Chapter

Green's Function Method for Electromagnetic and Acoustic Fields in Arbitrarily Inhomogeneous Media

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Abstract

An analytical method based on the Green's function for describing the electromagnetic field, scalar-vector and phase characteristics of the acoustic field in a stationary isotropic and arbitrarily inhomogeneous medium is proposed. The method uses, in the case of an electromagnetic field, the wave equation proposed by the author for the electric vector of the electromagnetic field, which is valid for dielectric and magnetic inhomogeneous media with conductivity. In the case of an acoustic field, the author uses the wave equation proposed by the author for the particle velocity vector and the well-known equation for acoustic pressure in an inhomogeneous stationary medium. The approach used allows one to reduce the problem of solving differential wave equations in an arbitrarily inhomogeneous medium to the problem of taking an integral.

Keywords: inhomogeneous media, Green's function, electromagnetic field, acoustic field, analytical method

1. Introduction

The chapter discusses the procedure for using the Green's function for the analytical description of electromagnetic and acoustic fields in a stationary isotropic and arbitrarily inhomogeneous medium. In the case of the electromagnetic field, the wave equation for the electric vector of the electromagnetic field in the inhomogeneous medium with conductivity, dielectric and magnetic permeability is used. In the case of the acoustic field, the wave equation proposed by the author for the vector of particle velocity and the well-known equation for acoustic pressure in an inhomogeneous stationary medium are used. Using the Green's function and the method of successive approximations makes it possible to achieve the required accuracy of calculating the electric and magnetic vectors of the electromagnetic field, as well as to calculate the vectors of complex intensity and intensity, density of energy, acoustic pressure and the particle velocity vector of the acoustic field in media with arbitrary spatial variability of the parameters. The approach used allows one to reduce the problem of solving differential wave equations in an arbitrarily inhomogeneous medium to integration. The chapter is divided into two parts. At the beginning of each part, the corresponding wave equations are derived and next, a method of using the Green's function is described and analytical expressions describing the fields are formulated. At the beginning we will describe the method as applied to the electromagnetic field, and then as applied to the acoustic field.

Research and modeling of the electromagnetic field in spatially inhomogeneous natural and composite media are actively developing in various fields of science and technology, ranging from systems of underground and underwater electromagnetic communication to photonics, metamaterials and metasurfaces [1]. Such a wide field of scientific research requires methods of mathematical modeling of the properties of the electromagnetic field in media with different spatial scales of conductivity, magnetic and dielectric permittivity. At present, analytical methods are applicable in a very limited range of environments. Among the methods of mathematical modeling of the electromagnetic field in the frequency range from fractions of the hertz to optical, various numerical methods and technologies are used [2, 3]. Numerical modeling uses a variety of methods and technologies, for example, parallel computing which are used in electrodynamic modeling programs. Among them, there are also direct and universal methods for solving boundary problems. The drawback of these methods is a large expenditure of computer resources, which leads to a significant simplification of physical models of the environment and mathematical approximations. There is a third class of methods, in which, at the initial stage, analytical methods are used, for example, the Green's function method, which brings the problem to a form that can be solved by fairly simple numerical methods. Below we will use exactly this approach using the Green's function. Green's function is actively used in a wide range of problems [4–6] of describing electromagnetic and other physical fields in various multilayer, chiral and anisotropic media, including inhomogeneous ones. The proposed procedure is also applicable to media with boundaries and arbitrary dependence on the coordinates of conductivity, magnetic and dielectric permittivity. The source of the field in the environment can be the electric current or an external field. The electric current can be located also inside the medium and outside it. The problem of descriptions the electric vector in an inhomogeneous medium by using the Green's function is formulated as the integral equation with its subsequent solution by the method of successive approximations. This procedure uses the equation for the vector of electric field strength in an inhomogeneous medium, with a certain conductivity, magnetic permeability, and dielectric constant.

The acoustic energy flux density vector (intensity vector), basically, until the beginning of the second half of the 20th century was only of theoretical interest. The second half of the 20th century brought about reliable means of synchronous measurement, practically at a single point, of the acoustic pressure and the components of particle velocity vector necessary to determine the intensity vector of acoustic field [7–14]. However, this did not lead to a significant increase in the number and quality of theoretical research methods and modeling of the intensity vector in an inhomogeneous medium. For the complete theoretical description of the acoustic field, knowledge of its acoustic pressure and the particle velocity vector is required. These two quantities make it possible to find the field of the acoustic intensity vector, to describe the energy and phase structure of the acoustic field. Knowledge of these quantities is useful for solving fundamental and applied problems of acoustic tomography and sounding of the geosphere, applied and fundamental hydroacoustics, creation of acoustic metamaterials, technical and architectural acoustics, noise control, etc. [15–17]. The acoustic pressure $\vec{P}_a\left(\vec{\mathbf{r}},t\right)$ and the particle velocity vector $\vec{\mathrm{V}}\left(\vec{\mathrm{r}},\mathrm{t}\right)$ are interrelated. This connection is obvious for a plane wave and, in the approximation of a continuous medium, has the following form: $\vec{\mathrm{V}}\left(\vec{\mathrm{r}},\mathrm{t}\right) = -\frac{1}{\rho_0\left(\vec{\mathrm{r}}\right)}\int \vec{P}_a\left(\vec{\mathbf{r}},t\right)dt$, where $\rho_0\left(\vec{\mathrm{r}}\right)$ is the density of the medium unperturbed by the acoustic field at the point $\vec{\mathbf{r}}$ and t is time, and ∇ is the Nabla operator. This relationship largely determined the development of the theory of sound as a scalar field of acoustic pressure. Currently, there are several directions for the development of methods of calculation and theoretical analysis of the characteristics of the intensity vector. In the first direction, the relationship between the acoustic pressure and the particle velocity vector is used. This approach is applicable when there are mathematical expressions for the acoustic pressure field. As a rule, this is only possible in a homogeneous medium or for simple waveguides [18]. The second direction requires the use of the continuity equation and the equation of state of the inhomogeneous medium, as well as dynamic equations of motion of elementary volumes or particles of the inhomogeneous medium, for example: the Euler or Navier-Stokes equations. These equations are viewed as a system of equations for determining the pressure and the particle velocity vector. This approach is used to model the propagation of waves in various environments, including plasma and stellar atmospheres [19–22]. These equations are widely known, but to find analytical wave solutions of such systems given an arbitrary dependence of the density and speed of sound on the coordinates is a very difficult task. The use of the acoustic energy transfer equation is the third approach [9]. This approach allows one to describe the energy structure of the acoustic field which makes it possible to study the statistical characteristics of the complex intensity vector in a Gaussian delta-correlated inhomogeneous medium with refraction [9]. It is a very difficult task to find solutions to the transport equation in an inhomogeneous media. In turn, numerical methods for modeling metamaterials and propagation of acoustic waves in a medium are usually limited to specific problems [23–25]. None of the listed approaches, including numerical ones, provides the possibility of a complete theoretical description of the characteristics of the acoustic field and their evolution during field propagation in an arbitrary inhomogeneous medium. One of the promising directions is to use two wave equations in an inhomogeneous medium: equations for the acoustic pressure and equations for the particle velocity vector. We use this very approach. It is based on the proposed by authors wave equation for the particle velocity vector and the well-known equation for acoustic pressure in an inhomogeneous stationary medium. The proposed wave equation for the vector of the particle velocity of the acoustic field in a stationary inhomogeneous and isotropic medium is much more complicated than for the acoustic pressure. This makes it difficult to find the analytical solution for inhomogeneous media with an arbitrary spatial dependence of the density of the medium and the speed of sound in it. However, in an inhomogeneous medium, in which the field of the acoustic intensity vector is weakly vortex, the use of the Green's tensor together with the method of successive approximations makes it possible to find analytical solutions for an arbitrary spatial dependence of the speed of sound and density of the inhomogeneous medium.

