be any arbitrary extensive property. Then,

$$N_t = (N_1)_t + (N_2)_t, \text{ and } N_{t+\Delta t} = (N_2)_{t+\Delta t} + (N_3)_{t+\Delta t}. \tag{6}$$

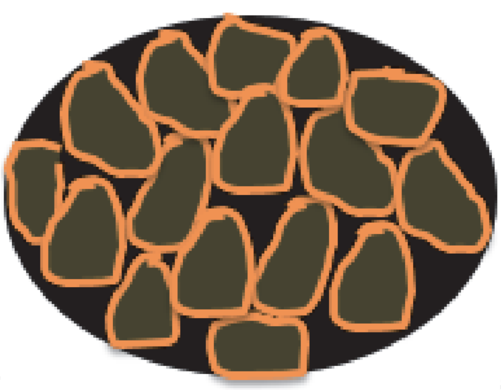

Figure 1.
Schematic diagram for fluid particle and fluid element.

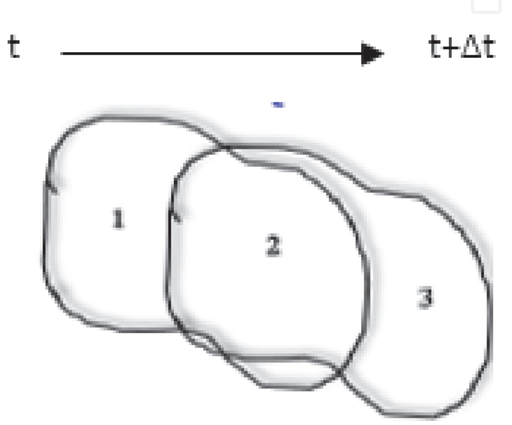

Figure 2.
Schematic of fluid particle motion to visualize Reynolds transport.

The rate of change of property N with respect to time t is given as

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N_{t+\Delta t} - N_t}{\Delta t}, \tag{7}$$

Solving Eq. (2), we can obtain

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{(N_2)_{t+\Delta t} - (N_2)_t}{\Delta t} + (N_3)_{t+\Delta t} - (N_1)_t \tag{8}$$

Here, region 2 is our control volume. And region 1 is property ready to enter the control volume and region 3 is a region leaving the control volume [7].

In Eq. (4), the first term on the right hand side is the rate of *change of material property N with respect to time. The second term is the outflow from the control volume and the third term is the inflow to the control volume.

Therefore, Eq. (4) can be read as the rate if change of a material property N with respect to time t is equal to the rate of increase of N in the control volume plus the net flux i.e. Outflow minus the inflow rate of the system.

Rate of change of N = Net rate of change w.r.t. time + Net flux into and out of C.V.

Net flux >0 if inflow is less than outflow and.

Net flux <0 if inflow is greater than outflow.

Now, let us introduce a derived property Φ, which is the rate of N per unit mass.

$$\mathrm{N} = \Phi * \mathrm{m}.$$

Flux is written in terms of control surface instead of control volume. Therefore, using divergence theorem, the net flux is given by

$$\int_{cs} \rho\phi u.nds \tag{9}$$

Where u is the velocity of the fluids and n is the outward unit normal.

The outward unit normal is normal to the surface since only the normal components of the flux terms enter and leave the control volume.

Hence, the general transport equation of extensive property Φ is

$$\frac{d}{dt}\int_{v} \rho\phi dv = \frac{\partial}{\partial t}\int_{cv} \rho\phi dv + \int_{cs} \rho\phi u.nds \tag{10}$$

2.2 Conservation of mass

The conservation of mass states that mass of fluids in a system is conserved [1, 2, 8–12].

$$\text{Rate of Mass increase in a fluid element} = \text{Net rate of flow of mass in the fluid.}$$

So using the transport equation, we can derive the conservation of mass general equation.

Now, let the property N be mass m. Therefore, our desired property Φ be m/m which is equal to 1. Therefore, plugging this into Eq. (10) yields,

$$\frac{d}{dt}\int_{v} \rho dv = \frac{\partial}{\partial t}\int_{cv} \rho dv + \int_{cs} \rho u.nds \tag{11}$$

But, since mass is conserved, the term on the left hand side is zero, i.e. the net rate of change of mass is zero. Therefore,

$$\frac{\partial}{\partial t}\int_{cv}\rho dv + \int_{cs}\rho u.nds = 0 \tag{12}$$

To write Eq. (12) in compact form, the flux term can be written in the form of control volume, instead of control surface. Hence,

$$\int_{cs}\rho u.nds = -\int_{cv}\nabla.\rho udv \tag{13}$$

$$\frac{\partial}{\partial t}\int_{cv}\rho dv + \int_{cv}\nabla.\rho udv = 0 \tag{14}$$

Finally, the compact form of conservation of mass is given by

$$\int_{cv}\left[\frac{\partial\rho}{\partial t} + \nabla.\rho u\right]dv = 0 \tag{15}$$

Physical Intuition method of deriving the conservation of Mass Equation.

