Chapter

Peculiarities of the Fundamental Solution of Parabolic Systems with a Negative Genus

Vladyslav Antonovich Litovchenko

Abstract

For the parabolic Shilov-type systems with a negative genus, a method of studying the properties of a fundamental solution of the Cauchy problem is proposed. This method allows to improve the known estimates of Zhitomirskii fundamental solution for systems with dissipative parabolicity and describe the features of this solution more accurately. It opens wide possibilities for constructing a classical theory of the Cauchy problem for parabolic systems with negative genus and variable coefficients.

Keywords: parabolic Shilov systems, negative genus, fundamental solution, Cauchy problem, matriciant, dissipative parabolicity

1. Introduction

The theory of parabolic equations dates back to the time of the classical equation of thermal conductivity [1]. However, it acquired its most distinct features from the fundamental work by I.G. Petrovskii [2] published in 1938. There he describes and investigates a fairly wide class of systems of linear equations with partial derivatives, the fundamental solution of which has typical properties of the fundamental solution of the thermal conductivity equation:

$$G_\circ(t-\tau;x) = \left(\sqrt{4\pi a(t-\tau)}\right)^{-n} e^{-\frac{\|x\|^2}{4a(t-\tau)}}, \quad t>\tau\geq 0, x\in\mathbb{R}^n \tag{1}$$

(here a – is the coefficient of thermal conductivity, and $\|\cdot\|$ – is the Euclidean norm in $\mathbb{R}^n$). These systems were later called “parabolic by Petrovskii” or “$2b$-parabolic” systems. Due to the efforts of many researchers, the theory of $2b$-parabolic systems developed rapidly throughout the second half of the 20th century. At that, there were considered the systems with both fixed and variable coefficients having different properties. Comprehensive results were obtained on the structure and properties of solutions, as well as on the correct solvability of boundary value problems, in particular, the Cauchy problem, in different functional spaces [3–13].

In 1955, G.Ye. Shilov formulates a new definition of parabolicity, which generalizes the concept of “$2b$ -parabolicity” and significantly expands the class of Petrovskii’s systems with constant coefficients by those systems, in which the order p is no longer necessarily even, and may not coincide with the parabolicity index h [14]. The parabolic Shilov-type systems, mostly with constant coefficients, were studied in [15–24].

The presence of a gap between p and h in such systems produces a peculiar "dissipation" effect, the measure of which may be a special characteristic of the system – its genus μ: $1-(p-h) \leq \mu \leq 1$. The parabolic systems, in which $p=h$, − the classical equation of thermal conductivity, in particular, as well as all $2b$-parabolic systems, − have the genus $\mu=1$, while for the systems with $p \neq h$, generally speaking, the genus is $\mu < 1$. Besides, the more the parabolicity index h deviates from the order of the system p, the more its genus μ, decreasing, gets further away from 1. In systems with such a dissipation, even with constant coefficients, deviations from the standards set by the classical thermal equation are observed. First of all, for their fundamental solution $G(t,\tau;\cdot)$, the analytic properties in the complex space $\mathbb{C}^n$ [15] are getting worse, and the order of exponential behavior on the real hyperplane $\mathbb{R}^n$ changes [16]:

$$|\partial_x^k G(t,\tau;x)| \leq A_k (t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} \begin{cases} e^{-\delta_0\left(\frac{\|x\|}{(t-\tau)^{\mu/p}}\right)^{\frac{p}{p-\mu}}}, & 0<\mu \leq 1, \\ e^{-\delta_0\left(\frac{\|x\|}{(t-\tau)^{\mu/h}}\right)^{\frac{h}{h-\mu}}}, & \mu \leq 0, \end{cases} \quad \gamma \geq 0. \tag{2}$$

Another anomalous phenomenon of the systems with "dissipative parabolicity" is their parabolic instability with respect to changes in the coefficients, even of those found at zero derivative. This fact was first pointed out by U Hou-Sin in 1960, who gave the example of a parabolically unstable system [17]. In this regard, the question of the study of parabolic Shilov-type systems with variable coefficients is problematic and still remains open.

Zhitomirskii's estimates (2) show that the fundamental solution of $G(t,\tau;x)$ parabolic systems with the positive genus μ on the set $(\tau;+\infty)\times\mathbb{R}^n$ shows the behavior typical for $G_\circ(t-\tau;x)$: it decreases exponentially and has a peculiarity at only one point $(t;x)=(\tau;0)$. This fact allowed us to successfully develop the classical theory of the Cauchy problem for parabolic systems with variable coefficients and non-negative genus μ in [25–28]. However, according to these estimates, in the case of $\mu<0$ the function $G(t,\tau;x)$ may have a peculiarity on the entire hyperplane $t=\tau$, $x \in \mathbb{R}^n$. This point significantly complicates the substantiation of the convergence of the process of successive approximations, in particular, while making the fundamental solution of the Cauchy problem for systems with variable coefficients using the Levy method. In this regard, a natural question arises: How accurate are the estimates (2) for systems of the genus $\mu<0$?

The answer to this question is given in this paper. A method for studying the function $G(t,\tau;x)$ for parabolic Shilov-type systems of genus $\mu<0$, which allows us to more accurately describe the behavior of this function in the vicinity of the point $(t;x)=(\tau;0)$ is also suggested in this research paper. In addition, one class of systems with dissipative parabolicity is also defined here. These systems are parabolically stable to changes in their lower coefficients.

The main content of the work is as follows. Section 2 contains the necessary information on the concept of parabolicity by Shilov. One class of systems with dissipative parabolicity and variable coefficients is described in Section 3. The study of the properties of the fundamental solution of the Cauchy problem for parabolic Shilov-type systems with a negative genus is carried out in Section 4. The final Section 5 is the conclusions.

