Chapter

Existence and Asymptotic Behaviors of Nonoscillatory Solutions of Third Order Time Scale Systems

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Abstract

Nonoscillation theory with asymptotic behaviors takes a significant role for the theory of three-dimensional (3D) systems dynamic equations on time scales in order to have information about the asymptotic properties of such solutions. Some applications of such systems in discrete and continuous cases arise in control theory, optimization theory, and robotics. We consider a third order dynamical systems on time scales and investigate the existence of nonoscillatory solutions and asymptotic behaviors of such solutions. Our main method is to use some well-known fixed point theorems and double/triple improper integrals by using the sign of solutions. We also provide examples on time scales to validate our theoretical claims.

Keywords: nonoscillation, three-dimensional time scale systems, dynamical systems, existence, fixed point theorems

1. Introduction

This chapter deals with the nonoscillatory solutions of 3D nonlinear dynamical systems on time scales. In addition, it is very critical to discuss whether or not there exist such solutions. Therefore, the existence along with limit behaviors are also studied in this chapter by using double/triple integrals and fixed point theorems. Stefan Hilger, a German mathematician, introduced a theory in his PhD thesis in 1988 [1] that unifies continuous and discrete analysis and extend it in one comprehensive theory, which is called the *time scale* theory. A *time scale*, symbolized by $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. After Hilger, the theory and its applications have been developed by many mathematicians and other researchers in Control Theory, Optimization, Population Dynamics and Economics, see [2–5]. In addition to those articles, two books were published by Bohner and Peterson in 2001 and 2003, see [6, 7].

Now we explain what we mean by continuous and discrete analysis in details. Assuming readers are all familiar with differential and difference equations; the results are valid for differential equations when $\mathbb{T} = \mathbb{R}$ (set of real numbers), while the results hold for difference equations when $\mathbb{T} = \mathbb{Z}$ (set of integers). So we might have two different proofs and maybe similar in most cases. In order to avoid repeating similarities, we combine continuous and discrete cases in one general theory and remove the duplication from both. For more details in the theory of differential and difference equations, we refer the books [8–10] to interested readers.

3D nonlinear dynamical systems on time scales have recently gotten a valuable attention because of its potential in applications of control theory, population dynamics and mathematical biology and Physics. For example, Akn, Güzey and Öztürk [3] considered a 3D dynamical system to control a wheeled mobile robots on time scales

$$\begin{cases} \alpha^{\Delta}(t) = -v(t)\cos\beta(t) \\ \beta^{\Delta}(t) = \dfrac{\sin\beta^{\sigma}(t)}{\alpha^{\sigma}(t)}v(t) - w(t) \\ \gamma^{\Delta}(t) = \dfrac{\sin\beta^{\sigma}(t)}{\alpha^{\sigma}(t)}v(t), \end{cases} \tag{1}$$

where α is the distance of the reference point from the origin, β is the angle of the pointing vector to the origin, γ is the angle with respect to the x axis, and v, w are controllers. They showed the asymptotic stability of the system above on time scales. Another example for $\mathbb{T} = \mathbb{R}$, Bernis and Peletier [11] considered an equation that can be written as the following system

$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = h(u) \end{cases} \tag{2}$$

to show the existence and uniqueness and properties of solutions for flows of thin viscous films over solid surfaces, where (u_1, u_2, u_3) is the film profile in a coordinate frame moving with the fluid.

We assume that readers may not be familiar with the time scale basics, so we give an introductory section to the time scale calculus. We refer the books [6, 7] for more details and information about time scales. Structure of the rest of this chapter is as follows: In Section 3.1 and 3.2 we consider a system with different values, 1 and -1, respectively, and show the qualitative behavior of solutions. In Section 4, we give some examples for readers to comprehend our theoretical results. Finally, we give a short conclusion about the summary of our results and open problems in the last section.

2. Time scale essentials

In the introduction section, we have only mentioned the time scales $\mathbb{R}$ and $\mathbb{Z}$. However, there are some other time scales in the literature, which also have gotten too much attention because of the applications of them. For example, when $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \cdots, \}, q > 1$, the results hold for so-called *q-difference* equations, see [12]. Another well-known time scale is $\mathbb{T} = h\mathbb{Z}, h > 0$.

Definition 2.1 Let $\mathbb{T}$ be a time scale. Then for all $t \in \mathbb{T}$,

1. $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is called *forward jump operator* $(\sigma(t) : \mathbb{T} \to \mathbb{T})$.

2. $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ is said to be *backward jump operator* $(\rho(t) : \mathbb{T} \to \mathbb{T})$.

3. $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$ is called the *graininess function* $((\mu(t) : \mathbb{T} \to [0, \infty))$.

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}_0}$	tq	$\frac{t}{q}$	$t(q-1)$

Table 1.
Some time scales with σ, ρ and μ.

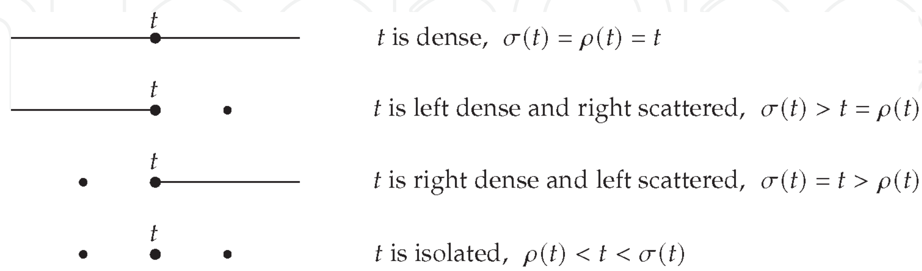

Figure 1.
Classification of points.

For the sake of the rest of the chapter, **Table 1** summarizes how σ, ρ and μ are defined for some time scales.

As we know, the set of real numbers are dense and set of integers are scattered. Now we show how we classify the points on general time scales. For any $t \in \mathbb{T}$, **Figure 1** shows the classification of points on time scales and how we represent those points by using σ, ρ and μ, see [6] for more details.

Now, let us introduce the derivative for general time scales. Note that

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.2 If there exists a $\delta > 0$ such that

$$|h(\sigma(t)) - h(s) - h^\Delta(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s| \text{ for all } s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

for any ε, then h is said to be *delta-differentiable* on $\mathbb{T}^\kappa$ and h^Δ is called the *delta derivative* of h.

Theorem 2.3 *Let $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ be functions with $t \in \mathbb{T}^\kappa$. Then.*

i. h_1 is said to be continuous at t if h_1 is differentiable at t.

ii. h_1 is differentiable at t and

$$h_1^\Delta(t) = \frac{h_1(\sigma(t)) - h_1(t)}{\mu(t)},$$

provided h_1 is continuous at t and t is right-scattered.

iii. Suppose t is right dense, then h_1 is differentiable at t if and only if

$$h_1^\Delta(t) = \lim_{s \to t} \frac{h_1(t) - h_1(s)}{t - s}$$

exists as a finite number.

iv. If $h_2(t)h_2(\sigma(t)) \neq 0$, then $\frac{h_1}{h_2}$ is differentiable at t with

$$\left(\frac{h_1}{h_2}\right)^{\Delta}(t) = \frac{h_1^{\Delta}(t)h_2(t) - h_1(t)h_2^{\Delta}(t)}{h_2(t)h_2(\sigma(t))}.$$

A function $h_1 : \mathbb{T} \to \mathbb{R}$ is called *right dense continuous* (rd-continuous) if it is continuous at right dense points in $\mathbb{T}$ and its left sided limits exist at left dense points in $\mathbb{T}$. We denote the set of rd-continuous functions with $C_{rd}(\mathbb{T}, \mathbb{R})$. On the other hand, the set of differentiable functions whose derivative is *rd-continuous* is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$. Finally, we use C for the set of continuous functions throughout this chapter.