We describe the behavior of the fluid in terms of macroscopic properties, such as velocity, pressure, density and temperature, and their space and time derivatives. These may be thought of as averages over suitably large numbers of molecules. A fluid particle or point in a fluid is then the smallest possible element of fluid whose macroscopic properties are not influenced by individual molecules. We consider such a small element of fluid with sides δ x, δ y and δ z.

From **Figure 3**, we can see that there are six faces labeled as N, S, W, E, T and B. The positive directions are given in the figure (**Figure 4**).

We should notice that all properties are functions of space coordinates X, Y, Z and time component t.

The element under consideration is so small that fluid properties at the faces can be expressed accurately enough by means of the first two terms of a Taylor series expansion. Let N be an arbitrary material property, then N at the W and E faces, which are both at a distance of 1/2 δ x from the element center, can be expressed as

$$N - \frac{\partial N}{\partial x} * \frac{1}{2}\delta x \text{ and } N + \frac{\partial N}{\partial x} * \frac{1}{2}\delta x. \tag{16}$$

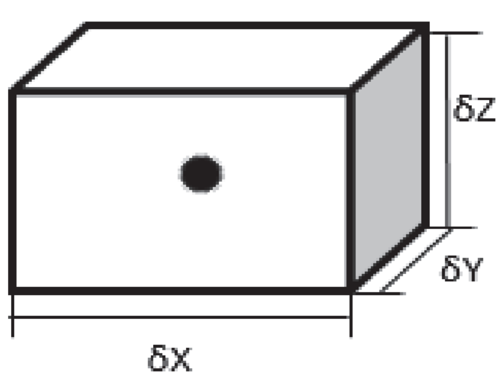

Figure 3.
Infinitesimal fluid element.

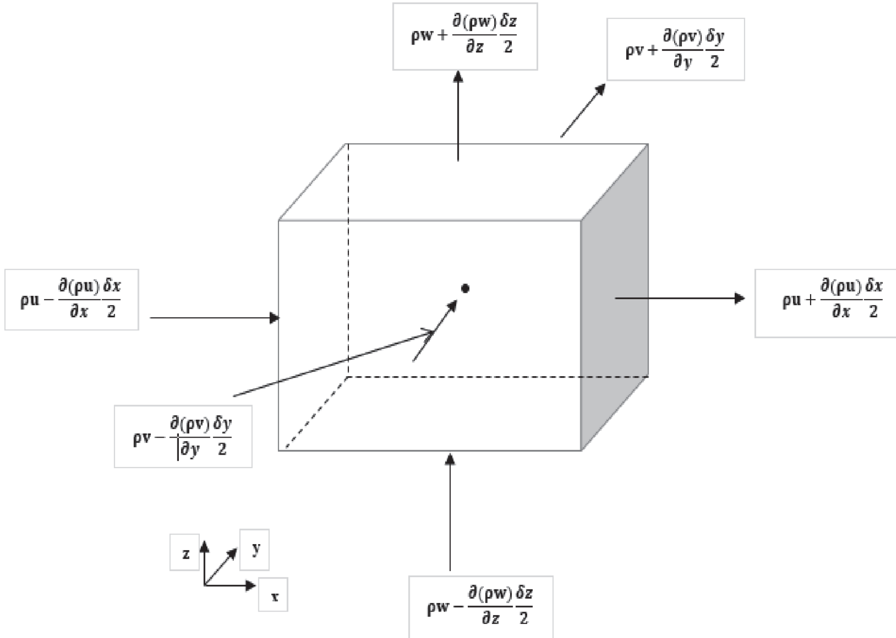

Figure 4.
Infinitesimal fluid element of mass transfer.

The net rate of increase of fluid element is given by

$$\frac{\partial m}{\partial t} = \frac{\partial(\rho dv)}{\partial t} \tag{17}$$

$$\frac{\partial(\rho dv)}{\partial t} = \frac{\partial \rho}{\partial t}(\delta x \delta y \delta z) \tag{18}$$

Assuming inflow to the fluid element to be positive and outflow to be negative. The net flow rate into and out of the fluid element is given as

$$\begin{aligned} &\left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z \\ &+\left[\rho v - \frac{\partial(\rho v)}{\partial y}\frac{\delta y}{2}\right]\delta x \delta z - \left[\rho v + \frac{\partial(\rho v)}{\partial y}\frac{\delta y}{2}\right]\delta x \delta z \\ &+\left[\rho w - \frac{\partial(\rho w)}{\partial z}\frac{\delta z}{2}\right]\delta x \delta y - \left[\rho w + \frac{\partial(\rho w)}{\partial z}\frac{\delta z}{2}\right]\delta x \delta y \end{aligned} \tag{19}$$

Therefore, equating Eq. (18) and (19), we obtain

$$-\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = \frac{\partial \rho}{\partial t} \tag{20}$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0 \tag{21}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0 \tag{22}$$

Eq. (21) is an unsteady, three dimensional mass conservation or continuity equation.

