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Chapter

Effect of Additive Perturbations on the Solution of Reflected Backward Stochastic Differential Equations

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Abstract

This chapter has as a topic large class of general, nonlinear reflected backward stochastic differential equations with a lower barrier, whose generator, final condition as well as barrier process arbitrarily depend on a small parameter. The solutions of these equations which are obtained by additive perturbations, named the perturbed equations, are compared in the L^p-sense, $p \in]1, 2[$, with the solutions of the appropriate equations of the equal type, independent of a small parameter and named the unperturbed equations. Conditions under which the solution of the unperturbed equation is L^p-stable are given. It is shown that for an arbitrary $a > 0$ there exists $t(a) \leq T$, such that the L^p-difference between the solutions of both the perturbed and unperturbed equations is less than a for every $t \in [t(a), T]$.

Keywords: reflected, backward, stochastic, perturbation, estimate

1. Introduction

This chapter is dedicated to the problem of additive perturbations of reflected backward stochastic differential equations (shorter RBSDEs) with one lower barrier. Motivation for the topic comes from a large application of perturbation problems in real life problems from one side, and reflected backward stochastic differential equations in finance from another. Perturbed stochastic differential equations are widely applied in theory and in applications. Randomness from the environment can be introduced via stochastic models with perturbations. In such manner, complex phenomena under perturbations in analytical mechanics, control theory, population dynamics or financial models, can be compared and approximated by appropriate unperturbed models of a simpler structure, i.e. the problems are translated on more simple and familiar cases which are easier to solve and investigate (see [1–3] for example). Problem of additively perturbed backward stochastic differential equations is analysed by Janković, M. Jovanović, J. Đorđević in [4], while generally perturbed reflected backward stochastic differential equations are already observed by Đorđević and Janković in [5]. Topic of this chapter is additive type of perturbations for reflected backward stochastic differential equations as a special type of mention general problem for reflected backward stochastic differential equations, and a more general one than the additive perturbation problem for simple backward stochastic differential equations. Finer and more precise estimates are deduced and generalizations emphasised.

Backward stochastic differential equations (BSDEs for short) was introduced and developed by Pardoux and Peng [6–8] in the 90s. Notation of nonlinear BSDE and proof of the existence and uniqueness of adapted solutions is given in their fundamental paper [6]. After that, many applications incited to introduction various types of BSDEs, in mathematical problems in finance (see [9]), stochastic control and stochastic games (see [10, 11]), stochastic partial differential equations, semi-linear parabolic partial differential equations (PDEs) (see [8, 12]) etc. (for further reading see also [13–17]).

Type of RBSDEs which is observed in this chapter have been first introduced in literature by El-Karoui *et al.* in [18]. Introduced RBSDEs with one lower barrier has following form,

$$\begin{aligned} &Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds + K_T - K_t - \int_t^T Z_s\,dBs, \quad 0 \le t \le T, \\ &Y_t \geqslant L_t, \quad t \le T \text{ and} \int_0^T (Y_s - L_s)\,dK_s = 0 \quad P - a.s., \end{aligned} \tag{1}$$

where one of the components of the solution is forced to stay above a given barrier/obstacle process $L = \{L_t, t \in [0, T]\}$. The solution is a triple of adapted processes $\{(Y_t, Z_t, K_t), t \in [0, T]\}$ which satisfies Eq. (1). The process $K = \{K_t, t \in [0, T]\}$ is nondecreasing and its purpose is to push upward the state process $Y = \{Y_t, t \in [0, T]\}$ in order to keep it above the obstacle L.

As it was already mentioned, RBSDEs are connected with a wide range of applications within which, the pricing of American options (constrained or not) in markets is most famous one. Further, the important applications of RBSDEs are in mixed control problems, partial differential variational inequalities, real options (see [9, 18–21] and the references therein) etc. El-Karoui *et al.* proved in [18] the existence and uniqueness of the solution to Eq. (1) under conditions of square integrability of the data and Lipschitz property for the coefficient (also called driver) f. Field of RBSDEs is developing in two directions, some authors deal with the issue of the existence and uniqueness results for RBSDEs under weaker assumptions (than the ones in [6] which are for the general BSDEs), while others are introducing some new types of those equations by adding jumps, introducing second barrier etc.

Systematization of the papers which are done in the framework of RBSDEs can be found in paper [5] by Đorđević and Janković.

Recently, Hamèdene and Popier in [22] proved that if ξ, $\sup_{t\in[0,T]}(L_t^+)$ and $\int_0^T |f(t,0,0)|dt$ belong to L^p for some $p \in]1,2[$, then RBSDE (1) with one reflecting barrier associated with (f, ξ, L) has a unique solution. Aman gave [23] a similar result for a class of generalized RBSDEs with Lipschitz condition on the coefficients, and he extended these results under non-Lipschitz condition in his paper [24]. There are several papers by Hamadène [25] and Hamadène and Ouknine [26], Matoussi [27], Lepeltier and Xu [28] and Ren *et al.* [29, 30] in which authors emphasise the significance of the case when the data are from L^p for some $p \in]1,2[$.

The aim of this chapter is a study Eq. (1) if the terminal condition ξ and generator f are p-integrable, $p \in]1,2[$. Regarding that in several applications such as in finance, control, games, PDEs, etc., data are not square integrable, and the influence of some random external factors on the system can be seen as perturbations of the solution of Eq.(1), it is natural to introduced additive perturbation in the parameters of equation ξ, f and barrier process L, in order to better describe the change of the system and find some measurement for the change.

This chapter is organized in following way; In Section 2 elementary notations, definitions and preliminary results regarding RBSDEs are introduced. Next section is dedicated to the formulation of the main problem, i.e. problem of additively perturbed RBSDEs with one lower obstacle is stated. Together with the set up for the problem, some auxiliary estimates are proved in this section. In Section 4, conditions under which the solutions are stable are given, and estimates for the stability are derived. Section 5 contains the most interesting result, i.e. the estimation of a time interval for a given closeness of the solutions. The chapter is finished with the Section 6, Conclusions remarks, where the highlights of the chapter are emphasised and ideas and open problems for the future research are stated.