After derivative and its properties, we also introduce integrals for any time scale $\mathbb{T}$. The Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for all } a, b \in \mathbb{T}.$$

Every rd-continuous function has an antiderivative. Moreover, F given by

$$F(t) = \int_{t_0}^{t} f(s)\Delta s \text{ for } t \in \mathbb{T}$$

is an antiderivative of f.

The following theorem leads us to the properties of integrals on time scales, which are similar to continuous case.

Theorem 2.4 Suppose that h_1 and h_2 are rd-continuous functions, $c, d, e \in \mathbb{T}$, and $\beta \in \mathbb{R}$,

a. h_1 is nondecreasing if $h_1^{\Delta} \geq 0$.

b. If $h_1(t) \geq 0$ for all $c \leq t \leq d$, then $\int_c^d h_1(t)\Delta t \geq 0$.

c. $\int_c^d [(\beta h_1(t)) + (\beta h_2(t))] = \beta \int_c^d h_1(t)\Delta t + \beta \int_a^b h_2(t)\Delta t$.

d. $\int_c^e h_1(t)\Delta t = \int_c^d h_1(t)\Delta t + \int_d^e h_1(t)\Delta t$.

e. $\int_c^d h_1(t)h_2^{\Delta}(t)\Delta t = (h_1h_2)(d) - (h_1h_2)(c) - \int_c^d h_1^{\Delta}(t)h_2(\sigma(t))\Delta t$

f. $\int_a^a h_1(t)\Delta t = 0$.

Table 2 shows how the derivative and integral are defined for some time scales for $a, b \in \mathbb{T}$.

$\mathbb{T}$	$f^{\Delta}(t)$	$\int_a^b f(t)\Delta t$
$\mathbb{R}$	$f'(t)$	$\int_a^b f(t)dt$
$\mathbb{Z}$	$\Delta f(t) = f(t+1) - f(t)$	$\sum_{t=a}^{b-1} f(t)$
$q^{\mathbb{N}_0}$	$\Delta_q f(t) = \frac{f(tq) - f(t)}{(q-1)t}$	$\sum_{[a,b)_{q^{\mathbb{N}_0}}} f(t)\mu(t)$

Table 2.
Derivative and integral for some time scales.

This chapter assumes that $\mathbb{T}$ is unbounded above and whenever it is written $t \geq t_1$, we mean $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. Finally, we provide Schauder's fixed point theorem, proved in 1930, see ([13], Theorem 2.A), the Knaster fixed point theorem, proved in 1928, see [14] and the following lemma, see [15], to show the existence of solutions.

Lemma 2.5 Let X be equi-continuous on $[t_0, t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$. In addition to that, let $X \subseteq BC[t_0, \infty)_{\mathbb{T}}$ be bounded and uniformly Cauchy. Then X is relatively compact.

Theorem 2.6 (Schauder's Fixed Point Theorem) Suppose that X is a Banach space and M is a nonempty, closed, bounded and convex subset of X. Also let $T : M \to M$ be a compact operator. Then, T has a fixed point such that $y = Ty$.

Theorem 2.7 (The Knaster Fixed Point Theorem) Supposing $(M, \leq)$ being a complete lattice and $F : M \to M$ is order-preserving, we have F has a fixed point so that $y = Fy$. In fact, the set of fixed points of F is a complete lattice.

3. Nonoscillatory solutions of nonlinear dynamical systems

Motivated by [16, 17], we deal with the nonlinear system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = q(t)g(z(t)) \\ z^{\Delta}(t) = \lambda r(t)h(x(t)), \end{cases} \tag{3}$$

where $p.q, r \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, $\lambda = \pm 1$, and f and g are nondecreasing functions such that $uf(u) > 0$, $ug(u) > 0$ and $uh(u) > 0$ for $u \neq 0$.

The other continuous and discrete cases of system (3) were studied in [18–20]. We first give the following definitions to help readers understand the terminology.

Definition 3.1 If (x, y, z), where $x, y, z \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})T \geq t_0$, satisfies system (3) for all large $t \geq T$, then we say (x, y, z) is a solution of (3).

Definition 3.2 By a proper solution (x, y, z), we mean a solution (x, y, z) of system (3) that holds

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$$

for $t \geq t_0$.

Finally, let us define nonoscillatory solutions of system (3).

Definition 3.3 By a nonoscillatory solution (x, y, z) of system (3), we mean a proper solution and the component functions x, y and z are all nonoscillatory. In other words, (x, y, z) is either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

For the sake of simplicity, let us set

$$P(t_0, t) = \int_{t_0}^{t} p(s)\Delta s, \quad Q(t_0, t) = \int_{t_0}^{t} q(s)\Delta s \quad \text{and} \quad R(t_0, t) = \int_{t_0}^{t} r(s)\Delta s,$$

where $s, t, t_0 \in \mathbb{T}$ and we assume that $P(t_0, \infty) = Q(t_0, \infty) = \infty$ throughout the chapter.

Suppose that N is the set of all nonoscillatory solutions (x, y, z) of system (3). Then according to the possible signs of solutions of system (3), we have the following classes:

$$N^a := \{(x,y,z) \in N : \quad \mathrm{sgn}x(t) = \mathrm{sgn}y(t) = \mathrm{sgn}z(t), \quad t \geq t_0\}$$
$$N^b := \{(x,y,z) \in N : \quad \mathrm{sgn}x(t) = \mathrm{sgn}z(t) \neq \mathrm{sgn}y(t), \quad t \geq t_0\}$$
$$N^c := \{(x,y,z) \in N : \quad \mathrm{sgn}x(t) = \mathrm{sgn}y(t) \neq \mathrm{sgn}z(t), \quad t \geq t_0\}.$$

It was shown in [21] that any nonoscillatory solution of system (3) for $\lambda = 1$ belongs to N^a or N^c, while it belongs to N^a or N^b for $\lambda = -1$. In the literature, solutions in N^a, N^b and N^c are also known as *Type* (a), *Type* (b) and *Type* (c) solutions, respectively.