In case of incompressible fluids like water, the density do not change with time and space and hence Eq. (35) can be reduced to

$$\nabla.\text{u} = 0 \tag{23}$$

In the long hand notation, the equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{24}$$

Rates of change following a fluid particle and for a fluid element.

Let the value of a property per unit mass be denoted by Φ. The total or substantive derivative of Φ with respect to time following a fluid particle, written as D Φ/Dt, is

$$\frac{D\phi}{Dt} = \frac{d\phi}{dt} + \frac{d\phi}{dx}\frac{dx}{dt} + \frac{d\phi}{dy}\frac{dy}{dt} + \frac{d\phi}{dz}\frac{dz}{dt} \tag{25}$$

Here, dx/dt = u, dy/dt = v, dz./dt = w. and hence,

$$\frac{D\phi}{Dt} = \frac{d\phi}{dt} + u\frac{d\phi}{dx} + v\frac{d\phi}{dy} + w\frac{d\phi}{dz} \tag{26}$$

$$\frac{D\phi}{Dt} = \frac{d\phi}{dt} + u.grad(\phi) \tag{27}$$

Where u, v and w are velocity in the x, y, and z direction.

$\frac{D\phi}{Dt}$ Is a property that is defined as the property per unit mass. But, we are interested in developing equations of rates of change per unit volume. Therefore. By multiplying the term $\frac{D\phi}{Dt}$ by density, we can obtain rates of change per unit volume.

Therefore,

$$\rho\frac{D\phi}{Dt} = \rho\left[\frac{d\phi}{dt} + u.grad(\phi)\right] \tag{28}$$

The generalization of these terms for an arbitrary conserved property is

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla.\rho\phi u = 0 \tag{29}$$

The above equation Eq. (41) expresses the rate of change in time of Φ per unit volume plus the net flow of Φ out of the fluid element per unit volume. It is now rewritten to illustrate its relationship with the substantive derivative of Φ.

$$\begin{gathered}\frac{\partial(\rho\phi)}{\partial t} + \nabla.\rho\phi u = \rho\left[\frac{\partial(\phi)}{\partial t} + \nabla.\phi u\right] + \phi\left[\frac{\partial(\rho)}{\partial t} + \nabla.\rho u\right] \\ \nabla.\phi u = \frac{\partial(\phi u)}{\partial x} + \frac{\partial(\phi v)}{\partial y} + \frac{\partial(\phi w)}{\partial z}\end{gathered} \tag{30}$$

$$\frac{\partial(\phi u)}{\partial x} + \frac{\partial(\phi v)}{\partial y} + \frac{\partial(\phi w)}{\partial z} = [u + v + w].\left[\frac{\partial(\phi)}{\partial x} + \frac{\partial(\phi)}{\partial y} + \frac{\partial(\phi)}{\partial z}\right]$$

$$\nabla.u\phi = u.\nabla\phi$$

But since the second term on the right hand side is the conservation of mass equation which is zero, therefore

$$\rho \frac{D\phi}{Dt} = \frac{\partial(\rho\phi)}{\partial t} + \nabla.\rho\phi u = \rho\left[\frac{\partial(\phi)}{\partial t} + u.\nabla\phi\right] \tag{31}$$

2.3 Conservation of momentum

The conservation of Momentum states that the sum of the rate of change of momentum on a fluid particle is equal to the sum of forces on the particle. This is basically Newton's second law.

$$\begin{aligned}&\text{Rate of change of Momentum On a fluid particle}\\&\quad = \text{Sum of forces on a fluid particle}\end{aligned}$$

Our property N is now momentum P. Therefore, P = m*u. Therefore, Φ is the velocity u since momentum is equal to mass times velocity.

From Reynolds's transport theorem, we can obtain

$$\begin{gathered}\frac{\partial}{\partial t}\int_{cv} \rho u dv + \int_{cs} \rho u.u.nds = 0\\ \int_{cv}\left[\frac{\partial(\rho u)}{\partial t} + \nabla.\rho uu\right] = 0\end{gathered} \tag{32}$$

Eq. (32), which is the integral form is used for the fluid element.

For fluid particle, we can use the differential form as.