2. Preliminary information

Let $\mathbb{N}$ – be the set of all natural numbers; $\mathbb{N}_m=\{1,\ldots;m\}$; $\mathbb{R}^n$ and $\mathbb{C}^n$ – real and complex space of $n \geq 1$ dimension respectively; $\mathbb{Z}_+^n$ – the set of all n-dimensional multi-indices; $\mathbb{R} := \mathbb{R}^1$, $\mathbb{C} := \mathbb{C}^1$, $\mathbb{Z}_+ := \mathbb{Z}_+^1$; i – imaginary unit; $(\cdot, \cdot)$ – scalar product in the space $\mathbb{R}^n$; $|x + iy| := (x^2 + y^2)^{\frac{1}{2}}$, if $\{x, y\} \subset \mathbb{R}$; $z^l := z_1^{l_1} \dots z_n^{l_n}$, $|z|^l := |z_1|^{l_1} \dots |z_n|^{l_n}$, $|z|_+^h := |z_1|^h + \dots + |z_n|^h$, $|z|_+ := |z|_+^1$, if $z := (z_1; \dots; z_n) \in \mathbb{C}^n$, $l := (l_1; \dots; l_n) \in \mathbb{Z}_+^n$, $h \in \mathbb{R}$; $\partial_\xi \cdot$ – is the partial derivative with the variable ξ.

Let us fix $\{m, p\} \subset \mathbb{N}$, $T \in (0; +\infty)$ arbitrarily and consider the system of partial differential equations of p order

$$\partial_t u(t; x) = A(t; i\partial_x) u(t; x), \ (t; x) \in \Pi_{(0; T]}, \tag{3}$$

in which $\Pi_{(0;T]} := (0; T] \times \mathbb{R}^n$, $u(t; x) := \mathrm{col}(u_1(t; x); \dots; u_m(t; x))$ – is an unknown vector-function and

$$A(t; i\partial_x) := \left(\sum_{|k|_+ \leq p} a_k^{jl}(t) i^{|k|_+} \partial_x^k \right)_{j,l=1}^m \tag{4}$$

matrix differential expression with coefficients $a_k^{jl}(\cdot)$.

Let us denote by $\mathcal{A}$ the matrix symbol of the differential expression $A(t; i\partial_x)$:

$$\mathcal{A}(t; s) = \left(\sum_{|k|_+ \leq p} a_k^{jl}(t) s^k \right)_{j,l=1}^m, t \in (0; T], s \in \mathbb{C}^n. \tag{5}$$

The Shilov-type parabolicity of the system (3) depending on the constancy or variability of its coefficients, is defined differently.

In the case when the coefficients a_k^{jl} are constant, i.e., when

$$A(t; i\partial_x) \equiv A(i\partial_x), \quad \mathcal{A}(t; \cdot) \equiv \mathcal{A}(\cdot), \tag{6}$$

the system (3) on the set $\Pi_{[0;T]}$ is referred to as *Shilov-type parabolic* system with the parabolicity index h, $0 < h \leq p$, if [15]

$$\exists \delta_0 > 0 \exists \delta \geq 0 \forall \xi \in \mathbb{R}^n : \quad \max_{j \in \mathbb{N}_m} \mathrm{Re}\, \lambda_j(\xi) \leq -\delta_0 \|\xi\|^h + \delta, \tag{7}$$

where $\lambda_j(s)$ - characteristic numbers of the matrix symbol $\mathcal{A}(s)$, $s \in \mathbb{C}^n$.

If the coefficients of the system (3) depend on t (continuously), then the Shilov-type parabolicity of this system is defined somewhat differently, using the concept of the matriciant of the linear differential equations system.

For the system (3) we shall write the corresponding dual by Fourier system

$$\partial_t v(t; \xi) = \mathcal{A}(t; \xi) v(t; \xi), \quad 0 \leq \tau < t \leq T, \xi \in \mathbb{R}^n. \tag{8}$$

The *matriciant of the system* (8) is such a matrix solution of the system $\Theta_\tau^t(\cdot)$, $0 \leq \tau < t \leq T$, that

$$\Theta_\tau^t(\cdot)\big|_{t=\tau} = E \quad (\forall \tau \in [0; T]) \tag{9}$$

(here E – a single matrix of m order).

Under the condition of continuity of the coefficients of the system (3), the matriciant $\Theta_\tau^t(\cdot)$ has the structure [29]

$$\Theta_\tau^t(\cdot) = E + \sum_{r=1}^{\infty} \int_\tau^t \int_\tau^{t_1} \dots \int_\tau^{t_{r-1}} \left(\prod_{j=1}^{r} \mathcal{A}(t_j; \cdot) \right) dt_r \dots dt_2 dt_1. \tag{10}$$

The system (3) with continuous coefficients on $[0; T]$ is called a *Shilov-type parabolic* system on the set $\Pi_{[0;T]}$ with parabolicity index h, $0 < h \leq p$, if for the matriciant $\Theta_\tau^t(\cdot), 0 \leq \tau < t \leq T,$ of the corresponding dual by Fourier system (8) the following estimation is performed [15]

$$|\Theta_\tau^t(\xi)| \leq c(1 + \|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, \quad (t; \xi) \in \Pi_{(\tau; T]}, \tag{11}$$

with some positive constants c and δ. Here

$$\gamma \coloneqq (p - h)(m - 1), \quad |(a_{jl})_{j=1,l=1}^{k,m}| \coloneqq \max_{jl} |a_{jl}|. \tag{12}$$

It should be noted that for Shilov-type parabolic systems with constant coefficients, the condition (11) is a direct consequence of the corresponding condition of parabolicity (7) [15]. For parabolic systems (3) with t-dependent coefficients at $p \neq h$, this fact generally cannot be confirmed by classical means of the theory of parabolic systems due to the parabolic instability of such systems to changing their coefficients.

