Chapter

Case Study: Coefficient Training in Paley-Wiener Space, FFT, and Wavelet Theory

Kayupe Kikodio Patrick

Abstract

Bessel functions form an important class of special functions and are applied almost everywhere in mathematical physics. They are also called cylindrical functions, or cylindrical harmonics. This chapter is devoted to the construction of the generalized coherent state (GCS) and the theory of Bessel wavelets. The GCS is built by replacing the coefficient $z^n/n!,\ \ z \in \mathbb{C}$ of the canonical CS by the cylindrical Bessel functions. Then, the Paley-Wiener space PW_1 is discussed in the framework of a set of GCS related to the cylindrical Bessel functions and to the Legendre oscillator. We prove that the kernel of the finite Fourier transform (FFT) of L^2-functions supported on $[-1, 1]$ form a set of GCS. Otherwise, the wavelet transform is the special case of CS associated respectively with the Weyl-Heisenberg group (which gives the canonical CS) and with the affine group on the line. We recall the wavelet theory on **R**. As an application, we discuss the continuous Bessel wavelet. Thus, coherent state transformation (CST) and continuous Bessel wavelet transformation (CBWT) are defined. This chapter is mainly devoted to the application of the Bessel function.

Keywords: coherent state, Hankel transformation, Bessel wavelet transformation

1. Introduction

Coherent state (CS) was originally introduced by Schrödinger in 1926 as a Gaussian wavepacket to describe the evolution of a harmonic oscillator [1].

The notion of coherence associated with these states of physics was first noticed by Glauber [2, 3] and then introduced by Klauder [4, 5]. Because of their important properties these states were then generalized to other systems either from a physical or mathematical point of view. As the electromagnetic field in free space can be regarded as a superposition of many classical modes, each one governed by the equation of simple harmonic oscillator, the CS became significant as the tool for connecting quantum and classical optics. For a review of all of these generalizations see [6–9].

Four main methods are well used in the literature to build CS, the so-called Schrödinger, Klauder-Perelomov, Barut-Girardello and Gazeau-Klauder approaches. The second and third approaches are based directly on the Lie algebra symmetries with their corresponding generators, the first is only established by means of an appropriate infinite superposition of wave functions associated with the harmonic oscillator whatever the Lie algebra symmetries. In [10–12] the authors introduced a new family of CS as a suitable superposition of the associated Bessel functions and in [13–15] the authors also use the generating function approach to construct a new type CS associated with Hermite polynomials and the associated Legendre functions, respectively. The important fact is that we do not use algebraic and group approaches (Barut-Girardello and Klauder-Perelomov) to construct generalized coherent states (GCS).

We first discuss GCS associated with a one-dimensional Schrödinger operator [16, 17] by following the work in [18, 19]. We build a family of GCS through superpositions of the corresponding eigenstates, say $\psi_n, n \in \mathbb{N}$, which are expressed in terms of the Legendre polynomial $P_n(x)$ [16]. The role of coefficients $z^n/\sqrt{n!}$ of the canonical CS is played by

$$\mathfrak{O}_n(\xi) := i^n \left(\frac{\pi(2n+1)}{2\xi}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(\xi), \quad n = 0, 1, 2, \dots, \tag{1}$$

where $\xi \in \mathbb{R}$ and $J_{n+\frac{1}{2}}(.)$ denotes the cylindrical Bessel function [20]. When $n = 0$, Eq. (1) becomes

$$\mathfrak{O}_0 = \mathscr{J}_0(\xi) = \frac{\sin(\xi)}{\xi} \tag{2}$$

where $\mathscr{J}_0(.)$ denotes the spherical Bessel function of order 0. The choosen coefficients (1) and eigenfunctions (27) (see below) have been used in ([21], p. 1625). We proceed by determining the wavefunctions of these GCS in a closed form. The latter gives the kernel of the associated CS transform which makes correspondence between the quantum states Hilbert space $L^2([-1,1], 2^{-1}dx)$ of the Legendre oscillator and a subspace of a Hilbert space of square integrable functions with respect to a suitable measure on the real line. We show that the kernel $e^{ix\xi}, \xi \in \mathbb{R}$, of the L^2-functions that are supported in $[-1,1]$ form a set of GCS.

There are in literature several approach to introducce Bessel Wavelets. We refer for instence to [22, 23]. Note that, for $[-1,1] \ni x \mapsto \cos(y/n), \ n \in \mathbb{N}$, the Legendre polynomial $P_n(x)$ and the Bessel function of order 0 are related by the Hansen's limit

$$\lim_{n\to\infty} P_n\left(\cos\frac{y}{n}\right) = \int_0^\pi e^{iy\cos\phi} d\phi = J_0(y),$$

and the integral

$$\int_0^\infty J_0(y) J_0(y) dy = \frac{\pi}{2}. \tag{3}$$

Note that in [22, 23] the authors have introduced the Bessel wavelet based on the Hankel transform. The notion of wavelets was first introduced by J. Morlet a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by A. Grossmann and J. Morlet [24]. Harmonic analyst Y. Meyer and other mathematicians understood the importance of this theory and they recognized many classical results within (see [25–27]). Classical wavelets have several applications ranging from geophysical and acoustic signal analysis to quantum theory and pure mathematics. A wavelet base is a family of functions obtained from a function known as mother wavelet, by translation and dilation. This tool permits the representation of L^2-functions in a basis well localized in time and in freqency. Wavelets are special functions with special properties which may not be satisfied by other functions. In the current context, our objective is to make a link between the construction of GCS and the theory of wavelets. Therefore, we will talk about coherent state transformation (CST) and the continuous Bessel wavelet transformation (CBWT).

The rest of this chapter is organized as follows: Section 2 is devoted to the generalized CS formalism that we are going to use. In Section 3, we briefly introduce the Paley-Wiener space PW_Ω and some notions on Legendre's Hamiltonian. We give in Section 4 a summary concept on the continuous wavelet transform on $\mathbb{R}$. In Section 5, we have constructed a class of GCS related to the Bessel cylindrical function for the legendre Hamiltonian. In Section 6, we discuss the theory of CBWT where we show as an example that the function $f \in L^2_\sigma(\mathbb{R}_+)$

$$f(t) := \frac{2w_0 - t^2}{2\left(w_0^2 + t^2\right)^{5/2}}, \quad w_0 > 0, \tag{4}$$

such that $\int_{\mathbb{R}_+} f(t) d\sigma(t) = 0$ is the mother wavelet where $d\sigma(t)$ is an appropriate Legesgue's measure on $\mathbb{R}$. Finally in Section 7. we gives some concluding remarks on the chapter.

2. Generalized coherent states formalism

We follow the generalization of canonical coherent states (CCS) introduced in [18, 19]. The definition of CS as a set of vectors associated with a reproducing kernel is general, it encompasses all the situations encountered in the physical literature. For applications we will work with normalized vectors. Let $(\mathscr{X}, \mu)$ be a measure space and let $\mathfrak{N}^2 \subset L^2(\mathscr{X}, \mu)$ be a sub-closed space of infinite dimension. Let $\{\mathscr{C}_n\}_{n=0}^{\infty}$ be a satisfactory orthogonal basis of $\mathfrak{N}^2$, for arbitrary $x \in \mathscr{X}$

$$\sum_{n=0}^{\infty} \rho_n^{-1} |\mathscr{C}_n(x)|^2 < +\infty \tag{5}$$

where $\rho_n := \|\mathscr{C}_n\|^2_{L^2(\mathscr{X},\mu)}$. Define the kernel

$$K(x, y) := \sum_{n=0}^{\infty} \rho_n^{-1} \mathscr{C}_n(x) \overline{\mathscr{C}_n(y)}, \quad x, y \in \mathscr{X}. \tag{6}$$

Then, the expression $K(x, y)$ is a reproducing kernel, $\mathfrak{N}^2$ is the corresponding kernel Hilbert space and $\mathscr{N}(x) := K(x, x), \quad x \in \mathscr{X}$. Define

$$\vartheta_x := (\mathscr{N}(x))^{-1/2} \sum_{n=0}^{\infty} \rho_n^{-1/2} \overline{\mathscr{C}_n(x)} \varphi_n.$$