2. Electromagnetic field

By an inhomogeneous medium, we mean a medium in which the conductivity $\boldsymbol{\sigma}\left(\vec{\mathbf{r}}\right)$, dielectric $\boldsymbol{\varepsilon}\left(\vec{\mathbf{r}}\right)$ and magnetic $\boldsymbol{\mu}\left(\vec{\mathbf{r}}\right)$ permittivity, and the current density $\vec{\mathbf{J}}\left(\vec{\mathbf{r}}\right)$ have an arbitrary, but differentiable, in the ordinary and in the generalized sense, dependence on coordinates points of the medium. Below we will not point out the explicit dependence of these and other quantities on time and coordinates, where this will not lead to misunderstanding. In an isotropic inhomogeneous medium, ε, μ, σ are scalar functions of coordinates. To derive the wave equation of the electromagnetic field in such a medium, we use the following well known fundamental and material Maxwell equations in a continuous isotropic and stationary medium:

$$\begin{aligned}&1.\ \nabla\times\vec{\mathrm{H}}=\vec{\mathrm{J}}+\frac{\partial\vec{\mathrm{D}}}{\partial \mathrm{t}};\quad 2.\ \nabla\times\vec{\mathrm{E}}=-+\frac{\partial\vec{\mathrm{B}}}{\partial \mathrm{t}};\quad 3.\ \vec{\mathrm{J}}=\sigma\vec{\mathrm{E}};\\&4.\ \nabla\vec{\mathrm{J}}=-\frac{\partial}{\partial \mathrm{t}}(\rho_{\mathrm{f}}+\rho_{\mathrm{ext}});\quad 5.\ \vec{\mathrm{J}}=\vec{\mathrm{J}}_{\mathrm{f}}+\vec{\mathrm{J}}_{\mathrm{ext}};\quad 6.\ \vec{\mathrm{D}}=\varepsilon\vec{\mathrm{E}};\\&7.\ \vec{\mathrm{B}}=\mathrm{u}\vec{\mathrm{H}};\quad 8.\ \varepsilon=\varepsilon_0\varepsilon_{\mathrm{r}};\quad 9.\ \mu=\mu_0\mu_{\mathrm{r}},\end{aligned}\tag{1}$$

where ρ_f is the density of free charges of the medium, and ρ_{ext} is the density of external charges introduced into the medium, $\boldsymbol{\varepsilon_r}$ and $\boldsymbol{\mu_r}$ are the relative dielectric and magnetic permeability of the medium, and $\vec{\mathbf{J}}_{ext}$ is the current density created by free and external charges. The wave equation for the electric vector in an inhomogeneous medium can be obtained, as for a homogeneous medium, excluding the vector of magnetic field strength from the system of Maxwell's equations. For this, we use the well-known vector analysis formulas [26] and take the rotor from the 2nd equation in system (Eq. (1)):

$$\nabla\times\nabla\times\vec{\mathbf{E}}=\nabla\cdot\left(\nabla\cdot\vec{\mathbf{E}}\right)-\Delta\cdot\vec{\mathbf{E}}=-\frac{\partial}{\partial t}\nabla\times\vec{\mathbf{B}}\tag{2}$$

In this case, the source of electromagnetic field is the electric current density $\vec{\mathbf{J}}$, so the divergence of the vector $\nabla\cdot\vec{\mathbf{E}}$, we need to associate with a current density in the medium. For this, we use (Eq. (5)) of the Maxwell system of equations (Eq. (1)) and obtain $\nabla\cdot\vec{\mathrm{E}}=\frac{1}{\sigma}\left[\left(\nabla\cdot\vec{\mathrm{J}}-\vec{\mathrm{E}}(\nabla\cdot\sigma)\right)\right]$. Using the vector analysis formulas, we find the following expression:

$$\begin{aligned}\nabla\left(\nabla\cdot\vec{\mathbf{E}}\right)=\ &\vec{\mathbf{E}}\frac{(\nabla\cdot\sigma)^2}{\sigma^2}-[(\nabla\cdot\ln\sigma)\nabla]\vec{\mathbf{E}}-\left(\vec{\mathbf{E}}\cdot\nabla\right)(\nabla\cdot\ln\sigma)-(\nabla\cdot\ln\sigma)\times\nabla\times\vec{\mathbf{E}}+\\&+\left(\nabla\cdot\vec{\mathbf{J}}\right)\left(\nabla\cdot\frac{1}{\sigma}\right)+\frac{1}{\sigma}\nabla\left(\nabla\cdot\vec{\mathbf{J}}\right)]\end{aligned}\tag{3}$$

The expression for the rotor of the magnetic field induction vector has the form

$$\nabla\times\vec{\mathbf{B}}=\nabla\times\left(\mu\vec{\mathbf{H}}\right)=\mu\left(\nabla\times\vec{\mathbf{H}}\right)+(\nabla\cdot\mu)\times\vec{\mathbf{H}}\tag{4}$$

Using equations 1 and 2 of the system of Maxwell equations and expressions (Eq. (2)) and (Eq. (3)), we find that $-\frac{\partial}{\partial t}\left(\nabla\times\vec{\mathbf{B}}\right)=-\mu\sigma\frac{\partial}{\partial t}\vec{\mathbf{E}}-\mu\frac{\partial}{\partial t}\vec{\mathbf{J}}-\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{\mathbf{E}}+(\nabla\cdot\ln\mu)\times\left(\nabla\times\vec{\mathbf{E}}\right)$, and the desired equation for an electric vector with a field source, in which the charge flux from the volume occupied by the current is not equal to zero, has the following form:

$$\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{\mathbf{E}}+\mu\sigma\frac{\partial}{\partial t}\vec{\mathbf{E}}-\Delta\vec{\mathbf{E}}-(\nabla\cdot\ln\mu\sigma)\times\left(\nabla\times\vec{\mathbf{E}}\right)+\vec{\mathbf{E}}\left(\frac{\nabla\cdot\sigma}{\sigma}\right)^2-[(\nabla\cdot\ln\sigma)\nabla]\vec{\mathbf{E}}-\left(\vec{\mathbf{E}}\cdot\nabla\right)(\nabla\cdot\ln\sigma)=$$
$$=-\mu\frac{\partial}{\partial t}\vec{\mathbf{J}}-\left(\nabla\cdot\vec{\mathbf{J}}\right)\left(\nabla\cdot\frac{1}{\sigma}\right)-\frac{1}{\sigma}\nabla\left(\nabla\cdot\vec{\mathbf{J}}\right) \tag{5}$$

If there is no injection of external charges into the medium, the value $\nabla\cdot\vec{\mathbf{J}}$ is equal to zero and $\nabla\vec{\mathbf{J}}=\nabla\vec{\mathbf{J}}_{ext}=-\frac{\partial}{\partial t}\rho_{ext}$ otherwise. When deriving (Eq. (5)), no conditions on the field frequency were used. Therefore, the equation is valid up to frequencies that correspond to wavelengths λ larger than the sizes of atoms or molecules. The smallness of the ratio of the first and second terms of equation (Eq. (5)) corresponds to the condition of quasi-stationarity of the electromagnetic field. For a monochromatic field with the angular frequency ω, the modulus of their ratio is equal $\frac{\varepsilon}{\sigma}\omega$ and small under conditions of high conductivity, low dielectric constant, or low the angular frequency. In this case, the propagation of the field in the medium will have a predominantly diffusion character and will be described by the following equation:

$$\mu\sigma\frac{\partial}{\partial t}\vec{\mathbf{E}}-\Delta\vec{\mathbf{E}}-(\nabla\cdot\ln\mu\sigma)\times\left(\nabla\times\vec{\mathbf{E}}\right)+\mathbf{E}\left(\frac{\nabla\cdot\sigma}{\sigma}\right)^2-[(\nabla\cdot\ln\sigma)\nabla]\vec{\mathbf{E}}-\left(\vec{\mathbf{E}}\cdot\nabla\right)(\nabla\cdot\ln\sigma)=$$
$$=-\mu\frac{\partial}{\partial t}\vec{\mathbf{J}}-\left(\nabla\cdot\vec{\mathbf{J}}\right)\left(\nabla\cdot\frac{1}{\sigma}\right)-\frac{1}{\sigma}\nabla\left(\nabla\cdot\vec{\mathbf{J}}\right) \tag{6}$$