Table 1. Conservation equations in vector form

$$\left[\frac{\partial(\rho u)}{\partial t} + \nabla.\rho uu\right] = 0 \tag{33}$$

But Eq. (33) deals with the rate of change of Momentum. Now we shall see the force components of the equation.

2.3.1 Types of forces on fluid particles

There are generally two types of forces on fluids.

Surface Forces: Are type of forces that are applied on surfaces (area). Some of the Surface forces pressure forces, viscous and the like [8].

Body Forces: Are type of forces that are applied on volumes. Some of the Body forces gravity forces, electromagnetic forces, centrifugal forces [1, 8].

There are nine viscous stress terms as state of stress and one pressure term as can be seen on **Figure 5**.

X-Momentum	U	$\rho\frac{Du}{Dt}$	$\left[\frac{\partial(\rho u)}{\partial t} + \nabla.\rho uu\right]$
Y-Momentum	V	$\rho\frac{Dv}{Dt}$	$\left[\frac{\partial(\rho v)}{\partial t} + \nabla.\rho vu\right]$
Z-Momentum	W	$\rho\frac{Dw}{Dt}$	$\left[\frac{\partial(\rho w)}{\partial t} + \nabla.\rho wu\right]$
Energy	E	$\rho\frac{DE}{Dt}$	$\left[\frac{\partial(\rho E)}{\partial t} + \nabla.\rho Eu\right]$

Table 1.
Vector notation of conservation Laws.

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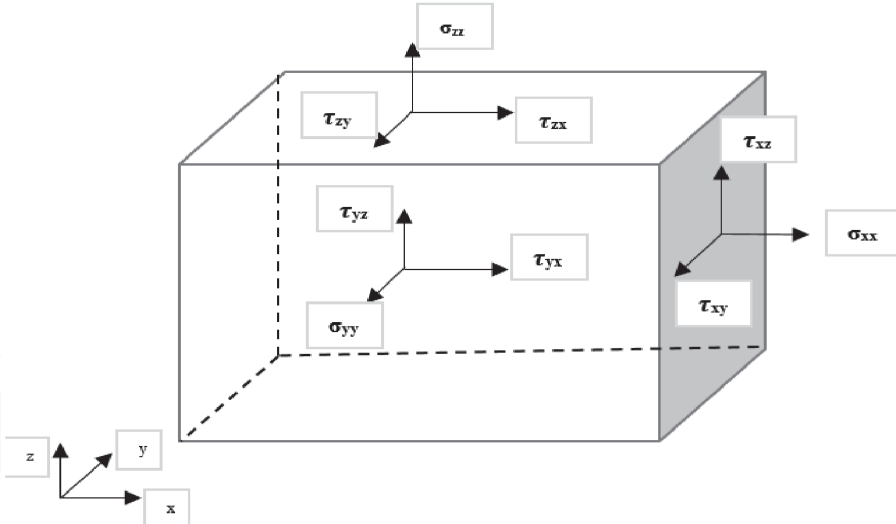

Figure 5.
Infinitesimal fluid element of momentum transfer with shear stress.

The pressure, a normal stress, is denoted by p. Viscous stresses are denoted by τ. The usual suffix notation τ_{ij} is applied to indicate the direction of the viscous stresses. The suffices i and j in τ_{ij} indicate that the stress component acts in the j direction on a surface normal to the I $_{th}$-direction [1, 8] (**Figure 6**).

Therefore, the summation of forces in the X direction is given below.

Summation for the Pressure term is given as

$$\left[\left[P - \frac{\partial P}{\partial x}\frac{\delta x}{2}\right] - \left[P + \frac{\partial P}{\partial x}\frac{\delta x}{2}\right]\right]\delta y \delta z \tag{34}$$

Eq. (46) can be reduced to

$$-\frac{\partial P}{\partial x}\delta x \delta y \delta z \tag{35}$$

Similarly, for the viscous shear stress along the X-direction at the east and west sides is given as

$$\left[\left[\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\frac{\delta x}{2}\right] - \left[\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}\frac{\delta x}{2}\right]\right]\delta y \delta z \tag{36}$$

$$\frac{\partial \tau_{xx}}{\partial x}\delta x \delta y \delta z \tag{37}$$

Similarly, for the viscous shear stress along the X-direction at the North and South sides is given as

$$\left[\left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right] - \left[\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right]\right]\delta x \delta z \tag{38}$$

$$\frac{\partial \tau_{yx}}{\partial y}\delta x \delta y \delta z \tag{39}$$

Again for the viscous shear stress along the X-direction at the Top and Bottom sides is given as

$$\left[\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right] - \left[\tau_{yx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]\right]\delta x \delta y \tag{40}$$