2. Preliminaries

All random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\{\mathcal{F}_t, t \in [0, T]\}$ is a natural filtration of a standard d-dimensional Brownian motion $B = \{B_t, t \in [0, T]\}$, that is, it is right continuous and complete. Also, all stochastic processes are defined for $t \in [0, T]$, where T is a positive, fixed, real constant, and they take values in $\mathbb{R}^n$ for some positive integer n. For any $k \in \mathbb{N}$ and $x \in \mathbb{R}^k$, $|x|$ denotes the Euclidean norm of x.

Further, for any real constant $p \in]1, 2[$, we recall on standard notations which will be used:

i. $\mathcal{S}^p(\mathbb{R})$ is the set of $\mathbb{R}$-valued, adapted and continuous processes $\{X_t, t \in [0, T]\}$ such that

$$\|X\|_{\mathcal{S}^p} = E\left[\sup_{t \in [0, T]} |X_t|^p\right]^{\frac{1}{p}} < \infty.$$

The space $\mathcal{S}^p(\mathbb{R})$ endowed with the norm $\|\cdot\|_{\mathcal{S}^p}$ is a Banach type.

ii. $\mathcal{M}^p$ is the set of predictable processes $\{Z_t, t \in [0, T]\}$ with values in $\mathbb{R}^d$ such that

$$\|X\|_{\mathcal{M}^p} = E\left[\left(\int_0^T |Z_t|^2 dt\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

Likewise, $\mathcal{M}^p(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\mathcal{M}^p}$ is a Banach space.

iii. The space $S^p \times \mathcal{M}^p$ will be denoted by B^p.

Let ξ be an $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variable and let a random function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}$ denotes the σ-field of progressive subsets of $[0, T] \times \Omega$, while $L := \{L_t, t \in [0, T]\}$ is a continuous progressively measurable $\mathbb{R}$-valued process.

The following hypothesis are introduced for ξ, f and L:

$(\mathbf{H_1})$ $\xi \in L^p(\Omega)$.

$(\mathbf{H_2})$

i. The process $\{ f(t, 0, 0), t \in [0, T]\}$ satisfies $E\left(\int_0^T |f(t, 0, 0)| dt\right)^p < \infty$;

ii. (ii) (Lipschitz condition) there exists a constant $k > 0$ such that for all $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,y,z)-f(t,y',z')| \leq k(|y-y'|+|z-z'|).$$

($\mathbf{H_3}$) The barrier process L satisfies:

i. $L_T \leq \xi$;

ii. $L^+ := L \vee 0 \in \mathcal{S}^p(\mathbb{R})$.

The definition of the unique solution to Eq. (1), associated with the triple (ξ, f, L), and the existence and uniqueness theorem under Lipschitz condition are given in [22].

Definition 1

I. (Existence of the solution.) The triple $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ is an L^p-solution to RBSDE (1) with a continuous lower reflecting barrier L, terminal condition ξ and drift/generator/driver f if:

1. $\{(Y_t, Z_t), t \in [0,T]\}$ belongs to $\mathcal{B}^p$;

2. $K = \{K_t, t \in [0,T]\}$ is an adapted continuous nondecreasing process such that $K_0 = 0$ and $K_T \in L^p(\Omega)$;

3. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s\, a.s., t \in [0,T]$;

4. $Y_t \geq L_t, t \in [0,T]$;

5. $\int_0^T (Y_s - L_s)dK_s = 0 \quad P-a.s.$

II. (Uniqueness of the solution.) The triple $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ is a unique L^p-solution to RBSDE (1) if for any other solution $\{(\overline{Y}_t, \overline{Z}_t, \overline{K}_t), t \in [0,T]\}$, the following holds,

$$\left\|Y_t - \overline{Y}_t\right\|_{\mathcal{S}^p} = 0, \quad \left\|Z_t - \overline{Z}_t\right\|_{\mathcal{M}^p} = 0, \quad \left\|K_t - \overline{K}_t\right\|_{\mathcal{S}^p} = 0. \tag{2}$$

Proposition 1 [Hamèdene, Popier [22]] Let $(\mathbf{H_1}) - (\mathbf{H_3})$ hold for ξ, f and L. Then, RBSDE (1) with one continuous lower reflecting barrier L associated with (ξ, f, L) has a unique L^p-solution, $p \in]1,2[$, i.e. there exists a triple of processes $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ satisfying Definition 1.

The following lemma is well known result and it is widely used in stability estimates.

Lemma 1 [Hamèdene, Popier [22]] Assume that $(Y, Z) \in \mathcal{B}^p$ is a solution of the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dB_s, \quad t \in [0,T],$$

where:

i. f is a function satisfying the previous assumptions;

ii. The process $\{A_t, t \in [0,T]\}$ is $P-a.s.$ of bounded variation.

Then, for any $0 \le t \le u \le T$ it follows that

$$\begin{aligned}
&|Y_t|^p + c(p)\int_t^u |Y_s|^{p-2}1_{Y_s\neq 0}|Z_s|^2 ds \\
&\quad \le |Y_u|^p + p\int_t^u |Y_s|^{p-1}\hat{Y}_s dA_s \\
&\qquad + p\int_t^u |Y_s|^{p-1}\hat{Y}_s f(s, Y_s, Z_s) ds - p\int_t^u |Y_s|^{p-1}\hat{Y}_s Z_s dB_s,
\end{aligned}$$

where $c(p) = \frac{p(p-1)}{2}$ and $\hat{y} = \frac{y}{|y|}1_{y\neq 0}$.

