Chapter

Averaged No-Regret Control for an Electromagnetic Wave Equation Depending upon a Parameter with Incomplete Initial Conditions

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Abstract

This chapter concerns the optimal control problem for an electromagnetic wave equation with a potential term depending on a real parameter and with missing initial conditions. By using both the average control notion introduced recently by E. Zuazua to control parameter depending systems and the no-regret method introduced for the optimal control of systems with missing data. The relaxation of averaged no-regret control by the averaged low-regret control sequence transforms the problem into a standard optimal control problem. We prove that the problem of average optimal control admits a unique averaged no-regret control that we characterize by means of optimality systems.

Keywords: optimal control, averaged no-regret control, electromagnetic wave equation, parameter depending equation, systems with missing data

1. Introduction

The research in the field of electromagnetism is set to become a vital factor in biomedical technologies. Those studies included several areas like the usage of electromagnetic waves for probing organs and advanced MRI techniques, microwave biosensors, non-invasive electromagnetic diagnostic tools, therapeutic applications of electromagnetic waves, radar technologies for biosensing, the adoption of electromagnetic waves in medical sensing, cancer detection using ultra-wideband signal, the interaction of electromagnetic waves with biological tissues and living systems, theoretical modeling of electromagnetic propagation through human body and tissues and imaging applications of electromagnetic.

Actually, the principal goal of the study is to control such electromagnetic waves to be compatible with some biomedical needs like X-rays in the framework of medical screening and wireless power transfer of electromagnetic waves through the human body [1] where we want to make waves closer to a desired distribution.

In this chapter, we consider a linear wave equation with a potential term $p(x, \sigma)$ supposed dependent on space variable x and real parameter $\sigma \in (0, 1)$, this term generally comprises the dielectric permittivity of the medium which has different properties and cannot be exactly presented, this is because of the difference or lack of knowledge of the physical properties of the material penetrated from the electromagnetic waves. The initial position and velocity are also supposed unknown.

In this study, we consider an optimal control problem for electromagnetic wave equation depending upon a parameter and with missing initial conditions. We use the method of no-regret control which was introduced firstly in statistics by Savage [2] and later by Lions [3, 4] where he used this concept in optimal control theory, and its related idea is "low-regret" control to apply it to control distributed systems of incomplete data which has the attention of many scholars [5–12], motivated by various applications in ecology, and economics as well [13]. Also, we use the notion of average control because our system depends upon a parameter, Zuazua was the first who introduced this new concept in [14].

The rest of this chapter is arranged as follows. Section 2, lists the definition of the problem we are studying. Section 3, is devoted to the study of the averaged no-regret control and the averaged low-regret control for the electromagnetic wave equation. Ultimately, we prove the existence of a unique average low-regret control, and the characterization of the average optimal is given in Section 4. Finally, we make a conclusion in Section 5.

2. Statement of the problem

Consider a bounded open domain Ω with a smooth boundary $\partial\Omega$. We set $\Sigma = (0, T) \times \partial\Omega$ and $Q = \Omega \times (0, T)$. We introduce the following linear electromagnetic wave equation depending on a parameter

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + p(x,\sigma)y = 0 & \text{in } Q \\ y = \begin{cases} v & \\ 0 & \end{cases} & \begin{array}{l}\text{on } \Sigma_0 \\ \text{on } \Sigma \backslash \Sigma_0\end{array} \\ y(x,0) = y_0(x); \ \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega \end{cases} \tag{1}$$

where $p \in L^\infty(\Omega)$ is the potential term supposed dependent on a real parameter $\sigma \in (0,1)$ presents the dielectric permittivity and permeability of the medium and such that $0 < \alpha_1 \le p(x,\sigma) \le \alpha_2$ *a.e.*in Ω, v is a boundary control in $L^2(\Sigma_0)$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$ are the initial position and velocity respectively, both supposed unknown. For all $\sigma \in (0,1)$, the wave Eq. (1) has a unique solution $y(v, y_0, y_1, \sigma)$ in $C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ [15].

