Chapter

Global Existence of Solutions to a Class of Reaction–Diffusion Systems on $\mathbb{R}^n$

Salah Badraoui

Abstract

We prove in this work the existence of a unique global nonnegative classical solution to the class of reaction–diffusion systems

$$u_t(t,x) = a\Delta u(t,x) - g(u)v^m,$$
$$v_t(t,x) = d\Delta v(t,x) + \lambda(t,x)g(u)v^m,$$

where $a > 0, d > 0, t > 0, x \in \mathbb{R}^n, n, m \in \mathbb{N}^*$, λ is a nonnegative bounded function with $\lambda(t,.) \in BUC(\mathbb{R}^n)$ for all $t \in \mathbb{R}_+$, the initial data $u_0, v_0 \in BUC(\mathbb{R}^n), g : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ is a of class C^1, $\frac{dg(u)}{du} \in L^\infty(\mathbb{R}), g(0) = 0$ and $g(u) \geq 0$ for all $u \geq 0$. The ideas of the proof is inspired from the work of Collet and Xin who proved the same result in the particular case $d > a = 1, \lambda = 1, g(u) = u$. Moreover, they showed that the L^∞-norm of v can not grow faster than $O(\ln \ln t)$ for any space dimension.

Keywords: reaction–diffusion systems, local existence, positivity, comparison principle, global existence

1. Introduction

In the sequel, we use the notations.

$\mathbb{R}_+ = [0,\infty[, \ \mathbb{R}_+^* =]0,\infty[$.

$\mathbb{N} = \{0, 1, \dots\}$ the set of natural numbers and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

For $p \in \mathbb{R}$: $[p]$ the integer part of p.

For $n \in \mathbb{N}^*$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|^2 = \sum_{j=1}^n x_j^2$.

$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ the set of integers.

For $x^{(0)} \in \mathbb{R}^n$ and $\rho \in \mathbb{R}_+^*$, :

$B'\left(x^{(0)}, \rho\right) = \left\{x \in \mathbb{R}^n : \left|x - x^{(0)}\right| \leq \rho\right\}$ the closed ball of center $x^{(0)}$ and radius ρ.

$S\left(x^{(0)}, \rho\right) = \left\{x \in \mathbb{R}^n : \left|x - x^{(0)}\right| = \rho\right\}$ the boundary of $B'\left(x^{(0)}, \rho\right)$.

Let $Q \subset \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ a subset. ∂Q denote the boundary of Q.

ln: the natural logaritm function.

$\omega_n(\rho) = \frac{2\pi^{n/2}\rho^{n-1}}{\Gamma(n/2)}$ the surface area of $S(0,\rho)$, where $\Gamma(x) = \int_0^\infty e^{-t}t^{-x}dt \ \left(x \in \mathbb{R}_+^*\right)$ is the Gamma function.

$BUC(\mathbb{R}^n)$ the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$ with the supremum norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^n}|u(x)|$.

$X = BUC(\mathbb{R}^n) \times BUC(\mathbb{R}^n)$ which is a Banach space endowed with the norm $\|(u,v)\|_X = \|u\|_\infty + \|v\|_\infty$.

For $u\in L^p(\mathbb{R}^n)$ $(p\in[1,\infty[)$, we denote by $\|u\|_p^p = \int_{\mathbb{R}^n}|u|^p dx$.

For $u, v : \mathbb{R}^n \to \mathbb{R}$ two regular functions, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ and $\nabla u.\nabla v = \sum_{j=1}^n \frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_j}$.

Reaction-Diffuison equations are nonlinear parabolic partial differential equations arises in many fields of sciences like chemistry, physics, biology, ecology and even medicine. It appears usually as coupled systems.

The somewhat general form of these systems of two equations is

$$\begin{cases} u_t(t,x) = a\Delta u(t,x) + f_1(t,x,u,v), \\ v_t(t,x) = d\Delta v(t,x) + f_2(t,x,u,v), \end{cases}$$

where $t>0$, $x\in\Omega$ with $\Omega\subset\mathbb{R}^n (n\in\mathbb{N}^*)$ is an open set, Δ is the Laplacian operator, a, d are two real positive constants called the coefficients of the diffusion. For a chemical reaction where two substances S_1 and S_2, u and v represent their concentrations at time t and position x respectively, and f_1 and f_2 represent the rate of production of these substances in the given order. For more details see [1, 2].

In this chapter, we are concerned with the existence of global solutions to the reaction–diffusion system

$$u_t(t,x) = a\Delta u(t,x) - g(u)v^m, \quad (t,x)\in\mathbb{R}_+^* \times \mathbb{R}^n, \tag{1}$$

$$v_t(t,x) = d\Delta v(t,x) + \lambda(t,x)g(u)v^m, \quad (t,x)\in\mathbb{R}_+^* \times \mathbb{R}^n, \tag{2}$$

with initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x\in\mathbb{R}^n. \tag{3}$$

Whe assume that.

(H1) The constants a, d are such that $a,\ d\in\mathbb{R}_+^*$.

(H2) $\lambda : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a non-null, nonnegative and bounded function on $\mathbb{R}^+ \times \mathbb{R}^n$ such that $\lambda(t,.)\in BUC(\mathbb{R}^n)$ for all $t\in\mathbb{R}_+$. We denote $\lambda_\infty = \sup_{t\geq 0}(\|\lambda(t)\|_\infty)$.

