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Chapter

Classical and Quantum Integrability: A Formulation That Admits Quantum Chaos

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Abstract

The concept of integrability of a quantum system is developed and studied. By formulating the concepts of quantum degree of freedom and quantum phase space, a realization of the dynamics is achieved. For a quantum system with a dynamical group G in one of its unitary irreducible representative carrier spaces, the quantum phase space is a finite topological space. It is isomorphic to a coset space G/R by means of the unitary exponential mapping, where R is the maximal stability subgroup of a fixed state in the carrier space. This approach has the distinct advantage of exhibiting consistency between classical and quantum integrability. The formalism will be illustrated by studying several quantum systems in detail after this development.

Keywords: classical, quantum, chaos, integrability, conservation law, algebra

1. Introduction

In classical mechanics, a Hamiltonian system with N degrees of freedom is defined to be integrable if a set of N constants of the motion U_i which are in involution exist, so their Poisson bracket satisfies $\{U_i, U_j\} = 0$, $i, j = 1, \ldots, N$. For an integrable system, the motion is confined to an invariant two-dimensional torus in $2N$-dimensional phase space. If the system is perturbed by a small nonintegrable term, the KAM theorem states that its motion may still be confined to the N-torus but deformed in some way [1–3]. The first computer simulation of nonequilibrium dynamics for a finite classical system was carried out by Fermi and his group. They considered a one-dimensional classical chain of anharmonic oscillators and found it did not equilibrate.

Classically, chaotic motion is longtime local exponential divergence with global confinement, a form of instability. Confinement with any kind of divergence is produced by repeatedly folding, a type of mixing that can only be analyzed by using probability theory. The motion of a Hamiltonian system is usually neither completely regular nor properly described by statistical mechanics, but shows both regular and chaotic motion for different sets of initial conditions. There exists generally a transition between the two types of motion as initial conditions are changed which may exhibit complicated behavior. As entropy or the phase space area quantifies the amount of decoherence, the rate of change of the phase space area quantifies the decoherence rate. In other words, the decoherence rate is the rate at which the phase space area changes.

It is important to extend the study of chaos into the quantum domain to better understand concepts such as equilibration and decoherence. Both integrable as well as nonintegrable finite quantum systems can equilibrate [4, 5]. Integrability does not seem to play a crucial role in the structure of the quasi-stationary state. This is in spite of the fact that integrable and nonintegrable quantum systems display different level-spacing statistics and react differently to external perturbation. Although integrable systems can equilibrate, the main difference from nonintegrable systems may be longer equilibration times. This kind of behavior is contrary to integrable classical finite systems that do not equilibrate at all. Nonintegrable classical systems can equilibrate provided they are chaotic.

The properties of a quantum system are governed by its Hamiltonian spectrum. Its form should be important for equilibration of a quantum system. The equilibration of a classical system depends on whether the system is integrable or not. Integrable classical systems do not tend to equilibrate, they have to be nonintegrable. Quantum integrability in n dimensions may be defined in an analogous way requiring the existence of n mutually commuting operators, but there is no corresponding theorem like the Liouville theorem. An integrable system in quantum mechanics is one in which the spectral problem can be solved exactly, and such systems are few in number [6, 7].

In closed classical systems, equilibration is usually accompanied by the appearance of chaos. Defining quantum chaos is somewhat of an active area of study now. The correspondence principle might suggest we conjecture quantum chaos exists provided the corresponding classical system is chaotic and the latter requires the system to be nonintegrable. Classical chaos does not necessarily imply quantum chaos, which seems to be more related to the properties of the energy spectrum.

It was proposed that the spectrum of integrable and nonintegrable quantum systems ought to be qualitatively different. This would be seen in the qualitative difference of the density of states. At a deeper level, one may suspect that changes in the energy spectrum as a whole may be connected to the breaking of some symmetry or dynamical symmetry. This is the direction taken here [8–10].

It is the objective to see how algebraic and geometric approaches to quantization can be used to give a precise definition of quantum degrees of freedom and quantum phase space. Thus a criterion can be formulated that permits the integrability of a given system to be defined in a mathematical way. It will appear that if the quantum system possesses dynamical symmetry, it is integrable. This suggests that dynamical symmetry breaking should be linked to nonintegrability and chaotic dynamics at the quantum level [11–13].

Algebraic methods first appeared in the context of the new matrix mechanics in 1925. The importance of the concept of angular momentum in quantum mechanics was soon appreciated and worked out by Wigner, Weyl and Racah [14–16]. The close relationship of the angular momentum and the $SO(3)$ algebra goes back to the prequantum era. The realization that $SO(4)$ is the symmetry group of the Kepler problem was first demonstrated by Fock. A summary of the investigation is as follows. To familiarize those who are not familiar with algebraic methods in solving quantum problems, an introduction to the algebraic solution of the hydrogen atom is presented as opposed to the Schrödinger picture. This approach provides a platform for which a definition of quantum integrability of quantum systems can be established. Thus, at least one approach is possible in which a definition of concepts such as quantum phase space, degrees of freedom as well as how an idea of quantum integrability and so forth can be formulated [17–21]. After these issues are addressed, a number of quantum models will be discussed in detail to show how the formalism is to be used [22–24].

1.1 The hydrogen atom

The hydrogen atom is a unique system. In this system, almost every quantity of physical interest can be computed analytically as it is a completely degenerate system. The classical trajectories are closed and the quantum energy levels only depend on the principle quantum number. This is a direct consequence of the symmetry properties of the Coulomb interaction. Moreover, the properties of the hydrogen atom in an external field can be understood using these symmetry properties. They allow a parallel treatment in the classical and quantum formalisms.

The Hamiltonian of the hydrogen atom in atomic units is

$$H_0 = \frac{\mathbf{p}^2}{2} - \frac{1}{r}. \tag{1}$$

The corresponding quantum operator is found by replacement of $\mathbf{p}$ by $-i\nabla$. Due to the spherical symmetry of the system, the angular momentum components are constants of the motion,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad [H_0, \mathbf{L}] = 0. \tag{2}$$

So $\{H_0, \mathbf{L}^2, L_z\}$ is a complete set of commuting operators classically, so three quantities in mutual involution, which implies integrability of the system.