The Eq. (10) allows us to extend the matriciant $\Theta_\tau^t(\cdot)$ into the complex space $\mathbb{C}^n$ to the complete analytical function. Taking into account the estimation

$$|\mathcal{A}(t; s)| \leq c(1 + \|s\|^p), \;\; 0 \leq t \leq T, s \in \mathbb{C}^n, \tag{13}$$

we find that

$$|\Theta_\tau^t(s)| \leq c_0 e^{\delta_0(t-\tau)\|s\|^p}, \;\; 0 \leq \tau < t \leq T, s \in \mathbb{C}^n \tag{14}$$

(here, a c_0 and δ_0 are positive constants independent of τ, t and s).

The smoothness of the matriciant $\Theta_\tau^t(\cdot)$ together with the estimates (11), (14), according to the statement of the theorem of the Phragmén-Lindelöf type [30, p. 247], ensure the existence of the area

$$\mathbb{K}_\nu = \{\xi + i\eta \in \mathbb{C}^n : \|\eta\| \leq K(1 + \|\xi\|)^\nu\} \tag{15}$$

from ν with $[1 - (p - h); 1]$, in which the following estimate is performed

$$|\Theta_\tau^t(\xi + i\eta)| \leq c_1(1 + \|\xi\|^\gamma) e^{-\delta_1(t-\tau)\|\xi\|^h}, \quad 0 \leq \tau < t \leq T. \tag{16}$$

The *genus* μ of the Shilov-type parabolic system (3) is the exact upper boundary of the indices ν, with which in the domain $\mathbb{K}_\nu$ for the matriciant $\Theta_\tau^t(\cdot)$ the estimate (16) is performed [15]

Similarly to $2b$-parabolicity, it is convenient to call the Shilov-type parabolicity a $\{p, h\}$*-parabolicity*.

It should be noted that the fundamental solution of the Cauchy problem for $\{p, h\}$-parabolic system (3) is represented by the function [15]

$$G(t, \tau; x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x,\xi)} \Theta_\tau^t(\xi) d\xi. \tag{17}$$

The following section gives an example of a $\{p, h\}$-parabolic system and defines a class of systems with dissipative parabolicity, each of which is a $\{p, h\}$-parabolic system with variable coefficients.

3. One class of parabolically resistant systems

Due to the difficulty of establishing the fundamental condition (11), for the system (3) with variable coefficients, the definition of parabolability according to Shilov is somewhat specific. It is known [4] that the corresponding condition (11) is satisfied for $2b$-parabolic systems (3) with continuous coefficients. However, it is impossible to confirm the fulfillment of this condition in a similar way for systems (3) with $p \neq h$ based on the condition (7). Therefore, it is important to be aware of the richness of the class of the Shilov-type systems with variable coefficients, in particular, of the examples of such systems that are not parabolic by Petrovskii.

Let us consider a system of Eq. (3), in which the differential expression $A(t; i\partial_x)$ allows an image

$$A(t; i\partial_x) = A_0(i\partial_x) + A_1(t; i\partial_x), \tag{18}$$

where

$$A_0(i\partial_x) := \left(\sum_{|k|_+ \leq p} a_k^{lj} i^{|k|_+} \partial_x^k \right)_{l,j=1}^{m}, \quad A_1(t; i\partial_x) := \left(\sum_{|k|_+ \leq p_1} a_k^{lj}(t) i^{|k|_+} \partial_x^k \right)_{l,j=1}^{m}. \tag{19}$$

Let us assume that the corresponding system

$$\partial_t u(t; x) = A_0(i\partial_x) u(t; x), \quad (t; x) \in \Pi_{(0;T]}, \tag{20}$$

is $\{p, h\}$-parabolic on the set $\Pi_{(\tau;T]}$, and the coefficients of the differential expression $A_1(t; i\partial_x)$ are continuous complex-valued functions defined on $[0; T]$, while the values p, p_1 and h satisfy the condition

(A): $\quad 0 \leq p_1 + (p - h)(m - 1) < h$.

Example of system (3) with condition (A). Let $n = 1$, $m = 2$, $a > 0$ and $c_j(\cdot)$, $j \in \mathbb{N}_5$, are some continuous on $[0; T]$ complex-valued functions. Then the system

$$\begin{cases} \partial_t u_1 = \{-a\partial_x^4 + c_1(t)\partial_x^2\} u_1 + \{\partial_x^5 - \partial_x^3 + c_2(t)\partial_x\} u_2, \\ \partial_t u_2 = \{c_3(t)\partial_x^2 - \partial_x^3\} u_1 - \{a\partial_x^4 - c_4(t)\partial_x^2 - c_5(t)\} u_2, \end{cases} \tag{21}$$

is the system of kind (3) with condition (A). Indeed, putting

$$A_0(i\partial_x) = \begin{pmatrix} -a\partial_x^4 & \partial_x^5 - \partial_x^3 \\ -\partial_x^3 & -a\partial_x^4 \end{pmatrix}, \tag{22}$$

$$A_1(t; i\partial_x) = \begin{pmatrix} c_1(t)\partial_x^2 & c_2(t)\partial_x \\ c_3(t)\partial_x^2 & c_4(t)\partial_x^2 + c_5(t) \end{pmatrix} \tag{23}$$

and solving the appropriate equation

$$\det(\mathcal{A}_0(s) - \lambda E) = 0, \quad s \in \mathbb{C}^n, \tag{24}$$

we obtain that $\lambda_{1,2}(s) = -as^4 \pm i\sqrt{s^8 + s^6}$, $p = 5$, $p_1 = 2$ and $h = 4$. For these values p, p_1 and h, obviously the condition (A) holds.

Theorem 1 *Let (3) be a system with continuous coefficients, for which the conditions formulated in this clause are satisfied. Then it is an* $\{p, h\}$*-parabolic system with variable coefficients.*

Proof. According to the definition of $\{p, h\}$-parabolicity for the system (3) with variable coefficients, it is enough to show that for the matrix $\Theta_\tau^t(\cdot)$ of the corresponding dual by Fourier system (8) on the set $\Pi_{[\tau;T]}$, $\tau \in [0;T)$, the estimate (11) is performed.