(H3) n and m are positive integers, i.e. $n, m\in\mathbb{N}^*$.

(H4) $g : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ is a function defined on $BUC(\mathbb{R}^n)$ such that:

i. $g(0) = 0$ and $g(u)\geq 0$ pour $u\geq 0$.

ii. g is of class C^1 and $\frac{dg(u)}{du}$ is bounded on $\mathbb{R}$.

(H5) The initial data u_0, v_0 are nonnegative and are in $BUC(\mathbb{R}^n)$.

One of the essential questions for (1)–(3) is the existence of global solutions and possibly bounds uniform in time. Recently, Collet and Xin in their paper [3] published in 1996 have studied the system (1)–(3) but with $a = \lambda = 1$, $d>1$ and $\varphi(u) = u$. In this particular case, this system describes the evolution of u the mass fraction of reactant A and that v of the product B for the autocatalytic chemical reaction of the form $A + mB \to (m+1)B$. They proved the existence of global solutions and showed that the L^∞ norm of v can not grow faster than $O(\ln\ln t)$ for any space dimension.

If we replace $g(u)v^m$ by $u\exp\{-E/v\}$ where $E>0$ is a constant and take $\lambda = 1$, there are many works on global solutions, see Avrin [4], Larrouturou [5] for results in one space dimension, among others.

It is worth mentioning here the result of S. Badraoui [6] who studied the system

$$u_t = a\Delta u - uv^m,$$

$$v_t = b\Delta u + d\Delta v + uv^m,$$

where $a > 0$, $d > 0$, $b \neq 0$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$, $m \in 2\mathbb{N}^*$ is an even positive integer. He has proved that if u_0, v_0 are nonnegative and are in $BUC(\mathbb{R}^n)$ that:

If $a > d,\ b > 0,\ v_0 \geq \frac{b}{a-d} u_0$ on $\mathbb{R}^n$, then the solution is global and uniformly bounded.

If $a < d,\ b < 0,\ v_0 \geq \frac{b}{a-d} u_0$ on $\mathbb{R}^n$, then the solution is global.

Our work here is a continuation of the work of Collet and Xin [3]. We treat the same question in a slightly general case. Inspired by the same ideas in [3] we prove that the system (1)–(3) under the assumptions (H1) to (H5) has a unique global nonnegative classical solution.

The chapter is organized as follows: In section 2, we treat the existence of local solution and reveal its positivity using the maximum principle.

In section 3, firstly, we prove by a simple comparison argument that if $a \geq d$, the solution is uniformly bounded and we give an upper bound of it. Afterwards, we attack the hard case in which $a < d$ where we used the Lyapunov functional $L(u, v) = [\alpha + 2u - \ln(1+u)]e^{\varepsilon v}$ $(\alpha,\ \varepsilon > 0)$ and the cut-off function $\varphi(x) = \left(1 + |x|^2\right)^{-n}$. We show that for α sufficiently large and ε small enough we can control the L^p-norms of v $(p > \max\{1, n/2\})$ on every unit spacial cub in $\mathbb{R}^n$ from which we deduce the L^∞-norm of v at any time $t > 0$.

We emphazise here that I have engaged to calculate the constants encountered in all equations and inequalities exactly.

2. Existence of a local solution and its positivity

We convert the system (1)–(3) to an abstract first order system in the Banach space $X := BUC(\mathbb{R}^n) \times BUC(\mathbb{R}^n)$ of the form

$$\begin{cases} w'(t) = Aw(t) + F(w(t)), \quad t > 0, \\ w(0) = w_0 \in X. \end{cases} \tag{4}$$

Here $w(t) = (u(t), v(t))$; the operator A is defined as

$$Aw := \begin{pmatrix} a\Delta & 0 \\ 0 & d\Delta \end{pmatrix} w = (a\Delta u, d\Delta v),$$

where $D(A) := \{w = (u, v) \in X : (\Delta u, \Delta v) \in X\}$. The function F is defined as $F(w(t)) = (-\varphi(u(t))v^m(t), \lambda(t)\varphi(u(t))v^m(t))$.

It is known that for $c > 0$ the operator $c\Delta$ generates an analytic semigroup $G(t)$ in the space $BUC(\mathbb{R}^n)$:

$$G(t)u = (4\pi ct)^{-n/2} \int_{\mathbb{R}^n} \exp\left\{-\frac{|x-y|^2}{4ct}\right\} u(y)dy. \tag{5}$$

Hence, the operator A generates an analytic semigroup defined by

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}, \tag{6}$$

where $S_1(t)$ is the semigroup generated by the operator $a\Delta$, and $S_2(t)$ is the semigroup generated by the operator $d\Delta$.

Since the map F is locally Lipschitz in w in the space X, then proving the existence of a loacl classical solution on $[0, t_1]$ where $t_1 \in \mathbb{R}_+^*$ is standard [7, 8].

For the positivity, let $w(t) = (u(t), v(t))$ is a local solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ on the interval $[0, t_1]$.