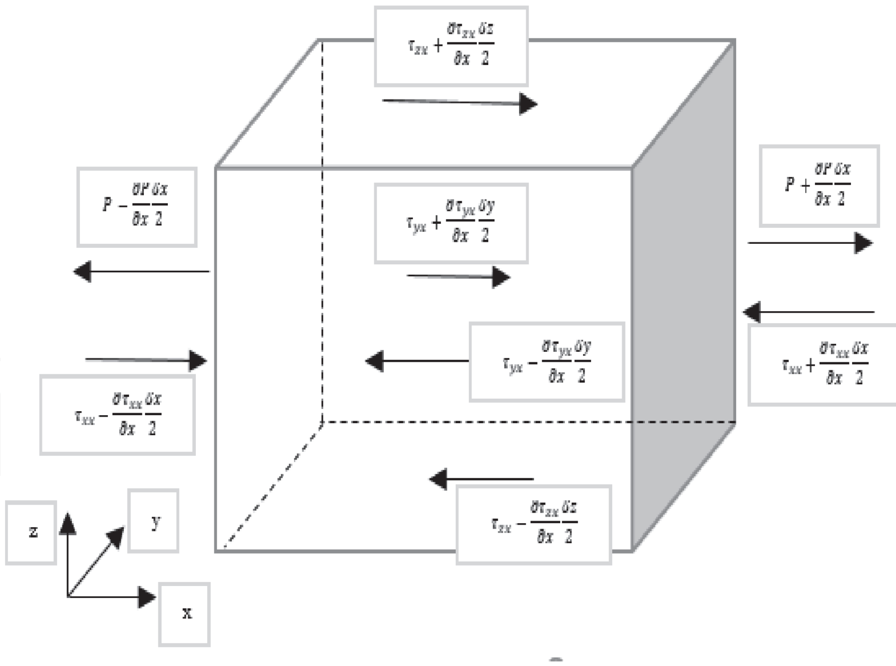

Figure 6.
Infinitesimal fluid element of momentum transfer with shear stress and pressure term.

$$\frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z \tag{41}$$

Therefore, the Total net force per unit volume along the X-axis is given as

$$\frac{\partial(-P+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \tag{42}$$

Finally, the General Conservation of Momentum Equation is given as

$$\frac{\partial(-P+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}=\rho\frac{Du}{Dx} \tag{43}$$

And the Final equation for X-Momentum is

$$\frac{\partial(-P+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+S_x=\frac{\partial(\rho u)}{\partial t}+\nabla.(\rho u u) \tag{44}$$

Similarly, Total Y-Momentum

$$\frac{\partial\left(\tau_{xy}\right)}{\partial x}+\frac{\partial\left(-P+\tau_{yy}\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+S_y=\frac{\partial(\rho v)}{\partial t}+\nabla.(\rho u v) \tag{45}$$

And for the Z-Momentum

$$\frac{\partial(\tau_{xz})}{\partial x}+\frac{\partial\left(\tau_{yz}\right)}{\partial y}+\frac{\partial(-P+\tau_{zz})}{\partial z}+S_z=\frac{\partial(\rho w)}{\partial t}+\nabla.(\rho u w) \tag{46}$$

The sign of the Pressure term is opposite to the stress term in the same direction because, normally, the sign convention for the normal tensile stress is positive. But, since the pressure term is compressive, their signs are opposite.

2.4 Conservation of energy equation in three dimension

The energy equation is derived from the first law of thermodynamics, which states that the rate of increase of energy on a particle is equal to the sum of the net rate of heat addition on to the fluid particle and the work done on the particle.

Rate of Energy on a fluid particle
=
Net rate of heat added
+
Work done on a fluid particle

The rate of increase of Energy of fluid per unit volume is given as

$$\left[\frac{\partial(\rho E)}{\partial t} + \nabla.\rho E u\right] = \rho\frac{DE}{Dt} \tag{47}$$

2.4.1 Work done on a fluid particle

Work done by the surface forces per unit volume are equal to the stress and pressure terms multiplied by the velocity. The sum of these terms (work done) can be obtained multiplying the terms we derived by the Momentum equation with the velocity components [8].

Pressure terms:

$$-\left[\frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z}\right] = -\nabla.(pu) \tag{48}$$

The work done due to the stresses is given as.