Therefore,

$$\langle \vartheta_x, \vartheta_x \rangle = \mathscr{N}(x)^{-1} \sum_{n=0}^{\infty} \rho_n^{-1} \mathscr{C}_n(x) \overline{\mathscr{C}_n(x)} = 1,$$

and

$$\mathscr{W} : \mathscr{H} \to \mathfrak{N}^2 \quad with \quad \mathscr{W}\phi = \mathscr{N}^{1/2}\langle \vartheta_x, \phi \rangle$$

is an isometry. For $\phi, \psi \in \mathscr{H}$, whe have

$$\langle \phi, \psi \rangle_{\mathscr{H}} = \langle \mathscr{W}\phi, \mathscr{W}\psi \rangle_{\mathfrak{N}^2} = \int_{\mathscr{X}} \overline{\mathscr{W}\phi(x)} \mathscr{W}\psi(x) d\mu(x) \tag{7}$$

$$= \int_{\mathscr{X}} \langle \phi, \vartheta_x \rangle \langle \vartheta_x, \psi \rangle \mathscr{N}(x) d\mu(x), \tag{8}$$

and

$$\int_{\mathscr{X}} |\vartheta_x\rangle\langle\vartheta_x| \mathscr{N}(x) d\mu(x) = I_{\mathscr{H}}, \tag{9}$$

where $\mathscr{N}(x)$ is a positive weight function.

Definition 1. *Let $\mathscr{H}$ be a Hilbert space with $\dim \mathscr{H} = \infty$ and $\{\varphi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathscr{H}$.The generalized coherent state (GCS) labeled by point $x \in \mathscr{X}$ are defined as the ket-vector $\vartheta_x \in \mathscr{H}$, such that*

$$\vartheta_x := (\mathscr{N}(x))^{-1/2} \sum_{n=0}^{\infty} \rho_n^{-1/2} \overline{\mathscr{C}_n(x)} \varphi_n. \tag{10}$$

By definition, it is straightforward to show that $\langle \vartheta_x, \vartheta_x \rangle_{\mathscr{H}} = 1$.

Definition 2. *For each function $f \in \mathscr{H}$, the coherent state transform (CST) associated to the set $(\vartheta_x)_{x \in \mathscr{X}}$ is the isometric map*

$$\mathscr{W}[f](x) := (\mathscr{N}(x))^{1/2} \langle f | \vartheta_x \rangle_{\mathscr{H}}. \tag{11}$$

Thereby, we have a resolution of the identity of $\mathscr{H}$ which can be expressed in Dirac's bra-ket notation as

$$\mathbf{1}_{\mathscr{H}} = \int_{\mathscr{X}} T_x \mathscr{N}(x) d\mu(x) \tag{12}$$

where the rank one operator $T_x := |\vartheta_x\rangle\langle\vartheta_x| : \mathscr{H} \to \mathscr{H}$ is define by

$$f \mapsto T_x[f] = \langle \vartheta_x | f \rangle \vartheta_x.$$

$\mathscr{N}(x)$ appears as a weight function.

Next, the reproducing kernel has the additional property of being square integrable, i.e.,

$$\int_{\mathscr{X}} K(x,z) K(z,y) \mathscr{N}(z) d\mu(z) = K(x,y). \tag{13}$$

Note that the formula (10) can be considered as generalization of the series expansion of the CCS [28].

$$\vartheta_z = \sqrt{\pi} e^{-\frac{z\bar{z}}{2}} \sum_{k=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n, \quad z \in \mathbb{C} \tag{14}$$

with $\{\phi_n\}_{n=0}^{\infty}$ being an orthonormal basis of eigenstates of the quantum harmonic oscillator. Then, the space $\mathfrak{N}^2$ is the *Fock space* $\mathscr{F}(\mathbb{C})$ and $\mathscr{N}(z) = \pi^{-1}e^{z\bar{z}},\ z \in \mathbb{C}$.

3. The Paley-wiener space PW_Ω and the Legendre Hamiltonian: a brief overview

3.1 The Paley-wiener space PW_Ω

The Paley-Wiener space is made up of all integer functions of exponential type whose restrictions on the real line is square integrable. We give in this Section a general overview on this notion ([29], pp. 45–47).

Definition 3. *Consider F as an entire function. Then, F is an entire function of exponential type if there exists constants* $A, B > 0$ *such that, for all* $z \in \mathbb{C}$

$$|F(z)| \leq Ae^{B|z|}. \tag{15}$$

Note that, if F satisfy Definition 3, we call Ω the type of F where

$$\Omega = \lim_{r \to +\infty} \sup \frac{\log M(r)}{r} \tag{16}$$

and where $M(r) = \sup_{|z|=r} |F(z)|$. The following conditions on an entire function F are verified:

1. For all $\varepsilon > 0$ there exists C_ε such that

$$|F(z)| \leq C_\varepsilon e^{(\Omega+\varepsilon)|z|};$$

2. There exists $C > 0$ such that

$$|F(z)| \leq Ce^{\Omega|z|};$$

3. as $|z| \to +\infty$

$$|F(z)| = o\left(e^{\Omega|z|}\right).$$

Then cleary, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ F is of exponential type at most Ω.

Definition 4. *Let* $\Omega > 0$ *and* $1 \leq p \leq \infty$. *The Paley-Wiener space* PW_Ω^p *is defined as*

$$PW_\Omega^p = \left\{ f \in L^2(\mathbb{R}) : f(x) = \int_{-\Omega}^{\Omega} g(y)e^{-ixy}dy, \ \text{where } g \in L^p(-\Omega, \Omega) \right\} \tag{17}$$

and we set

$$\|f\|_{PW_\Omega^p} = 2\pi \|g\|_{L^p}. \tag{18}$$

The Paley-Wiener PW_Ω^p is the image via the *Fourier transform* of the L^p-function that are supported in $[-\Omega, \Omega]$. We will be interested in the case $p = 2$, in which PW_Ω to denote the Paley-Wiener space PW_ω^2. From the *Plancherel formula* we have

$$\|f\|_{PW^2_\Omega} = \|\hat{g}\|_{PW^2_\Omega} = 2\pi\|g\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}. \tag{19}$$

Hence, by polarization, for $f, \varphi \in PW_\Omega$,

$$\langle f, \varphi\rangle_{PW_\Omega} = \langle f, \varphi\rangle_{L^2}. \tag{20}$$

Theorem 1.1 Let F be an entire function and $\Omega > 0$. Then the following are equivalent

- $F_{|\mathbb{R}} \in L^2(\mathbb{R})$ and

$$|F(z)| = o\left(e^{\Omega|z|}\right) \text{ as } |z| \to +\infty, \tag{21}$$

- there exists $f \in L^2(\mathbb{R})$ with $supp\hat{f} \subseteq [-\Omega, \Omega]$ such that

$$F(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{iz\xi}d\xi. \tag{22}$$

The function $f \in PW_\Omega$ if and only if $f \in L^2(\mathbb{R})$ and $f = F_{|\mathbb{R}}$ (that is, f is the restriction to the real line of a function F), where F is an entire function of exponential type such that $|F(z)| = o\left(e^{\Omega|z|}\right)$ for $|z| \to +\infty$.

Theorem 1.2 The Paley-Wiener space PW_Ω is a Hilbert space with reproducing kernel w.r.t the inner product (20). Its reproducing kernel is the function

$$K(x,y) = \frac{\Omega}{\pi} sinc(\Omega(x-y)), \tag{23}$$

where $sinct = sint/t$. Hence, for every $f \in PW_\Omega$

$$f(x) = \frac{\Omega}{\pi}\int_{\mathbb{R}} f(y) sinc(\Omega(x-y))dy, \tag{24}$$

where $x \in \mathbb{R}$.