We can write the first equation as

$$u_t - a\Delta u + \left[v^m \frac{d}{du} g(\xi)\right] u = 0, (t, x) \in]0, t_1] \times \mathbb{R}^n, \tag{7}$$

for some $\xi \in \mathbb{R}$. Thanks to the assumption (H4)-ii we deduce that $v^m \frac{\partial}{\partial u} g(\xi)$ is bounded on $[0, t_1] \times \mathbb{R}^n$. Whence, by the theorem 9 on page 43 in [9], we obtain that

$$u(t, x) \geq 0, \text{for all } (t, x) \in [0, t_1] \times \mathbb{R}^n, \tag{8}$$

The second equation can be written as

$$v_t - d\Delta v + \left[-\lambda g(u) v^{m-1}\right] v, \quad (t, x) \in]0, t_1] \times \mathbb{R}^n. \tag{9}$$

By the same theorem we get

$$v(t, x) \geq 0, \text{for all } (t, x) \in [0, t_1] \times \mathbb{R}^n. \tag{10}$$

For the existence of a global solution, we use the contraposed of the characterization of the maximal existence time $t_{\max}$ ([8] on page 193) as follows

$$\begin{bmatrix} \text{there exists a map } C : \mathbb{R}_+ \to \mathbb{R}_+ \text{such that :} \\ \|u(t)\|_\infty + \|v(t)\|_\infty \leq C(t) \text{ for all } t \in \mathbb{R}_+ \end{bmatrix} \Rightarrow t_{\max} = +\infty. \tag{11}$$

3. Existence of a global solution

For this task we will use the fact that the solution is nonnegative.

Theorem 3.1. Let (u, v) be the solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ and such that

$$a \geq d. \tag{12}$$

Then, the solution is global and uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^n$. More precisely, we have the estimates

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{for all } t \in \mathbb{R}_+, \tag{13}$$

$$\|v(t)\|_\infty \leq \|v_0\| + \lambda_\infty \left(\frac{a}{d}\right)^{n/2} \|u_0\|_\infty, \quad \text{for all } t \in \mathbb{R}_+. \tag{14}$$

Proof. By the comparison principle we get (13).

The solution (u, v) satisfies the integral equations

$$u(t,x) = S_1(t)u_0 - \int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau, \tag{15}$$

$$v(t,x) = S_2(t)v_0 + \int_0^t S_2(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau. \tag{16}$$

Here $S_1(t)$ and $S_2(t)$ are the semigroups generated by the operators $a\Delta$ and $d\Delta$ in the space $BUC(\mathbb{R}^n)$ respectively. As u is nonnegative, then from (15) we get

$$\int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau \le S_1(t)u_0. \tag{17}$$

Since $a \ge d$, using the explicit expression of $S_1(t-\tau)g(u(\tau))v^m(\tau)$ and $S_2(t-\tau)g(u(\tau))v^m(\tau)$, one can observe that (see [10])

$$\begin{aligned}\int_0^t S_2(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau &\le \left(\frac{a}{d}\right)^{n/2}\int_0^t S_1(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau \\ &\le \lambda_\infty \left(\frac{a}{d}\right)^{n/2}\int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau.\end{aligned} \tag{18}$$

From (17) and (18) into (16) we get

$$v(t) \le S_2(t)v_0 + \lambda_\infty \left(\frac{a}{d}\right)^{n/2} S_1(t)u_0. \tag{19}$$

This last inequality leads to the veracity of (14).

Thus, from (13) and (14), we deduce that the solution (u, v) is global and uniformly bounded on $\mathbb{R}_+ \times \mathbb{R}^n$. ◆

In the case where $d > a$, it seems that the idea of comparison cannot be applied. Nevertheless, we can prove the existence of global classical solutions; but it appears that their boundedness is not assured.

Theorem 3.2. Let (u, v) be the solution of the problem (1)–(3) with the assumptions $\{Hj\}_{j=1}^5$. If

$$a < d, \tag{20}$$

the solution (u, v) is global. More precisely we have the estimates (13) and (83).

Proof. In this case, it is not easy to prove global existence. But can derive estimates of solutions independent of t_1 by using the same method used in [3] and the same form of the functional used in [6] but with different coefficients.

We need some lemmas.

Lemma 3.3. Let (u, v) be the solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ on the local interval time $[0, t_1]$. Define the functional

$$L(u,v) = [\alpha + 2u - \ln(1+u)]e^{\varepsilon v} \quad \text{with } \alpha, \varepsilon \in \mathbb{R}_+^*. \tag{21}$$

Then for any $\varphi = \varphi(x)$ $(x \in \mathbb{R}^n)$ a smooth nonnegative function with exponential spacial decay at infinity, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx = d\int_{\mathbb{R}^n}\Delta\varphi L dx + (d-a)\int_{\mathbb{R}^n} L_1\nabla\varphi.\nabla u dx\\ -\int_{\mathbb{R}^n}\varphi\left[aL_{11}|\nabla u|^2+(a+d)L_{12}\nabla u\nabla v + dL_{22}|\nabla v|^2\right]dx\\ +\int_{\mathbb{R}^n}\varphi(\lambda L_2 - L_1)g(u)v^m dx,\end{aligned}\tag{22}$$

where

$$\begin{aligned}&L_1 \equiv \frac{\partial L}{\partial u} = \left(2-\frac{1}{1+u}\right)e^{\varepsilon v}, L_2 \equiv \frac{\partial L}{\partial v} = \varepsilon[\alpha + 2u - \ln(1+u)]e^{\varepsilon v},\\ &L_{11} \equiv \frac{\partial^2 L}{\partial u^2} = \frac{1}{(1+u)^2}e^{\varepsilon v}, L_{12} \equiv \frac{\partial^2 L}{\partial u\partial v} = \varepsilon\left(2-\frac{1}{1+u}\right)e^{\varepsilon v},\\ &L_{22} \equiv \frac{\partial^2 L}{\partial v^2} = \varepsilon^2[\alpha+2u - \ln(1+u)]e^{\varepsilon v}.\end{aligned}\tag{23}$$