Total surface stress = Stress in X-direction + Stress in the Y Direction + Stress in the Z Direction

$$\left[\frac{\partial(u.\tau_{xx})}{\partial x} + \frac{\partial(u.\tau_{yx})}{\partial y} + \frac{\partial(u.\tau_{zx})}{\partial z}\right] + \left[\frac{\partial(\mathrm{v}.\tau_{xy})}{\partial x} + \frac{\partial(\mathrm{v}.\tau_{yy})}{\partial y} + \frac{\partial(\mathrm{v}.\tau_{zy})}{\partial z}\right] + \left[\frac{\partial(\mathrm{w}.\tau_{xz})}{\partial x} + \frac{\partial(\mathrm{w}.\tau_{yz})}{\partial y} + \frac{\partial(\mathrm{w}.\tau_{zz})}{\partial z}\right] \tag{49}$$

Therefore, the total work done on a fluid particle can be given as

$$-\nabla.(pu) + \left[\frac{\partial(u.\tau_{xx})}{\partial x} + \frac{\partial(u.\tau_{yx})}{\partial y} + \frac{\partial(u.\tau_{zx})}{\partial z}\right] + \left[\frac{\partial(\mathrm{v}.\tau_{xy})}{\partial x} + \frac{\partial(\mathrm{v}.\tau_{yy})}{\partial y} + \frac{\partial(\mathrm{v}.\tau_{zy})}{\partial z}\right] + \left[\frac{\partial(\mathrm{w}.\tau_{xz})}{\partial x} + \frac{\partial(\mathrm{w}.\tau_{yz})}{\partial y} + \frac{\partial(\mathrm{w}.\tau_{zz})}{\partial z}\right] \tag{50}$$

The first term is Energy Flux due to heat Conduction.

The second component that contributes to the rate of Energy addition is the rate of heat flux.

From **Figure 7**, we can see the relation

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = -\nabla.q \tag{51}$$

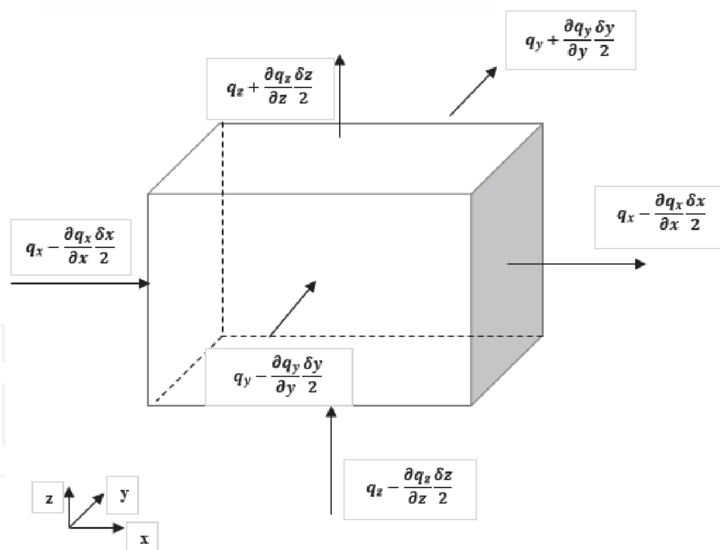

Figure 7.
Infinitesimal fluid element of heat energy.

But from Fourier's series, we can relate heat conduction to the temperature gradient as

$$q = -k\nabla T \tag{52}$$

Therefore, the energy addition due to heat is given as

$$div(k.grad.T) \tag{53}$$

2.4.2 Total energy equation

The energy equation mostly deals with the heat transfer analysis of a given system.

There are three main forms of energy namely, kinetic energy per unit mass 1/2 (u2 + v2 + w2), Internal (thermal) energy (i), and gravitational potential Energy [8]. The gravitational potential energy can be regarded as a source term as it does work when the fluid pass through a gravitational field.

As derived earlier, the energy equation is given as

$$\begin{aligned}\rho\frac{DE}{Dt} &= -\nabla.(pu) + \left[\frac{\partial(u.\tau_{xx})}{\partial x} + \frac{\partial(u.\tau_{yx})}{\partial y} + \frac{\partial(u.\tau_{zx})}{\partial z}\right] \\ &+ \left[\frac{\partial(\mathrm{v}.\tau_{xy})}{\partial x} + \frac{\partial(\mathrm{v}.\tau_{yy})}{\partial y} + \frac{\partial(\mathrm{v}.\tau_{zy})}{\partial z}\right] \\ &+ \left[\frac{\partial(\mathrm{w}.\tau_{xz})}{\partial x} + \frac{\partial(\mathrm{w}.\tau_{yz})}{\partial y} + \frac{\partial(\mathrm{w}.\tau_{zz})}{\partial z}\right] + div(k.grad.T) + \mathrm{SE}\end{aligned} \tag{54}$$

The mass and momentum conservation equations are solved for flow equations. Alongside them, energy equation is solved for heat transfer problems.

3. Navier–stokes equation

3.1 Basics of the Navier–stokes equation

The Navier- Stokes equation is a general equation that is used to understand properties of fluid flow. It is a core mathematical equation used to solve very important parameters like Pressure and velocity in the area of fluid Flow (fluid Engineering Simulation). It is basically the general conservation of momentum equation with the force (stress terms) more complicated and include viscosity terms as the viscosity induce stress on fluid particles.