Next, we consider system (3) for $\lambda = 1$ and $\lambda = -1$ separately in different subsections, split the classes N^a, N^b and N^c into some subclasses and show the existence of nonoscillatory solutions in those subclasses. To show the existence and limit behaviors, we use the following improper integrals:

$$Y_1 = \int_{t_0}^{\infty} r(t)h\left(\int_{t_0}^{t} p(s)f\left(k_1\int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_2 = \int_{t_0}^{\infty} p(t)f\left(k_2 + \int_{t}^{\infty} q(s)g\left(k_3\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_3 = \int_{t_0}^{\infty} q(t)g\left(k_4 + \int_{t}^{\infty} r(s)h\left(k_5\int_{t_0}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_4 = \int_{t_0}^{\infty} p(t)f\left(k_6 - \int_{t}^{\infty} q(s)g\left(k_7 + k_8\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_5 = \int_{t_0}^{\infty} p(t)f\left(\int_{t_0}^{t} q(s)g\left(k_9\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_6 = \int_{t_0}^{\infty} p(t)f\left(k_{10}\int_{t_0}^{t} q(s)\Delta s\right)\Delta t,$$
$$Y_7 = \int_{t_0}^{\infty} q(t)g\left(\int_{t}^{\infty} r(s)h\left(k_{11}\int_{s}^{\infty} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_8 = \int_{t_0}^{\infty} q(t)g\left(k_{12} + k_{13}\int_{t}^{\infty} r(s)\Delta s\right)\Delta t,$$
$$Y_9 = \int_{t_0}^{\infty} r(t)h\left(k_{14}\int_{t_0}^{t} p(s)\Delta s\right)\Delta t,$$

for some nonnegative k_i, $i = 1, \ldots, 14$.

3.1 The case $\lambda = 1$

In this section, we consider system (3) with $\lambda = 1$ and investigate the limit behaviors and the criteria for the existence of nonoscillatory solutions. The limit behaviors are characterized by Akin, Došla and Lawrence in the following lemma, see [21].

Lemma 3.4 Let (x,y,z) be any nonoscillatory solution of system (3). Then we have:

i. Nonoscillatory solutions in N^a satisfy

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty.$$

ii. Nonoscillatory solutions in N^c satisfy

$$\lim_{t\to\infty}|z(t)| = 0.$$

Therefore, for a nonoscillatory solution (x,y,z), we at least know that the components x and y tend to infinity while the other component z tends to 0 as $t \to \infty$.

3.1.1 Existence in N^a

Let (x,y,z) be a nonoscillatory solution of system (3) in N^a such that x is eventually positive. ($x<0$ can be repeated very similarly.) Then by System (3), we have that x, y and z are positive and increasing. Hence, one can have the following cases:

$$\text{(i) } x \to c_1 \text{ or } x \to \infty,\ \text{(ii) } y \to c_2 \text{ or } y \to \infty,\ \text{(iii) } z \to c_3 \text{ or } z \to \infty,.$$

where $0<c_1,c_2,c_3<\infty$. But, the cases $x \to c_1$ and $y \to c_2$ are impossible due to Lemma 3.4 (i). So we have that any nonoscillatory solution (x,y,z) of system (3) in N^a must be in one of the following subclasses:

$$N^a_{\infty,\infty,B} := \left\{(x,y,z)\in N^a: \quad \lim_{t\to\infty}|x(t)| = \lim_{t\to\infty}|y(t)| = \infty,\ \lim_{t\to\infty}|z(t)| = c_3\right\}$$

$$N^a_{\infty,\infty,\infty} := \left\{(x,y,z)\in N^a: \quad \lim_{t\to\infty}|x(t)| = \lim_{t\to\infty}|y(t)| = \lim_{t\to\infty}|z(t)| = \infty\right\}.$$

Now, we start with our first main result which shows that the existence of a nonoscillatory solution in $N^a_{\infty,\infty,B}$.

Theorem 3.5 $N^a_{\infty,\infty,B} \neq \varnothing$ if the improper integral Y_1 is finite for some $k_1>0$.

Proof: Suppose that $Y_1<\infty$. Then choose $t_1 \geq t_0$, $k_1>0$ such that

$$\int_{t_1}^{\infty} r(t)h\left(\int_{t_1}^{t} p(s)f\left(k_1\int_{t_1}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}, \quad t\geq t_1, \tag{4}$$

where $k_1 = g(1)$. Suppose that Φ is the partially ordered Banach space of all real-valued continuous functions with the norm $\|z\| = \sup_{t\geq t_1}|z(t)|$ and the usual pointwise ordering $\leq$. Let ϕ be a subset of Φ so that

$$\phi := \left\{z\in X: \quad \frac{1}{2}\leq z(t)\leq 1, \quad t\geq t_1\right\}$$

and define an operator $Tz : \Phi \to \Phi$ by

$$(Tz)(t) = \frac{1}{2} + \int_{t_1}^{t} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s \tag{5}$$

for $t\geq t_1$. First, it is trivial to show that T is increasing, hence let us prove that $Tz : \phi \to \phi$. Indeed,

$$\frac{1}{2}\leq (Tz)(t) \leq \frac{1}{2} + \int_{t_1}^{t} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(1)\Delta\tau\right)\Delta u\right)\Delta s \leq 1$$

by (2). Also, it is trivial to show that $\inf B \in \phi$ and $\sup B \in \phi$ for any subset B of ϕ, i.e., $(\phi, \leq)$ is a complete lattice. Therefore, by Theorem 2.7, we have that there exists $\bar{z} \in \phi$ such that $\bar{z} = T\bar{z}$, i.e.,

$$\bar{z}(t) = \frac{1}{2} + \int_{t_1}^{t} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right)\Delta s. \tag{6}$$

Then taking the derivative of (4) gives us

$$\bar{z}^{\Delta}(t) = r(t)h\left(\int_{t_1}^{t} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

By setting

$$\bar{x}(t) = \int_{t_1}^{t} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u \tag{7}$$

and taking the derivative of (5), we have

$$\bar{x}^{\Delta}(t) = p(t)f\left(\int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Finally letting

$$\bar{y}(t) = \int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\Delta\tau \tag{8}$$

and taking the derivative yield

$$\bar{y}^{\Delta}(t) = q(t)g(\bar{z}(t)), \quad t \geq t_1,$$

that leads us to $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (3). Thus, by taking the limit of (4)–(6) as $t \to \infty$, we have that $\bar{x}, \bar{y}$ tend to infinity and $\bar{z}$ tend to a finite number, i.e., $N^a_{\infty,\infty,B} \neq \varnothing$. This completes the proof.

Showing existence of a nonoscillatory solution in $N^a_{\infty,\infty,\infty}$ is not easy (left as an open problem in Conclusion section). So, we only provide the following result by assuming the existence of such solutions in N^a. We leave the proof to readers.

Theorem 3.6 Suppose that (x, y, z) is a nonoscillatory solution of system (3) in N^a with $C(t_0, \infty) = \infty$. Then any such solution belongs to $N^a_{\infty,\infty,\infty}$.

3.1.2 Existence in N^c

Similarly, for any nonoscillatory solution of system (3) in N^c with $x > 0$, we have x is positive increasing, z is negative increasing and y is positive decreasing, that results in the following cases:

(i) $x \to c_1$ or $x \to \infty$, (ii) $y \to c_2$ or $y \to 0$, (iii) $z \to c_3$ or $z \to 0$,

where $0 < c_1, c_2 < \infty$ and $-\infty < c_3 < 0$. However, the component function z cannot tend to c_3 by Lemma 3.4 (ii). Hence, any nonoscillatory solution of (3) in N^c must belong to one of the following sub-classes:

$$N^c_{B,B,0} := \left\{(x, y, z) \in N^c : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\right\}$$

$$N^c_{B,0,0} := \left\{ (x,y,z) \in N^c : \lim_{t\to\infty} |x(t)| = c_1 \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^c_{\infty,B,0} := \left\{ (x,y,z) \in N^c : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^c_{\infty,0,0} := \left\{ (x,y,z) \in N^c : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0 \right\},$$

where $0 < c_1, c_2 < \infty$.