Proof. Note that $L>0$, $L_1>0$, $L_2>0$, $L_{11}>0$, $L_{12}>0$ and $L_{22}>0$. We can differentiate under the integral symbol

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx = a\int_{\mathbb{R}^n}\varphi L_1 u dx + d\int_{\mathbb{R}^n}\varphi L_2\Delta v dx + \int_{\mathbb{R}^n}\varphi(\lambda L_2 - L_1)g(u)v^m dx.\tag{24}$$

Using integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi L_1\Delta u dx = \int_{\mathbb{R}^n}(\varphi L_1)\Delta u dx = -\int_{\mathbb{R}^n}\nabla(\varphi L_1)\nabla u dx = -\int_{\mathbb{R}^n}L_1\nabla\varphi\nabla u dx\\ -\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx - \int_{\mathbb{R}^n}\varphi L_{12}\nabla u\nabla v dx,\end{aligned}\tag{25}$$

In fact, let $\rho\in\mathbb{R}_+^*$, then we have by the Geen theorem

$$\begin{aligned}\int_{B'(0,\rho)}\varphi L_1\Delta u dx &= \int_{B'(0,\rho)}(\varphi L_1)\Delta u dx\\ &= -\int_{B'(0,\rho)}\nabla(\varphi L_1).\nabla u dx + \int_{S(0,\rho)}(\varphi L_1)\frac{\partial u}{\partial\nu}dx,\end{aligned}\tag{26}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative of u with respect to the unit outer normal ν to the boundary $S(0,\rho)$.

We have

$$\begin{aligned}\left|\int_{S(0,\rho)}(\varphi L_1(t))\frac{\partial u(t)}{\partial\nu}dx\right| &\le 2e^{\varepsilon\|v(t)\|_\infty}\left\|\frac{\partial u(t)}{\partial\nu}\right\|_\infty\int_{S(0,\rho)}\varphi dx\\ &\le 2e^{\varepsilon\|v(t)\|_\infty}\left\|\frac{\partial u(t)}{\partial\nu}\right\|_\infty\frac{1}{(1+\rho^2)^n}\frac{2\pi^{n/2}\rho^{n-1}}{\Gamma(n/2)}.\end{aligned}\tag{27}$$

From (27) we obtain

$$\lim_{\rho\to\infty}\int_{S(x_0,\rho)}[\varphi L_1(t)]\frac{\partial u(t)}{\partial\nu}dx = 0.\tag{28}$$

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We pass to the limit for $\rho \to \infty$ in (26) taking into account (28) we obtain (25). By the same way we get

$$\int_{\mathbb{R}^n} \varphi L_2 \Delta v dx = -\int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx - \int_{\mathbb{R}^n} \varphi L_{22} |\nabla v|^2 dx - \int_{\mathbb{R}^n} \varphi L_{12} \nabla u \nabla v dx, \tag{29}$$

$$\int_{\mathbb{R}^n} L \Delta\varphi dx = -\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx - \int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx. \tag{30}$$

From (30) we find that

$$\int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx = -\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx - \int_{\mathbb{R}^n} L \Delta\varphi dx. \tag{31}$$

From (25), (29) and (31) into (24) we get our basic identity (22). ◆

Lemma 3.4. There exist two positive real constants $\alpha = \alpha(a, d, \gamma_1, \|u_0\|_\infty)$ and $\varepsilon = \varepsilon(a, d, \gamma_1, \gamma_2, \lambda_\infty, \|u_0\|_\infty)$ such that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \varphi L dx \le & d\int_{\mathbb{R}^n} L\Delta\varphi dx + (d-a)\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx \\ & -\gamma_1 \int_{\mathbb{R}^n} \varphi \left[a L_{11} |\nabla u|^2 + d L_{22} |\nabla v|^2\right] dx - \gamma_2 \int_{\mathbb{R}^n} \varphi L_1 g(u) v^m dx, \end{aligned} \tag{32}$$

where $\gamma_1, \gamma_2 \in]0, 1[$ are two arbitrary constants.

Proof. We seek L such that

$$a L_{11} |\nabla u|^2 + (a+d) L_{12} \nabla u \nabla v + d L_{22} |\nabla v|^2 \ge \gamma_1 \left[a L_{11} |\nabla u|^2 + d L_{22} |\nabla v|^2\right] \tag{33}$$

and

$$\lambda L_2 - L_1 \le -\gamma_2 L_1 \tag{34}$$

for $\gamma_1, \gamma_2 \in]0, 1[$.