When $\frac{\varepsilon}{\sigma}\omega>>1$ the field propagation in the media is of the wave-type mainly. At the present time, there are no methods for finding exact solutions of equations of the type (Eq. (5)) and (Eq. (6)). Solutions satisfying a given accuracy can be obtained in two ways. The first is to use numerical methods. The second, which we will follow, consists in passing from the differential equation (5) to the integral equation using the tensor Green's function of the Helmholtz equation for the Fourier - the spectrum of the vector of the electric field strength. The solution to an integral equation can be written in the form of a sequence of approximate solutions, in which each subsequent term is more accurate. It is important that such a procedure for finding a solution is applicable for arbitrary differentiable, both in the usual and in the generalized sense, dependences of σ, ε, and μ on coordinates. For this, we express the vector of the electric field $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)$ and the current density $\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)$ through their Fourier spectra $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)$ and $\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)$

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)e^{i\omega t}d\omega,\ \vec{\mathbf{J}}(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)e^{i\omega t}d\omega. \tag{7}$$

For high frequencies, when the dependence of ε and μ from the field frequency cannot be neglected, but spatial dispersion and nonlinear effects can be neglected, $\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)=\sigma\left(\omega,\vec{r}\right)\vec{\mathbf{E}}\left(\omega,\vec{r}\right)$, $\vec{D}\left(\omega,\vec{r}\right)=\varepsilon\left(\omega,\vec{r}\right)\vec{\mathbf{E}}\left(\omega,\vec{r}\right)$ и $\vec{B}\left(\omega,\vec{r}\right)=\mu\left(\omega,\vec{r}\right)\vec{H}\left(\omega,\vec{r}\right)$. Spatial dispersion plays a minor role in comparison with temporal dispersion and is significant in media with the mean free path of the charge or its diffusion much longer than the field wavelength. Below we will not indicate the dependence of the conductivity and permeability on frequency. Let's consider conductivity and permittivity as a sum of a constant and a space-dependent variable:

$$\sigma\left(\vec{\mathbf{r}}\right) = \sigma_c + \sigma_1\left(\vec{\mathbf{r}}\right), \mu\left(\vec{\mathbf{r}}\right) = \mu_c + \mu_1\left(\vec{\mathbf{r}}\right), \varepsilon\left(\vec{\mathbf{r}}\right) = \varepsilon_c + \varepsilon_1\left(\vec{\mathbf{r}}\right) \tag{8}$$

Let's introduce the following notations:

$$\left(\omega^2\varepsilon\left(\vec{r}\right)\mu\left(\vec{r}\right) - i\omega\mu\left(\vec{r}\right)\sigma\left(\vec{r}\right) - \left(\frac{\nabla\cdot\sigma\left(\vec{r}\right)}{\sigma\left(\vec{r}\right)}\right)^2\right) = k^2\left(\vec{r},\omega\right),$$

$$\vec{\mathbf{f}}_{ext}\left(\omega,\vec{\mathbf{r}}\right) = i\omega\mu\left(\vec{\mathbf{r}}\right)\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right) + \left(\nabla\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)\right)\left(\frac{\nabla\sigma_1\left(\vec{\mathbf{r}}\right)}{\sigma^2\left(\vec{\mathbf{r}}\right)}\right) + \frac{1}{\sigma\left(\vec{\mathbf{r}}\right)}\nabla\left(\nabla\cdot\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)\right), \tag{9}$$

$\vec{\vec{f}}\left(\omega,\vec{r}\right) = -\left(\frac{\nabla\mu_1\left(\vec{r}\right)}{\mu\left(\vec{r}\right)} + \frac{\nabla\sigma_1\left(\vec{r}\right)}{\sigma\left(\vec{r}\right)}\right)\times\left(\nabla\times\vec{E}\left(\vec{r},\omega\right)\right) - \left(\frac{\nabla\sigma_1\left(\vec{r}\right)}{\sigma\left(\vec{r}\right)}\nabla\right)\vec{E}\left(\vec{r},\omega\right) - \left(\vec{E}\left(\vec{r},\omega\right)\cdot\nabla\right)\left(\frac{\nabla\sigma_1\left(\vec{r}\right)}{\sigma\left(\vec{r}\right)}\right)$. Substituting expressions (Eq. (8)) and (Eq. (9)) into equation (Eq. (5)), we arrive at the following equation:

$$\Delta\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right) + k^2\left(\vec{\mathbf{r}},\omega\right)\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right) = \vec{\mathbf{f}}\left(\vec{\mathbf{r}},\omega\right) + \vec{\mathbf{f}}_{ext}\left(\vec{\mathbf{r}},\omega\right) \tag{10}$$

Eq. (10) must be supplemented with boundary conditions. In an inhomogeneous medium, the interface between the media can be considered as an inhomogeneity with its dependence on coordinates, described by the corresponding functions, for example: Heaviside step function, etc. Therefore, the boundary conditions will be the conditions at infinity, where the field and its divergence must be equal to zero. In Eq. (10) the field source is not only the external currents (term $\vec{\mathbf{f}}_{ext}\left(\omega,\vec{\mathbf{r}}\right)$ but also the heterogeneity of the environment). These sources are described by $\vec{\mathbf{f}}\left(\omega,\vec{\mathbf{r}}\right)$. At present, there are no methods for the analytical solution of equations similar to (10) with an arbitrary dependence of the term $\vec{\mathbf{f}}\left(\omega,\vec{\mathbf{r}}\right)$ on coordinates. Nevertheless, using the Green's functions of the vector Helmholtz equation in a homogeneous isotropic medium we can reformulate (10) into the integral with respect to the vector $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)$, the solution of which can be found in an iterative way, for example, by the method of successive approximations.

Using (Eq. (8)) one can formulate $k^2\left(\vec{\mathbf{r}},\omega\right)$ as the sum of independent on coordinates $\mathbf{k}_{\mathbf{c}}^2$ function on coordinates and $\mathbf{k}_1^2\left(\vec{\mathbf{r}},\boldsymbol{\omega}\right)$

$$\begin{aligned} k^2\left(\vec{r}\right) &= \left(\omega^2\varepsilon_c\mu_c - i\omega\mu_c\sigma_c\right) + \omega^2\left(\varepsilon_c\mu_1\left(\vec{r}\right) + \varepsilon_1\left(\vec{r}\right)\mu_c + \varepsilon_1\left(\vec{r}\right)\mu\left(\vec{r}\right)\right) \\ &-i\omega\left(\sigma_c\mu_1\left(\vec{r}\right) + \sigma_1\left(\vec{r}\right)\mu_c + \sigma_1\left(\vec{r}\right)\mu\left(\vec{r}\right)\right) - \left(\frac{\nabla\cdot\sigma_1\left(\vec{r}\right)}{\sigma\left(\vec{r}\right)}\right)^2 = k_c^2 + k_1^2(r) \end{aligned} \tag{11}$$

Thus equation (Eq. (10)) can be written as:

$$\Delta\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)+k_c^2\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=\vec{\mathbf{f}}_l\left(\omega,\vec{\mathbf{r}}\right)+\vec{\mathbf{f}}_{ext}\left(\omega,\vec{\mathbf{r}}\right), \tag{12}$$

where $\vec{\mathrm{f}}_1\left(\omega,\vec{\mathrm{r}}\right)=\vec{\mathrm{f}}\left(\omega,\vec{\mathrm{r}}\right)-\mathrm{k}_1^2\left(\vec{\mathrm{r}},\omega\right)\vec{\mathrm{E}}\left(\vec{\mathrm{r}},\omega\right)$.