3.2 The Legendre Hamiltonian

The Legendre polynomials $P_n(x)$ and the Legendre function $\psi_n(x)$ are similar to the Hermite polynomials and the Hermite function in standard quantum mechanics. Based on the work of Borzov and Demaskinsky [16, 17] the Legendre Hamiltonian has the form

$$H = X^2 + P^2 = a^+a^- + a^-a^+, \tag{25}$$

where X and P denotes respectively the position and momentum operators, a^+ and a^- are the creation and annihilation operators. The eigenvalues of operators H are equal to

$$\lambda_0 = \frac{2}{3}, \quad \lambda_n = \frac{n(n+1) - \frac{1}{2}}{\left(n+\frac{3}{2}\right)\left(n-\frac{1}{2}\right)}, n = 1, 2, 3, \ldots\ , \tag{26}$$

and the corresponding eigenfunctions reads

$$\psi_n(x) = \sqrt{2n+1}P_n(x), \quad n = 0, 1, 2, 3, ..\ , \tag{27}$$

in terms of the Legendre polynomial $P_n(.)$, which form an orthonormal basis $\{\psi_n \equiv |n\rangle\}_{n=0}^{\infty}$ in the Hilbert space $\mathscr{H} := L^2\left([-1,1], 2^{-1}dx\right)$. These functions satisfy the recurrence relations

$$x\psi_n(x) = b_{n-1}\psi_{n-1}(x) + b_n\psi_{n+1}(x), \quad \psi_{-1}(x) = 0, \ \psi_0(x) = 1, \tag{28}$$

with coefficients

$$b_n = \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0. \tag{29}$$

The generalized position operator on the Hilbert space $\mathscr{H}$ connected with the Legendre polynomials $P_n(x)$ is an operator of multiplication by argument $X\psi_n = x\psi_n$. Taking into account of the relation (28), then

$$X\psi_n(x) = b_n\psi_{n+1}(x) + b_{n-1}\psi_{n-1}(x), \tag{30}$$

whee b_n are coefficients defined by Eq. (29). Because $\sum_{n=0}^{\infty} 1/b_n = +\infty$, X is a self-adjoint operator on the Hilbert space $\mathscr{H}$ (see [30–32]). The momentum operator P by the way described in ([17], p. 126) acts on the basis elements in $\mathscr{H}$, by the formula $P\psi_n = i\left(b_n\psi_{n+1} - b_{n-1}\psi_{n-1}\right)$. The usual commutator of operator X and P on the basis elements reads as

$$[X,P]\psi_n = 2i\left(b_n^2 - b_{n-1}^2\right)\psi_n = \frac{2i}{(2n-1)(2n+1)(2n+3)}\psi_n. \tag{31}$$

The creation and annihilation operators (25) are define by relations

$$a^+ = \frac{1}{\sqrt{2}}(X - iP); \quad a^- = \frac{1}{\sqrt{2}}(X + iP), \tag{32}$$

these operators act as $a^+\psi_n = \sqrt{2}b_n\psi_{n+1}$ and $a^-\psi_n = \sqrt{2}b_{n-1}\psi_{n-1}$. They satisfy $[a^-, a^+] = -i[X,P]$, the commutation relations.

4. Wavelet theory on $\mathbb{R}$ and the reproduction of kernels

We briefly describe below some basis definitions and properties of the one-dimensional wavelet transform on $\mathbb{R}_+$, we refer to [22, 23, 33]. In the Hilbert space $\mathfrak{N} = L^2(\mathbb{R}, dx)$, the function ψ satisfying the so-called admissibility condition

$$\mathscr{C}_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{\xi} d\xi < \infty, \tag{33}$$

where $\hat{\psi}$ being the Hankel transform of ψ. Not every vector in $\mathfrak{N}$ satisfies the above condition. A vector ψ satisfying (33) is called a mother wavelet. Combining dilatation and translation, one gets affine transformation

$$y = (b,a)x \equiv ax + b, \ a > 0, \ b \in \mathbb{R}, \ x \in \mathbb{R}_+. \tag{34}$$

Thus $\{(b,a)\}=:G_{aff}=\mathbb{R}\times(0,\infty)$, the affine group of the line. Specifically, for each pair (a,b) of the real numbers, with $a>0$, from translations and dilatations of the function ψ, we obtain a family of wavelets $\{\psi_{a,b}\}\in\mathfrak{N}$ as

$$\psi_{a,b}(x)=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right),\qquad \psi_{1,0}=\psi. \tag{35}$$

Here a is the parameter of dilation (or scale) and b is the parameter of translation (or position). It is then easily cheked that

$$\|\psi_{a,b}(x)\|^2_{\mathfrak{N}}=\|\psi(x)\|^2_{\mathfrak{N}},\quad for\quad all\quad a>0\quad and\ \ b\in\mathbb{R}. \tag{36}$$

Moreover, in terms of the Dirac's bracket notation it is an easy to show that the resolution of the identity

$$\frac{1}{\mathscr{C}_\psi}\int\int_{\mathbb{R}\times\mathbb{R}_+^*}|\psi_{a,b}\rangle\langle\psi_{a,b}|\frac{dbda}{a}=I_{\mathfrak{N}} \tag{37}$$

holds for these vectors (in the weak sense). Here $I_{\mathfrak{N}}$ is the identity operator on $\mathfrak{N}$. The *continuous wavelet transform* of an arbitrary vector (signal) $f\in\mathfrak{N}$ at the scale a and the position b is given by

$$\mathscr{S}_f(a,b)=\int_0^\infty f(t)\psi_{a,b}(t)dt. \tag{38}$$

The wavelet transform $\mathscr{S}_f(a,b)$ has several properties [34]:

- It is linear in the sense that:

$$\mathscr{S}_{\alpha f_1+\beta f_2}(a,b)=\alpha\mathscr{S}_{f_1}(a,b)+\beta\mathscr{S}_{f_2}(a,b),\quad \forall\alpha,\beta\in\mathbb{R}\ \ and\quad f_1,f_2\in L^2(\mathbb{R}_+).$$

- It is translation invariant:

$$\mathscr{S}_{\tau_{b'}f}(a,b)=\mathscr{S}_f(a,b-b')$$

where $\tau_{b'}$ refers to the translation of the function f by b' given

$$(\tau_{b'}f)(x)=f(x-b').$$

- It is dilatation-invariant, in the sense that, if f satisfies the invariance dilatation property $f(x)=\lambda f(rx)$ for some $\lambda,r>0$ fixed then

$$\mathscr{S}_f(a,b)=\lambda\mathscr{S}_f(ra,rb). \tag{39}$$

As in Fourier or Hilbert analysis, wavelet analysis provides a Plancherel type relation which permits itself the reconstruction of the analyzed function from its wavelet transform. More precisely we have

$$\langle f,g\rangle=\frac{1}{\mathscr{C}_\psi}\int_{a>0}\int_{b\in\mathbb{R}}\mathscr{S}_f(a,b)\overline{\mathscr{S}_g(a,b)}\frac{dadb}{a^2},\quad \forall f,g\in L^2(\mathbb{R}) \tag{40}$$

which in turns to reconstruct the analyzed function f in the L^2- sense from its wavelet transform as

$$f(x) = \frac{1}{\mathscr{C}_\psi}\int_{a>0}\int_{b\in\mathbb{R}} \mathscr{S}_f(a,b)\psi_{a,b}\frac{dadb}{a^2}, \quad where \quad \mathscr{S}_f(a,b) = \langle\psi_{a,b}|f\rangle. \tag{41}$$

The function $\mathscr{S}_f$ is the continuous wavelet transform of the signal f. The parameter $1/a$ represents the signal frequency of f and b its time. The conservation of the energy of the signal is due to the resolution of the identity (37), so

$$\mathscr{C}_\psi\|f\|^2 = \int\int_{\mathbb{R}\times\mathbb{R}_+} |\mathscr{S}_f(b,a)|^2\frac{dbda}{a^2}. \tag{42}$$

Then, the transform $\mathscr{S}_f$ is a fonction in the Hilbert space $L^2\left(\mathbb{R}\times\mathbb{R}_+^*, \frac{dbda}{a^2}\right)$. The reproducing kernel associated to the signal is

$$K_\psi(b,a,b',a') = \frac{1}{\mathscr{C}_\psi}\langle\psi_{a,b}|\psi_{a',b'}\rangle. \tag{43}$$

which satisfies the square integrability condition (13) with respect to the measure $dbda/a^2$. The corresponding reproducing kernel Hilbert space $\mathfrak{N}_\psi$, one see that this is the space of all signal transforms, corresponding to the mother wavelet ψ. If ψ and ψ' are two mother wavelets such that $\langle\psi'|\psi\rangle \neq 0$, then

$$\frac{1}{\langle\psi'|\psi\rangle}\int\int_{\mathbb{R}\times\mathbb{R}_+^*} |\psi_{a,b}\rangle\langle\psi'_{a,b}|\frac{dbda}{a^2} = I_{\mathfrak{N}}, \tag{44}$$