One basic assumption here is that the fluid is considered isotropic. It does not behave differently at different points in nature.

In many fluid flows the viscous stresses can be expressed as functions of the local deformation rate or strain rate. In three-dimensional flows the local rate of deformation is composed of the linear deformation rate and the volumetric deformation rate.

And hence, the deformations can be found as

$$S_{xx} = \frac{\partial u}{\partial x}, S_{yy} = \frac{\partial v}{\partial y}, S_{zz} = \frac{\partial w}{\partial z} \tag{55}$$

These are the main stress terms.

For the linear shearing terms, which we have six of them, we have

$$S_{xy} = S_{yx} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right], S_{yz} = S_{zy} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right], S_{xz} = S_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \tag{56}$$

The volumetric deformation can be obtained by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = div(u) \tag{57}$$

In fluid flow, there are two viscous term constants of proportionality, namely the linear viscous term (μ), and the volumetric viscous constant (λ). But the volumetric viscosity constant can be considered negligible.

$$\tau_{xx} = \left[2\mu\left[\frac{\partial u}{\partial x}\right]\right] + \lambda\nabla.u, \tau_{yy} = \left[2\mu\left[\frac{\partial v}{\partial y}\right]\right] + \lambda\nabla.u, \tau_{zz} = \left[2\mu\left[\frac{\partial w}{\partial z}\right]\right] + \lambda\nabla.u \tag{58}$$

And for the linear shearing viscous stresses, we obtain

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right], \tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right], \tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \tag{59}$$

Inserting the general terms into the Navier stokes equation in the three dimensions yields,

$$\begin{aligned}
\rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \left[\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda\nabla.u\right]\right] + \frac{\partial}{\partial y}\left[\mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]\right] + \frac{\partial}{\partial z}\left[\mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]\right] + S_x \\
\rho\frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]\right] + \left[\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda\nabla.u\right]\right] + \frac{\partial}{\partial z}\left[\mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]\right] + S_y \\
\rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]\right] + \frac{\partial}{\partial y}\left[\mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]\right] + \left[\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda\nabla.u\right]\right] + S_z
\end{aligned} \tag{60}$$

Therefore, from this equation, we can rearrange the terms to obtain

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial z}\left[\mu\frac{\partial u}{\partial z}\right]\\&+\\&\frac{\partial}{\partial x}\left[\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\frac{\partial w}{\partial x}\right]+\lambda\nabla.u=[S_x]\end{aligned} \tag{61}$$

Eq. (73) can be summarized as

$$= \nabla.(\mu\nabla U) + Sx \tag{62}$$

Therefore, the general Navier Stokes equations in three dimensions can be written as

$$\begin{aligned}\rho\frac{Du}{Dt}&=-\frac{\partial P}{dx}+\nabla.(\mu\nabla U)+S_x\\\rho\frac{Dv}{Dt}&=-\frac{\partial P}{dy}+\nabla.(\mu\nabla V)+S_y\\\rho\frac{Dw}{Dt}&=-\frac{\partial P}{dz}+\nabla.(\mu\nabla W)+S_z\end{aligned} \tag{63}$$

3.2 Basics of transport equation

The Transport equation traces the transport of a flow property Φ, which can be pollutants or temperature. The differential form of the Transport Equation can be written as [1, 8].

$$\frac{\partial(\rho\phi)}{\partial t}+\nabla.(\rho\phi u)=\nabla.(\Gamma\nabla\phi)+\mathrm{S_x} \tag{64}$$

The first term, $\frac{\partial(\rho\phi)}{\partial t}$ mentions the rate of increase of Φ in a fluid element. The second term is a convective term that represents convective outflow transport. The third term to the right of the equality sign is a diffusive transport term to mean rate of increase of ϕ due to diffusion.

The last term is the increase of ϕ in a fluid element due to production from the source.

3.3 Vorticity stream function

The vorticity stream function is a method of reducing unknowns and solving the Navier–Stokes equation.

Generally, for a two dimensional flow, we usually use three equations, two momentum equations in X and Y directions and one is Conservation of mass (Continuity equation) for case of incompressible flow [13].

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{65}$$

$$\frac{du}{dt}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{dx}+\nu\frac{\partial^2u}{\partial x^2}+\nu\frac{\partial^2u}{\partial y^2}+S_{mx} \tag{66}$$

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$$\frac{dv}{dt} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{dy} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2} + S_{my} \tag{67}$$

Here, in Eq. (63), we have three equations and three unknowns namely u,v and P. But, we can reduce the number of equations and the number of unknowns.

To do that, we can introduce a vorticity term.

$$\begin{gathered} \omega = \nabla \times U \\ U = f_n(u, v, w, t) \end{gathered} \tag{68}$$

The vorticity term has three components,

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \omega_y = -\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right], \text{and} \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{69}$$

Here, ω_x and ω_y contain terms varying with respect to Z, which we are not considering. Therefore, we shall take ω_z since it is valid and have variations X and Y, and not Z.