Formally, we can consider Eq. (12) as an inhomogeneous Helmholtz equation and using the Green's function for it, we can rewrite Eq. (12) in the form of an integral equation. For vector fields, the Green's function [7–10] is a tensor of the second rank. In an orthogonal coordinate system, Eq. (5) decomposes into a system of three scalar equations for the projections of the field E (r, ω) on the coordinate axis. This simplifies the form of the Green's tensor and it has only diagonal elements that are not equal to zero. It can be represented as the vector

$\vec{G}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)=\sum_{i=1}^3\vec{n}_iG_i\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)$, where $G_i\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)$ is the components of which are the Green's functions of the one-dimensional Helmholtz equation and $\mathbf{n}_i$ are the unit vectors of the coordinate axes. Let the area Ω in which we describe the field be large enough so that on its borders the field and its derivatives can be equated to zero. Using the Green tensor, we can rewrite Eq. (5) for the electric vector at the point $\vec{r}\in\Omega$ of the in the form of the following integral equation

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=\sum_{i=1}^{3}\mathbf{n}_i\int_{\Omega}G_i\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)\left[\mathbf{f}^i{}_{ext}\left(\omega,\vec{\mathbf{r}}_1\right)+\mathbf{f}^i{}_1\left(\omega,\vec{\mathbf{r}}_1\right)\right]d\vec{\mathbf{r}}_1 \tag{13}$$

where $\left(\vec{\mathbf{r}},\ \vec{\mathbf{r}}_1\right)\in\Omega$, $\mathbf{f}^i{}_{ext}(\omega,\mathbf{r}_1)$ and $\mathbf{f}^i{}_1(\omega,\mathbf{r}_1)$ are the projections of vectors $\vec{\mathbf{f}}_{ext}\left(\omega,\vec{\mathbf{r}}\right)$ and $\vec{\mathbf{f}}_1\left(\omega,\vec{\mathbf{r}}\right)$ on to the coordinate axes. Integration is performed over the volume occupied by inhomogeneities, which are secondary sources of the field. In practice, the volume should be chosen such that secondary and higher order sources make a noticeable contribution to the field. Due to the rapid decrease in the amplitude of the Green's function and, especially with a strong absorption of the electromagnetic field by the medium, the region of integration can be about 1/α where α is the absorption coefficient of the field.

The steps for finding $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)$ by the method of successive approximations can be as follows. We find the zeroth approximation $\vec{\mathbf{E}}_0\left(\vec{\mathbf{r}},\omega\right)$ for the field, which is valid in a homogeneous medium with parameters σ_c, μ_c, ε_c

$$\vec{\mathbf{E}}_0\left(\vec{\mathbf{r}},\omega\right)=\sum_{i-1}^{3}\vec{\mathbf{n}}_i\int_{\Omega_1}G\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)\left[\left(\vec{\mathbf{n}}_i\cdot\vec{\mathbf{f}}_{ext}\left(\omega,\vec{\mathbf{r}}_1\right)\right)\right]d\vec{\mathbf{r}}_1. \tag{14}$$

Integration is performed over the volume Ω_1 occupied by the external current (primary source of the field). This solution describes the primary field created by an external current. Using obtained by Eq. (14) expression $\vec{\mathbf{E}}_0\left(\vec{\mathbf{r}},\omega\right)$ and expression (9), we find $\mathbf{f}^i{}_1\left(\omega,\vec{\mathbf{r}}_1\right)$. Using (Eq. (13)) and integrating, we obtain a more accurate first $\vec{\mathbf{E}}_1\left(\vec{\mathbf{r}},\omega\right)$ approximation for $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)$, which takes into account the influence of medium inhomogeneities on the field. To find the second approximation, it is necessary to substitute $\vec{\mathbf{E}}_1\left(\vec{\mathbf{r}},\omega\right)$ into $\mathbf{f}^i{}_1\left(\omega,\vec{\mathbf{r}}_1\right)$ and using (Eq. (13)) to obtain the second more accurate approximation. Similarly, more accurate solutions are obtained that take into account multiple field scattering by medium inhomogeneities. At these stages, the integration is performed over the volume occupied by inhomogeneities, which are secondary sources of the field. If the source of the field in an inhomogeneous medium is an external field with an electric vector $\vec{\mathbf{E}}_{ext}\left(\vec{\mathbf{r}},\omega\right)$ it should be used as the vector $\vec{\mathbf{E}}_0\left(\vec{\mathbf{r}},\omega\right)$.

For determining the magnetic field component one uses Maxwell's equations and writes them in terms of magnetic and electric fields Fourier spectrums: $\nabla\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=-\omega\vec{B}\left(\vec{r},\omega\right)$. Substituting (Eq. (13)) in this equation one obtains

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)=i\frac{1}{\omega\mu\left(\vec{\mathbf{r}}\right)}\int_{\Omega}\sum_{i=1}^{3}\vec{\mathbf{n}}_i\times\nabla G\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1\right)\left[\mathbf{f}^i{}_{ext}\left(\omega,\vec{\mathbf{r}}_1\right)+\mathbf{f}^i{}_1\left(\omega,\vec{\mathbf{r}}_1\right)\right]d\vec{\mathbf{r}}_1. \quad (14a)$$

Using the Green's function, as the experience of its use shows (e.g. [7]) in such tasks, significantly reduces the requirements for computing resources and reduces the computation time. Note that the proposed procedure can be effective in simulating the optical properties of metamaterials, nanocomposites, and nanostructures.

3. Acoustic field

The wave equation for acoustic pressure $\mathrm{P}_a\left(\vec{r},\mathrm{t}\right)$ in a continuous inhomogeneous motionless and stationary medium is well known [16, 17]

$$\frac{1}{\mathrm{c}^2\left(\vec{r}\right)}\frac{\partial^2}{\partial\mathrm{t}^2}\mathrm{P}_a\left(\vec{r},\mathrm{t}\right)+\Delta\mathrm{P}_a\left(\vec{r},\mathrm{t}\right)+\left[\nabla\mathrm{P}_a\left(\vec{r},\mathrm{t}\right)+\vec{\mathbf{f}}\left(\vec{r},\mathrm{t}\right)\right]\nabla\ln\rho_0\left(\vec{r}\right)=\nabla\vec{f}\left(\vec{r},\mathrm{t}\right). \quad (15)$$

where $\vec{f}\left(\vec{r},\mathrm{t}\right)$ is the density of volumetric external forces that are the source of the acoustic field.

Eq. (1) is obtained by excluding the particle velocity vector from the linearized Euler equations, continuity and state of the medium. If we exclude acoustic pressure from these equations, then we get the equation for the vector of the particle velocity of the acoustic field. For this, we differentiate the equation of state in taking into account the smallness of the acoustic pressure, perturbation of the density of the medium by the $\rho_a\left(\vec{r},\mathrm{t}\right)$ in comparison with the background values $\rho_0\left(\vec{r}\right)$ and $\mathrm{P}_0\left(\vec{r}\right)$ the medium. In the inhomogeneous medium, the equation of state $\mathrm{P}_c\left[\rho\left(\vec{r},\mathrm{t}\right)\right]$ describes the relationship of the instantaneous local value of pressure and density of the medium. Therefore, it is necessary to use the total time derivative when differentiating the equation of state. Using it, we find in the linear approximation

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{P}_c\left[\rho\left(\vec{r},\mathrm{t}\right)\right]=\mathrm{c}^2\left(\vec{r},\right)\left[\frac{\partial}{\partial\mathrm{t}}\rho_a\left(\vec{r},\mathrm{t}\right)+\nabla\rho_0\left(\vec{r},\right)\vec{V}\left(\vec{r},\mathrm{t}\right)\right], \quad (16)$$

where $C\left(\vec{r},\right)=\sqrt{\frac{\partial \mathrm{P}_c\left[\rho\left(\vec{r},\mathrm{t}\right)\right]}{\partial\rho\left(\vec{r},\mathrm{t}\right)}}$ is the local phase speed of sound for the acoustic pressure wave. We used the expansion $\rho\left(\vec{r},\mathrm{t}\right)=\rho_0\left(\vec{r},t\right)+\rho_a\left(\vec{r},\mathrm{t}\right)$ and the condition $\nabla\rho_a\left(\vec{r},\mathrm{t}\right)V\left(\vec{r},\mathrm{t}\right)<<\nabla\rho_0\left(\vec{r}\right)V\left(\vec{r},\mathrm{t}\right)$. With the help of expression (2) and representation, $\mathrm{P}_c\left[\rho\left(\vec{r},\mathrm{t}\right)\right]=\mathrm{P}_0\left(\vec{r}\right)+\mathrm{P}_a\left(\vec{r},\mathrm{t}\right)$ the equation of continuity$\frac{\partial}{\partial \mathrm{t}}\rho\left(\vec{r},\mathrm{t}\right)+\nabla\left[\rho\left(\vec{r},\mathrm{t}\right)\vec{V}\left(\vec{r},\mathrm{t}\right)\right]=0$ is reduced to a linearized form

$$\frac{1}{\rho_0\left(\vec{r}\right)}\frac{1}{\mathrm{c}^2\left(\vec{r}\right)}\frac{\partial}{\partial \mathrm{t}}\mathrm{P}\left(\vec{r},\mathrm{t}\right)+\nabla V\left(\vec{r},\mathrm{t}\right)=0 \tag{17}$$