The Coulomb interaction has another constant of the motion associated with the Runge-Lenz vector $\mathbf{R}$. This has the symmetrized quantum definition

$$\mathbf{R} = \frac{1}{2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r}, \qquad [H_0, \mathbf{R}] = 0. \tag{3}$$

If the $\mathbf{R}$ direction is chosen as the reference axis of a polar coordinate system in the plane perpendicular to $\mathbf{L}$, one deduces the equation of the trajectory as

$$r = \frac{\mathbf{L}^2}{1 + \|\mathbf{R}\| \cos \vartheta}. \tag{4}$$

The modulus determines whether the trajectory is an ellipse, a parabola or a hyperbola.

There are then 7 constants of the motion $(\mathbf{L}, \mathbf{R}, H_0)$ are not independent and satisfy

$$\mathbf{R} \cdot \mathbf{L} = 0, \qquad \mathbf{L}^2 - \frac{\mathbf{R}^2}{2H_0} = -\frac{1}{2H_0} - 1. \tag{5}$$

The minus one on the right in (5) is not present in classical mechanics. The mutual commutation relations are given in terms of ε_{ijk}, the fully antisymmetric tensor as follows,

$$[L_i, L_j] = i\varepsilon_{ijk} L_k, \qquad [L_i, R_j] = i\varepsilon_{ijk} R_k, \qquad [R_i, R_j] = i\varepsilon_{ijk}(-2H_0)L_k, \tag{6}$$

Let us look at the symmetry group of the hydrogen atom. The symmetry group is the set of phase space transformations which preserve the Hamiltonian and the equations of motion. It can be identified from the commutation relations between constants of motion. For hydrogen, for negative energies, the group of rotations in 4-dimensional space is called $SO(4)$.

The generators of the rotation group in an n-dimensional space are the $(n-1)n/2$ components of the n-dimensional angular momentum

$$\mathcal{L}_{ij} = x_i p_j - x_j p_i, \quad 1 \le i, j \le n. \tag{7}$$

In (7), $\mathcal{L}_{ij}$ is the generator of the rotations in the (i,j)-plane and has the following commutation relations

$$\left[\mathcal{L}_{ij}, \mathcal{L}_{kl}\right] = 0, \qquad \left[\mathcal{L}_{ij}, \mathcal{L}_{ik}\right] = i\mathcal{L}_{jk}. \tag{8}$$

The first bracket in (8) holds if all four indices are different.

Define the reduced Runge-Lenz vector to be

$$\mathbf{R}' = \frac{\mathbf{R}}{\sqrt{-2H_0}}. \tag{9}$$

The commutation relations (6) are those of a four-dimensional angular momentum with the identification

$$\begin{aligned} &\mathcal{L}_{12} = \mathcal{L}_z \quad \mathcal{L}_{23} = \mathcal{L}_z \quad \mathcal{L}_{31} = \mathcal{L}_y \\ &\mathcal{L}_{14} = R_{z'} \quad \mathcal{L}_{24} = R_{y'} \quad \mathcal{L}_{34} = R_{z'}, \end{aligned} \tag{10}$$

and Casimir operator

$$\mathcal{L}^2 = \sum_{i<j} \left(\mathcal{L}_{ij}\right)^2 = \mathbf{L}^2 + \mathbf{R}^2 = -\frac{1}{2H_0} - 1. \tag{11}$$

The classical trajectory is thus uniquely defined with the 6 components of $\mathcal{L}$ and $\mathbf{L} \cdot \mathbf{R}' = 0$. Any trajectory can be transformed into any other one having the same energy by a 4-dimensional rotation. An explicit realization of this four-dimensional invariance is to use a stereographic projection from the momentum space onto the 4-dimensional sphere with radius $p_0 = \sqrt{-2H_0}$. On this sphere, the solutions to Schrödinger's equation as well as the classical equations of motion are those of the free motion. Schrödinger's equation on the four-dimensional sphere can be separated into six different types of coordinates each associated with a set of commuting operators.

Spherical coordinates correspond to the most natural set, and choosing the quantization axis in the 4 direction and inside the (1,2,3) subspace, the z-axis or usual 3-axis as reference axis, the three operators can be simultaneously diagonalized,

$$\begin{aligned} \mathcal{L}^2 &= -\frac{1}{2H_0} - 1, \\ \mathbf{L}^2 &= \mathcal{L}_{12}^2 + \mathcal{L}_{31}^2 + \mathcal{L}_{23}^2 = L_x^2 + L_y^2 + L_z^2, \\ L_z &= \mathcal{L}_{12}, \end{aligned} \tag{12}$$

The respective eigenvalues of these operators are $n^2 - 1$, $l(l+1)$ and M such that $|M| \le l \le n-1$, so the total degeneracy is n^2. It corresponds to a particular subgroup chain given by

$$SO(4)_n \supset SO(3)_l \supset SO(2)_M. \tag{13}$$

Other choices are possible, such as other spherical coordinates obtained from the previous by interchanging the role of the 3 and 4 axes. This simultaneously diagonalizes the three operators

$$\begin{gathered} \mathcal{L}^2 = -\frac{1}{2H_0} - 1, \\ \lambda^2 = \mathcal{L}_{12}^2 + \mathcal{L}_{14}^2 + \mathcal{L}_{24}^2 = R_x'^2 + R_y'^2 + L_z'^2, \\ L_z = \mathcal{L}_{12}. \end{gathered} \tag{14}$$

The respective eigenvalues of these operators are $n^2 - 1$, $\lambda(\lambda + 1)$ and M such that $|M| \leq \lambda \leq n - 1$. The subgroup chain for this situation is

$$SO(4)_n \supset SO(3)_\lambda \supset SO(2)_M. \tag{15}$$

Another relevant case is the adoption of cylindrical coordinates on the 4-dimensional sphere associated with the following set of commuting operators

$$\begin{gathered} \mathcal{L}^2 = -\frac{1}{2H_0} - 1, \\ \mathcal{L}_{12} = L_z, \\ \mathcal{L}_{34} = R_z'. \end{gathered} \tag{16}$$

This set has the following associated subgroup chain,

$$SO(4) \supset SO(2) \otimes SO(2). \tag{17}$$

In configuration space, this is associated with separability in parabolic coordinates. This is a specific system but it exhibits many of the mathematical and physical properties that will appear here.