The formula (41) generalizes to

$$f = \frac{1}{\langle\psi'|\psi\rangle}\int\int_{\mathbb{R}\times\mathbb{R}_+^*} \mathscr{S}'_f(b,a)\psi_{a,b}\frac{dbda}{a^2}, \quad where \quad \mathscr{S}'_f(a,b) = \langle\psi'_{a,b}|f\rangle. \tag{45}$$

The vector ψ' is called the analyzing wavelet and ψ the reconstructing wavelet. The repoducing kernel Hilbert space $\mathfrak{N}\subset L^2\left(\mathbb{R}\times\mathbb{R}_+^*\right)$, consisting of all signal transforms with respect to the mother wavelet ψ'. Then, we have

$$K_{\psi,\psi'}(b,a;b',a') = \frac{1}{\left[\mathscr{C}_\psi\mathscr{C}_{\psi'}\right]^{\frac{1}{2}}}\left\langle\psi_{a,b}|\psi'_{a',b'}\right\rangle \tag{46}$$

is the integral kernel of a unitary map between $\mathfrak{N}_{\psi'}$ and $\mathfrak{N}_\psi$. The properties of the wavelet transform can be understood in terms of the unitary irreductible representation of the one-dilensional affine group.It is important to note that the Wavelets built on the basis of the group representation theory have all the properties of CS. There is a wole body of work devoted to the study of CS arising from group representation theory [7, 33, 35].

5. Application 1: GCS for the Legendre Hamiltonian and CS transform

5.1 GCS for the Legendre Hamiltonian

By replacing the coefficients $z^n/\sqrt{n!}$ of the canonical CS by the function $\mathfrak{O}_n(\xi)$ in (1) as mentioned in the introduction. We construct in this section a class of GCS indexed by point $\xi\in\mathbb{R}$.

Definition 5. *The GCS labeled by points $\xi \in \mathbb{R}$ is defined by the following superposition*

$$\vartheta_\xi = \mathcal{N}(\xi)^{-1/2} \sum_{n=0}^{\infty} \mathfrak{O}_n(\xi) \psi_n, \quad \xi \in \mathbb{R} \tag{47}$$

here $\mathcal{N}(\xi)$ is a normalization factor, the function $\mathfrak{O}_n(\xi) := \Phi_n(\xi)\rho_n^{-1/2}$, with

$$\Phi_n(\xi) = i^n \sqrt{\frac{\pi}{2\xi}} J_{n+\frac{1}{2}}(\xi), \tag{48}$$

where $J_{n+1/2}(.)$ is the cylindrical Bessel function ([20], p. 626):

$$J_{n+\frac{1}{2}}(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n+1/2)!} \left(\frac{z}{2}\right)^{2s+n+\frac{1}{2}}, \ z \in \mathbb{C} \tag{49}$$

and ρ_n are positive numbers given by

$$\rho_n = \frac{1}{2n+1}, \ n = 0, 1, 2, \ldots \ , \tag{50}$$

and $\{\psi_n\}$ is an orthonormal basis of the Hilbert space $\mathcal{H} = L^2([-1,1], 2^{-1}dx)$ defined in (27).

Proposition 1. *The normalization factor defined by the GCS (47) reads as*

$$\mathcal{N}(\xi) = 1, \tag{51}$$

for every $\xi \in \mathbb{R}$.

Proof. From (47) and by using the orthonormality relation of basis elements $\{\psi_n\}_{n=0}^{+\infty}$ in (27), then

$$\langle \vartheta_\xi | \vartheta_\xi \rangle = \pi(\xi \mathcal{N}(\xi))^{-1} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) J_{n+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi). \tag{52}$$

In order to identify the above series, we make appeal to the formula ([36], p. 591):

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) J_{n+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi) = \pi^{-1}\xi, \tag{53}$$

we then obtain the result (51) by using the GCS condition $\langle \vartheta_\xi | \vartheta_\xi \rangle = 1$. □

Proposition 2. *The GCS defined in* (47) *satisfy the following resolution of the identity*

$$\int_{\mathbb{R}} T_\xi d\mu(\xi) = \mathbf{1}_{\mathcal{H}}, \tag{54}$$

(in the weak sense) in terms of an acceptable measure

$$d\mu(\xi) = \frac{1}{\pi} d\xi, \tag{55}$$

where $d\xi$ the Lebesgue's measure on $\mathbb{R}$. The rank one operator $T_\xi = |\vartheta_\xi\rangle\langle\vartheta_\xi| : \mathcal{H} \to \mathcal{H}$ is define as

$$\varphi \mapsto T_{\xi}[\varphi] = \langle \vartheta_{\xi} | \varphi \rangle \vartheta_{\xi}. \tag{56}$$

Proof. We need to determine the function $\sigma(\xi)$. Let

$$d\mu(\xi) = \sigma(\xi) d\xi, \tag{57}$$

where $\sigma(\xi)$ is an auxiliary function. Let us writte $T_{m,n} := |\psi_m\rangle\langle\psi_n|$, defined as in (56). According to (56) and by writing

$$\int_{\mathbb{R}} T_{\xi} d\mu(\xi)$$
$$= \sum_{n,m=0}^{\infty} \frac{\pi}{2}(-1)^n i^{n+m} \left(\int_{-\infty}^{\infty} \frac{J_{m+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi)}{\sqrt{\rho(m)\rho(n)}} \sigma(\xi) \frac{d\xi}{\xi} \right) T_{m,n} \tag{58}$$

$$= \sum_{n,m=0}^{\infty} \frac{\pi}{2}(-1)^n i^{n+m} \sqrt{(2m+1)(2n+1)} \left(\int_{-\infty}^{\infty} J_{m+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi) \sigma(\xi) \frac{d\xi}{\xi} \right) T_{m,n}. \tag{59}$$

Hence, we need $\sigma(\xi)$ such that

$$\int_{-\infty}^{\infty} J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi) \sigma(\xi) \frac{d\xi}{\xi} = \frac{2}{\pi(2n+1)} \delta_{m,n}. \tag{60}$$

We make appeal to the integral ([36], p. 211):

$$\int_{-\infty}^{\infty} \frac{1}{y} J_{m+\frac{1}{2}}(cy) J_{n+\frac{1}{2}}(cy) dy = \frac{2}{2n+1} \delta_{m,n}, \tag{61}$$

with condition $c > 0$. Then, for parameters $c = 1$, we have

$$\int_{-\infty}^{\infty} \frac{1}{\xi} J_{m+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi) d\xi = \frac{2}{2n+1} \delta_{m,n}. \tag{62}$$

By comparing (62) with (66) we obtain finally the desired weight function $\sigma(\xi) = 1/\pi$. Therefore, the measure (57) has the form (55) [37]. Indeed (59) reduces further to $\sum_{n=0}^{\infty} T_{n,n} = 1_{\mathscr{H}}$, in other words

$$\int_{\mathbb{R}} T_{\xi} d\mu(\xi) = \mathbf{1}_{\mathscr{H}}. \tag{63}$$

According to this construction, the state ϑ_{ξ} form an overcomplete basis in the Hilbert space $\mathscr{H}$ (**Figure 1**). □

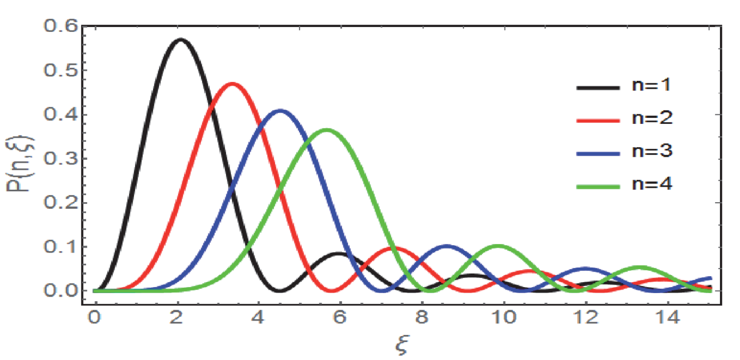

Figure 1.
Plots of the probability distribution $P(n,\xi)$ versus ξ for various values of n.