In the inhomogeneity of the medium $\nabla\vec{V}\left(\vec{r},\mathrm{t}\right)\neq 0$, even in the approximation of an incompressible medium. Let us Take the time derivative on the linearized Euler equation $\rho_0\left(\vec{r}\right)\frac{\partial}{\partial \mathrm{t}}\vec{V}\left(\vec{r},\mathrm{t}\right)+\nabla\mathrm{P}\left(\vec{r},\mathrm{t}\right)+\vec{f}\left(\vec{r},\mathrm{t}\right)=0$ and take the gradient of the equation of continuity (Eq. (17)). We exclude the acoustic pressure from the obtained expressions and find the equation for the particle velocity vector

$$\begin{aligned}&\frac{1}{\mathrm{c}^2\left(\vec{r}\right)}\frac{\partial^2}{\partial \mathrm{t}^2}\vec{V}\left(\vec{r},\mathrm{t}\right)-\Delta\vec{V}\left(\vec{r},\mathrm{t}\right)-\nabla\ln\left[\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)\right]\nabla\vec{V}\left(\vec{r},\mathrm{t}\right)-\nabla\times\nabla\times\vec{V}\left(\vec{r},\mathrm{t}\right)=\\&=-\frac{1}{\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)}\frac{\partial}{\partial \mathrm{t}}\vec{f}\left(\vec{r},\mathrm{t}\right)\end{aligned} \tag{18}$$

When $\vec{f}\left(\vec{r},\mathrm{t}\right)=0$ then $\nabla\times\nabla\times\vec{V}\left(\vec{r},\mathrm{t}\right)=-\nabla\times\left(\left(\nabla\ln\rho_0\left(\vec{r}\right)\right)\times\vec{V}\left(\vec{r},\mathrm{t}\right)\right)$ and can transformed equation (Eq. (18)) to the following form, which is valid in the absence of external forces

$$\begin{aligned}&\frac{1}{\mathrm{c}^2\left(\vec{r}\right)}\frac{\partial^2}{\partial \mathrm{t}^2}\vec{V}\left(\vec{r},\mathrm{t}\right)-\Delta\vec{V}\left(\vec{r},\mathrm{t}\right)-\nabla\ln\left[\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)\right]\nabla\vec{V}\left(\vec{r},\mathrm{t}\right)+\\&+\nabla\times\left(\left(\nabla\ln\rho_0\left(\vec{r}\right)\right)\times\vec{V}\left(\vec{r},\mathrm{t}\right)\right)=0\end{aligned} \tag{19}$$

Eqs. (18) and (19) are much more complicated than equation (Eq. (15)) due to the third and fourth vortex term. The estimate of their ratio is

$$\left|\frac{\nabla\ln\left(\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)\right)\nabla\vec{\mathbf{V}}\left(\vec{r},\mathrm{t}\right)}{\nabla\times\left(\left(\nabla\ln\rho_0\left(\vec{r}\right)\right)\times\vec{V}\left(\vec{r},\mathrm{t}\right)\right)}\right|\sim\left|\frac{\nabla\ln\left(\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)\right)}{\nabla\ln\rho_0\left(\vec{r}\right)}\right| \tag{20}$$

In areas of the medium where this ratio is greater than unity, the fourth term in equations (Eqs. (18) and (19)) can be neglected. As a rule, these are media with a large relative gradient of the speed of sound. One of the examples of such media can be the marine environment, in which the local gradient of the speed of sound is determined less by the change in water density than salinity and temperature [12, 24]. Directly near the surface and the ocean floor or the interface between the media, the relative density gradient of the medium can be large. In these regions, the field of the particle velocity vector and acoustic intensity can have a significant rotational (vortex) component.

At present, the solution of these equations is possible only by numerical methods. If the fourth term in the equations is small, the equations for the vector of particle velocity and acoustic pressure allows one to find analytical expressions connecting the phases and moduli of vector of complex intensity and particle velocity vector, pressure, density of acoustic energy with the density of the medium and the speed of sound in it. Let us do it for equation 19. Using both scalar function $\Psi\left(\vec{r},\mathrm{t}\right)$ and vector $\vec{U}\left(\vec{r},\mathrm{t}\right)$

$$P\left(\vec{r},\mathrm{t}\right)=\sqrt{\frac{\mathrm{Z_p}\left(\vec{r}\right)}{\mathrm{Z_p^0}}}\Psi\left(\vec{r},\mathrm{t}\right)\ \vec{V}\left(\vec{r},\mathrm{t}\right)=\sqrt{\frac{\mathrm{Z}_v\left(\vec{r}\right)}{\mathrm{Z}_v^0}}\vec{U}\left(\overrightarrow{r'},\mathrm{t}\right), \tag{21}$$

we rewrite equations (Eqs. (15) and (19)) in the following form:

$$\frac{1}{\mathrm{c}^2(r)}\frac{\partial^2}{\partial \mathrm{t}^2}\Psi\left(\vec{r},\mathrm{t}\right)-\Delta\Psi\left(\vec{r},\mathrm{t}\right)+\left[\frac{3\nabla \mathrm{Z_p}\left(\vec{r}\right)\nabla \mathrm{Z_p}\left(\vec{r}\right)}{4\mathrm{Z_p^2}\left(\vec{r}\right)}-\frac{\Delta \mathrm{Z_p}\left(\vec{r}\right)}{2\mathrm{Z_p}\left(\vec{r}\right)}\right]\Psi\left(\vec{r},\mathrm{t}\right)=0 \tag{22}$$

$$\frac{1}{\mathrm{c}^2(r)}\frac{\partial^2}{\partial \mathrm{t}^2}\vec{U}\left(\vec{r},\mathrm{t}\right)-\Delta\vec{U}\left(\vec{r},\mathrm{t}\right)+\left[\frac{3\nabla \mathrm{Z}_v\left(\vec{r}\right)\nabla \mathrm{Z}_v\left(\vec{r}\right)}{4\mathrm{Z}_v^2\left(\vec{r}\right)}-\frac{\Delta \mathrm{Z}_v\left(\vec{r}\right)}{2\mathrm{Z}_v\left(\vec{r}\right)}\right]\vec{U}\left(\vec{r},\mathrm{t}\right)=0, \tag{23}$$

where $\mathrm{Z_p}\left(\vec{r}\right)=\rho_0\left(\vec{r}\right)$ and $\mathrm{Z}_v\left(\vec{r}\right)=\frac{1}{\rho_0\left(\vec{r}\right)\mathrm{c}^2\left(\vec{r}\right)}$, and $\mathrm{Z_p^0}=\rho_0\left(\vec{r}_0\right)$, $\mathrm{Z}_v^0=\frac{1}{\rho_0\left(\vec{r}_0\right)\mathrm{c}^2\left(\vec{r}_0\right)}$ are values $\mathrm{Z}_p\left(\vec{r}\right)$ and $\mathrm{Z}_v\left(\vec{r}\right)$ at some point in space $\vec{r}_0$. For the spectral components $\Psi\left(\vec{r},\omega\right)$ and $\vec{U}\left(\vec{r},\omega\right)$ using the Fourier transform of equations (Eqs. (22) and (23)) with respect to the time variable, we obtain the following equations

$$\Delta\Psi\left(\vec{r},\omega\right)+\mathrm{k}_\psi^2\left(\vec{r}\right)\Psi\left(\vec{r},\omega\right)=0, \tag{24}$$

$$\Delta\vec{U}\left(\vec{r},\omega\right)+\mathrm{k}_U^2\left(\vec{r}\right)\vec{U}\left(\vec{r},\omega\right)=0, \tag{25}$$