2. Quantum degrees of freedom

The time evolution of a system in classical mechanics in time is usually represented by a trajectory in phase space and the dynamical variables are functions defined on this space. The dimension of phase space is twice the number of degrees of freedom, and a point represents a physical state. The space is even-dimensional and it is endowed with a symplectic Poisson bracket structure. Dynamical properties of the system are described completely by Hamilton's equations within this space.

For a quantum system, on the other hand, the dynamical properties are discussed in the setting of a Hilbert space. Dynamical observables are self-adjoint operators acting on elements of this space. A physical state is represented by a ray of the space, so the Hilbert space plays a role similar to phase space for a classical system. The Hilbert space cannot play the role of a quantum phase space since its dimension does not in general relate directly to degrees of freedom. Nor can it be directly reduced to classical phase space in the classical limit. Let us define first the quantum degrees of freedom as well as giving a suitable meaning to quantum phase space.

Suppose $\mathcal{H}$ is a Hilbert space of a system characterized completely by a complete set of observables denoted $\mathcal{C}$. Set $\mathcal{C}$ is composed of the basic physical observables, such as coordinates, momenta, spin and so forth, but excludes the Hamiltonian. The basis vectors of the space can be completely specified by a set of quantum numbers which are related to the eigenvalues of what are usually referred to as the fully non-degenerate commuting observables $\mathcal{C}'$ of $\mathcal{C}$. A fully degenerate operator or observable $O \in \mathcal{C}$ has for some constant λ the action

$$O|\psi_i\rangle = \lambda|\psi_i\rangle, \qquad |\psi_i\rangle \in \mathcal{H}. \tag{18}$$

Definition 1: (Quantum Dynamical Degrees of Freedom) Let $\mathcal{C}$: $\{O_j|[O_i, O_j] = 0; i, j = 1, \dots, N\}$ be a complete set of commuting observables of a quantum system. A basis set of its Hilbert space $\mathcal{H}$ can be labeled completely by M numbers $\{\alpha_i : i = 1, \dots, M\}$ called quantum numbers which are related to the eigenvalues of the non-fully degenerate observables $\{O_i : i = 1, \dots, M(M \le N)\}$, a subset of $\mathcal{C}$. Then the number M is defined to be the number of quantum dynamical degrees of freedom. □

Since the members of $\mathcal{C}$ are provided by the system, not including the Hamiltonian, it depends only on the structure of the system's dynamical group **G**. Thus the number of quantum dynamical degrees of freedom based on this definition is unique for a given system with a specific Hilbert space $\mathcal{H}$.

The physical and mathematical considerations for defining the dimension of the nonfully degenerate operator subset $\mathcal{C}'$ of $\mathcal{C}$, not the dimension of $\mathcal{C}$ itself, as the number of quantum dynamical degrees of freedom is as follows. In a given $\mathcal{H}$, all fully degenerate operators in $\mathcal{C}$ are equivalent to a constant multiple of the identity operator guaranteeing the irreducibility of $\mathcal{H}$. The expectation values of any fully degenerate operator is a constant and contains no dynamical information.

A given quantum system generally has associated with it a well-defined dynamical group structure due to the fact that the mathematical image of a quantum system is an operator algebra **g** in a linear Hilbert space. This was seen in the case of hydrogen. It comes about from the mathematical structure of quantum mechanics. The dynamical group G with algebra **g** is generated out of the basic physical variables, with the corresponding algebraic structure defined by the commutation relations.

The Hamiltonian $\mathcal{H}$ and all transition operators $\{O\}$ can be expressed as functions of a closed set of operators

$$H = H(T_i), \qquad O = O(T_i), \qquad [T_i, T_j] = \sum_k C_{ij}^k T_k. \tag{19}$$

The C_{ij}^k in (19) are called the structure constants of algebra **g**. The Hilbert space is decomposed into a direct sum of the carrier spaces of unitary irreducible (irrep) representations of the group. Consequently, the dynamical symmetry properties of the system can be restricted to an irreducible Hilbert space which acts as one of the irrep carrier spaces of G.

From group representation theory, it will be given that a total of σ subgroup chains exist for a given group

$$G^\alpha = \{G_{s^\alpha}^\alpha \supset G_{s^{\alpha-1}}^\alpha \supset \cdots \supset G_1^\alpha\}, \qquad \alpha = 1, \dots, \sigma. \tag{20}$$

For each subgroup chain G^α of G, there is a complete set of commuting operators **C** which specifies a basis set of its irreducible basis carrier space $\mathcal{H}$, so the dimension of **C** for all subgroup chains of G is the same. A subgroup chain of dynamical group G serves to determine the M quantum dynamical degrees of freedom for a given quantum system with Hilbert space $\mathcal{H}$ an irrep carrier space of G.

Definition 2: For a quantum system with M independent quantum dynamical degrees of freedom the quantum phase space is defined to be a $2M$-dimensional topological space. The space is isomorphic to the coset space G/R with explicit symplectic structure. Here G is the dynamical group of the system and $R \subset G$ is the maximal stability subgroup of the Hilbert space. □

3. Quantum integrability and dynamical symmetry

Quantum phase space defined here can be compact or noncompact depending on the finite or infinite nature of the Hilbert space. A consequence of this development is that the classical definition of integrability can in general be directly transferred to the quantum case.

Definition 3: (Quantum Integrability) A quantum system with M independent dynamical degrees of freedom, hence a $2M$-dimensional quantum phase space, is integrable if and only if there are M quantum constraints of motion, or good quantum numbers, which are related to the eigenvalues of M non-fully degenerate observables: $O_1, O_2, \ldots, O_n$. □

Any set of variables that commute may be put in the form of a complete set of commuting observables **C** by including certain additional observables with it. The definition then says that if the system is integrable, a complete set of commuting variables **C** can be found so that the Hamiltonian is always diagonal in the basis referred to by **C**. In the reverse sense, the definition implies that if the system is integrable, simultaneous accurate measurements of M non-fully degenerate observables in the energy eigenvalues can be carried out.