When the GCS (47) describes a quantum system, the probability of finding the state ψ_n in some normalized state ϑ_ξ of the state Hilbert space $\mathscr{H}$ is given by $P(n,\xi) \coloneqq \left|\langle\psi_n|\vartheta_\xi\rangle\right|^2$. For the GCS (47) the probability distribution function is given by

$$P(n,\xi) = \frac{\pi(2n+1)}{2|\xi|}\left|J_{n+\frac{1}{2}}(\xi)\right|^2, \quad \xi \in \mathbb{R}_+^*. \tag{64}$$

5.2 Coherent state transform

To discuss coherent state transforms (CST), we will start by establishing the kernel of this transformation by giving the closed form of the GCS (47).

Proposition 3. *For all $x \in [-1,1]$, the wave functions of GCS in (47) can be written as*

$$\vartheta_\xi(x) = e^{-ix\xi}, \tag{65}$$

for all $\xi \in \mathbb{R}$.

Proof. We start by the following expression

$$\vartheta_\xi(x) = \mathcal{N}(\xi)^{-1/2}\mathfrak{S}(x,\xi), \tag{66}$$

where the series

$$\mathfrak{S}(x,\xi) \coloneqq \sum_{n=0}^{\infty}\mathfrak{D}_n(\xi)\psi_n(x), \tag{67}$$

with the function $\mathfrak{D}_n(\xi) = \Phi_n(\xi)\rho_n^{-1/2}$, mentioned in Definition 5. To do this, we start by replacing the function $\Phi_n(\xi)$ and the positive sequences ρ_n by their expressions in (48) and (50) thus Eq. (67) reads

$$\mathfrak{S}(x,\xi) = \sqrt{\frac{\pi}{2\xi}}\sum_{n=0}^{\infty}(-1)^n i^n\sqrt{2n+1}J_{n+\frac{1}{2}}(\xi)\psi_n(x). \tag{68}$$

Making use the explicit expression (27) of the eigenstates $\psi_n(x)$, then the sum (68) becomes

$$\mathfrak{S}(x,\xi) = \sqrt{\frac{2\pi}{\xi}}\sum_{n=0}^{\infty}(-1)^n i^n\left(n+\frac{1}{2}\right)J_{n+\frac{1}{2}}(\xi)P_n(x). \tag{69}$$

We now appeal to the Gegenbauer's expansion of the plane wave in Gegenbauer polynomials and Bessel functions ([38], p. 116):

$$\mathrm{e}^{\mathrm{i}\xi\mathrm{x}} = \Gamma(\gamma)(\xi 2)^{-\gamma}\sum_{\mathrm{n}=0}^{\infty}\mathrm{i}^{\mathrm{n}}(\mathrm{n}+\gamma)\mathrm{J}_{\mathrm{n}+\gamma}(\xi)\mathrm{C}_{\mathrm{n}}^{\gamma}(\mathrm{x})$$

Then, for $\gamma = 1/2$, $y = x$ and by using the identity $\Gamma(1/2) = \sqrt{\pi}$, we arrive at (65). □

Corollary 1. *When the variable $\xi \ll 1$, the GCS in* (47) *becomes*

$$\vartheta_\xi \approx \mathcal{N}(\xi)^{-1/2}\sum_{n=0}^{\infty}\frac{\sqrt{2\pi}(-i\xi)^n}{\sqrt{2^{2n+1}(2n+1)\Gamma\left(n+\frac{1}{2}\right)}}\psi_n. \tag{70}$$

Proof. The result follows immediately by using the formula ([20], p. 647):

$$\mathcal{J}_n(\xi) \approx \frac{\xi^n}{(2n+1)!!}, \quad \xi \ll 1 \tag{71}$$

where

$$\mathcal{J}_n(\xi) = \sqrt{\frac{\pi}{2\xi}} J_{n+\frac{1}{2}}(\xi), \quad n = 0, 1, 2, \ldots \quad , \tag{72}$$

is the spherical Bessel function [20]. This ends the proof. □

The careful reader has certainly recognized in (70) the expression of nonlinear coherent states [38].

Let us note that, in view of the formula ([36], p. 667):

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) J_{n+\frac{1}{2}}(\eta) J_{n+\frac{1}{2}}(\xi) = \frac{\sqrt{\eta\xi}}{\pi(\eta - \xi)} \sin(\eta - \xi), \tag{73}$$

the reproducing kernel arising from GCS (47) can be written as

$$K(\eta, \xi) := \langle \vartheta_\eta | \vartheta_\xi \rangle \tag{74}$$

$$= \pi \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \frac{J_{n+\frac{1}{2}}(\eta)}{\sqrt{\eta}} \frac{J_{n+\frac{1}{2}}(\xi)}{\sqrt{\xi}} = \frac{\sin(\eta - \xi)}{\eta - \xi}, \tag{75}$$

denotes the Dyson's sine kernel, which is the reproducing kernel of the Paley-Wiener Hilbert space PW_1. Then, the family $\left\{ [\pi(n + 1/2)/\xi]^{1/2} J_{n+\frac{1}{2}}(\xi) \right\}; \; n \in \mathbb{N}_0$, forms an orthonormal basis of PW_1 [39].

Once we have a closed form of GCS, we can look for the associated CST, this transform should map the space $\mathcal{H} = L^2([-1, 1], 2^{-1}dx)$ spanned by eigenstates $\{\psi_n\}$ in (27) onto $PW_1 \subset L^2(\mathbb{R}, d\mu)$ as.

Proposition 4. *For $\varphi \in L^2([-1, 1], 2^{-1}dx)$, the CST is the unitary map*

$$\mathcal{W}\left(L^2([-1, 1], 2^{-1}dx\right) = PW_1, \tag{76}$$

defined by means of (65) *as*

$$\mathcal{W}[\varphi](\xi) = (\mathcal{N}(\xi))^{1/2} \langle \varphi | \xi \rangle_{\mathcal{H}} = \int_{-1}^{1} e^{-ix\xi} \overline{\varphi(x)} \frac{dx}{2}, \tag{77}$$

for all $\xi \in \mathbb{R}$.

Corollary 2. *The following integral*

$$\frac{(-i)^n}{\sqrt{\xi}} J_{n+\frac{1}{2}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} P_n(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}. \tag{78}$$

holds.

Proof. From (75), the image of the basis vector $\{\psi_n\}$ under the transform $\mathcal{W}$ should exactly be

$$\mathcal{W}[\psi_n](\xi) = (-i)^n \sqrt{\frac{\pi(2n+1)}{2\xi}} J_{n+\frac{1}{2}}(\xi). \tag{79}$$

Now, by writing (75) as

$$\mathscr{W}[\psi_n](\xi) = \int_{-1}^{1} e^{-ix\xi}\psi_n(x)\frac{dx}{2},$$

and replacing ψ_n by their values given in (27), we obtain

$$\mathscr{W}[\psi_n](\xi) = \frac{\sqrt{2n+1}}{2}\int_{-1}^{1} e^{-ix\xi}P_n(x)dx,$$

the integral (78) can be evaluated by the help of the formula ([40], p. 456):

$$\int_{-1}^{1} P_n(x)e^{i\xi x}dx = i^n\sqrt{\frac{2\pi}{\xi}}J_{n+\frac{1}{2}}(\xi), \tag{80}$$

this ends the proof. □

Note that, in view of ([28], p. 29), by considering $h_n(\xi) := \rho_n^{-1/2}\overline{\Phi_n(\xi)}$ and GCS $\mathscr{K}(\xi, x) := \langle x|\vartheta_\xi\rangle$, the basis element $\psi_n \in L^2([-1,1], 2^{-1}dx)$ has the integral representation

$$\psi_n(x) = \int_{-\infty}^{\infty} h_n(\xi)\overline{\mathscr{K}(\xi, x)}d\mu(\xi) \tag{81}$$

where the function $\Phi_n(\xi)$ and the positive sequences ρ_n are given in (48) and (50) respectively, the measure $d\mu(\xi)$ is given in (55), then the Legendre polynomial has the following integral representation

$$P_n(x) = \frac{(-i)^n}{\pi}\int_{-\infty}^{\infty} \mathscr{J}_n(t)e^{ix\xi}d\xi, \tag{82}$$

where the function $\mathscr{J}_n(.)$ is given in (72), which is recognized as the Fourier transform of the spherical Bessel function (72) (see [40], p. 267):

$$\int_{-\infty}^{\infty} e^{ixt}\mathscr{J}_n(t)dt = \begin{cases} \pi i^n P_n(x), & -1 < x < 1 \\ \frac{1}{2}\pi(\pm i)^n, & x = \pm 1, \\ 0, & \pm x > 1 \end{cases} \tag{83}$$

where $P_n(.)$ the Legendre's polynomial [40].