where $\mathrm{k}_\psi^2\left(\vec{r}\right)=\frac{\omega^2}{\mathrm{c}^2\left(\vec{r}\right)}-\frac{3}{4}\left[\frac{\nabla\rho_0\left(\vec{r}\right)}{\rho_0\left(\vec{r}\right)}\right]^2+\frac{\Delta\rho_0\left(\vec{r}\right)}{2\rho_0\left(\vec{r}\right)}$

$$\begin{gathered}\mathrm{k}_U^2\left(\vec{r}\right)=\frac{\omega^2}{\mathrm{c}^2\left(\vec{r}\right)}+\frac{5}{4}\left[\frac{\nabla\rho_0\left(\vec{r}\right)}{\rho_0\left(\vec{r}\right)}\right]^2+\frac{\nabla\rho_0\left(\vec{r}\right)\nabla\mathrm{c}\left(\vec{r}\right)}{\rho_0\left(\vec{r}\right)\mathrm{c}\left(\vec{r}\right)}\\+3\left[\frac{\nabla\mathrm{c}\left(\vec{r}\right)}{\mathrm{c}\left(\vec{r}\right)}\right]^2-\frac{\Delta\rho_0\left(\vec{r}\right)}{\rho_0\left(\vec{r}\right)}-2\frac{\Delta\mathrm{c}\left(\vec{r}\right)}{\mathrm{c}\left(\vec{r}\right)}\end{gathered} \tag{26}$$

From expressions (Eq. (26)) it follows that the gradient of the speed of sound affects only the vibrational speed in a medium with a small swirl of the particle velocity field. The acoustic pressure depends only on the gradient of the density of the medium and does not depend on the gradient of the speed of sound. This situation is valid for media in which the density gradient of the medium is less than the gradient of the speed of sound. These differences form the phase difference between the acoustic pressure and the vibrational velocity vector during the propagation of an acoustic wave in an inhomogeneous medium. Different field reactions $\vec{V}\left(\vec{r},\mathrm{t}\right)$ and $P\left(\vec{r},\mathrm{t}\right)$ to the density gradient of the medium and the gradient of the sound speed form the phase difference between the acoustic pressure $\Phi_p\left(\vec{\mathrm{r}},\mathrm{t}\right)$ and the particle velocity $\Phi_v\left(\vec{\mathrm{r}},\mathrm{t}\right)$ vector during the propagation of an acoustic wave in an inhomogeneous medium. Solutions to equations (Eqs. (24) and (25)) can be found using the method of successive approximations. For this, we represent these equations in the following form:

$$\Delta\Psi\left(\vec{r},\omega\right)+\mathrm{k}_0^2\Psi\left(\vec{r},\omega\right)=\mathrm{k}_{1\Psi}^{\ 2}\left(\vec{r}\right)\Psi\left(\vec{r},\omega\right) \tag{27}$$

$$\Delta\vec{U}\left(\vec{r},\omega\right)+\mathrm{k}_0^2\vec{U}\left(\vec{r},\omega\right)=\mathrm{k}_{1_v}^{\ 2}\left(\vec{r}\right)\overleftarrow{U}\left(\vec{r},\omega\right) \tag{28}$$

where $\mathrm{k}_0^2=\frac{\omega^2}{\mathrm{c}^2\left(\vec{r}_0\right)}$, $\mathrm{k}_{1\Psi}^2\left(\vec{r}\right)=\mathrm{k}_0^2-\mathrm{k}_\psi^2\left(\vec{r}\right)$ and $\mathrm{k}_{1U}^2\left(\vec{r}\right)=\mathrm{k}_0^2-\mathrm{k}_U^2\left(\vec{r}\right)$

Similarly to the case of an electromagnetic field in the inhomogeneous medium, using the scalar $\mathrm{G}\left(\vec{r}-\vec{r}_1\right)$ and vector $\vec{G}\left(\vec{r}-\vec{r}_1\right)$ Green's functions of the Helmholtz equation for a homogeneous unbounded medium. Eqs. (2) and (13) and Eq. (2.14) can be rewritten in the form of the following integral equations:

$$\Psi\left(\vec{r},\omega\right)=\Psi_0\left(\vec{r},\omega\right)+\int\limits_\Omega \mathrm{G}\left(\vec{r}-\vec{r}_1\right)\mathrm{k}_{1\Psi}^2\left(\vec{r}_1\right)\Psi\left(\vec{r}_1,\omega\right)d\vec{r}_1 \tag{29}$$

$$U_i\left(\vec{r},\omega\right)=U_{0i}\left(\vec{r},\omega\right)+\int\limits_\Omega G_i\left(\vec{r}-\vec{r}_1\right)\mathrm{k}_{1U}^2\left(\vec{r}_1\right)U_i\left(\vec{r}_1,\omega\right)d\vec{r}_1 \tag{30}$$

Here $U_i\left(\vec{r}_1,\omega\right)$ is the projections of the vector onto the coordinate axes and $\Psi_0\left(\vec{r},\omega\right)$ and $U_{0i}\left(\vec{r},\omega\right)$ are the solutions of equations (Eq. (27)) and (Eq. (28)) with the right-hand side equal to zero, and $G_i\left(\vec{r}-\vec{r}_1\right)$ are the components of the vector Green's function. The steps for finding a solution to equations (2.15) and (2.16) by the method of successive refinements can be as follows:

1. We find $\Psi_0\left(\vec{r},\omega\right)$ and $U_{0i}\left(\vec{r},\omega\right)$ which are the zeroth approximation for the field, valid in a homogeneous medium

2. We find an explicit form of dependence $\mathrm{k}_{1\Psi}^2\left(\vec{r}\right)=\mathrm{k}_0^2-\mathrm{k}_\psi^2\left(\vec{r}\right)$ and $\mathrm{k}_{1U}^2\left(\vec{r}\right)=\mathrm{k}_0^2-\mathrm{k}_U^2\left(\vec{r}\right)$ on coordinates

3. Using $\Psi_0\left(\vec{r},\omega\right)$ and $U_{0i}\left(\vec{r},\omega\right)$ and expressions for $\mathrm{k}_{1\Psi}^2\left(\vec{r}\right)$ and $\mathrm{k}_{1U}^2\left(\vec{r}\right)$with the help of (Eqs. (29) and (30)), we obtain more accurate first approximations that take into account single scattering of the primary field.

4. To obtain the second more accurate approximation, it is necessary to substitute the first approximations in expressions (Eqs. (29) and (30)) and obtain the second more accurate approximation of the solution.

Similarly, you can get solutions that are more accurate. Integration is performed over the volume of the inhomogeneous medium, the inhomogeneities of which will be secondary, etc. field sources. In a real situation, the volume should be chosen such that its secondary sources make a noticeable contribution to the field.

Let us consider an example that shows how the parameters of an inhomogeneous medium affect the characteristics of the acoustic field. We represent the acoustic pressure, and the vector of the particle velocity of the monochromatic acoustic field by the frequency ω in the following form