The inequality (33) is satisfied if

$$\frac{(a+d)^2 L_{12}^2}{4ad(1-\gamma_1)^2 L_{11} L_{12}} \le 1. \tag{35}$$

From (23); (35), then (33) is satisfied if

$$\alpha \ge \frac{(a+d)^2 \left[1 + 2\|u_0\|_\infty\right]^2}{4ad(1-\gamma_1)^2}. \tag{36}$$

Also, (34) is satisfied if $\frac{\varepsilon\lambda_\infty\left(\alpha + 2\|u_0\|_\infty\right)}{1-\gamma_2} \le 1$, i.e. $\varepsilon \le \frac{1-\gamma_2}{\lambda_\infty\left(\alpha + 2\|u_0\|_\infty\right)}$, and from (36) we get

$$0 < \varepsilon \le \frac{1-\gamma_2}{\lambda_\infty} \frac{4ad(1-\gamma_1)^2}{(a+d)^2\left[1 + 2\|u_0\|_\infty\right]^2 + 8ad(1-\gamma_1)^2 \|u_0\|_\infty}. \tag{37}$$

Whence, if α satisfies (36) and ε satisfies (37), we obtain (32). ◆

As a consequence of (33) we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le d\int_{\mathbb{R}^n} L\Delta\varphi dx + (d-a)\int_{\mathbb{R}^n} L_1\nabla\varphi.\nabla u dx - \gamma_1 a\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx. \tag{38}$$

Lemma 3.5. With the functional L defined in (21) and α, ε defined in (36) and (37) respectively and with the truncation function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{\left(1+|x-x_0|^2\right)^n}. \tag{39}$$

We have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le dk_1(n)\int_{\mathbb{R}^n}\varphi L dx + \frac{1}{4\gamma_1 a}(d-a)^2 k_2^2(n)\int_{\mathbb{R}^n}\varphi\frac{L_1^2}{L_{11}}dx, \tag{40}$$

where

$$k_1(n) = 2n(3n+2), k_2(n) = 2n. \tag{41}$$

Proof. Calulate $\Delta\varphi$ and estimate it

$$\Delta\varphi = -\frac{2n^2}{\left(1+|x-x_0|^2\right)^{n+1}} - \frac{4n(n+1)|x-x_0|^2}{\left(1+|x-x_0|^2\right)^{n+2}};$$

whence

$$|\Delta\varphi| \le 2n(3n+2)\varphi. \tag{42}$$

Calulate $\nabla\varphi$ and estimate it

$$|\nabla\varphi|^2 = 4n^2\frac{|x-x_0|^2}{\left(1+|x-x_0|^2\right)^{2(n+2)}};$$

whence

$$|\nabla\varphi| \le 2n\varphi. \tag{43}$$

Using the Cauchy-Schwarz inequality $\nabla\varphi.\nabla u \le |\nabla\varphi||\nabla u|$ and the inequalities (42) and (43) into (38) we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le dk_1(n)\int_{\mathbb{R}^n}\varphi L dx + (d-a)k_2(n)\int_{\mathbb{R}^n}\varphi L_1|\nabla\varphi| dx - \gamma_1 a\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx. \tag{44}$$

We pove that

$$(d-a)k_2(n)\varphi L_1|\nabla\varphi| - \gamma_1 a\varphi L_{11}|\nabla u|^2 \le \frac{1}{4\gamma_1}\frac{(d-a)^2}{a}k_2^2(n)\varphi\frac{L_1^2}{L_{11}}. \tag{45}$$

To do this, it sufficies to compute the discriminant of the trinoma in $|\nabla\varphi|$

$$\Delta = -\gamma_1 a\varphi L_{11}|\nabla u|^2 + (d-a)k_2(n)\varphi L_1|\nabla\varphi| - \frac{1}{4\gamma_1}\frac{(d-a)^2}{a}k_2^2(n)\varphi\frac{L_1^2}{L_{11}}.$$

From (45) into (42) we find the desired result (40). ◆

Lemma 3.6. For α and ε defined in (36) and (37) respectively and for all real constant γ

$$\gamma \geq \max\left\{\frac{1}{a}, 8\|u_0\|_\infty + 4\right\}, \tag{46}$$

we have

$$\int_{\mathbb{R}^n}\varphi L dx \leq \beta e^{\sigma t}, \text{for all} \, t \in \mathbb{R}_+; \tag{47}$$

where

$$\beta = \frac{2}{n}\left(\alpha + 2\|u_0\|_\infty\right)\omega_n e^{\varepsilon\|v_0\|_\infty}, \tag{48}$$

and

$$\sigma = dk_1(n) + \frac{\gamma}{4\gamma_1 a}(d-a)^2 k_2^2(n). \tag{49}$$

Proof. We seek a constant $\gamma \in \mathbb{R}_+^*$ such that

$$\frac{L_1^2}{L_{11}} \leq \gamma L, \text{for all } u \in \left[0, \|u_0\|_\infty\right]. \tag{50}$$

The inequality (50) is equivalent to $(2u+1)^2 e^{\varepsilon v} \leq \gamma[\alpha + 2u - \ln(1+u)]$. We prove that if γ satisfies (46) then (50) follows.