Denote by $g = (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ the initial data. We want to choose a control u independently of σ and g in a way such that the average state function y approaches a given observation $y_d \in L^2(Q)$. To achieve our goal, let' associate to (1) the following quadratic cost functional

$$J(v,g) = \left\| \int_0^1 y(v,g,\sigma)d\sigma - y_d \right\|^2_{L^2(Q)} + N\|v\|^2_{L^2(\Sigma_0)} \tag{2}$$

where $N \in \mathbb{R}_+^*$.

In this work, we aim to characterize the solution u of the optimal control problem with missing data given by

$$\inf_{v\in L^2(\Sigma_0)} J(v,g) \text{ subject to } (1) \tag{3}$$

independently of g and σ.

3. Averaged no-regret control and averaged low-regret control for the electromagnetic wave equation

A classical method to obtain the optimality system is then to solve the minmax problem

$$\inf_{v\in L^2(\Sigma_0)} \left(\sup_{g\in H_0^1(\Omega)\times L^2(\Omega)} (J(v,g)) \right), \tag{4}$$

but $J(v,g)$ is not upper bounded since $\sup_{g\in H_0^1(\Omega)\times L^2(\Omega)}(J(v,g)) = +\infty$. A natural idea of Lions [3] is to search for controls v such that

$$J(v,g) - J(0,g) \le 0, \forall g\in H_0^1(\Omega) \times L^2(\Omega) \tag{5}$$

Those controls v are called averaged no-regret controls.

As in [16, 17], we introduce the averaged no-regret control defined by.

Definition 1 [1] We say that $v\in L^2(\Sigma_0)$ is an averaged no-regret control for (1) if v is a solution of

$$\inf_{v\in L^2(\Sigma_0)} \left(\sup_{g\in H_0^1(\Omega)\times L^2(\Omega)} (J(v,g) - J(0,g)) \right). \tag{6}$$

Let us start by giving the following important lemma.

Lemma 1 For all $v\in L^2(\Sigma_0)$ and $g\in G$ we have

$$J(v,g) - J(0,g) \quad = \quad J(v,0) - J(0,0) \tag{7}$$

$$-2\int_\Omega y_0(x)\int_0^1 \frac{\partial\zeta}{\partial t}(x,0)d\sigma dx + 2\int_\Omega y_1(x)\int_0^1 \zeta(x,0)d\sigma dx \tag{8}$$

where ζ is given by the following backward wave equation

$$\begin{cases} \dfrac{\partial^2\zeta}{\partial t^2} - \Delta\zeta + p(x,\sigma)\zeta = \displaystyle\int_0^1 y(v,0,\sigma)d\sigma & \text{in } Q \\ \zeta = 0 & \text{on } \Sigma \\ \zeta(x,T) = 0,\ \dfrac{\partial\zeta}{\partial t}(x,T) = 0 & \text{in } \Omega \end{cases} \tag{9}$$

which has a unique solution in $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ [15].

Proof. It's easy to check that for all $(v,g) \in L^2(\Sigma_0) \times G$

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\int_Q \left(\int_0^1 y(v,0)d\sigma\right)\left(\int_0^1 y(0,g)d\sigma\right)dxdt. \tag{10}$$

Use (9) and apply Green formula to get

$$\int_Q \left(\int_0^1 y(v,0)d\sigma\right)\left(\int_0^1 y(0,g)d\sigma\right)dxdt = \int_0^T\int_\Omega \left(\frac{\partial^2\zeta}{\partial t^2} - \Delta\zeta + p(x,\sigma)\zeta\right)\left(\int_0^1 y(0,g)d\sigma\right)dxdt \tag{11}$$

$$= -2\int_\Omega y_0(x)\int_0^1 \frac{\partial\zeta}{\partial t}(v,0)d\sigma dx + 2\int_\Omega y_1(x)\int_0^1 \zeta(v,0)d\sigma dx. \tag{12}$$

■

The no-regret control seems to be hard to characterize (see [11]), for this.
reason we relax the no-regret control problem by making some quadratic perturbation as follows.