$P\left(\vec{r},t\right) = P_0\left(\vec{r}\right) \exp i\left[\omega t - \Phi_p\left(\vec{r}\right)\right]$ and $\vec{\mathbf{V}}\left(\vec{r},t\right) = \vec{\mathbf{V}}_0\left(\vec{r}\right) \exp i\left[\omega t - \Phi_v\left(\vec{r}\right)\right]$. Complex intensity vector will be written as $\vec{\mathbf{I}}\left(\vec{r}\right) = P\left(\vec{r},t\right)\vec{\mathbf{V}}^*\left(\vec{r},t\right) = \vec{\mathbf{I}}_0\left(\vec{r}\right) \exp i\left[\Phi_v\left(\vec{r}\right) - \Phi_p\left(\vec{r}\right)\right]$. In a medium without absorption of the acoustic energy, the phases $\Phi_p\left(\vec{r}\right)$ and $\Phi_v\left(\vec{r}\right)$, respectively are equal to the phases $\Psi\left(\vec{r},\mathrm{t}\right)$ and $\vec{U}\left(\vec{r},\mathrm{t}\right)$. The wave vector of a wave is normal to its phase surface and is determined by the wave phase gradient and the wave number by the modulus of this gradient. For the wavenumbers of the pressure $\mathrm{k}_p\left(\vec{r}\right)$ and the particle velocity vector $\mathrm{k}_v\left(\vec{r}\right)$, we can take, respectively, the quantities $k_\psi\left(\vec{r}\right)$ and $k_U\left(\vec{r}\right)$ if the inequalities $\left|\frac{\Delta\psi_0\left(\vec{r}\right)}{\psi_0\left(\vec{r}\right)}\right| < < \left|k_\psi^2\left(\vec{r}\right) - \left(\nabla\Phi_p\left(\vec{r}\right)\right)^2\right|$ and $\left|\frac{\Delta\vec{U}_0\left(\vec{r}\right)}{U_0\left(\vec{r}\right)}\right| < < \left|\mathrm{k}_U^2\left(\vec{r}\right) - \left(\nabla\Phi_v\left(\vec{r}\right)\right)^2\right|$. The refractive indices of the medium for the acoustic pressure and the particle velocity relative to the point $\vec{r}_0$ are different and accordingly, equal $n_p\left(\vec{r}\right) = \frac{k_p\left(\vec{r}\right)}{k_0}$ and $n_v\left(\vec{r}\right) = \frac{k_v\left(\vec{r}\right)}{k_0}$, where $\mathrm{k}_0 = \frac{\omega}{\mathrm{c}\left(\vec{r}_0\right)}$ and $C\left(\vec{r}_0\right)$ is the phase velocity of sound for the acoustic pressure wave at the point $\vec{r}_0$. The phase velocities of the acoustic pressure wave and the particle velocity vector become different, which leads to the inequality of the phases of the acoustic pressure and the particle velocity vector when the acoustic wave propagates in an inhomogeneous medium. The absorption of acoustic energy by the medium is taken into account by assuming the speed of sound and the density of the medium to be complex quantities, in which the imaginary part is responsible for the absorption of the energy of the acoustic field. In this case, the phases and acquire an additive equal to the phases of the values $\sqrt{\frac{\mathrm{Z}_p\left(\vec{r}\right)}{\mathrm{Z}_p^0}}$ and $\sqrt{\frac{\mathrm{Z}_v\left(\vec{r}\right)}{\mathrm{Z}_v^0}}$. To avoid cumbersome expressions, we restrict ourselves to the first approximation. The proposed example is often implemented in real measurements of the characteristics of the acoustic field. In practice, as a rule, the projections of the vibrational velocity vector and, accordingly, the intensity vector are measured in an orthogonal coordinate system, for example, in a Cartesian one. Consider the projection of a vector $\vec{U}\left(\vec{r},\omega\right)$ on the OX axis. In this case, the projection will be a function of only the x coordinate, and the Y and Z coordinates will act as parameters and determine the straight line parallel to the OX axis, along which the observation point x changes. The wave numbers $\mathrm{k}_{1\Psi}^2\left(\vec{r}\right)$ and $\mathrm{k}_{1U}^2\left(\vec{r}\right)$, accordingly, the solutions of equations (Eqs. (29) and (30)) depend on the values of these parameters. In fact, we turn to the case of one-dimensional propagation of acoustic radiation along the OX axis passing through the point $\mathbf{r}_0$ $(\boldsymbol{X}_0, \boldsymbol{Y}_0 \;, \boldsymbol{Z}_0)$. Let us choose $\Psi_0\left(\vec{r}, \omega\right)$ and $\vec{U}_0\left(\vec{r}, \omega\right)$ both in the form of plane waves and propagating along the X axis. We can put the moduli of these plane waves $\Psi_0(\omega)$ and $\vec{\mathrm{U}}_0(\omega)$ equal to the moduli of the acoustic pressure P_0 and the component $\vec{v}_x$ of the particle velocity vector on the OX axis. In this case, the component of the vector Green's function will be equal to the one-dimensional Green's function

$\mathrm{G(x - x_1)} = \frac{1}{2\mathrm{ik}_0} \exp\left[\mathrm{ik}_0|\mathrm{x} - \mathrm{x}_1|\right]$ we find the solution corresponding to the first approximation at the point X

$$\Psi(x, y_0, z_0, \omega) = P_0 \exp ik_0 x + \int\limits_{-\infty}^{x} \mathrm{k}_{1\Psi}^2(x_1, y_0, z_0) \frac{P_0}{2ik_0} \exp(ik_0 x) dx_1 +$$
$$+ \int\limits_{x}^{\infty} \mathrm{k}_{1\Psi}^2(x_1, y_0, z_0) \frac{P_0}{2ik_0} \exp(-ik_0 x + 2ik_0 x_1) dx_1 = P_0 \exp ik_0 x + \Psi_1(x) + \Psi_2(x) \tag{31}$$

$$\vec{U}_x(x, y_0, z_0, \omega) = \vec{V}_x \exp ik_0 x + \int\limits_{-\infty}^{x} k_{1U}^2(x, y_0, z_0) \frac{\vec{V}_x \exp(ik_0 x)}{2ik_0} dx_1 +$$
$$+ \int\limits_{x}^{\infty} k_{1U}^2(x_1, y_0, z_0) \frac{\vec{V}_x}{2ik_0} \exp(-ik_0 x + 2ik_0 x_1) dx_1 = \vec{V}_x \exp ik_0 x + \vec{U}_1(x) + \vec{U}_2(x) \tag{32}$$

Here $P_0 \exp ik_0 x$ и $\vec{\mathbf{U}}_0 \exp ik_0 x$ represent the primary radiation, the second terms are the radiation scattered forward in the region $-\infty \le x_1 \le x$, and the third terms are the radiation scattered back. Solutions (Eqs. (31) and (32)) and the relations (Eq. (21)) allow us, in the first approximation, to find an expression for the projection of the complex intensity vector on the on the OX axis

$$\vec{\mathbf{I}}_x(x, y_0, z_0, \omega) = \sqrt{\frac{Z_p(x, y_0, z_0) Z_v(x, y_0, z_0)}{Z_p^0 Z_v^0}} \begin{pmatrix} P_0 \vec{\mathbf{V}}_x + + P_0 \vec{\mathbf{U}}_1^*(x) + P_0 \vec{\mathbf{U}}_2^*(x) + \Psi_1(\mathbf{x}) \vec{\mathbf{V}}_x + \\ + \Psi_1(\mathbf{x}) \vec{\mathbf{U}}_1^*(x) + \Psi_1(\mathbf{x}) \vec{\mathbf{U}}_2^*(x) + \Psi_2(\mathbf{x}) \vec{\mathbf{V}}_x + \\ + \Psi_2(\mathbf{x}) \vec{\mathbf{U}}_1^*(x) + \Psi_2(\mathbf{x}) \vec{\mathbf{U}}_2^*(x) \end{pmatrix} \tag{33}$$

In this expression, the first term describes the complex intensity vector of the primary radiation, the fifth term corresponds to the forward propagating secondary radiation, and the ninth term corresponds to the backscattered radiation. The other terms describe the mutual energy of the primary and scattered radiation. If the field is measured arriving at a point x only from the region, $x_1 \le x$ then the dependence of the projection of the complex intensity vector on the OX axis takes the following form:

$$\vec{\mathbf{I}}_x(x, y_0, z_0 \omega) \exp i\Phi(x, y_0, z_0) = \frac{C_0(x_0, y_0, z_0)}{C(x, y_0, z_0)} P_0 \vec{\mathbf{V}}_x \begin{bmatrix} 1 + \frac{i}{2k_0}(\alpha_v(x_0, y_0, z_0) - \alpha_p(x, y_0, z_0)) + \\ + \frac{1}{4k_0^2} \alpha_v(x, y_0, z_0) \alpha_p(x, y_0, z_0) \end{bmatrix}. \tag{34}$$