Whence, from (50) into (40) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \leq \left[dk_1(n) + \frac{\gamma}{4\gamma_1 a}(d-a)^2 k_2^2(n)\right]\int_{\mathbb{R}^n}\varphi L dx, \text{forall} \, t \in \mathbb{R}_+. \tag{51}$$

As

$$\int_{\mathbb{R}^n}\varphi L(t=0)dx = \int_{\mathbb{R}^n}\varphi[\alpha + 2u_0 - \ln(1+u_0)]e^{\varepsilon v_0}dx; \tag{52}$$

then, from (51) and (52) we get

$$\int_{\mathbb{R}^n}\varphi L dx \leq \left(\alpha + 2\|u_0\|_\infty\right)\|\varphi\|_1\left[\exp\left(\varepsilon\|v_0\|_\infty\right)\right]e^{\sigma t}, \text{for all} \, t \in \mathbb{R}_+, \tag{53}$$

where σ is defined by (49).

Now, let us estimate $\|\varphi\|_1$. We have ([11] on page 485)

$$\|\varphi\|_1 = \int_{\mathbb{R}^n}\varphi dx = \int_{\mathbb{R}^n}\frac{1}{\left(1+|x|^2\right)^n}dx = \omega_n\int_0^\infty r^{n-1}\frac{1}{(1+r^2)^n}dr.$$

As

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}dr = \int_0^1 \frac{r^{n-1}}{(1+r^2)^n}dr + \int_1^\infty \frac{r^{n-1}}{(1+r^2)^n}dr$$
$$\leq \int_0^1 r^{n-1}dr + \int_1^\infty \frac{1}{r^{n+1}}dr \leq \frac{2}{n},$$

then

$$\|\varphi\|_1 \leq \frac{2}{n}\omega_n. \tag{54}$$

Thus, from (54) in (53) we get the estimate (47) with β and σ given by (48) and (49). ◆

In the following step we trie to control the second component v of the solution on any unit spacial cube in the L^p- norms with $p \in [1, \infty[$.

Let $x^{(0)} = \left(x_1^{(0)}, \dots, x_n^{(0)}\right) \in \mathbb{R}^n$ be an arbitrary fixed point and

$$Q = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left|x_k - x_k^{(0)}\right| \leq \frac{1}{2}, \text{for all} \, k = 1, \dots, n \right\}. \tag{55}$$

Lemma 3.7. Let (u, v) be the solution of the problem in consideration. For α and ε satisfying (36) above and (63) below respectively, then for any unit cube Q of $\mathbb{R}^n$ of the form (55) we have

$$\int_Q v^p dx \leq \frac{\beta(p+1)^{p+1}}{\alpha\varepsilon^{[p]+1}}\left(\frac{4+n}{4}\right)^n e^{\sigma t}, \text{for all} (p, t) \in [1, \infty[\times \mathbb{R}_+. \tag{56}$$

Proof. It's obvious that

$$\varphi(x) \geq \left(\frac{4}{4+n}\right)^n, \text{for all} \, x \in \mathbb{R}^n, \tag{57}$$

and

$$e^{\varepsilon v} \geq \frac{\varepsilon^k}{k!}v^k, \text{for all} \, k \in \mathbb{N}^*. \tag{58}$$

Then

$$\int_{\mathbb{R}^n} \varphi L dx \geq \frac{\alpha\varepsilon^k}{k!}\left(\frac{4}{4+n}\right)^n \int_Q v^k dx. \tag{59}$$

Let us combine (47) and (59)

$$\int_Q v^k dx \leq \frac{\beta k!}{\alpha\varepsilon^k}\left(\frac{4+n}{4}\right)^n e^{\sigma t}, \text{for all} (k, t) \in \mathbb{N}^* \times \mathbb{R}_+. \tag{60}$$

By induction we prove that

$$k! \leq p^p, \text{for all} \, k \in \mathbb{N}^* \text{and} \, p \geq k \;. \tag{61}$$

Let $p \geq 1$ and $k = [p] + 1$, then we have by the imbedding theorem for L^p −spaces

$$\int_Q v^p dx \leq \left(\int_Q v^k dx\right)^{p/k}. \tag{62}$$

Taking ε enough small such that $\frac{\beta k!}{\alpha \varepsilon^k}\left(\frac{4+n}{4}\right)^n \geq 1$. Combining this with (37)

$$0 < \varepsilon \leq \min \left\{ \begin{array}{c} \frac{1-\gamma_2}{\lambda_\infty} \frac{4ad(1-\gamma_1)^2}{(a+d)^2\left[1+2\|u_0\|_\infty\right]^2 + 8ad(1-\gamma_1)^2\|u_0\|_\infty}, \\ \left[\frac{1}{\alpha}\beta k!\left(\frac{4+n}{4}\right)^n\right]^{1/k} \end{array} \right\}. \tag{63}$$

From (60), (61) and (63) into (62) we get (56). ◆

Lemma 3.8. Let Q_i et Q_j be two different unit cubes of center $x^{(i)} = \left(x_1^{(i)}, \dots, x_n^{(i)}\right)$ and $x^{(j)} = \left(x_1^{(j)}, \dots, x_n^{(j)}\right)$ respectively of the form

$$\begin{aligned} Q_i &= \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left|x_k - x_k^{(i)}\right| \leq 1/2\right\}, \text{for all } k = 1, \dots, n, \\ Q_j &= \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left|x_k - x_k^{(j)}\right| \leq 1/2\right\}, \text{for all } k = 1, \dots, n, \end{aligned} \tag{64}$$

with $x^{(j)} = x^{(i)} + l$, where $l = (l_1, \dots, l_n) \in \mathbb{Z}^n \backslash 0_{\mathbb{Z}^n}$. Then, there exists a positive constant

$$\delta(n) = \left(2 + \sqrt{n}\right)^2, \tag{65}$$

such that

$$dist\left(x^{(i)}, Q_j\right)^2 \leq \left|x^{(i)} - y\right|^2 \leq \delta(n) dist\left(x^{(i)}, Q_j\right)^2, \text{for all } y \in Q_j. \tag{66}$$