Definition 2 [17] We say that $u_\gamma \in L^2(\Sigma_0)$ is an averaged low-regret control for (1) if u_γ is a solution of

$$\inf_{v\in L^2(\Sigma_0)}\left(\sup_{g\in H_0^1(\Omega)\times L^2(\Omega)} (J(v,g) - J(0,g) - \gamma\|y_0\|^2_{H_0^1(\Omega)} - \gamma\|y_1\|^2_{L^2(\Omega)})\right), \gamma > 0. \tag{13}$$

Using (9) the problem (13) can be written as

$$\inf_{v\in L^2(\Sigma_0)}\left(\begin{array}{c} J(v,0) - J(0,0) + \sup\limits_{g\in G} 2(\int_\Omega y_1(x)\int_0^1\zeta(v,\sigma)(x,0)d\sigma dx - \int_\Omega y_0(x)\int_0^1\frac{\partial\zeta}{\partial t}(v,\sigma)(x,0)d\sigma dx \\ -\gamma\|y_0\|^2_{H_0^1(\Omega)} - \gamma\|y_1\|^2_{L^2(\Omega)}). \end{array}\right), \gamma > 0. \tag{14}$$

And thanks to Legendre transform (see [18, 19]), we have

$$\sup_{g\in G} 2\left(-\int_\Omega y_0(x)\int_0^1 \frac{\partial\zeta}{\partial t}(v,0)d\sigma dx + \int_\Omega y_1(x)\int_0^1\zeta(v,0)d\sigma dx - \gamma\|y_0\|^2_{H_0^1(\Omega)} - \gamma\|y_1\|^2_{L^2(\Omega)}\right) \tag{15}$$

$$= \frac{1}{\gamma}\left\|\int_0^1 \frac{\partial\zeta(v,\sigma)}{\partial t}(x,0)d\sigma\right\|^2_{H_0^1(\Omega)} + \frac{1}{\gamma}\left\|\int_0^1 \zeta(v,\sigma)(x,0)d\sigma\right\|^2_{L^2(\Omega)}. \tag{16}$$

Then, the averaged low-regret control problem (9) is equivalent to the following classical optimal control problem

$$\inf_{v\in L^2(\Sigma_0)} J_\gamma(v) \tag{17}$$

where

$$J_\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|\int_0^1 \frac{\partial\zeta(v,\sigma)}{\partial t}(x,0)d\sigma\right\|^2_{H_0^1(\Omega)} + \frac{1}{\gamma}\left\|\int_0^1\zeta(v,\sigma)(x,0)d\sigma\right\|^2_{L^2(\Omega)}. \tag{18}$$

4. Characterizations

In the recent section, we aim to find a full characterization for the averaged no-regret control and averaged low-regret control via optimality systems.

Theorem 1.1 There exists a unique averaged low-regret control u_γ solution to (17), (18).

Proof. We have for every $v \in L^2(\Sigma_0) : J_\gamma(v) \geq -J(0,0)$, this means that (17), (18) has a solution.

Let $(v_n^\gamma) \in L^2(\Sigma_0)$ be a minimizing sequence such that

$$\lim_{n\to\infty} J_\gamma(v_n^\gamma) = J_\gamma(u_\gamma) = d_\gamma. \tag{19}$$

We know that

$$J_\gamma(v_n^\gamma) = J(v_n^\gamma, 0) - J(0,0) + \frac{1}{\gamma}\left\| \int_0^1 \frac{\partial \zeta(v_n^\gamma, \sigma)}{\partial t}(x,0)d\sigma \right\|^2_{H_0^1(\Omega)} + \frac{1}{\gamma}\left\| \int_0^1 \zeta(v_n^\gamma, \sigma)(x,0)d\sigma \right\|^2_{L^2(\Omega)} \leq d_\gamma + 1. \tag{20}$$

This implies the following bounds

$$\|v_n^\gamma\|_{L^2(\Sigma_0)} \leq C_\gamma, \tag{21}$$

$$\left\| \int_0^1 y(v_n^\gamma, 0, \sigma) d\sigma \right\|_{L^2(Q)} \leq C_\gamma, \tag{22}$$

$$\left\| \frac{\partial^2 \zeta_n}{\partial t^2} - \Delta\zeta_n + p(x,\sigma)\zeta_n \right\|_{L^2(Q)} \leq C_\gamma, \tag{23}$$

where C_γ is a positive constant independent of n. Moreover, by continuity w.r.t. data and (21) we get

$$\|y(v_n^\gamma, 0, \sigma)\|_{L^2(Q)} \leq C_\gamma. \tag{24}$$

By similar way an by using (22) we obtain

$$\|\zeta(v_n^\gamma, \sigma)\|_{L^2(0,T;H_0^1(\Omega))} \leq C_\gamma. \tag{25}$$