In this expression $\alpha_p(x) = \int\limits_{-\infty}^{x} k_{1\Psi}^2(x_1)dx_1$ and $\alpha_v(x) = \int\limits_{-\infty}^{x} k_{1U}^2(x_1)dx_1$. The modulus $\mathbf{I_0}(\mathbf{x}, \boldsymbol{y_0}, \boldsymbol{z_0}, \boldsymbol{\omega})$ and phase $\boldsymbol{\Phi}(\mathbf{x}, \boldsymbol{y_0}, \boldsymbol{z_0}, \boldsymbol{\omega})$ of the complex of acoustic intensity vector are respectively equal:

$$\mathbf{I}_0(x,\omega) = \frac{C_0 P_0 |\mathbf{V}_x|}{4k_0^2 C(x,y_0,z_0)} \sqrt{\left(4k_0^2 + \alpha_p(x,y_0,z_0)^2\right)\left(4k_0^2 + \alpha_v(x,y_0,z_0)^2\right)}, \Phi(x,y_0,z_0,\omega) = arctg \frac{2k_0\left[\alpha_v(x,y_0,z_0) - \alpha_p(x,y_0,z_0)\right]}{4k_0^2 + \alpha_v(x,y_0,z_0)\alpha_p(x,y_0,z_0)}. \tag{35}$$

The field intensity vector is

$$\mathrm{Re}\,\frac{1}{2}\vec{\mathbf{I}}_x(x,y_0,z_0,\omega) = \frac{1}{2}\frac{C_0}{C(x,y_0,z_0)} P_0 \vec{\mathbf{V}}_x \left[1 + \frac{1}{4k_0^2}\alpha_v(x,y_0,z_0)\alpha_p(x,y_0,z_0)\right], \tag{36}$$

and for the average field energy density we have the following expression:

$$\varepsilon(x,y_0,z_0\omega) = \frac{P(x,y_0,z_0\omega)P^*(x,y_0,z_0\omega)}{\rho(x,y_0,z_0)C^2(x,y_0,z_0)} = \frac{P_0^2}{\rho_0 C^2(x,y_0,z_0)}\left[1 + \frac{1}{4k_0^2}\alpha_p(x,y_0,z_0)\alpha_p(x,y_0,z_0)\right]. \tag{37}$$

From expressions (Eqs. (22) and (23)) it is seen that the inhomogeneous nature of the speed of sound will have a more significant effect on the particle velocity vector than on the acoustic pressure. This makes it possible in principle to create methods for separately measuring the contribution to the acoustic field in an inhomogeneous medium of the density of the medium and the speed of sound in it.

In conclusion, we note that the proposed method makes it possible to analytically and numerically solve the problems of mathematical modeling of a shallow sea, remote sensing of natural media, problems of acoustics of a shallow sea, modeling acoustic and optical metamaterials, etc. Note that for applied problems of acoustics, both fields of the particle velocity vector and the intensity vector in any inhomogeneous medium have a vortex character. Therefore, the algorithms for solving applied problems of ocean and especially shallow sea acoustics, problems of modeling the propagation of acoustic energy in composite media and metamaterials should take into account the vortex component of the vector acoustic field intensity and curvature of the streamlines of the acoustic field.

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References

[1] Caloz Ch., Itoh T. Electromagnetic metamaterials: transmission line theory and microwave applications (the engineering approach). A John Wiley & Sons, Inc., 2006. 352 p.

[2] Prather, D. W., S.Shi. Formulation and application of the finite-difference time-domain method for the analysis of axially symmetric diffractive. J. Opt. Soc. Am. A.1999. Vol.16, Iss. 5. P. 1131-1142. Doi:10.1364/ JOSAA.16.001131

[3] Taflove A. and Hagness S. C. Computational Electrodynamics: The Finite-Difference Time-Domain Method. Publisher: Artech House; 3 edition ,2005. 1038 pages.

[4] C.T. Tai, Diadic Green, Function in Electromagnetic Teory, 2nd ed. Piscataway, NJ, USA IEEEPress,1994.

[5] George W. Hanson Dyadic Green's Function for a Multilayered Planar Medium—A Dyadic Eigenfunction Approach. IEEE Transactions on Antennas and Propagation 2005, 52(12), P.3350 – 3356. DOI:10.1109/ TAP.2004.836409

[6] Dzyuba, V.P., Zapolsky, A.M. The G, reen function for the elastic half-space with a curvilinear boundary. Doklady Earth Sciences V. 372, May 2000, Pages 709-711.

[7] V.A. Gordienko Vector-phase methods in acoustics. Moscow: Fizmatlit; 2007. 480 p. ISBN978-5-92210864

[8] F. J. Fahy Sound intensity. 2nd ed. London: Elsevier; 1989.

[9] Dzyuba V.P. Scalar-vector methods of theoretical acoustics. Vladivostok: Dalnauka; 2006. 192 p. ISBN 5-8044-0559-4.

[10] Akulichev V.A., Dzyuba V.P., Gladkov P. V. On Acoustic Tomography Scheme for Reconstruction of Hydrophysical Parameters for Marine Environment. Theoretical and Computational Acoustics. Ed. Er- Chang Shang and Qihi Li. Published by World Scientific; 2002.P.107 -114.

[11] Baster K.J. , Lanchle L.C., Mc. ConnelJ.A. Development of a Velocity Gradient Underwater Acoustic Intensity Sensor. JASA .1999.Vol. 106. P. 3178-3188. doi:10.1121/1.428172.

[12] Mann J.A., Tichy J., Romano A.J. Instantaneous and time-average energy transfer in acoustic fields. JASA. 1987. Vol. 82. №1. P. 17-30. DOI:10.1121/ 1.395562.

[13] K. J. Taylor Absolute measurement of acoustic particle velocity. The Journal of the Acoustical Society of America. 1976 Vol. 59, 691 DOI:10.1121/1.380896

[14] Nelson, P.A. and Elliott, S.J. (1991) Active control of sound: Academic Press; 1993. 250pp. ISBN: 0125154259.

[15] Medwin H.. Sounds in the sea: from ocean acoustics to acoustical oceanography. Cambridge, UK: Cambridge Univ. Press. 2005. 643 pp.

[16] David R. Dall'Ostoa and Peter H. Dahl Properties of the acoustic intensity vector field in a shallow water waveguide. The Journal of the Acoustical Society of America 2012. 131, 2023; DOI: 10.1121/1.3682063

[17] Woon Siong Gan New Acoustics Based on Metamaterials . Springer Nature Singapore Pte Ltd. 2018. P. 314. DOI:10.1007/978-981-10-6376-3.

[18] Morse, P.N., Ingard, K.U. Theoretical Acoustics. New York: McGraw-Hill; 1987. 949P.

[19] Jan-Niklas Hau1, and Björn Müller Acoustic wave propagation in a temporal evolving shear-layer for low-Mach number perturbations. Physics of Fluids 2018; 30, 016105. DOI:10.1063/1.4999044.

[20] L. Friedland, G. Marcus, J. S. Wurtele, and P. Michel3, Excitation and control of large amplitude standing ion acoustic waves. Phys. Plasmas 2019; 26, 092109 DOI: 10.1063/1.5122300

[21] Zhu, X., Li, K., Zhang, P. *et al*. Implementation of dispersion-free slow acoustic wave propagation and phase engineering with helical-structured metamaterials. Nat Commun. 2016; 11731. DOI:10.1038/ncomms11731

[22] N. Anders Petersson & Björn Sjögreen High Order Accurate Finite Difference Modeling of Seismo-Acoustic Wave Propagation in a Moving Atmosphere and a Heterogeneous Earth Model Coupled Across a Realistic Topography.Journal of Scientific Computing 2018; volume 74, pages 290–323. DOI:10.1016/j.jcp.2020.109386

[23] S.MishrabCh.SchwabaJ. ŠukyscMulti- level Monte Carlo finite volume methods for uncertainty quantification of acoustic wave propagation in random heterogeneous layered medium. Journal of Computational Physics. Volume 312, 1 May 2016, Pages 192-217. DOI:10.1016/j.jcp.2016.02.014.

[24] Elias Perras1, and Chuanzeng Zhang1 Analysis of acoustic wave propagation in composite laminates via aspectral element method// PAMM • Proc. Appl. Math. Mech. 2019;19: e201900282. DOI:10.1002/pamm.201900282

[25] Yijun Mao, Jiancheng Cai, Yuanyuan Gu and Datong Qi Direct Evaluation of Acoustic-Intensity Vector Field Around an Impedance Scattering Body AIAA JOURNA. 2015; Vol. 53, No. 5, May. DOI:10.2514/1.J053431.

[26] Granino Arthur Korn, Theresa M. Korn Mathematical handbook for scientists and engineers, McGraw-Hill, 1968. P. 1130