When the model of some phenomenon is described by RBSDE, than, some change of the system can be treated as additive perturbation of the initial equation. The size of the change could be estimated as the difference between the solutions of the initial equation and the perturbed one. In view of this direction, together with Eq. (1), we study the following perturbed RBSDE,

$$\begin{aligned}
&Y_t = \xi^\varepsilon + \int_t^T f^\varepsilon\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds + K_T^\varepsilon - K_t^\varepsilon - \int_t^T Z_s^\varepsilon dB_s, \quad 0 \;\; t \in [0, T], \\
&Y_t^\varepsilon \ge L_t^\varepsilon, \;\; t \le T \; and \int_0^T \left(Y_s^\varepsilon - L_s^\varepsilon\right) dK_s^\varepsilon = 0, \;\; P - a.s,
\end{aligned} \tag{3}$$

where $\xi^\varepsilon, f^\varepsilon$ and the barrier L^ε are defined as ξ, f and L, respectively, they depend on a small parameter $\varepsilon \in (0, 1)$, and they are of a special additive form

$$\begin{aligned}
\xi^\varepsilon &= \xi + \beta(T, \varepsilon), \\
f^\varepsilon(t, y, z, \varepsilon) &= f(t, y, z) + \alpha(t, y, z, \varepsilon), \\
L_t^\varepsilon &= L_t + l_t^\varepsilon.
\end{aligned}$$

For a given $(f^\varepsilon, \xi^\varepsilon, L^\varepsilon)$, a triple of adapted processes $\left\{\left(Y_t^\varepsilon, Z_t^\varepsilon, K_t^\varepsilon\right), t \in [0, T]\right\}$ is a solution to Eq. (3). In the sequel Eq. (1) will be named the *unperturbed* equation, while Eq. (3) a *additively perturbed* one. It is usually expected that the additively perturbed Eq. (3) is more general and more complexed than the unperturbed one. Furthermore, it is obvious that in case when $\beta(\varepsilon) \equiv \alpha(t, y, z, \varepsilon) \equiv l_t^\varepsilon \equiv 0$, additively perturbed equation reduces to unperturbed equation. This fact is a basic motivation for us to introduce conditions guaranteeing the closeness of the solutions of the additively perturbed and unperturbed equations in the L^p-sense, and to estimate the conditions for the additive parameters in order for the solutions of these equations to stay close in the L^p-sense in some way.

After basic notations, definitions and results are present, the formulation of the main problem is given in following section.

3. Formulation of the problem of additively perturbed RBSDEs with one lower obstacle & and auxiliary results

In order to deduce estimates for the closeness of the solutions of additively perturbed and unperturbed equations, following assumptions are introduced;

$(\mathcal{A}\mathbf{0})$ For the additional part in final condition of perturbed equation $\beta(T, \varepsilon)$, such that $\xi^\varepsilon = \xi + \beta(T, \varepsilon)$, while $\xi^\varepsilon, \xi \in L^p(\Omega)$, there exists a non-random function $\beta_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$E|\beta(T, \varepsilon)|^p \le \beta_1(\varepsilon).$$

$(\mathcal{A}\mathbf{1})$ For the additional part in the generator/driver integral, $\alpha(t, y, z, \varepsilon)$, there exists a non-random function $\alpha_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$\sup_{(t,y,z) \in [0,T] \times B^p} |\alpha(t, y, z, \varepsilon)| \leq \alpha_1(\varepsilon) \quad a.s.$$

$(\mathcal{A}\mathbf{2})$ For the additional part in barrier processes l_t^ε, there exists a non-random function $l_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$E \sup_{t \in [0,T]} |l_t^\varepsilon|^p \leq l_1(\varepsilon).$$

We give first an auxiliary result for the stability of the solutions which we will use to prove main result.

Proposition 2 Let $p \in]1, 2[$ and let $\{(Y_t, Z_t, K_t), t \in [0, T]\}$ and $\{(Y_t^\varepsilon, Z_t^\varepsilon, K_t^\varepsilon), t \in [0, T]\}$ be the solutions to additively unperturbed and perturbed Eqs. (1) and (3), respectively. Let also assumptions $(\mathcal{A}0) - (\mathcal{A}2)$ and conditions $(H_1) - (H_3)$ be satisfied. Then,

$$E|Y_t^\varepsilon - Y_t|^p \leq C_1 e^{c_1(T-t)}, \quad t \in [0, T], \tag{4}$$

where $c_1 = p - 1 + pk + \frac{pk^2}{p-1}$ and $C_1 = \beta_1(\varepsilon) + \alpha_1^p(\varepsilon)T + l_1^{\frac{p-1}{p}}(\varepsilon)\left(E|\hat{K}_T|^p\right)^{\frac{1}{p}}$.

Proof: Let us denote for $t \in [0, T]$ the differences of the processes of the solutions,

$$\hat{Y}_t = Y_t^\varepsilon - Y_t, \quad \hat{Z}_t = Z_t^\varepsilon - Z_t, \; \hat{K}_t = K_t^\varepsilon - K_t.$$

If we subtract Eqs. (1) and (3), we obtain

$$\hat{Y}_t = \beta(T, \varepsilon) + \int_t^T \alpha(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon)\, ds + \hat{K}_T - \hat{K}_t - \int_t^T \hat{Z}_s dB_s, \; t \in [0, T]. \tag{5}$$