Proof. By Pythagorean theorem we have

$$\left|x^{(j)} - y\right| \leq \frac{\sqrt{n}}{2}. \tag{67}$$

As $\left|x^{(i)} - x^{(j)}\right| \geq 1$, then from (67)

$$\left|x^{(j)} - y\right| \leq \frac{\sqrt{n}}{2}\left|x^{(i)} - x^{(j)}\right|. \tag{68}$$

Also, it's clear that $dist\left(x^{(i)}, Q_j\right) = dist\left(x^{(i)}, \partial Q_j\right)$, but every point $z = (z_1, \dots, z_n) \in \partial Q_j$ is of the form

$$z = x^{(j)} + s, \tag{69}$$

where $s = (s_1, \dots, s_n) \neq 0$ and $s_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, for all $k = 1, \dots, n$ with at least one of the $s_k \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

It's easy to prove that

$$\left|x_k^{(j)} - x_k^{(i)}\right| \leq 2\left|x_k^{(j)} - x_k^{(i)} + s_k\right|, \text{for all } k = 1, \dots, n. \tag{70}$$

Then

$$\left|x^{(j)} - x^{(i)}\right| \leq 2dist\left(x^{(i)}, Q_j\right). \tag{71}$$

As $\left|x^{(i)} - y\right| \leq \left|x^{(i)} - x^{(j)}\right| + \left|x^{(j)} - y\right|$ we get from (68) and (71) the estimate

$$\begin{aligned}\left|x^{(i)} - y\right| &\leq 2dist\left(x^{(i)}, Q_j\right) + \frac{\sqrt{n}}{2}\left|x^{(i)} - x^{(j)}\right| \\ &\leq \left(2 + \sqrt{n}\right)dist\left(x^{(i)}, Q_j\right).\end{aligned} \tag{72}$$

We have obviously

$$\left|x^{(i)} - y\right| \geq dist\left(x^{(i)}, Q_j\right). \tag{73}$$

From (71) and (73) we get (66). ◆

Proof of theorem 3.2.

Let $x \in \mathbb{R}^n$ an arbitrary point and $\left\{Q_j\right\}_{j \in \mathbb{N}}$ be the family of pairwise disjoint measurable cubes of the form (64) covering $\mathbb{R}^n$ such that the center of Q_0 is $x^{(0)} = x$.

Firstly, using the fact that $\mathbb{R}^n = \cup_{j=0}^{\infty} Q_j$ and applying the left-hand inequality in (66)

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4d(t-s)}}\lambda g(u)v^m dy &= \sum_{j=0}^{\infty}\int_{Q_j} e^{-\frac{|x-y|^2}{8d(t-s)}}e^{-\frac{|x-y|^2}{8d(t-s)}}\lambda g(u)v^m dy \\ &\leq \sum_{j=0}^{\infty}\left\{e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}}\int_{Q_j}^{-\frac{|x-y|^2}{8d(t-s)}}\lambda g(u)v^m dy\right\}.\end{aligned} \tag{74}$$

By Hölder ineguality with $p > \max\left\{1, \frac{n}{2}\right\}$ and $q = 1 - \frac{1}{p}$

$$\int_{Q_j}^{-\frac{|x-y|^2}{8d(t-s)}}\lambda g(u)v^m dy \leq \left[\int_{Q_j}^{-\frac{q|x-y|^2}{8d(t-s)}}dy\right]^{1/q}\left[\int_{Q_j}\lambda^p g^p(u)v^{pm}dy\right]^{1/p}. \tag{75}$$

As

$$\int_{Q_j} e^{-\frac{q|x-y|^2}{8d(t-s)}}dy \leq \int_{\mathbb{R}^n}^{-\frac{q|x-y|^2}{8d(t-s)}}dy = \left(\frac{8\pi d}{q}\right)^{n/2}(t-s)^{n/2} \tag{76}$$

and by (56) we have

$$\int_{Q_j}\lambda^p g^p(u)v^{pm}dy \leq \lambda_\infty^p g_\infty^p \beta\frac{(pm+1)^{pm+1}}{\alpha\varepsilon^{[pm]+1}}\left(\frac{4+n}{4}\right)^n e^{\sigma t}, \tag{77}$$

where

$$g_\infty = \sup_{u \in \left[0, \|u_0\|_\infty\right]} g(u). \tag{78}$$

Then, from (76) and (77) into (75)

$$\int_{Q_j} {}^{-\frac{|x-y|^2}{8d(t-s)}} \lambda g(u) v^m dy \le K \left(\frac{8\pi d}{q} \right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} (t-s)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \lambda_\infty g_\infty e^{(\sigma/p)t}, \tag{79}$$

where

$$K = K(p, m, n, \alpha, \varepsilon) = \left[\beta \frac{(pm+1)^{pm+1}}{\alpha \varepsilon^{[pm]+1}} \left(\frac{4+n}{4} \right)^n \right]^{1/p}. \tag{80}$$