Then, from (21) we deduce that there exists a subsequence still denoted (v_n^γ) such that $v_n^\gamma \rightharpoonup u_\gamma$weakly in $L^2(\Sigma_0)$, and from (22) we get

$$y(v_n^\gamma, 0, \sigma) \rightharpoonup y_\gamma \text{ weakly in } L^2(Q). \tag{26}$$

Also, because of continuity w.r.t. data we have $y(v_n^\gamma, 0, \sigma) \rightharpoonup y(u_\gamma, 0, \sigma)$ weakly in $L^2(Q)$, by limit uniqueness $y_\gamma = y(u_\gamma, 0, \sigma)$ solution to

$$\begin{cases} \dfrac{\partial^2 y_\gamma}{\partial t^2} - \Delta y_\gamma + p(x,\sigma)y_\gamma = 0 & \text{in } Q, \\ y_\gamma = \begin{cases} u_\gamma \\ 0 \end{cases} & \begin{array}{l} \text{on } \Sigma_0, \\ \text{on } \Sigma\backslash\Sigma_0, \end{array} \\ y_\gamma(x,0) = y_0(x);\ \dfrac{\partial y_\gamma}{\partial t}(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{27}$$

In other hand, use (24) and (22) to apply the convergence dominated theorem and, we have

$$\int_0^1 y(v_n^\gamma, 0, \sigma) d\sigma \rightharpoonup \int_0^1 y(u_\gamma, 0, \sigma) d\sigma \text{ weakly in } L^2(Q). \tag{28}$$

From (25) we deduce the existence of a subsequence still be denoted by $\zeta\left(v_n^\gamma, \sigma\right)(x, 0)$ such that

$$\zeta\left(v_n^\gamma, \sigma\right)(x, 0) \rightharpoonup \zeta\left(u_\gamma, \sigma\right)(x, 0) \text{ weakly in } H_0^1(\Omega), \tag{29}$$

then

$$\frac{\partial^2 \zeta_n}{\partial t^2} - \Delta\zeta_n + p(x,\sigma)\zeta_n \rightarrow \frac{\partial^2 \zeta_\gamma}{\partial t^2} - \Delta\zeta_\gamma + p(x,\sigma)\zeta_\gamma \text{ in} D'(Q), \tag{30}$$

where $D(Q) = C_0^\infty(Q)$, and (23) leads to

$$\frac{\partial^2 \zeta_n}{\partial t^2} - \Delta\zeta_n + p(x,\sigma)\zeta_n \rightharpoonup f \ \text{ weakly in} L^2(Q). \tag{31}$$

Again, by limit uniqueness $\frac{\partial^2 \zeta_\gamma}{\partial t^2} - \Delta\zeta_\gamma + p(x,\sigma)\zeta_\gamma = \int_0^1 y\left(u_\gamma, 0, \sigma\right)d\sigma$ in $L^2(Q)$. Finally, ζ_γ is a solution to

$$\begin{cases} \dfrac{\partial^2 \zeta_\gamma}{\partial t^2} - \Delta\zeta_\gamma + p(x,\sigma)\zeta_\gamma = \displaystyle\int_0^1 y\left(u_\gamma, 0, \sigma\right)d\sigma & \text{in } Q, \\ \zeta_\gamma = 0 & \text{on } \Sigma, \\ \zeta_\gamma(x,T) = 0, \dfrac{\partial \zeta_\gamma}{\partial t}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{32}$$

The uniqueness of u_γ follows from strict convexity and weak lower semi-continuity of the functional $J_\gamma(v)$.■

After proving existence and uniqueness, we aim in the next theorem to give a full description to the average low-regret control for the electromagnetic wave equation.