Next, we show the existence of nonoscillatory solutions of (3) in those subclasses by using fixed point theorems. Observe that we have some additional assumption in theorems such that g is an odd function. This assumption is very critical and cannot show the existence without it.

Theorem 3.7 Let g be an odd function. Then $N^c_{B,B,0} \neq \varnothing$ if $Y_2 < \infty$ for some $k_2, k_3 > 0$.

Proof: Supposing $Y_2 < \infty$ and g is odd lead us to that we can choose $k_2, k_3 > 0$ and $t_1 \ge t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_2 + \int_t^{\infty} q(s) g\left(k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{4}, \tag{9}$$

where $k_2 = \frac{1}{2}$ and $k_3 = h\left(\frac{1}{2}\right)$. Suppose Φ is the space of all bounded, continuous and real-valued functions with $\|x\| = \sup_{t\ge t_1} |x(t)|$. It is easy to show that Φ is a Banach space, see [22]. Let ϕ be a subset of Φ so that

$$\phi := \left\{ x \in X : \quad \frac{1}{4} \le x(t) \le \frac{1}{2}, \quad t \ge t_1 \right\}.$$

Set an operator $Tx : \Phi \to \Phi$ such that

$$(Tx)(t) = \frac{1}{4} + \int_{t_1}^{t} p(s) f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

One can show that ϕ is bounded, closed and convex. So, we first prove that $Tx : \phi \to \phi$. Indeed,

$$\frac{1}{4} \le (Tx)(t) \le \frac{1}{4} + \int_{t_1}^{t} p(s) f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(h\left(\frac{1}{2}\right)\int_u^{\infty} r(\tau)\Delta\tau\right)\Delta u\right)\Delta s \le \frac{1}{2}.$$

Second, we need to show T is continuous on ϕ. Supposing x_n is a sequence in ϕ such that $x_n \to x \in \phi = \overline{\phi}$ gives us

$$\begin{aligned} &\|(Tx_n)(t) - (Tx)(t)\| \\ &\le \int_{t_1}^{t} p(s) \left| f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x_n(\tau))\Delta\tau\right)\Delta u\right) - \right. \\ &\left. f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right) \right| \Delta s. \end{aligned}$$

So the Lebesgue dominated convergence theorem, continuity of f, g and h lead us to that T is continuous on ϕ. As a last step, we prove that T is relatively compact, i.e., equibounded and equicontinuous. Since

$$(Tx)^{\Delta}(t) = p(t)f\left(\frac{1}{2} + \int_t^{\infty} q(u)g\left(\int_u^{\infty} r(\tau)h(x(\tau))\Delta\tau\right)\Delta u\right) < \infty,$$

we have that T is relatively compact by Lemma 2.5 and the mean value theorem. So, there does exist $\overline{x} \in \phi$ such that $\overline{x} = T\overline{x}$ by Theorem 2.6. In addition to that, convergence of $\overline{x}(t)$ to a finite number as $t \to \infty$ is so easy to show. Therefore, setting

$$\overline{y}(t) = \frac{1}{2} + \int_t^{\infty} q(u)g\left(\int_u^{\infty} r(\tau)h(\overline{x}(\tau))\Delta\tau\right)\Delta u > 0, \quad t \geq t_1$$

and

$$\overline{z}(t) = -\int_t^{\infty} r(\tau)h(\overline{x}(\tau))\Delta\tau < 0, \quad t \geq t_1,$$

and by a similar discussion as in Theorem 3.5, we get $\overline{y}(t) \to \frac{1}{2}$ and $\overline{z}(t) \to 0$. So we conclude that $(\overline{x}, \overline{y}, \overline{z})$ is a nonoscillatory solution of system (3) in $N^c_{B,B,0}$..

Next, we focus on the existence of nonoscillatory solutions in $N^c_{\infty,B,0}$ and $N^c_{B,0,0}$. In other words, we will show there exists such a solution (x, y, z) such that x tend to infinity while y and z tend to a finite number. After that, we provide the fact that it is possible to have such a solution whose limit is finite for all component functions x, y and z. Since the following theorems can be proved similar to the previous theorem, the proofs are skipped.

Theorem 3.8 Let g be an odd function. Then we have the followings:

i. There does exist a nonoscillatory solution in $N^c_{\infty,B,0}$ if Y_3 is finite for $k_4 = 0$ and some $k_5 > 0$.

ii. There does exist a nonoscillatory solution in $N^c_{B,0,0}$ if $Y_2 < \infty$ for $k_2 = 0$ and $k_3 > 0$.

Finally, the last theorem in this section leads us to the fact that there must be a solution such that $x \to \infty$ while the other components converge to zero according to the convergence and divergence of the improper integrals of Y_2 and Y_3.

Theorem 3.9 Supposing the fact that g is an odd function, $N^c_{\infty,0,0} \neq \varnothing$ if $Y_2 = \infty$ and $Y_3 < \infty$ for $k_2 = k_4 = 0$ and $k_3, k_5 > 0$.

Proof: Suppose that $Y_2 = \infty$ and $Y_3 < \infty$. Then choose $t_1 \geq t_0$ and $k_3, k_5 > 0$ such that

$$\int_{t_1}^{\infty} q(t)g\left(\int_t^{\infty} r(s)h\left(k_5\int_{t_1}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta s < \frac{1}{2}, \quad t \geq t_1. \tag{10}$$

and

$$\int_{t_1}^{\infty} p(t)f\left(\int_t^{\infty} q(s)g\left(k_3\int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t > \frac{1}{2}, \quad t \geq t_1, \tag{11}$$

where $k_5 = \frac{1}{2}$ and $k_3 = h\left(\frac{1}{2}\right)$. Let Φ be the partially ordered Banach space of all continuous functions with the supremum norm $\|x\| = \sup_{t \geq t_1} \frac{x(t)}{P(t_1, t)}$ and usual pointwise ordering $\leq$. Define a subset ϕ of Φ such that

$$\phi := \left\{ x \in \Phi : \quad \frac{1}{2} \leq x(t) \leq \frac{1}{2} \int_{t_1}^{t} p(s) \Delta s, \quad t \geq t_1 \right\}$$

and an operator $Tx : \Phi \to \Phi$ by

$$(Tx)(t) = \int_{t_1}^{t} p(s) f\left(\int_{s}^{\infty} q(\tau) g\left(\int_{\tau}^{\infty} r(u) h(x(u)) \Delta u \right) \Delta \tau \right) \Delta s.$$

One can easily show that $T : \phi \to \phi$ is an increasing mapping and $(\phi, \leq)$ is a complete lattice. So by Theorem 2.7, there does exist $\overline{x} \in \phi$ such that $\overline{x} = T\overline{x}$. So $\overline{x}(t) \to \infty$ as $t \to \infty$. By setting

$$\overline{y}(t) = \int_{t}^{\infty} q(\tau) g\left(\int_{\tau}^{\infty} r(u) h(\overline{x}(u)) \Delta u \right) \Delta \tau, \quad t \geq t_1$$

and

$$\overline{z}(t) = -\int_{t}^{\infty} r(u) h(\overline{x}(u)) \Delta u, \quad t \geq t_1,$$

one can have $\overline{y}(t) > 0$ and $\overline{z}(t) < 0$ for $t \geq t_1$ so that $\overline{y}(t) \to 0$ and $\overline{z}(t) \to 0$ as $t \to \infty$. This proves the assertion.