Applying Lemma 1 on $|\hat{Y}_t|^p$, we have

$$\begin{aligned}
&|\hat{Y}_t|^p + c(p)\int_t^T |\hat{Y}_s|^{p-2} 1_{\hat{Y}_s \neq 0} |\hat{Z}_s|^2 ds \\
&\quad \leq |\beta(T, \varepsilon)|^p + p\int_t^T |\hat{Y}_s|^{p-1} sgn\,(\hat{Y}_s)\alpha(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon)\, ds \\
&\quad + p\int_t^T |\hat{Y}_s|^{p-1} sgn\,(\hat{Y}_s) d(\Delta\hat{K}_s) - p\int_t^T |\hat{Y}_s|^{p-1} sgn\,(\hat{Y}_s)\hat{Z}_s dB_s \\
&\quad \leq |\beta(T, \varepsilon)|^p + p\int_t^T |\hat{Y}_s|^{p-1} |\alpha(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon)|\, ds \\
&\quad + pk\int_t^T |\hat{Y}_s|^{p-1} |Y_s^\varepsilon - Y_s| ds + pk\int_t^T |\hat{Y}_s|^{p-1} |Z_s^\varepsilon - Z_s| ds \\
&\quad + p\int_t^T |\hat{Y}_s|^{p-1} sgn\,(\hat{Y}_s) d(\Delta\hat{K}_s) - p\int_t^T |\hat{Y}_s|^{p-1} sgn\,(\hat{Y}_s)\hat{Z}_s dB_s \\
&\quad := |\beta(T, \varepsilon)|^p + I_1(t) + pk\int_t^T |\hat{Y}_s|^p ds + I_2(t) + I_3(t) + I_4(t),
\end{aligned} \tag{6}$$

where $I_i(t), i = 1, 2, 3, 4$ are the appropriate integrals. In order to estimate $I_1(t)$, we apply the elementary inequality $a^{p-1}b \leq \frac{p-1}{p}a^p + \frac{1}{p}b^p, a, b \geq 0$ and assumption $(\mathcal{A}1)$. Then,

$$
\begin{aligned}
I_1(t) &= p\int_t^T \left|\hat{Y}_s\right|^{p-1} |\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right)| ds \\
&\leq (p-1)\int_t^T \left|\hat{Y}_s\right|^p ds + \alpha_1^p(\varepsilon)(T-t).
\end{aligned}
\tag{7}
$$

In order to estimate $I_2(t)$, we use the elementary inequality $2ab \leq \frac{a^2}{2} + 2b^2$,

$$
I_2(t) \leq \frac{pk^2}{p-1}\int_t^T \left|\hat{Y}_s\right|^p ds + \frac{c(p)}{2}\int_t^T \left|\hat{Y}_s\right|^{p-2} I_{\{\hat{Y}_s \neq 0\}} \left|\hat{Z}_s\right|^2 ds,
$$

where $c(p) = p(p-1)/2$.

For estimation of member $I_3(t)$, we will use mapping $(x, a) \to \tilde{\theta}(x, a) = |x-a|^{p-2} 1_{x\neq a}(x-a)$, $(x, a) \in \mathbb{R} \times \mathbb{R}$. Function $x \to \tilde{\theta}(x, a)$ is non-decreasing, while the function $a \to \tilde{\theta}(x, a)$ is non-increasing. As it is known, $l_s^\varepsilon = L_s^\varepsilon - L_s$ and since $Y_s^\varepsilon \geq L_s^\varepsilon$, $Y_s \geq L_s$, then

$$
\begin{aligned}
I_3(t) &= p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right) d\left(\Delta \hat{K}_s\right) \\
&= p\int_t^T \tilde{\theta}\left(Y_s^\varepsilon, Y_s\right) I_{\{Y_s^\varepsilon = L_s^\varepsilon\}} dK_s^\varepsilon - p\int_t^T \tilde{\theta}\left(Y_s^\varepsilon, Y_s\right) I_{\{Y_s = L_s\}} dK_s \\
&\leq p\int_t^T \left|L_s^\varepsilon - L_s\right|^{p-2} I_{\{L_s^\varepsilon - L_s \neq 0\}} \left(L_s^\varepsilon - L_s\right) dK_s^\varepsilon - p\int_t^T \left|L_s^\varepsilon - L_s\right|^{p-2} I_{\{L_s^\varepsilon - L_s \neq 0\}} \left(L_s^\varepsilon - L_s\right) dK_s \\
&= p\int_t^T \left|l_s^\varepsilon\right|^{p-1} d\left(\hat{K}_s\right).
\end{aligned}
\tag{8}
$$

Substituting estimates for $I_i(t), i = 1, 2, 3$ in in (6), we obtain

$$
\begin{aligned}
&\left|\hat{Y}_t\right|^p + \frac{c(p)}{2}\int_t^T \left|\hat{Y}_s\right|^{p-2} I_{\{\hat{Y}_s \neq 0\}} \left|\hat{Z}_s\right|^2 ds \\
&\quad \leq |\beta(T, \varepsilon)|^p + \left(p - 1 + pk + \frac{pk^2}{p-1}\right)\int_t^T \left|\hat{Y}_s\right|^p ds + p\int_t^T \left|l_s^\varepsilon\right|^{p-1} d\left(\hat{K}_s\right), \\
&\quad + \alpha_1^p(\varepsilon)(T-t) + I_4(t).
\end{aligned}
$$

Taking expectation on last inequality, and taking into account that expectation of I_4 is 0, we have

$$
\begin{aligned}
E\left|\hat{Y}_t\right|^p &\leq \beta_1(\varepsilon) + \alpha_1^p(\varepsilon)T + l_1^{\frac{p-1}{p}}(\varepsilon)\left(E\left|\hat{K}_T\right|^p\right)^{\frac{1}{p}} \\
&\quad + \left(p - 1 + pk + \frac{pk^2}{p-1}\right)\int_t^T E\left|\hat{Y}_s\right|^p ds.
\end{aligned}
$$

As $K_T, K_T^\varepsilon \in L^p(\Omega)$, it follows that $E\left|\hat{K}_T\right|^p < \infty$. So (4) holds straightforwardly by applying the Gronwall-Bellman inequality ([31], Theorem 1.5):

Let $u(t)$ be a continuous function in $[a, b], f(t)$ be Riemann integrable function in $[a, b]$ and $c = const > 0$. If $u(t) = f(t) + c\int_t^b u(s)ds, t \in [a, b]$, then $u(t) \leq f(t) + c\int_t^b f(s)e^{c(s-t)}ds, t \in [a, b]$.