Theorem 1.2 For all $\gamma > 0$, the average low-regret control u_γ is characterized by the following optimality system

$$\begin{cases} \dfrac{\partial^2 y_\gamma}{\partial t^2} - \Delta y_\gamma + p(x,\sigma)y_\gamma = 0, & \\ \dfrac{\partial^2 \zeta_\gamma}{\partial t^2} - \Delta\zeta_\gamma + p(x,\sigma)\zeta_\gamma = \displaystyle\int_0^1 y\left(u_\gamma, 0, \sigma\right)d\sigma, & \\ \dfrac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta\rho_\gamma + p(x,\sigma)\rho_\gamma = 0, & \\ \dfrac{\partial^2 q_\gamma}{\partial t^2} - \Delta q_\gamma + p(x,\sigma)q_\gamma = \displaystyle\int_0^1 \left(\rho_\gamma + y\left(u_\gamma, 0, \sigma\right)\right)d\sigma - y_d & \text{in } Q, \\ y_\gamma = \begin{cases} u_\gamma & \text{on } \Sigma_0 \\ 0 & \text{on} \Sigma \backslash \Sigma_0 \end{cases}, \zeta_\gamma = 0, & \\ \rho_\gamma = 0, q_\gamma = 0 & \text{on } \Sigma, \\ y_\gamma(0,x) = 0, \dfrac{\partial y_\gamma}{\partial t}(0,x) = 0, & \\ \zeta_\gamma(x,T) = 0, \dfrac{\partial \zeta_\gamma}{\partial t}(x,T) = 0, & \\ \rho_\gamma(x,0) = -\dfrac{1}{\gamma}\displaystyle\int_0^1 \dfrac{\partial \zeta\left(u_\gamma\right)}{\partial t}(x,0)d\sigma, \dfrac{\partial \rho_\gamma}{\partial t}(x,0) = \dfrac{1}{\gamma}\displaystyle\int_0^1 \zeta\left(u_\gamma\right)(x,0)d\sigma, & \\ q_\gamma(T,x) = 0, \dfrac{\partial q_\gamma}{\partial t}(T,x) = 0 & \text{in } \Omega, \end{cases} \tag{33}$$

with

$$u_\gamma = \frac{1}{N}\int_0^1 \frac{\partial p_\gamma}{\partial \eta} d\sigma \, \text{in} \, L^2(\Sigma_0). \tag{34}$$

Proof. From the first order necessary optimality conditions, we have

$$J^{\gamma'}(u_\gamma)(w) = \left(\int_0^1 y(u_\gamma,0)d\sigma - y_d, \int_0^1 y(w,0)d\sigma\right)_{L^2(Q)} + N(u_\gamma,w)_{L^2(\Sigma_0)} + \left(\frac{1}{\gamma}\int_0^1 \zeta'(u_\gamma)(0)d\sigma, \int_0^1 \zeta'(w)(0)d\sigma\right)_{L^2(\Omega)} + \left(\frac{1}{\gamma}\int_0^1 \zeta(u_\gamma)(0)d\sigma, \int_0^1 \zeta(w)(0)d\sigma\right)_{L^2(\Omega)} = 0 \tag{35}$$

for all $w \in L^2(\Sigma_0)$.

Now, let us introduce $\rho_\gamma = \rho(u_\gamma, 0)$ unique solution to

$$\begin{cases} \dfrac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta\rho_\gamma + p(x,\sigma)\rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma = 0 & \text{on } \Sigma, \\ \rho_\gamma(x,0) = -\dfrac{1}{\gamma}\displaystyle\int_0^1 \frac{\partial \zeta(u_\gamma)}{\partial t}(x,0)d\sigma; \ \frac{\partial \rho_\gamma}{\partial t}(x,0) = \frac{1}{\gamma}\int_0^1 \zeta(u_\gamma)(x,0)d\sigma & \text{in } \Omega. \end{cases} \tag{36}$$

So that for every $w \in L^2(\Sigma_0)$, we obtain

$$J^{\gamma'}(u_\gamma)(w) = \left(\int_0^1 y(u_\gamma,0) + \rho_\gamma d\sigma - y_d, \int_0^1 y(w,0)d\sigma - y(0,0)\right)_{L^2(Q)} + N(u_\gamma,w)_{L^2(\Sigma_0)} = 0 \tag{37}$$