3.2 The case $\lambda = -1$

This section deals with system (3) for $\lambda = -1$. The assumptions on f, g and h are the same assumptions with the previous section. The following lemma describes the long-term behavior of two of the components of a nonoscillatory solution, see ([21], Lemma 4.2).

Lemma 3.10 Supposing (x, y, z) is a nonoscillatory solution in N^b, we have

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) = 0.$$

In the next section, we examine the solutions in each class N^a and N^b. We used fixed-point theorems to establish our results.

3.2.1 Existence in N^a

For any nonoscillatory solution (x, y, z) of system (3) in N^a with $x > 0$ eventually, one has the following subclasses by using the same arguments as in Section 3.1.1:

$$N^a_{B,B,B} := \left\{ (x, y, z) \in N^a : \lim_{t \to \infty} |x(t)| = c_1, \lim_{t \to \infty} |y(t)| = c_2, \lim_{t \to \infty} |z(t)| = c_3 \right\}$$

$$N^a_{B,B,0} := \left\{ (x, y, z) \in N^a : \lim_{t \to \infty} |x(t)| = c_1, \lim_{t \to \infty} |y(t)| = c_2, \lim_{t \to \infty} |z(t)| = 0 \right\}$$

$$N^a_{B,\infty,B} := \left\{ (x, y, z) \in N^a : \lim_{t \to \infty} |x(t)| = c_1, \lim_{t \to \infty} |y(t)| = \infty, \lim_{t \to \infty} |z(t)| = c_3 \right\}$$

$$N^a_{B,\infty,0} := \left\{ (x, y, z) \in N^a : \lim_{t \to \infty} |x(t)| = c_1, \lim_{t \to \infty} |y(t)| = \infty, \lim_{t \to \infty} |z(t)| = 0 \right\}$$

$$N^a_{\infty,B,B} := \left\{(x,y,z) \in N^a : \lim_{t\to\infty}|x(t)| = \infty,\ \lim_{t\to\infty}|y(t)| = c_2,\ \lim_{t\to\infty}|z(t)| = c_3\right\}$$

$$N^a_{\infty,B,0} := \left\{(x,y,z) \in N^a : \lim_{t\to\infty}|x(t)| = \infty,\ \lim_{t\to\infty}|y(t)| = c_2,\ \lim_{t\to\infty}|z(t)| = 0\right\}$$

$$N^a_{\infty,\infty,B} := \left\{(x,y,z) \in N^a : \lim_{t\to\infty}|x(t)| = \lim_{t\to\infty}|y(t)| = \infty,\ \lim_{t\to\infty}|z(t)| = c_3\right\}$$

$$N^a_{\infty,\infty,0} := \left\{(x,y,z) \in N^a : \lim_{t\to\infty}|x(t)| = \lim_{t\to\infty}|y(t)| = \infty,\ \lim_{t\to\infty}|z(t)| = 0\right\},$$

where c_1, c_2 and c_3 are positive constants. Finally, we have the following results:

Theorem 3.11 Suppose $R(t_0,\infty) < \infty$. If $Y_4 < \infty$ and $Y_8 < \infty$ for all positive constants $k_6, k_7, k_8, k_{12}, k_{13}$, then $N^a_{B,B,B} \neq \emptyset$.

Proof: Assume $Y_4 < \infty$ and $Y_8 < \infty$ for all $k_6, k_7, k_8, k_{12}, k_{13} > 0$. Choose $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_6 - \int_t^{\infty} q(s) g\left(k_7 + k_8 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} q(s) g\left(k_{12} + k_{13}\int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s < k_6,$$

where $k_8 = k_{13} = h\left(\frac{1}{2}\right) > 0$ and $k_7 = k_{12}$ for $t \geq t_1$.

Let $\mathbb{X}$ be the set of all continuous and bounded functions with the norm $\|x\| = \sup_{t\geq t_1}|x(t)|$. Then $\mathbb{X}$ is a Banach space ([22]). Define a subset Ω of $\mathbb{X}$ such that

$$\Omega := \left\{x \in \mathbb{X} : \quad \frac{1}{2} \leq x(t) \leq 1, \quad t \geq t_1\right\}$$

and an operator $Fx : \mathbb{X} \to \mathbb{X}$ by

$$(Fx)(t) = \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_6 - \int_s^{\infty} q(u) g\left(k_7 + \int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s$$

for $t \geq t_1$. First, for every $x \in \Omega$, $\|x\| = \sup_{t\geq t_1}|x(t)|$, we have $\frac{1}{2} \leq \|x(t)\| \leq 1$ for $t \geq t_1$, which implies Ω is bounded. For showing that Ω is closed, it is enough to show that it includes all limit points. So let x_n be a sequence in Ω converging to x as $n \to \infty$. Then $\frac{1}{2} \leq x_n(t) \leq 1$ for $t \geq t_1$. Taking the limit of x_n as $n \to \infty$, we have $\frac{1}{2} \leq x(t) \leq 1$ for $t \geq t_1$, which implies $x \in \Omega$. Since x_n is any sequence in Ω, it follows that Ω is closed. Now let us show Ω is also convex. For $x_1, x_2 \in \Omega$ and $\alpha \in [0,1]$, we have

$$\frac{1}{2} = \frac{\alpha}{2} + (1-\alpha)\frac{1}{2} \leq \alpha x_1 + (1-\alpha)x_2 \leq \alpha + (1-\alpha) = 1,$$

where $\frac{1}{2} \leq x_1, x_2 \leq 1$, i.e., Ω is convex. Also, because

$$\begin{aligned}\frac{1}{2} \leq (Fx)(t) &\leq \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_6 - \int_s^{\infty} q(u) g\left(k_7 + h\left(\frac{1}{2}\right)\int_u^{\infty} r(\tau)\Delta\tau\right)\Delta u\right)\Delta s \\ &\leq 1,\end{aligned}$$

i.e., $F : \Omega \to \Omega$. Let us now show that F is continuous on Ω. Let $\{x_n\}$ be a sequence in Ω such that $x_n \to x \in \Omega$ as $n \to \infty$. Then

$$\begin{aligned}
&|(Fx_n - Fx)(t)| \\
&\leq \int_{t_1}^{t} p(s) \left| f\left(k_6 - \int_s^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x_n(\tau)) \Delta\tau\right) \Delta u\right) \right. \\
&\left. - f\left(k_6 - \int_s^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x(\tau)) \Delta\tau\right) \Delta u\right) \right| \Delta s.
\end{aligned}$$