Remark 1. *Also note that:*

- *The usefulness expansion of GCS was made very clear in a paper authored by Ismail and Zhang, where it was used to solve the eigenvalue problem for the left inverse of the differential operator, on L^2-spaces with ultraspherical weights [41, 42].*

- *For $x, \xi \in \mathbb{R}$, the function $\varphi_\xi(x) = e^{ix\xi}$, is known as the Gabor's coherent states introduced in signal theory where the property $\psi_\xi = \hat{T}(\xi)\psi$, with $\psi \in L^2(\mathbb{R})$, and $\hat{T}(\xi)$ the unitary transformation, is obtained by using the standard representation of the Heisenberg group in three dimensions, in $L^2(\mathbb{R})$, for more information (see [43]).*

Exercise 1. *Show that the vectors*

$$\vartheta_\xi = \mathcal{N}(\xi)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{2\pi}(-i\xi)^n}{\sqrt{2^{2n+1}(2n+1)\Gamma\left(n+\frac{1}{2}\right)}} \psi_n. \tag{84}$$

forms a set of GCS and gives the associated GCS transform.

6. Application 2: continuous Bessel wavelet transform

The continuous wavelet transform (CWT) is used to decompose a signal into wavelets. In mathematics, the CWT is a formal tool that provides an overcomplete representation of a signal by letting the translation and scale parameter of the wavelets vary continuously. There are several ways to introduce the Bessel wavelet [22, 23]. For $1 \leq p \leq \infty$ and $\mu > 0$, denote

$$L^p_\sigma(\mathbb{R}_+) := \left\{ \psi \;\; such \;\; as \;\; \|\psi\|^p_{p,\sigma} = \int_0^\infty |\psi(x)|^p d\sigma(x) < \infty \right\}$$

and $\|\psi\|_{\infty,\sigma} = \;\; ess_{0<x<\infty} \;\; \sup|\psi(x)| < \infty$ and $d\sigma(x)$ is the measure defined as

$$d\sigma(x) = \frac{x^{2\mu}}{2^{\mu+\frac{1}{2}}\Gamma\left(\mu+\frac{3}{2}\right)} dx. \tag{85}$$

Now, let us consider the function

$$j(x) = 2^{\mu-\frac{1}{2}}\Gamma\left(\mu+\frac{1}{2}\right) x^{\frac{1}{2}-\mu} J_{\mu-\frac{1}{2}}(x), \tag{86}$$

where $J_{\mu-\frac{1}{2}}(x)$ is the Bessel function of order $l := \mu - 1/2$ given by

$$J_l(x) = \left(\frac{x}{2}\right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+l+1)} \left(\frac{x}{2}\right)^{2k}. \tag{87}$$

For $\mu = 1$, the function $j(x) = \mathfrak{O}_0(x)$ coincides with equation (2) discussed in the introduction. For each function $\phi \in L_{1,\sigma}(0,\infty)$, the Hankel transform of order μ is defined by

$$\hat{\phi}(x) := \int_0^\infty j(xt)\phi(t)d\sigma(t), \quad 0 \leq x < \infty. \tag{88}$$

We know that from ([44], p. 316) that $\hat{\phi}(x)$ is bounded and continuous on $[0,\infty)$ and $\|\hat{\phi}\|_{\infty,\sigma} \leq \|\phi\|_{1,\sigma}$. If $\phi, \hat{\phi} \in L_{1,\sigma}(0,\infty)$, then by inversion, we have

$$\phi(x) = \int_0^\infty j(xt)\hat{\phi}(t)d\sigma(t). \tag{89}$$

From ([45], p. 127) if $\phi(x)$ and $\Phi(x)$ are in $L_{1,\sigma}(0,\infty)$, then the following Parseval formula also holds

$$\int_0^\infty \hat{\phi}(t)\hat{\Phi}(t)d\sigma(t) = \int_0^\infty \phi(x)\Phi(x)d\sigma(x). \tag{90}$$

Denoting therefore by

$$\mathscr{D}(x,y,z) = \int_0^\infty j(xt)j(yt)j(zt)d\sigma(t). \tag{91}$$

For a 1-variable function $\psi \in L^2_\sigma(\mathbb{R}_+)$, we define the Hankel translation operator

$$\tau_y\psi(x) := \psi(x,y) = \int_0^\infty \mathscr{D}(x,y,z)\psi(z)d\sigma(z), \quad \forall x > 0, \ y < \infty. \tag{92}$$

Trime'che ([46], p. 177) has shown that the integral is convergent for almost all y and for each fixed x, and

$$\|\psi(x,.)\|_{2,\sigma} \le \|\psi\|_{2,\sigma}. \tag{93}$$

The map $y \mapsto \tau_y\psi$ is continuous from $[0,\infty)$ into $(0,\infty)$. For a 2-variables the function ψ, we define a dilatation operator

$$D_a\psi(x,y) = a^{-2\mu-1}\psi\left(\frac{x}{a},\frac{y}{a}\right). \tag{94}$$

From the inversion formula in (89), we have

$$\int_0^\infty j(zt)\mathscr{D}(x,y,z)d\sigma(z) = j(xt)j(yt), \quad \forall 0 < x,y < \infty, \quad 0 \le t < \infty,$$

for $t = 0$ and $\mu - 1/2 = 0$, we arrive at

$$\int_0^\infty \mathscr{D}(x,y,z)d\sigma(z) = 1. \tag{95}$$

The Bessel Wavelet copy $\psi_{a,b}$ are defined from the Bessel wavelet mother $\psi \in L^2_\sigma(\mathbb{R}_+)$ by

$$\psi_{a,b}(x) := D_a\tau_b\psi(x) = D_a\psi(b,x) \tag{96}$$

$$= a^{-2\mu-1}\int_0^\infty \mathscr{D}\left(\frac{b}{a},\frac{x}{a};z\right)\psi(z)d\sigma(z), \quad \forall \ a > 0, \ b \in \mathbb{R}, \tag{97}$$

the integral being convergent by virtue of (92). As in the classical wavelet theory on $\mathbb{R}$, let us define the continuous Bessel Wavelet transform (CBWT) of a function $f \in L^2_\sigma(\mathbb{R}_+)$, at the scale a and the position b by

$$\mathscr{B}(b,a) := \left(\mathscr{B}_\psi f\right)(b,a) = \left\langle f(t),\psi_{b,a}(t)\right\rangle \tag{98}$$

$$= \int_0^\infty f(t)\overline{\psi_{a,b}(t)}d\sigma(t) \tag{99}$$

$$= a^{-2\mu-1}\int_0^\infty\int_0^\infty f(t)\overline{\psi(z)}\mathscr{D}\left(\frac{b}{a},\frac{t}{a},z\right)d\sigma(z)d\sigma(t). \tag{100}$$

The continuity of the Bessel wavelet follows from the boundedness property of the Hankel translation ([46], (104), p. 177). The following result is due to [22]:

Theorem 1.3 Let $\psi \in L^2_\sigma(\mathbb{R}_+)$ and $f,g \in L^2_\sigma(\mathbb{R}_+)$. Then

$$\int_0^\infty \int_0^\infty \left(\mathscr{B}_\psi f\right)(b,a)\overline{\left(\mathscr{B}_\psi g\right)(b,a)}d\sigma(a)d\sigma(b) = \mathscr{C}_\psi \langle f,g\rangle \tag{101}$$

whenever

$$\mathscr{C}_\psi = \int_0^\infty t^{-2\mu-1}|\hat{\psi}(t)|^2 d\sigma(t) < \infty. \tag{102}$$

For all $\mu > 0$.