We finally define another adjoint state $q_\gamma = q(u_\gamma)$ as the unique solution of

$$\begin{cases} \dfrac{\partial^2 q_\gamma}{\partial t^2} - \Delta q_\gamma + p(x,\sigma)q_\gamma = \displaystyle\int_0^1 (\rho_\gamma + y(u_\gamma,0,\sigma))d\sigma - y_d & \text{in } Q, \\ q_\gamma = 0 & \text{on } \Sigma, \\ q_\gamma(x,T) = 0, \ \dfrac{\partial q_\gamma}{\partial t}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{38}$$

Then (35) becomes

$$u_\gamma = \frac{1}{N}\int_0^1 \frac{\partial q_\gamma}{\partial \eta} d\sigma \, \text{in} \, L^2(\Sigma_0). \tag{39}$$

■

The previous Theorem gives a low-regret control characterization. For the no-regret control, we need to prove the convergence of the sequence of averaged low-regret control to the averaged no-regret control. Then, we announce the following Proposition.

For some constant C independent of γ, we have

$$\|u_\gamma\|_{L^2(\Sigma_0)} \le C, \tag{40}$$

$$\left\| \int_0^1 y(u_\gamma, 0, \sigma) d\sigma \right\|_{L^2(Q)} \leq C, \tag{41}$$

$$\| y(u_\gamma, 0, \sigma) d\sigma \|_{L^2(Q)} \leq C, \tag{42}$$

$$\left\| \int_0^1 \frac{\partial \zeta(u_\gamma, \sigma)}{\partial t}(x, 0) d\sigma \right\|_{H^{-1}(\Omega)} \leq C\sqrt{\gamma}, \tag{43}$$

$$\left\| \int_0^1 \zeta(u_\gamma, \sigma)(x, 0) d\sigma \right\|_{L^2(\Omega)} \leq C\sqrt{\gamma}, \tag{44}$$

$$\| \rho_\gamma \|_{L^\infty\left(0,T; H_0^1(\Omega)\right)} \leq C, \tag{45}$$

$$\left\| q_\gamma \right\|_{L^\infty\left(0,T; H_0^1(\Omega)\right)} \leq C. \tag{46}$$

Proof. Since u_γ is a solution to (17) and (18), we get

$$J_\gamma(u_\gamma) \leq J_\gamma(0), \tag{47}$$

then

$$J(u_\gamma, 0) + \frac{1}{\gamma}\left\| \int_0^1 \frac{\partial \zeta(u_\gamma, \sigma)}{\partial t}(x, 0) d\sigma \right\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\left\| \int_0^1 \zeta(u_\gamma, \sigma)(x, 0) d\sigma \right\|^2_{L^2(\Omega)} \leq J(0, 0), \tag{48}$$

this gives (40), (41), (42) and (43). The bound (43) follows by a way similar to (24).

From energy conservation property with (43) and (44).

$$E_{\rho_\gamma}(t) = \frac{1}{2\Omega}\left[\left| \frac{\partial \rho_\gamma}{\partial t} \right|^2 + \left| \nabla \rho_\gamma \right|^2 + q(x, \sigma) \left| \rho_\gamma \right|^2 \right] dx = E_{\rho_\gamma}(0) \leq C, \tag{49}$$

we find (45).

To get q_γ estimates, just reverse the time variable by taking $s = T - t$ to find (46).

Lemma 2 The averaged low-regret control u_γ tends weakly to the averaged no-regret control u when $\gamma \to 0$.

Proof. From (40) we deduce the existence of a subsequence still be denoted u_γ such that

$$u_\gamma \rightharpoonup u \text{ weakly in } L^2(\Sigma_0), \tag{50}$$

let us prove u is an averaged no-regret control. We have for all $v \in L^2(\Sigma_0)$

$$J(u_\gamma, g) - J(0, g) - \gamma \| y_0 \|^2_{H_0^1(\Omega)} - \gamma \| y_1 \|^2_{L^2(\Omega)} \leq \sup_{g \in H_0^1(\Omega) \times L^2(\Omega)} (J(v, g) - J(0, g)), \tag{51}$$

take $\gamma \to 0$ to find

$$J(u, g) - J(0, g) \leq \sup_{g \in H_0^1(\Omega) \times L^2(\Omega)} (J(v, g) - J(0, g)), \tag{52}$$

i.e. is an averaged no-regret control. ■.