The theorem is proved.

★

For the introduced problem, conditions for the stability of the solutions and estimates for the stability of the solutions are derived In next section.

4. Stability estimates for additive perturbations

For the estimate of L^p-difference between the solutions to Eqs. (1) and (3), the L^p-stability of the solution to Eq. (1) is necessary.

Following theorem provides the result, that in case of small additive perturbations case, we can expect that the difference of the solutions of perturbed and unperturbed equations tends to zero, when the perturbations are sufficiently small.

Theorem 1 Let all the conditions of Proposition 2 be satisfied and let the functions $\beta_1(\varepsilon), \alpha_1(\varepsilon), l_1(\varepsilon)$ tend to zero as ε tends to zero, uniformly in $t \in [0, T]$. Then it follows that

$$E \sup_{t\in[0,T]} \left|Y_t^\varepsilon - Y_t\right|^p \to 0, \ \varepsilon \to 0,$$

$$E\left(\int_0^T \left|Z_s^\varepsilon - Z_s\right|^2 ds\right)^{\frac{p}{2}} \to 0, \varepsilon \to 0,$$

$$E \sup_{t\in[0,T]} E\left|K_t^\varepsilon - K_t\right|^p \to 0, \quad \varepsilon \to 0.$$

Proof: Let us define

$$\phi(\varepsilon) := \max\left\{\beta_1, (\varepsilon), \alpha_1^p(\varepsilon), l_1^{\frac{p-1}{p}}(\varepsilon)\right\}. \tag{9}$$

From Proposition 2, we have that $C_1 \le \phi(\varepsilon)\tilde{C}$, where $\tilde{C} = 1 + T + \left(E\left|\hat{K}_T\right|^p\right)^{\frac{1}{p}}$ and, therefore,

$$E\left|\hat{Y}_t\right|^p \le \phi(\varepsilon)\tilde{C}e^{c_1(T-t)}, \quad t \in [0, T].$$

Since $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, then for every $t_0 \in [0, T]$,

$$\sup_{t\in[t_0,T]} E\left|\hat{Y}_t\right|^p \le \phi(\varepsilon)\tilde{C}e^{c_1(T-t_0)} \to 0, \quad \varepsilon \to 0. \tag{10}$$

In order to estimate the L^p-closeness between the processes Z_t and $Z^\varepsilon(t)$, as well as K_t and $K^\varepsilon(T)$, we need estimate $E\sup_{t\in[0,T]}\left|\hat{Y}_t\right|^p$, that is to estimate $I_4(t)$. By applying the Burkholder-Davis-Guandy inequality [32] and Young inequality, $u^\alpha v^{1-\alpha} \le \alpha u + (1-\alpha)v, v \ge 0, \alpha \in [0, 1]$, we have

$$\begin{aligned}
E \sup_{t\in[t_0,T]} I_4(t) &\le 4\sqrt{2}pE\left(\int_{t_0}^T \left|\hat{Y}_s\right|^{2p-2}\left|\hat{Z}_s\right|^2 ds\right)^{\frac{1}{2}} \\
&\le 4\sqrt{2}pE\left(\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p \int_{t_0}^T \left|\hat{Y}_s\right|^{p-2}\left|\hat{Z}_s\right|^2 ds\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}E\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p + 16p^2E\int_{t_0}^T \left|\hat{Y}_s\right|^{p-2}\left|\hat{Z}_s\right|^2 ds.
\end{aligned}$$

We can conclude that

$$E \sup_{t\in[t_0,T]} I_4(t) \le \frac{1}{2} E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p + \frac{32p^2}{c(p)}\,\phi(\varepsilon)\,\tilde{C}e^{c_1(T-t_0)}. \tag{11}$$

It follows that

$$E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p \le \frac{1}{2} E \sup_{t\in[t_0,T]} \left|\hat{Y}_s\right|^p + \frac{32p^2}{c(p)}\,\phi(\varepsilon)\,\tilde{C}e^{c_1(T-t_0)} + c_1\int_{t_0}^{T} E\left|\hat{Y}_s\right|^p ds + \phi(\varepsilon)\,\tilde{C}.$$

Hence,

$$E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p \le 2\phi(\varepsilon)\,\tilde{C}\left[e^{c_1(T-t_0)}\left(1+\frac{32p^2}{c(p)}\right)+1\right] \equiv \phi(\varepsilon)A_1(t_0), \tag{12}$$

where $A_1(t_0)$ is a generic positive constant. By the assumption of the theorem, $\phi(\varepsilon) \to$ as $\varepsilon \to 0$, then $E\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p \to 0, as\ \ \varepsilon \to 0$. The desired estimate holds if we take $t_0 = 0$.

Now we can estimate the other two parts.

For every $i \in \{0, 1, 2, \dots\}$ and arbitrary $t_0 \in [0, T]$, let us define stopping times

$$\tau_i = \inf\left\{t\in[0,T], \int_{t_0}^{t} \left\|\hat{Z}_s\right\|^2 ds \ge i\right\} \wedge T.$$

Clearly, $\tau_i \uparrow T \quad a.s.$ when $i \to \infty$. If we apply the Ito formula to $e^{kt}\left|\hat{Y}_t\right|^2, t\in[t_0,\tau_i]$, we find that

$$\begin{aligned}
&\left|\hat{Y}_{t_0}\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\left|\hat{Z}_s\right|^2 ds \\
&= e^{k\tau_i}\left|\hat{Y}_{\tau_i}\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\left[2\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) - k\hat{Y}_s\right] + 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s d\hat{K}_s - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\
&:= e^{k\tau_i}\left|\hat{Y}_{\tau_i}\right|^2 + J_1 + J_2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s,
\end{aligned} \tag{13}$$

where estimates J_1 and J_2 are the appropriate integrals. For $\lambda_1 > 0$ that

$$\begin{aligned}
J_1 &= 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds - k\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s^2 ds \\
&\le 2 \sup_{s\in[t_0,\tau_i]} e^{ks}|\hat{Y}_s|\alpha_1(\varepsilon)(T-t_0) - k\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s^2 ds \\
&\le \frac{1}{\lambda_1}\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_1(T-t_0)^2\alpha_1^2(\varepsilon) - k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds.
\end{aligned} \tag{14}$$