On condition of continuity of the coefficients, the matriciant $\Theta_\tau^t(\cdot)$ is the only solution of the Cauchy problem for the system (8) with the initial condition

$$v(t;\cdot)|_{t=\tau} = E. \tag{25}$$

Thus, the correct equality

$$\partial_t \Theta_\tau^t(\xi) = A_0(\xi)\Theta_\tau^t(\xi) + Q(\tau, t; \xi), \tag{26}$$

in which

$$Q(\tau, t; \xi) := A_1(t; \xi)\Theta_\tau^t(\xi). \tag{27}$$

Having solved the Cauchy problem (26), (25), we obtain the image

$$\Theta_\tau^t(\xi) = e^{(t-\tau)P_0(\xi)} + \int_\tau^t e^{(t-\beta)P_0(\xi)} Q(\tau, \beta; \xi) d\beta, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T). \tag{28}$$

It should be noted that $e^{(t-\tau)P_0(\cdot)}$ is the matriciant of the dual by Fourier system to $\{p, h\}$-parabolic system (20), therefore, the estimate (11) is performed for it. Hence, considering the inequality

$$|Q(\tau, t; \xi)| \le c_0(1 + \|\xi\|^{p_1})|\Theta_\tau^t(\xi)|, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T) \tag{29}$$

(here the positive constant c_0 in independent of τ, t and ξ), the next estimate is obtained

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-\delta(t-\tau)\|\xi\|^h} + c_1(1 + \|\xi\|^\gamma)(1 + \|\xi\|^{p_1}) \int_\tau^t e^{-\delta(t-\beta)\|\xi\|^h} |\Theta_\tau^\beta(\xi)| d\beta, \tag{30}$$

from which we come to the ratio

$$\frac{|\Theta_\tau^t(\xi)| e^{\delta(t-\tau)\|\xi\|^h}}{(1 + \|\xi\|^\gamma)} \le c + c_1(1 + \|\xi\|^\gamma)(1 + \|\xi\|^{p_1}) \int_\tau^t \frac{|\Theta_\tau^\beta(\xi)| e^{\delta(\beta-\tau)\|\xi\|^h}}{(1 + \|\xi\|^\gamma)} d\beta. \tag{31}$$

Using now the classic Grönwall's lemma [4], we get

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-(t-\tau)\left(\delta\|\xi\|^h - c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\right)}, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T). \tag{32}$$

This inequality, in combination with condition (A), ensures the existence of positive constants c and δ, with which for all $(t;\xi) \in \Pi_{(\tau;T]}$, $\tau \in [0;T)$, the estimate (11) is performed.

DOI: http://dx.doi.org/10.5772/intechopen.95024

The theorem is proved.

Remark 1 *The proof of Theorem 1 is based on the classical idea of establishing an estimate (11) for* $2b$*-parabolic systems with the coefficients continuously depending on* t*. Therefore, analyzing this proof, especially its last part, we can understand why, in contrast to the* $2b$*-parabolicity, in the case of* $p \neq h$ *the difficulties in establishing the condition (11).*

The study of the properties of the matriciant $\Theta_\tau^t(\cdot)$ for systems with a negative genus μ will be continued in the next section.

4. Properties of fundamental solution

Let us move on to the search for an answer to the question posed in Section 1 concerning the accuracy of Zhitomirskii's estimates (2) in the case of a system (3) of genus $\mu < 0$.

Theorem 2 *Let the system (3)* $\{p, h\}$ *be parabolic with the negative genus* μ*, and let* $l \geq 0$ *and* $\alpha \geq 0$ *be such arbitrarily fixed numbers that* $l \leq 1 + \alpha h$ *and* $(\alpha h - l)\mu \geq \alpha h$*. Then*

$$\exists \{c, \delta, A, B\} \subset (0; +\infty) \forall k \in \mathbb{Z}_+^n \forall q \in \mathbb{Z}_+ \forall x \in \mathbb{R}^n \backslash \{0\} \forall \tau \in [0; T) \forall t \in (\tau; T] :$$

$$|\partial_x^k G(t, \tau; x)| \leq \frac{cA^q B^{|k|_+}}{\|x\|^q} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} (t-\tau)^{\frac{(l+\mu)q - n - |k|_+ - l_0\gamma}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{(l+\mu)/h}}\right)^{\frac{1}{1-\mu/h}}}, \tag{33}$$

where $l_0 := \max\{1; l\}$.

Proof. To simplify the calculations, we put $\tau = 0$. The general case $\tau > 0$ is realized similarly.

Let us consider the functional matrix

$$\mathfrak{J}_l(t; \xi) := \Theta_0^t \left(t^{-l/h} \xi\right), \quad l \geq 0, t \in (0; T], \xi \in \mathbb{R}^n, \tag{34}$$

for which, according to the definition of the genus μ of the system (3), on the set

$$\mathbb{K}_\mu = \left\{\xi + i\eta \in \mathbb{C}^n : t^{-l/h} \|\eta\| \leq K_0 \left(1 + t^{-l/h} \|\xi\|\right)^\mu\right\} \tag{35}$$

the estimate is performed

$$|\mathfrak{J}_l(t; \xi + i\eta)| \leq c\left(1 + t^{-l/h} \|\xi\|\right)^\gamma e^{-\delta t^{1-l} |\xi|_+^h}, \quad t \in (0; T], \tag{36}$$

with positive values c and δ, independent of t, ξ and η.