Proof. For the function $f \in L^2_\sigma(\mathbb{R}_+)$, let us write the Bessel wavelet by using Eq. (38) as

$$\left(\mathscr{B}_\psi f\right)(b,a) = \int_0^\infty f(t)\psi_{a,b}(t)d\sigma(t) \tag{103}$$

$$= \frac{1}{a^{2\mu+1}}\int_0^\infty\int_0^\infty f(t)\overline{\psi}(z)\mathscr{D}\left(\frac{b}{a},\frac{t}{a},z\right)d\sigma(z)d\sigma(t). \tag{104}$$

Now observe that

$$\mathscr{D}\left(\frac{b}{a},\frac{t}{a},z\right) = \int_0^\infty j\left(\frac{bu}{a}\right)j\left(\frac{tu}{a}\right)j(zu)d\sigma(u). \tag{105}$$

Hence whe have that

$$\left(\mathscr{B}_\psi f\right)(b,a) = \frac{1}{a^{2\mu+1}}\int_{\mathbb{R}^3_+} f(t)\psi(z)j\left(\frac{bu}{a}\right)j\left(\frac{tu}{a}\right)j(zu)d\sigma(u)d\sigma(z)d\sigma(t) \tag{106}$$

$$= \frac{1}{a^{2\mu+1}}\int_{\mathbb{R}^2_+} \hat{f}\left(\frac{u}{a}\right)\psi(z)j\left(\frac{bu}{a}\right)j(zu)d\sigma(u)d\sigma(z) \tag{107}$$

$$= \frac{1}{a^{2\mu+1}}\int_{\mathbb{R}_+} \hat{f}\left(\frac{u}{a}\right)\hat{\psi}(u)j\left(\frac{bu}{a}\right)d\sigma(u) \tag{108}$$

$$= \int_{\mathbb{R}_+} \hat{f}(v)\hat{\psi}(av)j(bv)d\sigma(v) \tag{109}$$

$$= \left(\hat{f}(v)\hat{\psi}(av)\right)^{\frown}(b). \tag{110}$$

In terms of the Parseval formula (90), we obtain

$$\int_{\mathbb{R}_+} \left(\mathscr{B}_\psi f\right)(b,a)\overline{\left(\mathscr{B}_\psi f\right)}(b,a)d\sigma(b)$$

$$= \int_0^\infty \left(\hat{f}(v)\hat{\psi}(av)\right)^{\frown}(b)\overline{\left(\hat{g}(v)\overline{\hat{\psi}(av)}\right)^{\frown}}(b)d\sigma(u) \tag{111}$$

$$= \int_0^\infty \hat{f}(u)\overline{\hat{\psi}(au)}\hat{g}(u)\overline{\hat{\psi}(au)}d\sigma(u) \tag{112}$$

Now multiplying by $a^{-2\mu-1}d\sigma(a)$ and integrating, we get

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \left(\mathscr{B}_\psi f\right)(b,a)\overline{\left(\mathscr{B}_\psi f\right)}(b,a)a^{-2\mu-1}d\sigma(a)d\sigma(b) \tag{113}$$

$$= \int\int_0^\infty \hat{f}(u)\overline{\hat{\psi}(au)}\overline{\hat{g}(u)\overline{\hat{\psi}(au)}}\frac{d\sigma(a)}{a^{2\mu+1}}d\sigma(u) \tag{114}$$

$$= \int_{\mathbb{R}} \hat{f}(u)\overline{\hat{g}(u)}\left(\int_{\mathbb{R}} |\hat{\Psi}(au)|^2 \frac{d\sigma(a)}{a^{2\mu+1}}\right)d\sigma(u) = C_\psi \int_{\mathbb{R}} \hat{f}(u)\overline{\hat{g}(u)}d\sigma(u) \tag{115}$$

$$= \mathscr{C}_\psi \langle f, g\rangle. \tag{116}$$

The *admissible condition* (102) requires that $\hat{\psi}(0) = 0$. If $\hat{\psi}$ is continuous then from (88) it follows that

$$\int_0^\infty \psi(x)d\sigma(x) = 0. \tag{117}$$

6.1 Example

Let us consider the function

$$f(t) = \frac{2w_0^2 - t^2}{2(w_0^2 + t^2)^{5/2}}, \quad w_0 > 0, \quad t \in \mathbb{R}_+. \tag{118}$$

In the case $\mu = 1/2$, the measure (85) takes the form

$$d\sigma(t) = \frac{t}{2}dt \tag{119}$$

and the function (86) reduces to

$$j(t) = J_0(t), \tag{120}$$

where $J_0(x)$ the Bessel's function of the first kind. Also note that

$$\int_0^\infty \frac{(2w_0^2 - t^2)^2}{2(w_0^2 + t^2)^5}d\sigma(t) < \infty. \tag{121}$$

The Bessel wavelet transform of $f(t)$ is given by

$$\left\{\mathscr{B}_\psi\left(\frac{2w_0^2 - t^2}{2(w_0^2 + t^2)^{5/2}}\right)\right\}(b, a) = a^{-2}\int_0^\infty \frac{2w_0^2 - t^2}{2(w_0^2 + t^2)^{5/2}}\psi\left(\frac{b}{a}, \frac{t}{a}\right)d\sigma(t) \tag{122}$$

$$= a^{-2}\int_0^\infty \psi(z)\left(\int_0^\infty \frac{2w_0^2 - t^2}{2(w_0^2 + t^2)^{5/2}}\mathscr{D}\left(\frac{b}{a}, \frac{t}{a}, z\right)d\sigma(t)\right)d\sigma(z) \tag{123}$$

Using the representation

$$\mathscr{D}\left(\frac{b}{a}, \frac{t}{a}, z\right) = \int_0^\infty J_0\left(\frac{b}{a}u\right)J_0\left(\frac{t}{a}u\right)J_0(zu)d\sigma(u) \tag{124}$$

then (122) becomes

$$a^{-2}\int_0^\infty \Psi(z)\left(\int_0^\infty J_0\left(\frac{b}{a}u\right)J_0(zu)\mathfrak{O}_{a,w_0}(u)d\sigma(u)\right)d\sigma(z)$$

Where the integral

$$\mathfrak{O}_{a,w_0}(u) = \int_0^\infty \frac{2w_0^2 - t^2}{2\left(w_0^2 + t^2\right)^{5/2}} J_0\left(\frac{t}{a}u\right) d\sigma(t). \tag{125}$$

In terms of the Legendre polynomial $P_2(t)$, the function

$$\frac{2w_0^2 - t^2}{2\left(w_0^2 + t^2\right)^{5/2}} = \left(w_0^2 + t^2\right)^{-3/2} P_2\left[w_0\left(w_0^2 + t^2\right)^{-1/2}\right]. \tag{126}$$

Then (125) reads

$$\mathfrak{O}_{a,w_0}(u) = \int_0^\infty \left(w_0^2 + t^2\right)^{-3/2} P_2\left[w_0\left(w_0^2 + t^2\right)^{-1/2}\right] J_0\left(\frac{t}{a}u\right) d\sigma(t). \tag{127}$$

The above equation can be evaluated by means of the formula ([47], p. 13):

$$\frac{1}{n!} y^{n-1/2} e^{-py} = \int_0^\infty x^{1/2}\left(p^2 + x^2\right)^{-\frac{1}{2}n - \frac{1}{2}} P_n\left[p\left(p^2 + x^2\right)^{-1/2}\right](xy)^{1/2} J_0(xy) dx. \tag{128}$$

For parameters $n = 2$ and $p = w_0$, we find that

$$\mathfrak{O}_{a,w_0}(u) = \frac{1}{4} u \exp\left(-w_0 \frac{u}{a}\right). \tag{129}$$

In terms of the above result, the CBWT read as

$$\left\{\mathscr{B}_\psi\left(\frac{2w_0^2 - t^2}{2\left(w_0^2 + t^2\right)^{5/2}}\right)\right\}(b,a) = a^{-2}\int_0^\infty \psi(z)\mathfrak{M}_{a,w_0}(z) d\sigma(z) \tag{130}$$

where

$$\mathfrak{M}_{a,w_0}(z) = \int_0^\infty 8^{-1} u^2 e^{-\frac{w_0}{a}u} J_0\left(\frac{b}{a}u\right) J_0(zu) du. \tag{131}$$