Finally, we can present the following theorem giving a full characterization the average no-regret control.

Theorem 1.3 The average no-regret control u is characterized by the following optimality system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + p(x,\sigma) y = 0, & \\ \dfrac{\partial^2 \zeta}{\partial t^2} - \Delta \zeta + p(x,\sigma) \zeta = \displaystyle\int_0^1 y(u,0,\sigma) d\sigma, & \\ \dfrac{\partial^2 \rho}{\partial t^2} - \Delta \rho + p(x,\sigma) \rho = 0, & \\ \dfrac{\partial^2 q}{\partial t^2} - \Delta q + p(x,\sigma) q = \displaystyle\int_0^1 (\rho + y(u,0,\sigma)) d\sigma - y_d & \text{in } Q, \\ y = \begin{cases} u & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma \backslash \Sigma_0 \end{cases}, \zeta = 0 & \\ \rho = 0; q = 0 & \text{on } \Sigma, \\ y(0,x) = 0, \dfrac{\partial y}{\partial t}(0,x) = 0, & \\ \zeta(x,T) = 0, \dfrac{\partial \zeta}{\partial t}(x,T) = 0, & \\ \rho(x,0) = \lambda_1(x), \dfrac{\partial \rho}{\partial t}(x,0) = \lambda_2(x), & \\ q(T,x) = 0, \dfrac{\partial q}{\partial t}(T,x) = 0 & \text{in } \Omega, \end{cases} \tag{53}$$

with

$$u = \frac{1}{N} \int_0^1 \frac{\partial p}{\partial \eta} d\sigma \text{ in } L^2(\Sigma_0), \tag{54}$$

and

$\lambda_1(x) = \lim\limits_{\gamma \to 0} -\frac{1}{\gamma} \int_0^1 \frac{\partial \zeta(u_\gamma)}{\partial t}(x,0) d\sigma$ weakly in $H_0^1(\Omega)$,

$\lambda_2(x) = \lim\limits_{\gamma \to 0} \frac{1}{\gamma} \int_0^1 \zeta(u_\gamma)(x,0) d\sigma$ weakly in $L^2(\Omega)$.

Proof. From (42) continuity w.r.t data, we can deduce that

$$y(u_\gamma, 0, \sigma) \rightharpoonup y(u,0,\sigma) \text{ weakly in } L^2(\Omega), \tag{55}$$

solution to

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + p(x,\sigma) y = 0 & \text{in } Q, \\ y = \begin{cases} u \\ 0 \end{cases} & \begin{array}{l} \text{on } \Sigma_0, \\ \text{on } \Sigma \backslash \Sigma_0, \end{array} \\ y(x,0) = y_0(x); \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{56}$$

Again, by (41) and dominated convergence theorem

$$\int_0^1 y(u_\gamma, 0, \sigma)d\sigma \rightharpoonup \int_0^1 y(u, 0, \sigma)d\sigma \,\text{weakly in}\, L^2(\Sigma_0). \tag{57}$$

The rest of equations in (53) leads by a similar way, except the convergences of initial data $\rho(x, 0), \frac{\partial\rho}{\partial t}(x, 0)$ which will be as follows.

From (43) and (44) we deduce the convergences of

$$-\frac{1}{\gamma}\frac{\partial\zeta(u_\gamma, \sigma)}{\partial t}(x, 0) \rightharpoonup \lambda_1(x) \text{ weakly in } H_0^1(\Omega), \tag{58}$$

and

$$\frac{1}{\gamma}\zeta(u_\gamma, \sigma)(x, 0) \rightharpoonup \lambda_2(x) \text{ weakly in } L^2(\Omega). \tag{59}$$

5. Conclusion

As we have seen, the averaged no-regret control method allows us to find a control that will optimize the situation of the electromagnetic waves with missing initial conditions and depending upon a parameter. The method presented in the paper is quite general and covers a wide class of systems, hence, we could generalize the situation to more control positions (regional, punctual, …) and different kinds of missing data (source term, boundary conditions, …).

The results presented above can also be generalized to the case of other systems which has many biomedical applications. This problem is still under consideration and the results will appear in upcoming works.

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