Similarly, for $\lambda_2 > 0$,

$$\begin{aligned} J_2 = 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s d\hat{K}_s &\leq 2 \sup_{s\in[t_0,\tau_i]} e^{ks}|\hat{Y}_s|\int_{t_0}^{\tau_i} d\hat{K}_s \\ &\leq \frac{1}{\lambda_2}\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_2\left(\int_{t_0}^{\tau_i} d\hat{K}_s\right)^2 \\ &= \frac{1}{\lambda_2}\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_2\left(\hat{K}_{\tau_i} - \hat{K}_{t_0}\right)^2. \end{aligned} \tag{15}$$

Also,

$$\begin{aligned} \left(\hat{K}_{\tau_i} - \hat{K}_{t_0}\right)^2 &= \left(\hat{Y}_{\tau_i} - \hat{Y}_{t_0} - \int_{t_0}^{\tau_i}\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right)ds + \int_{t_0}^{\tau_i}\hat{Z}_s dB_s\right)^2 \\ &\leq 4\left[\left|\hat{Y}_{\tau_i}\right|^2 + \left|\hat{Y}_{t_0}\right|^2 + \left|\int_{t_0}^{T}\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right)1_{\{s=T\wedge\tau_i\}}ds\right|^2 + \left|\int_{t_0}^{\tau_i}\hat{Z}_s dB_s\right|^2\right] \\ &\leq 4\left[\left|\hat{Y}_{\tau_i}\right|^2 + \left|\hat{Y}_{t_0}\right|^2 + 2(T-t_0)^2\alpha_1^2(\varepsilon) + \left|\int_{t_0}^{\tau_i}\hat{Z}_s dB_s\right|^2\right]. \end{aligned} \tag{16}$$

Substituting (14), (15) and (16) in (13) yields

$$\begin{aligned} &(1-4\lambda_2)\left|\hat{Y}_0\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\left|\hat{Z}_s\right|^2 ds \\ &\quad\leq \left(e^{k\tau_i} + 4\lambda_2\right)\left|\hat{Y}_{\tau_i}\right|^2 + \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 \\ &\qquad+4\lambda_2\left|\int_{t_0}^{\tau_i}\hat{Z}_s dBs\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s - k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds + (\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon). \end{aligned}$$

It follows that

$$\begin{aligned} \int_{t_0}^{\tau_i}\left|\hat{Z}_s\right|^2 ds \leq &\left(e^{k\tau_i} + 4\lambda_2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 \\ &+4\lambda_2\left|\int_{t_0}^{\tau_i}\hat{Z}_s dB_s\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\ &-k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds + (\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon). \end{aligned} \tag{17}$$

The last inequality can be written as

$$\begin{aligned} \int_{t_0}^{\tau_i}\left|\hat{Z}_s\right|^2 ds \leq c_2(t_0)\sup_{s\in[t_0,\tau_i]}\left|\hat{Y}_s\right|^2 + 4\lambda_3\left|\int_{t_0}^{\tau_i}\hat{Z}_s dB_s\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\ +(\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon), \end{aligned} \tag{18}$$

where

$$c_2(t_0) = \left[e^{kT} + 4\lambda_2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - k(T-t_0)\right]e^{2kT}.$$

By applying the inequality $\left(\sum_{i=1}^{m} a_i\right)^k \le \left(m^{k-1} \vee 1\right)\sum_{i=1}^{m} a_i^k, a_i \ge 0, k \ge 0$ on (18) and by taking expectation, we obtain

$$\begin{aligned} E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \le & c_2^{\frac{p}{2}}(t_0) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + 4^{\frac{p}{2}} \lambda_2^{\frac{p}{2}} E \left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^p \\ & + 2^{\frac{p}{2}} E \left|\int_{t_0}^{\tau_i} e^{ks} \hat{Y}_s \hat{Z}_s dB_s\right|^{\frac{p}{2}} + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}} (T - t_0)^p \phi(\varepsilon). \end{aligned} \tag{19}$$

It is left to estimate two integrals with respect to Brownian motion, which will be done by applying the Burkholder–Davis–Guandy inequality,

$$\begin{aligned} E\left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^p &\le C_p E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}}, \\ 2^{\frac{p}{2}} E\left|\int_{t_0}^{\tau_i} e^{ks} \hat{Y}_s \hat{Z}_s dB_s\right|^{\frac{p}{2}} &\le C_{\frac{p}{2}} 2^{\frac{p}{2}} e^{\frac{pkT}{2}} E\left[\left(\int_{t_0}^{\tau_i} \left|\hat{Y}_s\right|^2 \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{4}}\right] \\ &\le c_3 E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_t\right|^p + \lambda_3 \left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}}, \end{aligned}$$

where $\lambda_3 > 0$, $c_3 = \frac{1}{\lambda_3} C_{\frac{p}{2}}^2 2^{p-2} e^{pkT}$, and $C_p = (32/p)^{p/2}$ and $C_{\frac{p}{2}} = (64/p)^{p/4}$ are the universal constants. Substituting previous estimates in (19), it follows that

$$\begin{aligned} E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \le & c_2^{\frac{p}{2}}(t_0) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + 4^{\frac{p}{2}} \lambda_2^{\frac{p}{2}} C_p E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \\ & + c_3 E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_t\right|^p + \lambda_3 E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \\ & + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}} (T - t_0)^p \phi(\varepsilon), \end{aligned}$$

i.e.