The link with the dynamical group structure can be developed. This specifies exactly the integrability of a quantum system. To this end the definition of dynamical symmetry is needed.

Definition 4: (Dynamical Symmetry) A quantum system with dynamical group G possesses a dynamical symmetry if and only if the Hamiltonian operator of the system can be written and presented in terms of the Casimir operators of any specific chain with α fixed

$$H = \mathcal{F}\left(C_{kj}^{\alpha}\right) \tag{21}$$

The index of a particular subgroup chain C_{kj}^{α} the i-th Casimir operator of subgroup G_k^{α}, $k = s^{\alpha}, \ldots, 1$, $i = 1, \ldots, l_k^{\alpha}$ and l_k^{α} denotes that the rank of subgroup G_k^{α} is l. It is now possible to state a theorem which gives a condition for integrability to apply.

Proposition 1: (Quantum Integrability) A quantum system with dynamical group G is said to be integrable if it possesses a dynamical symmetry of G.

To prove this, note that it can be broken down into two cases or subgroup classes for a given dynamical group G and are referred to as canonical and noncanonical.

First consider the case in which G^{α} is a canonical subgroup chain of G. The Casimir operators of G, $\{C_{Gi}\}$ and all Casimir operators $\left\{C_{ki}^{\alpha}\right\}$ corresponding to the subgroups in chain G^{α} form a complete set of commuting operators C^{α} of any carrier irrep space H of G^{α} so for fixed α,

$$C^{\alpha} : \{C_{Gi}\} \cup \left\{C_{ki}^{\alpha}\right\} \equiv \left\{Q_j, j = 1, \cdots, N\right\}. \tag{22}$$

When G^{α} is the dynamical symmetry of the system, all operators in $\mathcal{C}^{\alpha}$ are constants of motion

$$\left[H, Q_j\right] = 0. \tag{23}$$

There are always M nonfully dynamical operators in $\mathcal{C}^{\alpha}$. By the third definition, the system is integrable.

For a non-canonical subgroup chain G^{α} the number of Casimir operators $\{C_{Gi}\}$ of G and all Casimir operators of $\left\{C_{ki}^{\alpha}\right\}$ of G^{α} is less than the number of the complete set of commuting operators $\mathcal{C}$ of any irrep carrier space of G. By definition, of any complete set of commuting operators, there must exist other commuting operators $\left\{O_j\right\}$ that commute with $\{C_{Gi}\}$ and $\left\{C_{ki}^{\alpha}\right\}$. These have to be included in the union as well when putting together $\mathcal{C}^{\alpha}$

$$\mathcal{C}^{\alpha} : \{C_{Gi}\} \cup \left\{C_{ki}^{\alpha}\right\} \cup \left\{O_j\right\} \equiv \left\{Q_j : j = 1, \ldots, N\right\}. \tag{24}$$

When the system is characterized by the dynamical symmetry of G^{α}, the operators in (24) satisfy relation (23) as well. In this case as well, there must exist M non-fully degenerate operators of constants of motion as in the previous case.

Based on this proposition, it can be stated that nonintegrability of a quantum system involves the breaking of the dynamical symmetry of the system. It may be concluded that dynamical symmetry breaking can be said to be a property which characterizes quantum nonintegrability. □

Let us summarize what has been found as to what quantum mechanics tells us. In a given quantum system with dynamical Lie group G which is of rank l and dimension n, the dimension of a complete set of commuting operators $\mathcal{C}$ of G with any particular subgroup chain is $d = l + (n - l)/2$ in which the l operators are Casimirs of G and are fully degenerate for any given irrep of G. The number M of the non-fully degenerate operators in $\mathcal{C}$ for a given irrep of G cannot exceed $M \leq (n - l)/2$. When dynamical symmetry is broken such that any of the M constants of the motion for the system is destroyed the system becomes nonintegrable.

4. Quantum phase space

It is of interest then to develop a model for phase space for quantum mechanics which may be regarded as an analogue to classical physics. By what has been said so far, the Hilbert space H of the system can be broken up into a direct sum of the unitary irreps carrier spaces of G,

$$H = \sum_{\Lambda} \oplus Y_{\Lambda} H_{\Lambda}. \tag{25}$$

In (25), the subscript Λ labels a particular irrep of Lie group G, Λ is the largest weight of the irrep and Y_{Λ} the degeneracy of Λ in H with no correlations existing between various H_{Λ}. The study of the dynamical properties of the system can be located on one particular irreducible subspace H_{Λ} of H. For a quantum system with M_{Λ} independent quantum dynamical degrees of freedom, the corresponding quantum phase space should be a $2M_{\Lambda}$-dimensional, topological phase space without additional constraints.

To construct the quantum phase space from the quantum dynamical degrees of freedom for an arbitrary quantum system, the elementary excitation operators can be obtained from the structures of G and H_{Λ}. Let $\left\{a_i^{\dagger}\right\}$ be a subset of generators of G such that any states $|\Psi\rangle$ of the system are generated for all $|\Psi\rangle \in H_{\Lambda}$ by means of

$$|\Psi\rangle = F\left(a_i^\dagger\right)|0\rangle. \tag{26}$$

Moreover $F\left(a_i^\dagger\right)$ is a polynomial in the operators $\left\{a_i^\dagger\right\}$ and $|0\rangle \in H_\Lambda$ is the reference state. The requirement placed on state $|0\rangle$ is that one can use a minimum subset of $\mathbf{g}$ to generate the entire subspace H_Λ from $|0\rangle$. In this event, the collection $\left\{a_i^\dagger\right\}$ is called the set of elementary excitation operators of the quantum dynamical degrees of freedom. If G is compact, $|0\rangle$ is the lowest $|\Lambda, -\Lambda\rangle$ or highest weight $|\Lambda, \Lambda\rangle$ state of H_Λ. If G is noncompact, it is merely the lowest state. The number of $\left\{a_i^\dagger\right\}$ is the same as the number of quantum dynamical degrees of freedom. Physically this has to be the case since that is how the operators are defined. Thus the set $\left\{a_i^\dagger\right\}$ and Hermitian conjugate $\{a_i\}$ in $\mathbf{g}_\Lambda$ form a dynamical variable subspace μ of g so we can write

$$\mu : \left\{a_i^\dagger, a_i; i = 1, \dots, M_\Lambda\right\}. \tag{27}$$

With respect to μ there exists a manifold whose dimension is twice that of the quantum dynamical degrees. It can be realized by means of a unitary exponential mapping of the dynamical variable operator subspace μ

$$\Omega = \exp\left(\sum_{i=1}^{M_\Lambda}\left(\eta_i a_i^\dagger - \eta_i^* a_i\right)\right) \in \coprod. \tag{28}$$