Therefore, differentiating the whole Y momentum equation with respect to X and the X momentum equation with respect to Y,

$$\frac{\partial(\mathrm{x} - momentum)}{\partial y} - \frac{\partial(\mathrm{y} - momentum)}{\partial x} \tag{70}$$

and finally subtracting it yields,

$$\frac{dw}{dt} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + S_{mx} \tag{71}$$

Now let us introduce stream function to reduce the number of equations and unknowns. The Stream function combines velocity components u and v into one variable ψ. Let

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \tag{72}$$

Substituting Eq. (72) into (71), we obtain a single equation

$$\frac{\partial(\nabla^3 \psi)}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial(\nabla^2 \psi)}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial(\nabla^2 \psi)}{\partial y} = \nu[\nabla^4 \psi] \tag{73}$$

Which is a final vorticity stream function.

3.4 Classification of simple partial differential equations

Independent variable in a PDE can be either Temporal or Spatial, or only spatial in more than one dimension [8].

Let a PDE be of form

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} + D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial x} + F\phi + G = 0 \tag{74}$$

Let coefficients be a,b,c,d,e,f and g be constants to maintain Linearity to make it simple.

The behavior of PDEs can be determined from its higher derivative terms, as in this case, the second order derivatives (**Table 2**).

It can be seen that the discriminant of the higher order terms are more determinant factor as to how a physical phenomenon modeled by a PDE behave [8].

Reduced form of Transport, Advection equation and Navier Stokes equations can be put in a form of a Matrix. Determinants and Eigenvalues can then be obtained from the Matrix, which can inform about the type of equation (PDE).

$$\mathrm{Det}|A - \lambda I| = 0 \tag{75}$$

If $\lambda = 0$, the equation is Parabolic,

$\lambda \neq 0$ and all are of the same sign, Elliptic,

$\lambda \neq 0$ and all but one are of the same sign, Hyperbolic.

3.5 Classification of fluid flow

Different flow types can be categorized, based on their properties, into different PDEs. Some of the flow types and their corresponding equation types are defined below.

As can be seen from **Table 3**, type of equation of Inviscid flow is different from that of the Navier Stokes equation because of the absence of a higher order term (Viscosity term).

Mach number is considered the measure of Compressibility of a fluid. It is the ratio of Speed of the Fluid to that of Speed of sound.

Flow with Mach number greater than one is termed as Supersonic flow and subsonic flow if less than one.

b^2-4 ac	Equation Type	Characteristics
>0	Hyperbolic	Two Real Characteristics
<0	Elliptic	No Real Characteristics
=0	Parabolic	One Real Characteristic

Table 2.
Equation types and their real characterstics.

Flow Type	Steady Flow	Unsteady Flow
Viscous Flow	Elliptic	Parabolic
Thin Shear Layer Flow	Parabolic	Parabolic
Inviscid Flow	$M > 1$, Hyperbolic $M < 1$,Elliptic	Hyperbolic
Navier Stokes Equation	Elliptic	Parabolic
Energy Equation	Elliptic	Parabolic

Table 3.
Flow types and their corresponding equations.

4. Conclusion

Fundamental derivation and discussions of Fluid dynamics is discussed in this chapter.

Initially, Mathematical background of fluid to help the reader sprint toward the area of fluids is discussed.

Green's theorem, Stoke 's theorem and Divergence theorem, which are basic tools to manipulate and convert among Line, surface and Volume integral was discussed briefly. The Divergence theorem is widely applicable to convert flux terms (Volume to surface integral) is discussed.

Leibniz's Rule, which helps solve Integro-differential equation was also included in this section.

Consequently, background on fluids was highlighted, followed by basic concepts that helps to understand fluids fundamentally, including appropriate scale of study, frame of reference and basic flow types.

In the main part of this chapter, Reynold's transport equation, which helps customize Lagrangian physical laws to Eulerian frame of reference is discussed.

Conservation of Mass and Momentum, which are fundamentals of fluid dynamics are discussed.

Energy equation, which is used to study heat flow within fluids are then discussed.

Considering Incompressible viscous flow, the famous Navier–stokes equation is then derived and discussed.

Application of Vorticity stream function, which is mostly used to solve the Navier–Stokes equation by reducing variables was derived and discussed.

Consequently, classification and characteristics of Partial Differential equations namely Parabolic, Elliptic and Hyperbolic equations are discussed.

Finally, classification of flows and their corresponding types like the Navier-Stokes equation, Inviscid and viscous flows, Compressible and Incompressible flows are then discussed.

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Conflict of interest

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