$$\begin{aligned} & \left(1 - 4^{\frac{p}{2}} \lambda_2^{\frac{p}{2}} C_p - \lambda_3\right) E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \\ & \quad \le \left(c_2^{\frac{p}{2}}(t_0) + c_3\right) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}} (T - t_0)^p \phi(\varepsilon). \end{aligned} \tag{20}$$

The constants λ_2, λ_3 can be chosen such that $1 - 4^{\frac{p}{2}} \lambda_2^{\frac{p}{2}} C_p - \lambda_3 > 0$, then, from (12) and (20) it follows that

$$\begin{aligned} E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} &\le \frac{\left(c_2^{\frac{p}{2}}(t_0) + c_3\right) A_1(t_0) + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}} (T - t_0)^p}{1 - 4^{\frac{p}{2}} \lambda_2^{\frac{p}{2}} C_p - \lambda_3} \phi(\varepsilon) \\ &\equiv A_2(t_0) \phi(\varepsilon), \end{aligned} \tag{21}$$

where $A_2(t_0)$ is a positive generic constant. By the Fatou's Lemma,

$$E\left(\int_{t_0}^{T} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \to 0, \quad \varepsilon \to 0.$$

Then, second estimate holds if we take $t_0 = 0$.

It is left to estimate the difference between the processes K and K^ε. From (5), we have

$$\hat{K}_t = \beta(T,\varepsilon) - \hat{Y}_t + \int_t^T \alpha\big(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\big)\,ds - \int_t^T \hat{Z}_s\,dB_s + \hat{K}_T.$$

In view of (12) and (21), we derive that

$$\begin{aligned}
&E\sup_{t\in[t_0,T]}\big|\hat{K}_t\big|^p \\
&\le 5^{p-1}\Bigg\{E|\beta(T,\varepsilon)|^p + E\sup_{t\in[t_0,T]}\big|\hat{Y}_t\big|^p + \bigg|\int_{t_0}^T \alpha\big(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\big)\,ds\bigg|^p \\
&\quad + E\sup_{t\in[t_0,T]}\bigg|\int_t^T \hat{Z}_s\,dB_s\bigg|^p + E\big|\hat{K}_T\big|^p\Bigg\} \\
&\le 5^{p-1}\Big\{1 + A_1(t_0) + (T-t_0)^{\frac{p}{2}}2^{\frac{p}{2}} + C_pA_2(t_0)\Big\}\phi(\varepsilon) + 5^{p-1}E\big|\hat{K}_T\big|^p.
\end{aligned} \tag{22}$$

Since

$$\hat{K}_T = \hat{Y}_0 - \beta(T,\varepsilon) - \int_0^T \alpha\big(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\big)\,ds + \int_0^T \hat{Z}_s\,dB_s,$$

in accordance with the last estimate, we have that

$$\begin{aligned}
E\big|\hat{K}_T\big|^p &\le 4^{p-1}\Bigg[E\big|\hat{Y}_0\big|^p + E|\beta(T,\varepsilon)|^p + E\bigg|\int_0^T \alpha\big(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\big)\,ds\bigg|^p + E\bigg|\int_0^T \hat{Z}_s\,dB_s\bigg|^p\Bigg] \\
&\le 4^{p-1}\Big[\tilde{C}e^{c_1T} + 1 + T^{\frac{p}{2}}2^{\frac{p}{2}} + A_2(0)\Big]\phi(\varepsilon).
\end{aligned}$$

Hence, it follows from that there exists a generic constant $A_3(t_0) > 0$ such that

$$E\sup_{t\in[0,T]}\big|\hat{K}_t\big|^p \le A_3(t_0)\phi(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{23}$$

Then, the last estimate of the theorem holds if we take $t_0 = 0$, which completes the proof.

★

In this section complete proof for the stability of the solutions is given, which as a strong result and it enable us to estimate the time interval for a given closeness of the solutions. This result is proved in next section.

5. Time interval for a given closeness of the solutions

Theorem 1 provides that the state processes Y_t^ε and Y_t, the control processes Z_t^ε and Z_t, as well as K^ε and K could be arbitrarily close for ε sufficiently small. I.e., if perturbations are small enough, closeness of the solutions can be provided. But, from the perspective of applications and modelling, it is usually important to study the closeness between Y_t^ε and Y_t near to the terminal values ξ^ε and ξ. Per example, for the application in pricing American options, an agent would be interested how will the price behave near the exercise time. It is interesting and useful to find the time interval on which we could preserve the wanted closeness, i.e. that for some permissible $a > 0$ and ε sufficiently small, find $\bar{t}(a) = \bar{t} \in [0, T]$ so that the rate of the closeness between Y_t^ε and Y_t does not exceed a on $[\bar{t}, T]$. Even-more, estimate of the closeness between the control processes Z_t^ε and Z_t on $[\bar{t}, T]$ can be estimated.

Theorem 2 Let all the conditions of Theorem 1 hold. Also, let the function $\phi(\varepsilon), \varepsilon \in (0,1)$ defined with (9) be continuous and monotone increasing. Then, for an arbitrary constant $a > 0$ and $\varepsilon \in \left(0, \Phi^{-1}(a)\right]$, there exists $\bar{t} \in [0, T]$, where

$$\bar{t} = \max\left\{0, T - \frac{1}{c_1} \ln \frac{\eta}{\phi(\varepsilon)\tilde{C}}\right\},$$

such that

$$\sup_{t \in [\bar{t}, T]} E\left|Y_t^\varepsilon - Y_t\right|^p \leq a, \tag{24}$$

$$E\left(\int_{\bar{t}}^T \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \leq A_2(\bar{t})\,\phi(\varepsilon), \tag{25}$$

$$E \sup_{t \in [\bar{t}, T]} \left|\hat{K}_t\right|^p \leq A_3(\bar{t})\,\phi(\varepsilon), \tag{26}$$

and $A_2(\bar{t})$ and $A_3(\bar{t})$ are constants defined in (21) and (23), respectively.