The η_i are complex parameters and $i = 1, \dots, M_\Lambda$. In fact, Ω is a unitary coset representation of G/R, where $R \subset G$ is generated by the subalgebra $\kappa = \mathbf{g} - \mu$. Thus (28) shows that q is isomorphic to the $2M_\Lambda$-dimensional coset space G/R, and will be denoted this way from now on. The discussion will apply just to semi-simple Lie groups whose $\mathbf{g}$ satisfies the usual Cartan decomposition $\mathbf{g} = \kappa + \mu$ and $[\kappa, \kappa] \subset \kappa$, $[\kappa, \mu] \subset \mu$ and $[\mu, \mu] \subset \kappa$. Thus G/R will be a complex homogeneous space with topology and a group transformation acting on G/R is a homomorphic mapping of G/R into itself.

The homogeneous space G/R has a Riemannian structure with metric

$$g_{ij} = \frac{\partial^2 \log \mathcal{K}(z, \bar{z})}{\partial z_i \partial \bar{z}_j} \tag{29}$$

The function $\mathcal{K}(z, \bar{z})$ is called the Bergmann kernel of G/R and can be represented as

$$\mathcal{K}(z, \bar{z}) = \sum_\lambda f_\lambda(z) f_\lambda^*(\bar{z}). \tag{30}$$

The functions $f_\lambda(z)$ in (30) constitute an orthogonal basis for a closed linear subspace $\mathcal{L}^2(G/R)$ of $L^2(G/R)$ such that

$$\int_{G/R} f_\lambda(z) f_{\lambda'}^*(\bar{z}) \mathcal{K}^{-1}(z, \bar{z}) d\nu(z, \bar{z}) = \delta_{\lambda\lambda'}, \tag{31}$$

and $d\nu(z, \bar{z})$ is the group invariant measure on the space G/R. It will be written

$$d\nu(z, \bar{z}) = \zeta\left[\det\left(g_{ij}\right)\right] \prod_{i=1}^{M_\lambda} \frac{dz_i d\bar{z}_i}{\pi}. \tag{32}$$

In (32) ζ is a normalization factor given by the condition that (32) integrated over the space G/R is equal to one. There is also a closed, nondegenerate two-form on G/R which is expressed as,

$$\omega = i\hbar \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j. \tag{33}$$

Corresponding to this two form there is a Poisson bracket which is given by

$$\{f, h\} = \frac{1}{i\hbar} \sum_{i,j} g^{ij} \left[\frac{\partial f}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j} \frac{\partial h}{\partial z_i} \right]. \tag{34}$$

In (34) f and h are functions defined on G/R. By introducing canonical coordinates $(\mathbf{q}, \mathbf{p})$ these quantities can be rewritten in terms of these coordinates.

4.1 Phase space quantum dynamics

Based on what has been stated about G/R, it would be useful to describe the quantum phase space. This means for a given quantum system a phase space representation must exist. Such a representation can be found if there exists an explicit mapping such that

$$O(T_i) \to U(\mathbf{q}, \mathbf{p}), \quad |\Psi\rangle \to \rho(q + ip). \tag{35}$$

Here O is given by (19), and $\rho(\mathbf{q}, \mathbf{p}) \in L^2$. For a quantum system with a quantum phase space G/R, this mapping can be realized by coherent states. To construct coherent states of G and H_Λ defined on G/R, the fixed state $|0\rangle$ is chosen as the initial state

$$g|0\rangle = \Omega \mathbf{r}|0\rangle = |\Lambda, \Omega\rangle e^{i\varphi(\mathbf{r})}, \quad g \in G, \quad \mathbf{r} \in R, \quad \Omega \in G/R. \tag{36}$$

Then R is the maximal stability subgroup of $|0\rangle$ so any $\mathbf{r} \in R$ acting on $|0\rangle$ will leave $|0\rangle$ invariant up to a phase factor

$$\mathbf{r}|0\rangle = e^{i\varphi(\mathbf{r})}|0\rangle. \tag{37}$$

The $|\Lambda, \Omega\rangle$ are the coherent states which are isomorphic to G/R. Therefore,

$$\begin{aligned} |\Lambda, \Omega\rangle \equiv \Omega|0\rangle &= \exp\left(\sum_{i=1}^{M_\Lambda} \left(\eta_i a_i^\dagger - \eta_i^* a_i \right) \right) |0\rangle = \mathcal{K}^{1/2}(z, \bar{z}) \exp\left(\sum_{i=1}^{M_\Lambda} z_i a_i^\dagger \right) |0\rangle \\ &= \mathcal{K}^{-1/2}(z, \bar{z}) \| \Lambda, z\rangle. \end{aligned} \tag{38}$$

$$\begin{aligned} \mathcal{K}(z, \bar{z}) &= \left\langle 0| \exp\left(\sum_{i=1}^{M_\Lambda} \bar{z}_i a_i \right) \exp\left(\sum_{i=1}^{M_\Lambda} z_i a_i^\dagger \right) |0 \right\rangle = \langle \Lambda, z \| \Lambda, z\rangle = |\rangle\, 0|\Lambda, 0\rangle|^2 \\ &= \sum_\lambda f_{\Lambda\lambda}(z) f_{\Lambda\lambda}^*(z). \end{aligned}$$

The Bargmann kernel was introduced in (30), and for a semisimple Lie group, the parameters z_i are given by

$$z = \begin{cases} \eta \dfrac{\tan (\eta^\dagger \eta)^{1/2}}{(\eta^\dagger \eta)^{1/2}}, & G\,\text{compact}, \\[2ex] \eta \dfrac{\tanh (\eta^\dagger \eta)^{1/2}}{(\eta^\dagger \eta)^{1/2}}, & G\,\text{noncompact}. \end{cases} \tag{39}$$