To estimate the derivatives $\partial_\xi^q \mathfrak{J}_l$ we use the Cauchy integral formula

$$\partial_\xi^q \mathfrak{J}_l(t; \xi) = \prod_{j=1}^n \frac{q_j!}{2\pi i} \int\limits_{\Gamma_{R_j}} \frac{\mathfrak{J}_l(t; \sigma) d\sigma_j}{\left(\sigma_j - \xi_j\right)^{q_j+1}}, \quad q \in \mathbb{Z}_+^n, \xi \in \mathbb{R}^n, \ t \in (0; T], \tag{37}$$

in which Γ_{R_j} – is a circle with the center in the point $\xi_j + i0$ of the radius

$$R_j = K_0 \left(1 + t^{-l/h} |\xi_j|\right)^\mu, \quad 0 < K_0 < < 1. \tag{38}$$

Let us put $\Gamma_R := \Gamma_{R_1} \times \ldots \times \Gamma_{R_n}$ and fix a fairly small positive K_0 so that $\Gamma_R \subset \mathbb{K}_\mu$ (the existence of such K_0 is substantiated in ([30], p. 287) when proving the theorem 4 of the Phragmén-Lindelöf type in the case of n independent variables). Then, according to the estimate (36), we have

$$|\partial_\xi^q \mathfrak{I}_l(t;\xi)| \le c\left(1+t^{-l/h}\|\hat{\xi}\|\right)^\gamma e^{-\delta t^{1-l}|\check{\xi}|_+^h} \prod_{j=1}^n \frac{q_j!}{R_j^{q_j}}, \tag{39}$$

where $\{\hat{\xi};\check{\xi}\}\subset\mathbb{R}^n$ – fixed points with such coordinates

$$\left\{\hat{\xi}_j;\check{\xi}_j\right\}\subset\left[\xi_j-R_j;\xi_j+R_j\right], \quad j\in\mathbb{N}_n, \tag{40}$$

that

$$\hat{\xi}_j^{\,2} = \max_{[\xi_j-R_j;\xi_j+R_j]} x_j^2, \quad |\check{\xi}_j| = \min_{[\xi_j-R_j;\xi_j+R_j]} |x_j|, \tag{41}$$

that is

$$\hat{\xi}_j = \xi_j + \chi_j R_j, \quad \check{\xi}_j = \xi_j + \zeta_j R_j, \tag{42}$$

at some $\left\{\chi_j,\zeta_j\right\}\subset[-1;1]$.

First of all it should be noted that

$$R_j = \frac{K_0}{\left(1+t^{-l/h}|\xi_j|\right)^{|\mu|}} \le K_0, \quad \xi_j\in\mathbb{R}, t\in(0;T]. \tag{43}$$

Since

$$\|\xi\|\le\sqrt{n}|\xi|_+, \quad \xi\in\mathbb{R}^n, \tag{44}$$

then

$$\begin{aligned}\|\hat{\xi}\| &\le \sqrt{n}\sum_{j=1}^n|\xi_j+\chi_j R_j| \le \sqrt{n}\sum_{j=1}^n\left(|\xi_j|+R_j\right) \le \sqrt{n}\sum_{j=1}^n\left(|\xi_j|+K_0\right) \le \\ &\le \sqrt{n}\left(1+|\xi|_+\right), \quad K_0\le 1/n, \xi\in\mathbb{R}^n, t\in(0;T].\end{aligned} \tag{45}$$

Now let us estimate the value $e^{-\delta t^{1-l}|\check{\xi}|_+^h}$.

Let us start with the simpler case when $t\in[1;T]$.

We assume that $|\xi_j|\ge 2K_0$, then

$$\left|\check{\xi}_j\right|^h = \left(|\xi_j|-R_j\right)^h \ge \left(|\xi_j|-K_0\right)^h \ge 2^{-h}\left|\xi_j\right|^h. \tag{46}$$

If $|\xi_j|<2K_0$, then

$$e^{-\delta t^{1-l}\left|\check{\xi}_j\right|^h} \le 1 = e^{\delta_0 t^{1-l}\left|\xi_j\right|^h} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} \le e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h+a} \quad (\forall\delta_0>0), \tag{47}$$

where $a = \delta_0(2K_0)^h \max\limits_{t\in[1;T]} t^{1-l}$.

Therefore, for each $\delta > 0$ there are such positive constants c_0 and δ_0 that for all $\xi_j \in \mathbb{R}$ and $t \in [1; T]$ the estimate is performed

$$e^{-\delta t^{1-l}\left|\breve{\xi}_j\right|^h} \le c_0 e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h}. \tag{48}$$

We show that the statement (48) is also true in the case of $t \in (0; 1)$.

We shall fix arbitrarily $\alpha \ge 0$ and further consider that $l \le 1 + \alpha h$. Then for $|\xi_j| < t^\alpha$, we have:

$$e^{-\delta t^{1-l}\left|\breve{\xi}_j\right|^h} \le e^{\delta_0 t^{1-l}\left|\xi_j\right|^h - \delta_0 t^{1-l}\left|\xi_j\right|^h} \le e^{-\delta_0\left(t^{1-l}\left|\xi_j\right|^h - t^{1+\alpha h - l}\right)} \le e^{-\delta_0\left(t^{1-l}\left|\xi_j\right|^h - 1\right)} \quad (\forall \delta_0 > 0). \tag{49}$$

Now let $t^\alpha \le |\xi_j|$, and α be such that the condition: $(l - \alpha h)|\mu| \ge \alpha h$ is satisfied. Taking into consideration that

$$R_j \le \frac{K_0}{\left(1 + t^{\alpha - l/h}\right)^{|\mu|}} \le K_0 t^{(l/h - \alpha)|\mu|} \le K_0 t^\alpha, \tag{50}$$

we obtain:

$$\begin{aligned}\left|\breve{\xi}_j\right|^h &= \left(|\xi_j| - R_j\right)^h \ge \left(|\xi_j| - K_0 t^\alpha\right)^h = \left|\xi_j\right|^h\left(1 - K_0 t^\alpha/|\xi_j|\right)^h \ge \\ &\ge \left|\xi_j\right|^h (1 - K_0)^h \ge 2^{-h}\left|\xi_j\right|^h.\end{aligned} \tag{51}$$

Hence we arrive at performing (48) at $t \in (0; 1)$.