Then the continuity of f, g and h and Lebesgue Dominated Convergence theorem imply that F is continuous on Ω. Finally, since

$$(Fx)^\Delta(t) = p(t) f\left(k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x(\tau)) \Delta\tau\right) \Delta u\right) < \infty,$$

we have F is relatively compact by the Mean Value theorem and Arzelà-Ascoli theorem. So, by Theorem 2.6, we have there exists $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. Then by taking the derivative of $\overline{x}$, we obtain

$$\overline{x}^\Delta(t) = p(t) f\left(k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(\overline{x}(\tau)) \Delta\tau\right) \Delta u\right), \quad t \geq t_1.$$

Setting

$$\overline{y}(t) := k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(\overline{x}(\tau)) \Delta\tau\right) \Delta u$$

for $k_6 > 0$ and taking its derivative yields

$$\overline{y}^\Delta(t) = q(t) g\left(k_7 + \int_t^\infty r(\tau) h(\overline{x}(\tau)) \Delta\tau\right), \quad t \geq t_1.$$

Finally, differentiating

$$\overline{z}(t) := k_7 + \int_t^\infty r(\tau) h(\overline{x}(\tau)) \Delta\tau$$

gives

$$\overline{z}^\Delta(t) = -r(t) h(\overline{x}(t)), \quad t \geq t_1.$$

Consequently $(\overline{x}, \overline{y}, \overline{z})$ is a solution of system (3) such that $\overline{x}(t) \to \alpha$, $\overline{y}(t) \to k_6$ and $\overline{z}(t) \to k_7$, where $0 < \alpha < \infty$, i.e., $N^a_{B,B,B} \neq \emptyset$.

The following theorems can be proven very similarly to Theorem 3.11 with appropriate operators. Therefore, the proof is left to the reader, see [17].

Theorem 3.12 We have the following results:

i. Suppose $R(t_0, \infty) < \infty$. If $Y_4 < \infty$ and $Y_8 < \infty$ for $k_7 = k_{12} = 0$ and for all $k_6, k_8, k_{13} > 0$, then $N^a_{B,B,0} \neq \emptyset$.

ii. If both Y_3 and Y_9 are finite for $k_4 = 0$ and for all $k_5, k_{14} > 0$, then $N^+_{\infty,B,0} \neq \emptyset$.

iii. If $Y_3 < \infty$ and $Y_9 < \infty$ for all $k_4, k_5, k_{14} > 0$, then $N^a_{\infty,B,B} \neq \varnothing$.

iv. If $Y_1 < \infty$ and $Y_6 = \infty$ for all $k_1, k_{10} > 0$, then $N^a_{\infty,\infty,B} \neq \varnothing$.

We continue with the case when $z(t)$ converges to 0 while other components $x(t)$ and $y(t)$ of solution (x,y,z) tend to infinity as $t \to \infty$.

Theorem 3.13 Suppose $R(t_0,\infty) < \infty$. If $Y_1 < \infty$ and $Y_5 = Y_8 = \infty$ for all positive constants k_1, k_9, k_{13} and $k_{12} = 0$, then $N^a_{\infty,\infty,0} \neq \varnothing$..

Proof: Suppose $Y_1 < \infty$ and $Y_5 = Y_8 = \infty$ for $k_1, k_9, k_{13} > 0, k_{12} = 0$. Then choose a $t_1 \ge t_0$ such that

$$\int_{t_1}^{\infty} r(t)h\left(\int_{t_1}^{t} p(s)f\left(k_1\int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} p(s)f\left(\int_{t_1}^{s} q(\tau)g\left(k_9\int_{\tau}^{\infty} r(v)\Delta v\right)\Delta\tau\right)\Delta s > 1, \quad t \ge t_1,$$

where $k_1 = g\left(\frac{1}{2}\right)$ and $k_9 = k_{13} = h(1)$. Suppose that Φ is a space of real-valued continuous functions and partially ordered Banach space with $\|y\| = \sup_{t \ge t_1}|y(t)|$ and the usual pointwise ordering $\le$. Let ϕ be a subset of Φ such that

$$\phi := \left\{z \in \Phi : \quad h(1)\int_{t}^{\infty} r(s)\Delta s \le z(t) \le \frac{d_1}{2}, \quad t \ge t_1\right\}.$$

and set an operator $F : \Phi \to \Phi$ such that

$$(Fz)(t) = \int_{t}^{\infty} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

The rest of the proof can be done as in proofs of the previous theorems by using the fact $Y_5 = Y_8 = \infty$, and therefore, $N^a_{\infty,\infty,0} \neq \varnothing$.

3.2.2 Existence in N^b

Assuming (x,y,z) is a nonoscillatory solution of system (3) in N^b such that $x > 0$ eventually and by a similar discussion as in the previous section, and by Lemma 3.10, we have the following subclasses:

$$N^b_{B,0,0} := \left\{(x,y,z) \in N^b : \lim_{t\to\infty}|x(t)| = c_1 \lim_{t\to\infty}|y(t)| = 0, \lim_{t\to\infty}|z(t)| = 0\right\}$$

$$N^b_{0,0,0} := \left\{(x,y,z) \in N^b : \lim_{t\to\infty}|x(t)| = 0, \lim_{t\to\infty}|y(t)| = 0, \lim_{t\to\infty}|z(t)| = 0\right\},$$

where $0 < c_1 < \infty$.

The first result of this section considers the case when each of the component solutions converges.

Theorem 3.14 Suppose $R(t_0,\infty) < \infty$ and f is odd. Then $N^b_{B,0,0} \neq \varnothing$ if $Y_2 < \infty$ and $Y_8 < \infty$ for all $k_3 = k_{13} > 0$ and $k_{12} = 0$.

Proof: Suppose that $Y_2 < \infty$ and $Y_8 < \infty$ for all $k_3 = k_{13} > 0$ and $k_{12} = 0$. Then choose $k_3, k_{13} > 0$ and $t_1 \geq t_0$ sufficiently large such that

$$\int_{t_1}^{\infty} p(t) f\left(\int_t^{\infty} q(s) g\left(k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2},$$

where $k_3 = h\left(\frac{3}{2}\right)$. Let Φ be a partially ordered Banach space of real-valued continuous functions with $\|x\| = \sup_{t \geq t_1} |x(t)|$ and the usual pointwise ordering $\leq$. Let us set a subset ϕ of Φ such that

$$\phi := \left\{x \in \Phi : \quad 1 \leq x(t) \leq \frac{3}{2}, \quad t \geq t_1\right\}$$

and an operator $Fx : \Phi \to \Phi$ by

$$(Fx)(t) = 1 + \int_t^{\infty} p(s) f\left(\int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

One can prove that F is an increasing mapping into itself and $(\Omega, \leq)$ is a complete lattice. Therefore, by Theorem 2.7, there does exist $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. It follows that $\overline{x}(t) > 0$ for $t \geq t_1$ and converges to 1 as t approaches infinity. Also,

$$\overline{x}^{\Delta}(t) = -p(t) f\left(\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Now for $t \geq t_1$, set

$$\overline{y}(t) = -\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u$$

and

$$\overline{z}(t) = \int_t^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau.$$

Then, since f is odd, we have

$$\overline{x}^{\Delta}(t) = p(t) f(\overline{y}(t))$$
$$\overline{y}^{\Delta}(t) = q(t) g(\overline{z}(t))$$
$$\overline{z}^{\Delta}(t) = -r(t) h(\overline{x}(t)).$$

Consequently $(\overline{x}, \overline{y}, \overline{z})$ is a solution of system (3). Since both $\overline{y}(t)$ and $\overline{z}(t)$ converge to 0 as t approaches infinity, $N_{B,0,0}^{b} \neq \emptyset$..