Here η represents the nonzero $k \times p$ block matrix of the operator $\sum_{i=1}^{M_\Lambda} \left(\eta_i a_i - \eta_i^* a_i^\dagger \right)$. The state $\|\Lambda, z\rangle$ in (38) is an unnormalized form of $|\Lambda, \Omega\rangle$ and $f_{\Lambda,\lambda}(z)$ is the orthogonal basis of $\mathcal{L}^2(G/R)$ the function space

$$f_{\Lambda,\lambda}(z) = \langle \Lambda, \lambda \| \Lambda, \lambda \rangle, \tag{40}$$

where $|\Lambda, \lambda\rangle$ is a basis for H_Λ, a particular irreducible subspace of the Hilbert space. The coherent states of (38) are over-complete

$$\int_{G/R} |\Lambda, \Omega\rangle \langle \Lambda, \Omega| d\nu(z) = I. \tag{41}$$

A classical-like framework or analogy has been established in the form of a quantum phase space specified by G and H_Λ. Variables which reside in this classical analogy are denoted thus $\tilde{c}$. The $2M_\Lambda$-dimensional quantum phase space G/R has all the required structures of a classical mechanical system. It is always possible a classical dynamical theory can be established in G/R whose motion is confined to G/R and is determined by the following equations of motion

$$\frac{d\tilde{\mathcal{U}}}{dt} = \{ \tilde{\mathcal{U}}(q,p), \tilde{H}(q,p) \}, \qquad q, p \in G/R. \tag{42}$$

This equation can be replaced by Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial \tilde{H}(q,p)}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \tilde{H}(q,p)}{\partial q_i}. \tag{43}$$

In (42) and (43), $\tilde{H}(q,p)$ is the Hamiltonian of the system, and $\tilde{U}(q,p)$ is a physical observable. A correspondence principle is implied here and requires that suitable conditions can be found such that the quantum dynamical Heisenberg equations can be written this way.

Clearly, if suitable conditions hold the phase space representation of the commutator of any two operators is equal to the Poisson bracket of the phase space representation of these two operators so that

$$\frac{1}{i\hbar} \langle \Lambda, \Omega | [A_H, B_H] | \Lambda, \Omega \rangle = \left\{ \tilde{\mathcal{A}}, \tilde{\mathcal{B}} \right\}. \tag{44}$$

Then the phase space representation of the Heisenberg equation

$$\frac{dA_H}{dt} = \frac{1}{i\hbar} [A_H, H_H], \tag{45}$$

given by (42) is therefore equivalent to (43). In (45), A_H is the Heisenberg operator

$$A_H = U A U^{-1}, \qquad U = e^{iHt/\hbar}, \tag{46}$$

and A is time-independent in the Schrödinger picture. The coherent state on the left of (44) is time-independent. Observables on the right side are the expectation values of the Schrödinger operators in the time-dependent coherent state. The quantum phase space maintains many of the quantum properties which are important, such as internal degrees of freedom, the Pauli principle, statistical properties and dynamical symmetry. Formally the equation of motion is classical. The phase space representation is based on the whole quantum structure of the coset space G/R.

Let us discuss integrability and dynamical symmetry. A quantum system with M_Λ independent degrees of freedom is integrable if and only if the M_Λ non-fully degenerate observables can simultaneously be measured in the energy representation. There exist non-fully degenerate observables $\{C_i : i = 1, \ldots, M_\Lambda - 1\}$ which commute with each other and H

$$\left[C_i, C_j\right] = 0, \qquad [C_i, H] = 0. \tag{47}$$

It follows that in the classical limit which has been formulated,

$$\left\{\tilde{C}_i, \tilde{C}_j\right\} = 0, \qquad \left\{\tilde{C}_i, \tilde{H}\right\} = 0. \tag{48}$$

Together with the Hamilton equations, (47) also formally defines classical integrability, so quantum integrability is completely consistent with the classical theory. In the classical analogy, the group structure of the system is defined by Poisson brackets. The concept of dynamical symmetry is naturally preserved in the classical analogy, so the theorem on dynamical symmetry and integrability is also meaningful for the classical analogy. If the Hamiltonian has the symmetry S, then its phase space picture representation has the same symmetry. To see this, if

$$SHS^{-1} = H, \tag{49}$$

in the phase space representation, it holds that

$$\langle\Lambda, \Omega|H|\Lambda, \Omega\rangle = \left\langle\Lambda, \Omega|SHS^{-1}|\Lambda, \Omega\right\rangle = \langle\Lambda, \Omega'|H|\Lambda, \Omega'\rangle. \tag{50}$$

To put this concisely, we write

$$\tilde{H}(q, p) = \tilde{H}(q', p'), \tag{51}$$

where $S^{-1}|\Lambda, \Omega\rangle = S^{-1}\Omega|0\rangle = |\Lambda, \Omega'\rangle e^{i\varphi(h)}$.

5. Applications to physical systems

5.1 Harmonic oscillator

The harmonic oscillator has dynamical group H_4 and is a single-degree of freedom system [13–15, 23]. To the dynamical group corresponds the algebra h_4 defined by the set $\{a^\dagger, a, a^\dagger a, I\}$ with Hilbert space the Fock space V^F : $\{|n\rangle, n = 1, 2, \ldots\}$, so the fixed state is the ground state $|0\rangle$, and elementary excitation operator $a^\dagger$. The quantum phase space is constructed from the unitary exponential mapping of the subspace $\mu : \{a^\dagger, a\}$ of h_4,

$$\Omega(z) = \exp\left(za^\dagger - \bar{z}a\right) \in H_4/U(1) \otimes U(1). \tag{52}$$