According to the estimates (45), (48) and equality

$$\sup_{y \ge 0}\left\{y^\beta e^{-\delta y}\right\} = \left(\frac{\beta}{e\delta}\right)^\beta, \quad \beta > 0, \delta > 0, \tag{52}$$

we find:

$$\begin{aligned}&c_0^{-1}\left(1 + t^{-l/h}\|\hat{\xi}\|\right)^\gamma e^{-\frac{\delta}{3}t^{1-l}\left|\breve{\xi}\right|_+^h} \le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + |\xi|_+\right)^\gamma e^{-\delta_0 t^{1-l}|\xi|_+^h} \le \\ &\le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + |\xi|_+ e^{-\frac{\delta_0}{\gamma}t^{1-l}|\xi|_+^h}\right)^\gamma \le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + n\left(\frac{\gamma t^{l-1}}{he\delta_0}\right)^{1/h}\right)^\gamma; \\ &c_0^{-1} R_j^{-q_j} e^{-\frac{\delta}{3n}t^{1-l}\left|\breve{\xi}\right|_+^h} \le R_j^{-q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} = K_0^{-q_j}\left(1 + t^{-l/h}|\xi_j|\right)^{|\mu| q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} \le \\ &\le \left(2^\mu K_0\right)^{-q_j}\left(1 + t^{\mu l q_j/h}\left|\xi_j\right|^{|\mu| q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h}\right) \le \left(2^\mu K_0\right)^{-q_j}\left(1 + \left(\frac{|\mu| q_j}{he\delta_0 t}\right)^{|\mu| q_j/h}\right).\end{aligned} \tag{53}$$

Together with (39), these estimates ensure the existence of such positive constants c, A and δ that for all $\xi \in \mathbb{R}^n$, $t \in (0; T]$ and $q \in \mathbb{Z}_+^n$ the following inequality is true

$$|\partial_\xi^q \mathfrak{I}_l(t; \xi)| \le c A^{|q|_+} q^{\left(1 - \frac{\mu}{h}\right) q} t^{\frac{\mu|q|_+ - l_0\gamma}{h}} e^{-\delta t^{1-l}|\xi|_+^h}, \tag{54}$$

in which $l_0 = \max\{1; l\}$.

Next, we shall use the image

$$G(t,0;x) = (2\pi)^{-n} t^{-nl/h} \int\limits_{\mathbb{R}^n} e^{-i\left(x,t^{-l/h}\xi\right)} \mathfrak{I}(t;\xi)d\xi, \quad (t;x) \in \Pi_{(0;T]}. \tag{55}$$

Identity

$$t^{l/h} L_{\xi;x}\left[e^{-i\left(x,t^{-l/h}\xi\right)}\right] = e^{-i\left(x,t^{-l/h}\xi\right)}, \tag{56}$$

in which

$$L_{\xi;x} = i\|x\|^{-2} \sum_{j=1}^{n} x_j \partial_{\xi_j}, \tag{57}$$

at $x \neq 0$ allows to write the previous equality in the form

$$G(t,0;x) = (2\pi)^{-n} t^{l(q-n)/h} \int\limits_{\mathbb{R}^n} L^q_{\xi;x}\left[e^{-i\left(x,t^{-l/h}\xi\right)}\right] \mathfrak{I}(t;\xi)d\xi \quad (\forall q \in \mathbb{Z}_+). \tag{58}$$

Hence, after integrating by parts q times, we arrive at the relation

$$G(t,0;x) = (-1)^q (2\pi)^{-n} t^{l(q-n)/h} \int\limits_{\mathbb{R}^n} e^{-i\left(x,t^{-l/h}\xi\right)} L^q_{\xi;x}[\mathfrak{I}(t;\xi)]d\xi \quad (\forall q \in \mathbb{Z}_+), \tag{59}$$

from which we obtain that

$$|x^r \partial_x^k G(t,0;x)| \leq (2\pi)^{-n} t^{\frac{l\left(q-n-|k-r|_+\right)}{h}} \int\limits_{\mathbb{R}^n} |\xi|^k \left|\partial_\xi^r \left(L^q_{\xi;x}[\mathfrak{I}(t;\xi)]\right)\right| d\xi, \tag{60}$$

for all $\{r,k\} \subset \mathbb{Z}^n_+$ and $q \in \mathbb{Z}_+$.
Having considered the estimate (54), for $(t;\xi) \in \Pi_{(0;T]}$ and $x \neq 0$ we find:

$$\left|\partial_\xi^r \left(L^q_{\xi;x}[\mathfrak{I}(t;\xi)]\right)\right| \leq cA^{q+|r|_+} \|x\|^{-q} t^{\frac{\mu\left(q+|r|_+\right)-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} e^{-\delta t^{1-l}|\xi|_+^h}. \tag{61}$$

Then

$$\begin{aligned} |x^r \partial_x^k G(t,0;x)| &\leq c_1 A^{q+|r|_+} \|x\|^{-q} t^{\frac{(l+\mu)\left(q+|r|_+\right)-l\left(n+|k|_+\right)-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} \times \\ &\times \int\limits_{\mathbb{R}^n} |\xi|^k e^{-\delta l^{1-l}|\xi|_+^h} d\xi \leq c_1 A^{q+|r|_+} \|x\|^{-q} t^{\frac{(l+\mu)\left(q+|r|_+\right)-n-|k|_+-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} \times \\ &\times \left(\prod_{j=1}^{n} \sup_{y\geq 0}\left\{y^{\frac{k_j}{h}} e^{-\frac{\delta}{2}y}\right\}\right) \int\limits_{\mathbb{R}^n} e^{-\frac{\delta}{2}\left|\zeta^h\right|_+} d\zeta \leq c_2 A^{q+|r|_+} B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)\left(q+|r|_+\right)-n-|k|_+-l_0\gamma}{h}} \times \\ &\times r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}}, \quad t \in (0;T], x \neq 0, q \in \mathbb{Z}_+, \{k,r\} \subset \mathbb{Z}^n_+ \end{aligned} \tag{62}$$

(here positive values c_2, A and B do not depend on t, x, q, k and r).