On the other hand, we deduce from the right-hand inequality in (66) that

$$\int_{Q_j} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \ge e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}}, \text{for all} j \in \mathbb{N}^*. \tag{81}$$

Then

$$\begin{aligned} \sum_{j=0}^{\infty} e^{-\frac{dist\left(x,Q_j\right)^2}{8d(t-s)}} &\le 1 + \sum_{j=1}^{\infty} e^{-\frac{dist\left(x,Q_j\right)^2}{8d(t-s)}} \le 1 + \sum_{j=1}^{\infty} \int_{Q_j} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \\ &\le 1 + \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \le 1 + [8\pi d\delta(n)]^{n/2} (t-s)^{n/2}. \end{aligned} \tag{82}$$

We have from (79) and (82) into (74)

$$\frac{1}{[4\pi d(t-s)]^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4d(t-s)}} \lambda g(u) v^m dy$$

$$\le 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} (t-s)^{-\frac{n}{2p}} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left\{ 1 + [8\pi d\delta(n)]^{n/2} (t-s)^{n/2} \right\}$$

$$\le 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left\{ (t-s)^{-\frac{n}{2p}} + [8\pi d\delta(n)]^{n/2} (t-s)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \right\}.$$

Whence

$$\int_0^t S_2(t-s) \lambda g(u) v^m ds$$

$$\le 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left[\frac{2p}{2p-n} t^{1-\frac{n}{2p}} + [8\pi d\delta(n)]^{n/2} \frac{2p}{p(n+2)+2p} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)+1} \right]$$

and finally we have for all $t \in \mathbb{R}_+$

$$\begin{aligned} &\|v(t)\|_\infty \le \|v_0\|_\infty \\ &+ 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left[\begin{array}{c} \frac{2p}{2p-n} t^{1-\frac{n}{2p}} \\ +[8\pi d\delta(n)]^{n/2} \frac{2p}{p(n+2)-n} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)+1} \end{array} \right] \end{aligned} \tag{83}$$

As $p > \max\left\{1, \frac{n}{2}\right\}$, the function in t on the right-hand side of the estimate (83) is continuous on $\mathbb{R}_+$. As $\|u(t)\|_\infty \leq \|u_0\|_\infty$ on $[0, t_{\max}[$ and v satisfied (83), we conclude from (11) that $t_{\max} = +\infty$. Whence, the solution is global. ♦

Remark. We can extend the system to the case where instead of v^m we put $vh(v)$ provided that.

i. $h : BUC(\mathbb{R}) \to BUC(\mathbb{R})$ is a locally continuous Lipschitz function, namely: for all constant $\rho \in \mathbb{R}_+$, there exists a constant $c(\rho) \in \mathbb{R}_+^*$ such that for all $u, v \in BUC(\mathbb{R}^n)$ with $\|u\|_\infty \leq \rho$ and $\|v\|_\infty \leq \rho$ we have

$$\|h(u) - g(v)\|_\infty \leq c(\rho)\|u - v\|_\infty.$$

ii. There exist two constants $M \in \mathbb{R}_+^*$ and $r \in \mathbb{N}$ such that:

$$0 \leq h(v) \leq Mv^r, \text{for all}\, v \in \mathbb{R}_+.$$

In this more general case, by examining the proof of the theorem 3.2; we see that under the same assumptions above, the system has also a global nonnegative classical solution. ♦

4. Illustrative example

To illustrate the previous study about global existence, we give the following reaction–diffusion system

$$\begin{cases} u_t(t,x) = a\Delta u(t,x) - \dfrac{c_1 u^3}{c_2 + c_3 u^2} v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ v_t(t,x) = d\Delta v(t,x) + c_4 e^{-c_5 t|x|^2} \dfrac{u^3}{c_2 + c_3 u^2} v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n \quad , \\ u(0,x) = u_0(x), v(0,x) = v_0(x), x \in \mathbb{R}^n, \end{cases} \tag{84}$$

where c_k, $k = 1, \dots, 4$ are real positive constants and c_5 is a real nonnegative constant. If $a, b \in \mathbb{R}_+$, $n, m \in \mathbb{N}^*$, $u_0, v_0 \in BUC(\mathbb{R}^n)$ and are nonnegative; the system (84) admits a unique global nonnegative classical solution $(u,v) \in C(\mathbb{R}_+; X) \cap C^1(\mathbb{R}_+^*; X)$. ♦

5. Conclusion and perspectives

We have prouved in the case where $a < d$ that the solution is global, but it remains an interesting question that if it is uniformly bounded or not.

As perspectives, we will replace the function $g = g(u)$ satisfying the hypothesis (H4) by the function $g(u) = u^r$ with $r \geq 1$ is a real constant and replace the term v^m by $e^{\alpha v}$ with $\alpha > 0$; namely that reaction term is of exponential growth. The system was studied on bounded domain by J. I. Kanel and M. Mokhtar in [12]. ♦

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Author details

Salah Badraoui
Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, University of 8 May 1945 Guelma, Guelma, Algeria

*Address all correspondence to: badraoui.salah@univ-guelma.dz

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