With generators $a^\dagger a$ and I, $U(1)\otimes U(1)$ in (52) is the maximal stability subgroup of $|0\rangle$. As $H_4/U(1)\otimes U(1)$ is isomorphic to the one-dimensional complex plane, the quantum phase space has metric $g_{ij}=\delta_{ij}$ and $d\nu(z)=dzd\overline{z}/\pi$. It is noncompact due to the infiniteness of the Fock space. There is a well-known symplectic structure on the complex plane with Poisson bracket of two functions $\tilde{F}_1, \tilde{F}_2$ defined by

$$\{\tilde{F}_1,\tilde{F}_2\}=\frac{1}{i\hbar}\left(\frac{\partial\tilde{F}_1}{\partial z}\frac{\partial\tilde{F}_2}{\partial\overline{z}}-\frac{\partial\tilde{F}_1}{\partial\overline{z}}\frac{\partial\tilde{F}_2}{\partial z}\right). \tag{53}$$

It is useful to introduce the standard canonical position and momentum coordinates

$$z=\frac{1}{\sqrt{2\hbar}}(q+ip),\qquad \overline{z}=\frac{1}{\sqrt{2\hbar}}(q-ip). \tag{54}$$

The Glauber coherent states can be realized by the states $|z\rangle$ with the set of these states isomorphic to $H_4/U(1)\otimes U(1)$ and given as

$$|z\rangle\equiv\Omega(z)|0\rangle=\exp\left(za^\dagger-\overline{z}a\right)|0\rangle=e^{-|z|^2/2}\exp\left(za^\dagger\right)|0\rangle. \tag{55}$$

The normalization constant in (55) is the Bargmann kernel

$$\mathcal{K}(z,\overline{z})=e^{|z|^2}. \tag{56}$$

The phase space representation of the wavefunction $|\Psi\rangle\in V^F$ is

$$f(z)=\langle\Psi||z\rangle=\sum_{n=0}^{\infty}f_n\frac{z^n}{\sqrt{n!}}. \tag{57}$$

By Wick's Theorem, it is always possible to write an operator A in normal product form

$$A=A\left(a^\dagger,a\right)=\sum_{k,l}A^n_{k,l}\left(a^\dagger\right)^k(a)^l. \tag{58}$$

The phase space representation of A is just

$$\tilde{U}(z,\overline{z})=\langle z|A|z\rangle=\sum_{k,l}A^n_{k,l}\overline{z}^kz^l. \tag{59}$$

In the case A is simply a generator of H_4, we can write (59) as

$$\begin{aligned}\tilde{a}^\dagger&=\langle z|a^\dagger|z\rangle, & \tilde{a}&=\langle z|a|z\rangle,\\ \tilde{a}^\dagger\tilde{a}&=\langle z|a^\dagger a|z\rangle=|z|^2, & \tilde{I}&=\langle z|I|z\rangle=I.\end{aligned} \tag{60}$$

The corresponding algebraic structure of H_4 in the phase-space representation is

$$i\hbar_c\{\tilde{a},\tilde{a}^\dagger\}=\tilde{I},\qquad i\hbar_c\{\tilde{a}^\dagger\tilde{a}\}=-\tilde{a},\qquad i\hbar_c\{\tilde{a}^\dagger\tilde{a},\tilde{a}^\dagger\}=\tilde{a}^\dagger. \tag{61}$$

Here $\hbar_c$ is used in the classical analogy. The algebraic structure of the H_4 generators is preserved when commutators are replaced by Poisson brackets in phase space. Using (54) the Dirac quantization condition and $\tilde{H}$ are given by

$$[q,p] = i\hbar_c\{q,p\}, \qquad \tilde{H}(q,p) = \langle z|H|z\rangle. \tag{62}$$

For the forced harmonic oscillator, the classical analogy of the Hamiltonian is given by

$$\begin{aligned}\tilde{H}(q,p) &= \frac{\omega}{2}\left(p^2+q^2\right) + i\sqrt{2}\mathfrak{R}(\lambda(t)q) - \sqrt{2}\mathfrak{I}(\lambda(t)p) \\ &= \frac{\omega}{2}\left(p^2+q^2\right) + \frac{1}{\sqrt{2}}\left(\lambda(t)+\bar{\lambda}(t)\right)q\big) + \frac{1}{\sqrt{2}i}\left(\left(\lambda(t)-\bar{\lambda}(t)\right)p\right).\end{aligned} \tag{63}$$

Hamilton's equations in (44) can be used to evaluate the t derivatives of q and p:

$$\frac{dq}{dt} = \omega p + \frac{1}{\sqrt{2}i}\left(\lambda(t)-\bar{\lambda}(t)\right), \qquad \frac{dp}{dt} = -\omega q - \frac{1}{\sqrt{2}}\left(\lambda(t)+\bar{\lambda}(t)\right). \tag{64}$$

Hence combining these two derivatives, we obtain

$$\frac{d}{dt}(q+ip) = -i\omega(q+ip) - \sqrt{2}i\,\lambda(t). \tag{65}$$

Multiplying both sides by the integrating factor $e^{i\omega t}$ and then integrating with respect to t, the solution is

$$q(t)+ip(t) = e^{-i\omega t}(q(0)+ip(0)) - i\sqrt{2}e^{-i\omega t}\int_0^t \lambda(\tau)e^{i\omega\tau}\,d\tau = z(t)\sqrt{2\hbar_c}. \tag{66}$$

If the initial state is $|0\rangle$ or a coherent state $|z(0)\rangle$, then the exact quantum solution is

$$|\psi(t)\rangle = |z(t)\rangle e^{i\varphi(t)} \tag{67}$$

and $z(t)$ is given by (66). The phase φ is a quantum effect obtained from $z(t)$

$$\varphi(t) = -\frac{1}{2}\omega t - \int_0^t \mathfrak{R}[\lambda(\tau)z(\tau)]\,d\tau. \tag{68}$$

This seems to imply the classical analogy provides an exact quantum solution if the Hamiltonian is a linear function of the generators of G.

5.2 $SU(2)$ spin system

The phase space structure of a spin system will be constructed and as well the phase-space distribution and classical analogy.