Thus, for all $t \in (0; T]$, $x \in \mathbb{R}^n \setminus \{0\}$, $q \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+^n$ the correct estimates are

$$|\partial_x^k G(t, 0; x)| \le c_2 A^q B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)(q+|r|_+)-n-|k|_+-l_0\gamma}{h}} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} \times$$

$$\times \left(\prod_{j=1}^{n} \inf_{r_j} \left\{ \left(t^{\frac{l+\mu}{h}} A \right)^{r_j} r_j^{r_j\left(1-\frac{\mu}{h}\right)} |x_j|^{-r_j} \right\} \right) \le \tag{63}$$

$$\le c A^q B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)q-n-|k|_+-l_0\gamma}{h}} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} e^{-\delta\left(\frac{|x|_+}{t^{(l+\mu)/h}}\right)^{\frac{1}{1-\mu/h}}},$$

in which the values $c > 0$, $A > 0$, $B > 0$ and $\delta > 0$ do not depend on k, q, t and x. The theorem is proved.

Remark 2 *Zhitomirskii's estimates (2) are obtained from (33) for $q = 0$, $l = 0$ and $\alpha = 0$.*

Given that $l = 1 + \alpha h$, $(\alpha h - l)\mu = \alpha h$ and $q = 0$, from the theorem 2 we arrive at the following statement.

Corollary 1 *For $\{p, h\}$-parabolic system (3) with genus $\mu < 0$ there are such positive constants c, B and δ that for all $k \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$, $\tau \in [0; T)$ and $t \in (\tau; T]$ the next estimate is performed*

$$|\partial_x^k G(t, \tau; x)| \le c B^{|k|_+} k^{\frac{k}{h}} (t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{1/h}}\right)^{\frac{1}{1-\mu/h}}}. \tag{64}$$

Therefore, according to the corollary 1, the fundamental solution G in the case of negative genus μ also has a singularity only at the point $(t; x) = (\tau; 0)$.

Corollary 2 *Let (3) $\{p, h\}$ be a parabolic system with negative genus μ, then for all $t \in (\tau; T]$, $\tau \in [0; T)$, $x \in \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{Z}_+^n$ estimate is performed*

$$|\partial_x^k G(t, \tau; x)| \le \frac{c B^{|k|_+} k^{\frac{k}{h}} |k|_+^{\left(1-\frac{\mu}{h}\right)|k|_+}}{\|x\|^{n+[\gamma]+1+|k|_+}} (t-\tau)^{\frac{1-\{\gamma\}}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{1/h}}\right)^{\frac{1}{1-\mu/h}}}, \tag{65}$$

in which the positive values c, δ and B do not depend on t, τ, x and k; $[\cdot]$ and $\{\cdot\}$ are integer and fractional parts of the number respectively.

Proof. Estimates (65) are obtained directly from (33) at $l = 1 + \alpha h$, $(\alpha h - l)\mu = \alpha h$ and $q = n + [\gamma] + 1 + |k|_+$.

The established estimates (65) provide exponential decrease when changing $t \to \tau + 0$ on the set $\mathbb{R}^n \setminus \{0\}$ derivatives of the function $G(t, \tau; \cdot)$ in case $\mu < 0$. Similarly to the case $\mu \ge 0$ considered in [25–28], this will allow us to successfully study the Cauchy problem for wide classes of $\{p, h\}$-parabolic systems (3) with negative genus μ and variable coefficients $a_k^{jl}(t; x)$. Moreover, this will also allow us to describe in a similar way the sets of classical solutions of such systems with generalized limit values f on the initial hyperplane and to study the local behavior of these solutions when changing $t \to \tau + 0$ on that part of $\mathbb{R}^n$ where the functional f has good properties etc.

5. Conclusions

The class of systems with dissipative parabolicity and variable coefficients defined in Section 3 proves that the class of parabolic Shilov-type systems with coefficients $a_k^{jl}(t)$ is quite broad and cannot be confined to the class of $2b$-parabolic systems (3) with continuous coefficients only.

Analyzing the obtained estimates (33) of the fundamental solution of the systems (3) with dissipative parabolicity, we conclude that in the case of the negative genus μ the function $G(t,\tau;x)$ on the set $(\tau;T] \times \mathbb{R}^n$ has only one singular point $(t;x) = (\tau;0)$. Similarly to the case $\mu \geq 0$, these estimates allow to perform the expansion of the Shilov class $\{p,h\}$-parabolic systems by supplementing it with the systems with negative genus μ and coefficients depending on space variable, and to successfully develop the theory of the Cauchy problem for it using the classical means. Moreover, the estimates (33) open wide possibilities for studying the properties of solutions of parabolic systems of the genus $\mu < 0$ at the approximation of the initial hyperplane.

Author details

Vladyslav Antonovich Litovchenko
Department of Differential Equations, Faculty of Mathematics and Informatics, Yu. Fed'kovych Chernivtsi National University, Ukraine

*Address all correspondence to: vladlit4@mail.ru

References

[1] *Fourier J.* Théorie analytique de la chaleur.– Paris: Firmin Didot, père et fils, 1822.

[2] *I. G. Petrovskii* On the Cauchy problem for systems of equations with partial derivatives in a domain of nonanalytic functions // Bull. MGU. Mat. Mekh. – 1938. – **1**, No. 7. – P. 1–72 (in Russian).

[3] *Friedman A.* Partial Differential Equations of Parabolic Type. – Englewood Cliffs: Prentice-Hall, 1964.

[4] *Eidelman S. D.* Parabolic Systems. – North-Holland, Amsterdam, 1969.

[5] *Gorbachuk V. I., Gorbachuk M. L.* Boundary values of solutions of some classes of differential equations // Mat. Sb. – 1977. – **102**, No. 1. – P. 124–150 (in Russian).

[6] *Eidelman S. D., Ivasyshen S. D., Kochubei A. N.* Analytic methods in the theory of differential and pseudo-differential equations of parabolic type. – Basel–Boston–Berlin : Birkhäser, 2004. – **152**. – 390 p.