To evaluated (131) we make appeal to the Lipschitz-Hankel integrals ([48], p. 389):

$$\int_0^\infty e^{-pt} J_\nu(qt) J_\nu(rt) t^{\mu-1} dt \tag{132}$$

$$= \frac{(qr)^\nu}{\pi p^{\mu+2\nu}} \frac{\Gamma(\mu+2\nu)}{2\nu+1} \int_{{}_2F_1}^{\pi} \left(\frac{\mu+2\nu}{2}, \frac{\mu+2\nu+1}{2}; \nu+1; -\frac{\zeta^2}{p^2}\right) \sin^{2\nu}\phi \, d\phi$$

with conditions $\Re(p \pm iq \pm ir) > 0$ and $\Re(\mu + 2\nu) > 0$, while ζ is written in place of $\left(q^2 + r^2 - 2qr\cos\phi\right)^{1/2}$, where ${}_2F_1$ denotes the hypergeometric function. For parameters $p = w_0/a, q = \mathrm{b/a}, r = z, \mu = 3$ and $n = 0$, we arrive at

$$\mathfrak{M}_{a,w_0}(z) = \frac{a^3}{4\pi w_0^3} \int_{{}_2F_1}^{\pi} \left(\frac{3}{2}, 2; 1; -\left(aw_0^{-1}\zeta\right)^2\right) d\phi \tag{133}$$

where $\zeta = \left[\left(a^{-1}b\right)^2 + z^2 - 2a^{-1}bz\cos\phi\right]^{1/2}$.

Next, by using the representation of the hypergeometric ${}_2F_1$-sum ([49], p. 404, Eq. 209) (**Figure 2**):

$$ {}_2F_1\left(\frac{3}{2}, 2; 1; z\right) = \frac{1}{2}(2+z)(1-z)^{-5/2}. \tag{134}$$

Then (131) takes the form

$$\mathfrak{M}_{a,w_0}(z) = \frac{a^3}{8\pi w_0^3}\int_0^{\pi}\left(2-\left(w_0^{-1}a\zeta\right)^2\right)\left(1+\left(w_0^{-1}a\zeta\right)^2\right)^{-5/2}d\zeta, \tag{135}$$

This leads to the following CBWT

$$\left\{\mathscr{B}_\psi\left(\frac{2w_0^2-t^2}{2\left(w_0^2+t^2\right)^{5/2}}\right)\right\}(b,a) = \frac{a}{4\pi}\int_0^\infty \psi(z)\int_0^\pi \frac{2w_0^2-(a\zeta)^2}{2\left(w_0^2+(a\zeta)^2\right)^{5/2}}d\phi d\sigma(z). \tag{136}$$

We have given an example of a signal $f(t)\in L_\sigma^2(0,\infty)$ such that the CBWT is written as

$$\{\mathscr{B}_\psi(f(t))\}(b,a) = \frac{a}{4\pi}\int_0^\pi\int_0^\infty \psi(z)f(a\zeta)d\sigma(z)d\phi. \tag{137}$$

According to Theorem 1.3, let $\psi\in L_\sigma^2(\mathbb{R}_+)$ and $f,g\in L_\sigma^2(\mathbb{R}_+)$, then

$$\int_0^\infty\int_0^\infty (\mathscr{B}_\psi f)(b,a)\overline{(\mathscr{B}_\psi g)(b,a)}d\sigma(a)d\sigma(b) = \frac{1}{128w_0^2}\langle f,g\rangle. \tag{138}$$

Note that, for all $w_0>0$, the given function

$$f(t) = \frac{2w_0^2-t^2}{2\left(w_0^2+t^2\right)^{5/2}},\quad t\in\mathbb{R}_+, \tag{139}$$

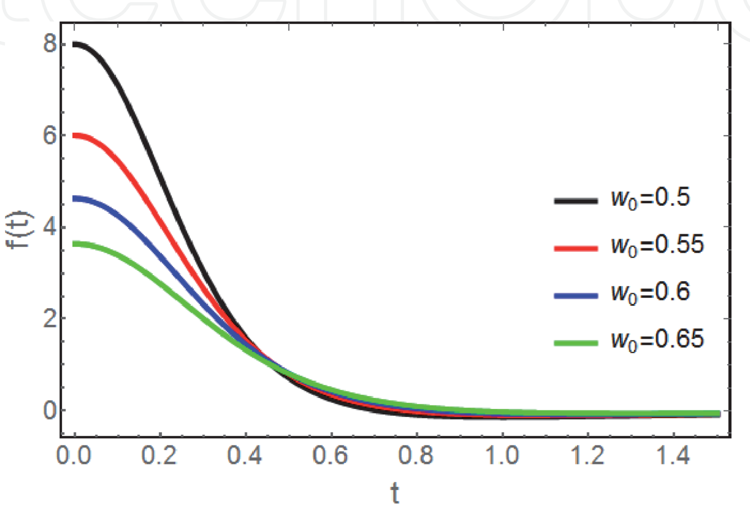

Figure 2.
Plots of the mother wavelet $f(t)$ defined in (6.34) versus t, for various values of the parameters w_0.

is the mother wavelet. The Hankel transform of $f(t)$ is given by

$$\hat{f}(y) = \int_0^\infty \frac{2w_0^2 - t^2}{2\left(w_0^2 + t^2\right)^{5/2}} J_0(xy) d\sigma(t) = \frac{1}{4} y e^{-w_0 y}, \quad \forall \quad 0 \le y < \infty. \tag{140}$$

and satisfy the admissible condition

$$\mathscr{C}_f = \frac{1}{2} \int_0^\infty \frac{\left|\hat{f}(\xi)\right|^2}{\xi} d\xi \tag{141}$$

$$= \frac{1}{128 w_0^2}, \quad w_0 > 0. \tag{142}$$

The Hankel transformation $\hat{f}(0) = 0$, so by the help of (140) we obtain

$$\int_0^\infty \frac{t\left(2w_0^2 - t^2\right)}{\left(w_0^2 + t^2\right)^{5/2}} dt = 0. \tag{143}$$

Exercise 2
For which numbers $n \in \mathbb{N}$, the following function

$$f_n(t) = \left(w_0^2 + x^2\right)^{-\frac{1}{2}n - \frac{1}{2}} P_n\left[w_0\left(w_0^2 + x^2\right)^{-1/2}\right] \tag{144}$$

Is the mother wavelet where $P_n(.)$ the Legendre's polynomial.

7. Conclusions

In this chapter we are interested in the construction of the generalized coherent state (GCS) and the theory of wavelets. As it is well know wavelets constructed on the basis of group representation theory have the same properties as coherent states. In other words, the wavelets can actually be thought of as the coherent state associated with these groups. Coherent state is very important because of three properties they have: coherence, overcompleteness, intrinsic geometrization. We have seen that it is possible to construct coherent states without taking into account the theory of group representation. Throughout this chapter we have used the Bessel function to construct the coherent state transform and Bessel continuous wavelets transform. We have prove that the kernel of the finite Fourier transform (FFT) of L^2-functions supported on $[-1, 1]$ form a set of GCS. We therefore discussed another way of building a set of coherent states based on Wavelet's theory makes it easier.

Building coherent states in this chapter is always not easy because it is necessary to find coefficients which will make it possible to find vectors which will certainly satisfy certain conditions but the procedure based on Wavelet's theory makes it easier.

It should be noted that the theory of classical wavelets finds several applications ranging from the analysis of geophysical and acoustic signals to quantum theory. This theory solves difficult problems in mathematics, physics and engineering, with several modern applications such as data compression, wave propagation, signal processing, computer graphics, pattern recognition, pattern processing. Wavelet analysis is a robust technique used for investigative methods in quantifying the timing of measurements in Hamiltonian systems.

Conflict of interest

The authors declare no conflict of interest.

Author details

Kayupe Kikodio Patrick
Faculty of Sciences, Lubumbashi University, DRC

*Address all correspondence to: kayupepatrick@gmail.com

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