Proof: Let us introduce function $S(\varepsilon, T - t), t \in [0, T]$, such that

$$S(\varepsilon, T - t) = \phi(\varepsilon)\,\tilde{C} e^{c_1 (T - t)},$$

where c_1 is given in Proposition 1 and $\tilde{C}$ in Theorem 1. For an arbitrary $a > 0$, it must be

$$S(\varepsilon, 0) \leq a \leq S(\varepsilon, T),$$

that is,

$$\phi(\varepsilon)\tilde{C} \leq a \leq \phi(\varepsilon)\tilde{C} e^{c_1 T}.$$

Since $\phi(\varepsilon)$ decreases if ε decreases, it follows that

$$\varepsilon_1 = \phi^{-1}\left(\frac{a}{\tilde{C} e^{c_1 T}}\right) \leq \varepsilon \leq \phi^{-1}\left(\frac{a}{\tilde{C}}\right) = \varepsilon_2,$$

where ϕ^{-1} is the inverse function of ϕ. For every $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, it is now easy to determine $\hat{t}$ from the relation $S(\varepsilon, T - \hat{t}) = a$, that is,

$$\hat{t} = T - \frac{1}{c_1} \ln \frac{\eta}{\phi(\varepsilon)\tilde{C}}.$$

If $\varepsilon \in (0, \varepsilon_1)$, then $a > S(\varepsilon, T)$. If $\varepsilon \in (0, \varepsilon_2)$, let us take

$$\bar{t} = \max\{0, \hat{t}\} = \max\left\{0, T - \frac{1}{c_1} \ln \frac{\eta}{\phi(\varepsilon)\tilde{C}}\right\}.$$

Hence, for every $\varepsilon \in (0, \varepsilon_2)$, it is easy to see that

$$\sup_{t \in [\bar{t}, T]} E\left|Y_s^\varepsilon - Y_s\right|^p \leq S(\varepsilon, T - \bar{t}) = a.$$

Clearly, $\bar{t} \uparrow T$ as $\varepsilon \uparrow \varepsilon_2$ and $\bar{t} \downarrow 0$ as $\varepsilon \downarrow \varepsilon_1$, that is, $\bar{t} \downarrow 0$ as $\varepsilon \downarrow 0$.

⋆

This section illustrates the most important result of the chapter. Indeed, estimate of a time interval, for the given, precise closeness of the solutions is very important in the applications. Per example, if some random observation is modelled by RBSDE, and its behaviour (value) on fixed time T is familiar, as well as its change up to some other value in capital moment, and if the driver of the model is supposed to linearly change, it is interesting to estimate the time interval on which we could. "control" the observations, i.e. under which our change under linearisation of final value and the drift will remain within the boundaries we impose.

6. Conclusions and remarks

It should be noted that this is a special case of generally perturbed problem observed by Đorđević and Janković in [5], but we have provided and explicit, concrete estimates for the additive type of perturbations. Interesting in this case also is, that even-though we introduce the hypothesis (H2), i.e. Lipschitz condition for the drift/driver/generator function, this hypothesis is not explicitly used in the estimates for perturbations. It is necessary to have it in order to have the existence of the solutions for perturbed and unperturbed equations, but it is not necessary for the perturbation estimates with the given assumptions ($\mathcal{A}\mathbf{0}$)–($\mathcal{A}\mathbf{2}$). It follows that results from this chapter can be generalized in several ways:

I. assumption ($\mathcal{A}\mathbf{1}$) can be weaken in the sense that it can be per example of the form:

i. Lipschitz condition

$$\left|\alpha(t, y, z, \varepsilon) - \alpha(t, y_1, z_1, \varepsilon)\right|^2 \leq L\left(\left|y - y_1\right|^2 + \|z - z_1\|^2\right) + \alpha_1(t, \varepsilon), \quad a.s.$$

for some Lipschitz constant L and nonrandom function $\alpha_1(t, \varepsilon)$.

ii. Non-Lipschitz condition.

There exist constants $C > 0$ such that for any $(\omega, t) \in \Omega \times [0, T]$ and $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$\left|\alpha(t, y_1, z_1, \varepsilon) - \alpha(t, y_2, z_2, \varepsilon)\right|^2 \leq \rho\left(t, \left|y_1 - y_2\right|^2\right) + C\|z_1 - z_2\|^2 + \alpha_1(t, \varepsilon),$$

where $\rho : [0, T] \times R^+ \to R^+$ satisfies: For fixed $t \in [0, T]$, $\rho(t, \cdot)$ is: a concave and non-decreasing function with $\rho(t, 0) \equiv 0$;

- for fixed u, $\int_0^T \rho(t, u)dt < \infty$;
- for any $M > 0$, the ODE

$$u' = -M\rho(t,u), \quad u(T) = 0$$

has a unique solution $u(t) \equiv 0, t \in [0,T]$.

iii. Linear growth condition

$$|\alpha(t,y,z,\varepsilon)| \leq K(|y|+\|z\|) + \alpha_1(t,\varepsilon), \ \ a.s.$$

for some constant K and nonrandom function $\alpha_1(t,\varepsilon)$.

In all alternatives, further assumption is that there exist nonrandom function $\overline{\alpha}(\varepsilon)$ such that

$$\sup_{t\in[0,T]} \alpha_1(t,\varepsilon) = \overline{\alpha}(\varepsilon).$$

II. Conditions of existence and uniqueness of the solutions of perturbed and unperturbed equations can be generalized in a sense for the driver f, f_ε of Eqs. (1) and (3) to satisfy some of mentioned conditions: non-Lipschitz or linear growth one. In this manner, these assumptions would hold for the additional function α in the perturbed driver also.

In the case when we change the initial conditions and assumptions, the steps will be similar, while the main inequality at the end of the estimates will be established by applying Bihari inequality and not Gronwall-Bellman one.

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