[7] *Jing Kang and Changzheng Qu* Symmetry groups and fundamental solutions for systems of parabolic equations // J. Math. Phys. – 2012. – **53**, No. 2.– P. 186–198. https://doi.org/10.1063/1.3684749

[8] *Baderko E.A., Cherepova M.F.* Uniqueness theorem for parabolic Cauchy problem // Applicable Analysis. – 2016. – **95**, No. 7. – P. 1570–1580. doi: 10.1080/00036811.2015.1114608

[9] *Arkhipova A.A., Stará J.* Regularity problem for $2m$-order quasilinear parabolic systems with non smooth in time principal matrix. $(a(t), m)$-caloric approximation method //Topol. Methods Nonlinear Anal. – 2018. – **52**, No. 1. – P. 111–146. doi:10.12775/TMNA.2018.006

[10] *Baderko E.A., Cherepova M.F.* Uniqueness of the solution of the Cauchy problem for parabolic systems // Diff. Equat. – 2019. – **55**, No. 6. - P. 806–814. doi: 10.1134/S0012266119060077

[11] *Melnikova A.A., Deryugina N.N.* Existence of a periodic solution in the form of a two-dimencional front in a system of parabolic equations // Diff. Equat. – 2020. – **56**, No. 4. – P. 462–477. doi: 10.1134/S0012266120040060

[12] *Bögelein V., Dacorogna B., Duzaar F., Marcellini P., Scheven C.* Integral convexity and parabolic systems //SIAM J. Math. Anal. – 2020. – **52**, No. 2. – P. 1489–1525. https://doi.org/10.1137/19M1287870

[13] *Giannetti F., Di Napoli A.P., Scheven C.* On higher differentiability of solutions of parabolic systems with discontinuous coefficients and (p, q)-growth // Proceedings of the Royal Society of Edinburgh Section A: Mathematics. – 2020. – **150**, No. 1. – P. 419–451. doi: https://doi.org/10.1017/prm.2018.63

[14] *Shilov G. E.* On conditions of correctness of Cauchy's problem for systems of partial differential equations with constant coefficients // Uspekhi Mat. Nauk. – 1955. – **10**, No. 4. – P. 89–100 (in Russian).

[15] *I. Gel'fand and G. Shilov* Generalized Functions. Vol. 3. Theory of Differential Equations. Boston, MA: Academic Press, 1967.

[16] *Zhitomirskii Ya. I.* Cauchy problem for some types of systems of linear equations, which are parabolic by G.E. Shilov, with partial derivatives with continuous coefficients // Izv. AN SSSR. Ser. Mat. – 1959. – **23**. – P. 925–932 (in Russian).

[17] *U. Hou-Sin* On the definition of parabolicity of systems of equations

with partial derivatives // Uspekhi Mat. Nauk. – 1960. – **15**, No. 6. – P. 157–161 (in Russian).

[18] *S. D. Eidelman, S. D. Ivasishen, and F. O. Porper* The Liouville theorems for systems parabolic by Shilov // Izv. Vyzov. Mat. – 1961. – No. 6. – P. 169–179 (in Russian).

[19] *Gorodetskii V. V.* Localization principle for solutions of the Cauchy problem for parabolic systems in the class of generalized functions of infinite order // Differ. Uravn. – 1985. – **21**, No. 6. – P. 1077–1079 (in Russian).

[20] *V. Litovchenko* The Cauchy problem for parabolic equations by Shilov // Siber. Mat. Zh. – 2004. – **45**, No. 4. – P. 809–821. doi: 10.1023/B:SIMJ.0000035831.63036.bb

[21] *V. Litovchenko* Cauchy problem for $\left\{\overleftarrow{p}; \overleftarrow{h}\right\}$-parabolic equations with time-dependent coefficients // Math. Notes. – 2005. – **77**, No. 3–4. – P. 364–379. doi: 10.1007/s11006-005-0036-9

[22] *Q. Zheng* Matrices of operators and regularized cosine functions // J. Math. Anal. Appl. – 2006. – **315**. – P. 68–75. doi: 10.1016/j.jmaa.2005.10.007

[23] *M. Kostić* Shilov parabolic systems // Bulletin (Acadtmie serbe des sciences et des arts. Classe des Sciences Mathematiques et Naturelles, Sciences mathematiques). – 2012. – No. 37. – P. 19–39. https://www.jstor.org/stable/44095570

[24] *Kuź A.M.* A problem with integral conditions with respect to time for shilov parabolic systems of equations // J. Math. Sci. – 2016. – **217**. – P. 149–165. https://doi.org/10.1007/s10958-016-2963-2

[25] *Litovchenko V. A., Dovzhitska I. M.* The fundamental matrix of solutions of the Cauchy problem for a class of parabolic systems of the Shilov type with variable coefficients // J. Math. Sci. – 2011. – **175**, No. 4. – P. 450–476. doi: 10.1007/s10958-011-0356-0

[26] *V. Litovchenko and I. Dovzhytska* Cauchy problem for a class of parabolic systems of Shilov type with variable coefficients // Cent. Eur. J. Math. – 2012. – **10**, No. 3. – P. 1084–1102. doi: 10.2478/s11533-012-0025-7

[27] *Litovchenko V. A., Unguryan G. M.* Conjugate Cauchy Problem for Parabolic Shilov Type Systems with Nonnegative Genus // Diff. Equat. – 2018. – **54**. – P. 335–351. doi: 10.1134/S0012266118030060

[28] *V. Litovchenko and G. Unguryan* Some properties of Green's functions of Shilov-type parabolic systems // Miskolc Math. Notes. – 2019. – **20**, No. 1. – P. 365–379. doi: 10.18514/MMN.2019.2089

[29] *Gantmacher F. R.* Matrix theory. – Moscow: Fiz. Mat. Lit., 2004 (in Russian).

[30] *I. M. Gel'fand and G. E. Shilov* Spaces of Basic and Generalized Functions. – Moscow: Gos. Izd. Fiz. Mat. Lit., 1958 (in Russian).