4. Examples

In this section, we provide some examples to highlight our theoretical claims. The following theorem help us evaluate the integrals on a specific time scale, see ([6] Theorem 1.79 (ii)).

Theorem 4.1 Suppose that $[a, b]$ has only isolated points with $a < b$. Then

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t).$$

Example 4.2 Let $\mathbb{T} = 3^{\mathbb{N}}, k_5 = 1 = k_{14}$ and consider the following system

$$\begin{cases} \Delta_3 x(t) = \left(\frac{t}{t-1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t) \\ \Delta_3 y(t) = \dfrac{1}{3t^{\frac{1}{5}}} z^{\frac{3}{5}}(t) \\ \Delta_3 z(t) = -\dfrac{26}{54t^{\frac{21}{5}}} x^{\frac{1}{5}}(t), \end{cases} \tag{12}$$

where

$$\Delta_3 k(t) = \frac{k(\sigma(t)) - k(t)}{\mu(t)} \quad \text{for} \quad \sigma(t) = 3t \quad \text{and} \quad \mu(t) = 2t, \quad t\in\mathbb{T}.$$

First we show $P(t_0,\infty) = Q(t_0,\infty) = \infty$. If $s = 3^m$ and $t = 3^n$, $m,n\in\mathbb{N}$, we have

$$\int_3^\infty p(s)\Delta s = \lim_{t\to\infty}\int_3^t p(s)\Delta s = 2\lim_{n\to\infty}\sum_{s=3}^{\rho(3^n)}\left(\frac{s^4}{s-1}\right)^{\frac{1}{3}} > 2\lim_{n\to\infty}\sum_{m=1}^{n-1} 3^m = \infty.$$

Similarly one can obtain $\int_3^\infty q(s)\Delta s = \infty$.
Now we consider Y_3. With $\tau = 3^m$ and $s = 3^n$, $m,n\in\mathbb{N}$, we have

$$\int_3^s \left(\frac{\tau}{\tau-1}\right)^{\frac{1}{3}}\Delta\tau = 2\sum_{m=1}^{n-1}\left(\frac{3^{4m}}{3^m-1}\right)^{\frac{1}{3}} < 2\sum_{m=1}^{n-1}(3^m)^{\frac{4}{3}}$$

since $3^m - 1 > 1$ on $\mathbb{N}$. We claim that

$$\sum_{m=1}^{n-1}(3^m)^{\frac{4}{3}} < (3^n)^{\frac{4}{3}}.$$

The sum formula for a finite geometric series, $1 - 3^{\frac{4}{3}} < 0$, and.
$\left(3^{\frac{4}{3}}\right)^{1-n} - 1 < 1$ for $n\in\mathbb{N}$ yield

$$0 \le \frac{\left(3^{\frac{4}{3}}\right)^{1-n} - 1}{1 - 3^{\frac{4}{3}}} < 1.$$

So the claim indeed holds, and consequently we have

$$\int_3^s \left(\frac{\tau}{\tau-1}\right)^{\frac{1}{3}}\Delta\tau < 2s^{\frac{4}{3}}. \tag{13}$$

Also, we obtain

$$\int_t^T r(s)h\left(\int_3^s p(\tau)\Delta\tau\right)\Delta s < \int_t^T \frac{26}{54}\frac{1}{s^{\frac{21}{5}}}\left(2s^{\frac{4}{3}}\right)^{\frac{1}{5}}\Delta s = \frac{26\cdot 2^{\frac{6}{5}}}{54}\sum_{s\in[t,T)_{3^{\mathbb{N}}}}\frac{1}{s^{\frac{44}{15}}} < 2\sum_{s\in[t,T)_{3^{\mathbb{N}}}}\frac{1}{s^{\frac{44}{15}}}$$

by (11). Therefore, as $T\to\infty$, we obtain

$$\sum_{s\in[t,\infty)_{3^{\mathbb{N}}}} \frac{1}{s^{\frac{44}{15}}} = \alpha \cdot \frac{1}{t^{\frac{44}{15}}}, \tag{14}$$

where $\alpha = 1 - \frac{1}{3^{\frac{44}{15}}}$. Finally, with $t = 3^m$ and $T = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_{t_0}^{T} q(t)g\left(\int_t^\infty r(s)h\left(\int_{t_0}^s p(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T \frac{1}{t^{\frac{1}{5}}}\left(\frac{1}{t^{\frac{44}{15}}}\right)^{\frac{3}{5}}\Delta t = \frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T \frac{1}{t^{\frac{49}{25}}}\Delta t$$
$$= \frac{(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1} 2\frac{1}{(3^m)^{\frac{49}{25}}}3^m = \frac{2(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1}\left(\frac{1}{3^{\frac{24}{25}}}\right)^m$$

by (12). Since the above integral converges as T approaches infinity, we have $Y_3 < \infty$. By using a similar discussion and (12), it is shown $Y_9 < \infty$. One can also show that $\left(t, 1 - \frac{1}{t}, \frac{1}{t^3}\right)$ is a nonoscillatory solution of system (10). Hence $N^a_{\infty,B,0} \neq \varnothing$ by Theorem 3.12 (ii).

Example 4.3 Let $\mathbb{T} = q^{\mathbb{N}_0}$. Consider the system

$$\begin{cases} x^\Delta(t) = \dfrac{1}{(1+t)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ y^\Delta(t) = \left(\frac{t}{2t-1}\right)^{\frac{1}{5}} z^{\frac{1}{5}}(t) \\ z^\Delta(t) = \dfrac{1}{qt^3} x(t). \end{cases} \tag{15}$$

We show that $N^a_{\infty,\infty,B} \neq \varnothing$ by Theorem 3.5 for $s = q^m, t = q^n, k_1 = 1$ and $t_0 = 1$. So we need to show $P(t_0, \infty) = Q(t_0, \infty) = \infty$ and $Y_1 < \infty$. Indeed,

$$\int_1^T p(t)\Delta t = \sum_{t\in[1,\rho(T)]_{q^{\mathbb{N}_0}}} \frac{1}{(1+t)^{\frac{1}{3}}} \cdot (q-1)t.$$

So as $T \to \infty$, we have

$$P(1, \infty) = (q-1)\sum_{n=0}^{\infty} \frac{q^n}{(1+q^n)^{\frac{1}{3}}} = \infty$$

by the ratio test. We can also easily show $Q(1, \infty) = \infty$. As the final step, let us show $Y_1 < \infty$ holds. Indeed,

$$\int_1^T r(t)h\left(\int_1^t p(s)f\left(\int_1^s q(\tau)\Delta\tau\right)\Delta s\right)\Delta t$$
$$= \int_1^T r(t)h\left(\int_1^t p(s)\left(\sum_{\tau\in[1,\rho(s)]_{q^{\mathbb{N}_0}}} \left(\frac{t}{2t-1}\right)^{\frac{1}{5}} \cdot (q-1)t\right)^{\frac{1}{3}} \Delta s\right)\Delta t$$
$$\leq \int_1^T r(t)h\left(\int_1^t p(s)\cdot s^{\frac{1}{3}}\right)\Delta t = (q-1)\int_1^T r(t)\left(\sum_{s\in[1,\rho(t)]_{q^{\mathbb{N}_0}}} \frac{1}{(1+s)^{\frac{1}{3}}}\cdot s^{\frac{1}{3}}\cdot s\right)\Delta t$$
$$\leq (q-1)\sum_{t\in[1,\rho(T)]_{q^{\mathbb{N}_0}}} \frac{1}{t}.$$

Hence, by the geometric series, and taking the limit of the latter inequality as $T \to \infty$ yield us

$$\sum_{n=0}^{\infty} \frac{1}{q^n} < \infty.$$

Therefore, we have $Y_1 < \infty$. One can also show that $\left(t, 1+t, 2-\frac{1}{t}\right)$ is a solution of system (13) in $N^a_{\infty,\infty,B}$..

Exercise 4.4 Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Show that $\left(1+t, \frac{3t+1}{t}, \frac{-1}{t^2}\right)$ is a solution of

$$\begin{cases} x^{\Delta}(t) = \left(\frac{t}{3t+1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t) \\ y^{\Delta}(t) = \dfrac{1}{2} z(t) \\ z^{\Delta}(t) = \dfrac{3}{4(1+t)t^3} x(t) \end{cases} \tag{16}$$

in N^c such that $x(t) \to \infty, y(t) \to 3$ and $z(t) \to 0$ as $t \to \infty$, i.e., $N^c_{\infty,B,0} \neq \emptyset$ by Theorem 3.8 (i).

5. Conclusion and open problems

In this chapter, we consider a 3D time scale system and show the asymptotic properties of the nonoscillatory solutions along with the existence of such solutions. We are able to show the existence of solutions in most subclasses. On the other hand, it is still an open problem to show the existence in $N^a_{\infty,\infty,\infty}$ for system (3), where $\lambda = 1$. In addition to that, there is one more open problem that also can be considered as a future work, which is to find the criteria for the existence of a nonoscillatory solution in $N^b_{0,0,0}$ of system (3), where $\lambda = -1$.

Another significance of our system that we consider in this chapter is the following system

$$\begin{cases} x^{\Delta}(t) = p(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) \\ y^{\Delta}(t) = q(t)|z(t)|^{\beta} \operatorname{sgn} z(t) \\ z^{\Delta}(t) = -r(t)|x^{\sigma}(t)|^{\gamma} \operatorname{sgn} x^{\sigma}(t), \end{cases} \tag{17}$$

which is known as the third order Emden-Fowler system. Here, p, q and r have the same properties as System (3) and α, β, γ are positive constants. Emden-Fowler equation has a lot of applications in fluid mechanics, astrophysics and gas dynamics. It would be very interesting to investigate the characteristics of solutions because of its potential in applications.

Notes/thanks/other declarations

I would like to dedicate this chapter to my beloved friend Dr. Serdar Çağlak, who always will be remembered as a fighter for his life. Also, I would like to thank to my wife for her tremendous support for writing this chapter.

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References

[1] Hilger S. Ein Maß kettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten [thesis]. Universität Würzburg; 1988.

[2] Adıvar M, Fang S. C. Convex optimization on mixed domains. J. Ind. Manag. Optim. 2012; 8(1), 189–227.

[3] Akin E, Güzey H, Öztürk Ö: Control of wheeled mobile robots on time scales. Differ. Equ. Dyn. Syst. (2019). doi: 10.1007/s12591-019-00492-w.

[4] Atıcı FM, Biles DC, Lebedinsky A: An application of time scales to economics. Mathematical and Computer Modelling 2006; 43(7–8), 718–726.

[5] Berger H, Henrich S, Jessee J, Mikels B, Mullen J, Meyer CK, Beresford D: Application of time scales calculus to the growth and development in populations of stomoxys calcitrans (Diptera: Muscidae). Int. J. Differ. Equ., 2013; 8(2), 125–134.

[6] Bohner M, Peterson A. Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston, 2001. 358 p. DOI: 10.1007/978-1-4612-0201-1.

[7] Bohner M, Peterson A. Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, 2003.348 p. DOI: 10.1007/978-0-8176-8230-9.

[8] Kelley W. G, Peterson A. C. The Theory of Differential Equations: Classical and Qualitative, Springer, 2010.413 p. DOI:10.1007/978-1-4419-5783-2

[9] Kelley W. G, Peterson A. C. Difference Equations, Second Edition: An Introduction with Applications, Academic Press, 2001.403 p.

[10] Agarwal R. P. Difference Equations and Inequalities: Theory, Methods, and Applications, Marcel Dekker, 2000.971 p.

[11] F. Bernis and L.A. Peletier: Two Problems from Draining Flows Involving Third – Order Ordinary Differential Equations. SIAM J. Math. Anal., 1996; 27:2, 515–527.

[12] Kac V, Cheung P. Quantum calculus. Universitext, Springer-Verlag, New York, 2002.112 p. DOI:10.1007/978-1-4613-0071-7

[13] Zeidler E. Nonlinear functional analysis and its applications - I: Fixed point theorems. Springer Verlag New York Inc, 1986.909 p. DOI:10.1007/978-1-4612-4838-5

[14] Knaster B. Un the′or`eme sur les fonctions d'ensembles. Ann. Soc. Polon. Math., 1928; 6, 133–134.

[15] Deng X, Wang Q, and Agarwal R. P: Oscillation and nonoscillation for second order neutral dynamic equations with positive and negative coefficients on time scales. Adv. in Difference Equations, 2014; 2014:115, 1–22.

[16] Öztürk Ö: On the existence of nonoscillatory solutions of three-dimensional time scale systems. Journal of Fixed Point Theory and Applications, 2017; 19:(2017) ,2617–2628. DOI 10.1007/s11784-017-0454-9.

[17] Öztürk Ö, Higgins R: Limit behaviors of nonoscillatory solutions of three-dimensional time scale system. Turkish Journal of Mathematics 2018; 42, 2576–2587. DOI:10.3906/mat-1802-104.

[18] Chanturia T. A: On oscillatory properties of systems of nonlinear ordinary differential equations. TProc I. N. Vekua Inst. Appl. Math. Tbilisi 1983; 14, 163–204 (in Russian).

DOI: http://dx.doi.org/10.5772/intechopen.94921

[19] Schmeidel E: Oscillation of nonlinear three-dimensional difference systems. Mathematica Bohemica, 2010; 135, 163–170.

[20] Schmeidel E: Boundedness of solutions of nonlinear three-dimensional difference systems with delays. Polytechnica Posnaniensis, 2010; 44, 109–113.

[21] Akn E, Došlá Z, Lawrence B: Almost oscillatory three-dimensional dynamical system, Adv. in Difference Equations, 2012;2012:46, 1–14.

[22] Ciarlet P. G: Linear and Nonlinear Functional Analysis with Applications. Siam, 2013, 849 p.