Since the dynamical group of the spin system is $SU(2)$ and the Hilbert space is described by the states $V^{2j+1} = \{|j,m\rangle\}$ where $m = -j,\ -j+1,\ \dots, j$ and j is an integer or half-integer, the fixed state is $|j,\ -j\rangle$. This is the lowest weight state of V^{2j+1}. Thus the elementary excitation operator of the spin system is J_+ and the explicit form of $|j,m\rangle$ is

$$|j,m\rangle = \frac{1}{(j+m)!}\binom{2j}{j+m}^{-1/2}(J_+)^{j+m}|j,\ -j\rangle. \tag{69}$$

Any state $|\Psi\rangle = \sum_{m=-j}^{j} f_m |j,m\rangle \in V^{2j+1}$ can be generated by a polynomial of J_+ acting on $|j,\ -j\rangle$. This means the number of quantum dynamical degrees of freedom is equal to the number of elementary excitation operators. The quantum phase space can be found by mapping $\mu : \{J_+, J_-\}$ to the coset space $SU(2)/U(1)$ by means of $(\eta J_+ - \overline{\eta} J_-) \to \exp(\eta J_+ - \overline{\eta} J_-)$ where $\eta = (\vartheta/2)e^{-i\varphi}$, $0 \le \vartheta \le \pi$, $0 \le \varphi \le 2\pi$. The coset space $SU(2)/U(1)$ is isomorphic to a two-dimensional sphere. The coherent states of $SU(2)/U(1)$ are well known

$$|j\Omega\rangle = \exp(\eta J_+ - \overline{\eta} J_-)|j,\ -j\rangle = \left(1+|z|^2\right)^{-j} \exp(zJ_+)|j,\ -j\rangle = \left(1+|z|^2\right)^{-j} ||jz\rangle,$$
$$z = \tan\frac{\vartheta}{2} e^{-i\varphi}. \tag{70}$$

The generalized Bargmann kernel on S^2 is $\mathcal{K}(z,\overline{z}) = \left(1+|z|^2\right)^{2j}$. Then the metric g_{ij} and measure are given by

$$g_{ij} = \delta_{ij} \frac{2j}{\left(1+|z|^2\right)^2}, \qquad d\nu = \frac{1}{\pi}(2j+1)\frac{dzd\overline{z}}{1+|z|^2)^2}. \tag{71}$$

Given the canonical coordinates

$$\frac{1}{\sqrt{4j\hbar}}(q+ip) = \frac{z}{\sqrt{1+|z|^2}} = \sin\left(\frac{\vartheta}{2}\right)e^{-i\varphi}, \tag{72}$$

there obtains the bracket

$$\{\tilde{F}_1, \tilde{F}_2\} = \frac{\partial \tilde{F}_1}{\partial q}\frac{\partial \tilde{F}_2}{\partial p} - \frac{\partial \tilde{F}_1}{\partial p}\frac{\partial \tilde{F}_2}{\partial q}, \tag{73}$$

where $q^2 + p^2 \le 4j\hbar$, which implies the phase space of a spin system is compact. The phase space representation of the state $|\Psi\rangle \in V^{2j+1}$ is for $f \in L^2(S^2)$,

$$f(z) = \langle\Psi||j\Omega\rangle = \sum_{n=0}^{\infty} f_n \binom{2j}{j+m}^{1/2} z^{j+m}, \tag{74}$$

The phase space representation of an operator $B = B(J_i)$ is

$$\tilde{B}(z,\overline{z}) = \langle j\Omega|B|j\Omega\rangle. \tag{75}$$

When the operator B in (75) is chosen to be one of the three operators J_+, J_- or J_0, the results are

$$\tilde{J}_+ = \langle j\Omega|J_+|j\Omega\rangle = \frac{2j\overline{z}}{1+|z|^2}, \quad \tilde{J}_- = \langle j\Omega|J_-|j\Omega\rangle = \frac{2jz}{\left(1+|z|^2\right)},$$
$$\tilde{J}_0 = \langle j\Omega|J_0|j\Omega\rangle = j\frac{|z|^2-1}{1+|z|^2}. \tag{76}$$

These can also be given in terms of q, p by using (72). The algebraic structure of $SU(2)$ in the phase space representation is given by the Poisson bracket

$$i\hbar_c\{\tilde{J}_-, \tilde{J}_+\} = -2\tilde{J}_0, \qquad i\hbar_c\{\tilde{J}_0, \tilde{J}_\pm\} = \pm\tilde{J}_\pm. \tag{77}$$

The classical analogy of an observable $B(J_i)$ is given by the following expression

$$\tilde{B}(q,p) = \langle j, \Omega | B(J_i) | j, \Omega \rangle. \tag{78}$$

The classical limit is found by taking $j \to \infty$ and the classical Hamiltonian function is

$$\tilde{H}_C(q,p) = H(\langle j, \Omega | J_i | j, \Omega \rangle) = H(\tilde{J}_+, \tilde{J}_-, \tilde{J}_0). \tag{79}$$

5.3 $SU(1,1)$ quantum systems and a two-level atom

A two-level atom is considered which interacts with two coupled quantum systems that can be represented in terms of a $su(1,1)$ Lie algebra. When for example mixed four-waves are injected into a cavity containing a single two level-atom an interaction occurs between the four waves and the atom that is electromagnetic radiation and matter. The Hamiltonian has the form

$$\frac{1}{\hbar}H = \sum_{i=1}^{2} \omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right) + \frac{1}{2}\omega_0 \sigma_z + \lambda \left(a_1^2 a_2^2 \sigma_+ + a_1^{\dagger 2} a_2^{\dagger 2} \sigma_- \right). \tag{80}$$

It is similar to **5.1**, so we sketch the physical situation. The $\sigma_\pm$, σ_z are raising lowering and inversion operators which satisfy the commutation relations $[\sigma_z, \sigma_\pm] = 2\sigma_\pm$, $[\sigma_+, \sigma_-] = \sigma_z$, whereas the $a_i^\dagger$, a_i are basic creation and annihilation operators with $\left[a_j, a_j^\dagger\right] = \delta_{ij}$. The interaction term in (80) can be thought of as the interaction between two different second harmonic modes. This can be cast in terms of three $su(1,1)$ Lie algebra generators K_+, K_- and K_z which satisfy the commutation relations,

$$[K_z, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_z. \tag{81}$$

The corresponding Casimir K which has eigenvalue $k(k-1)$ given by

$$K^2 = K_z^2 - \frac{1}{2}(K_+K_- + K_-K_+). \tag{82}$$

Given that this is the Lie algebra, it can be said that the Fock space is spanned by the set of vectors $V^F : \{|m;k\rangle\}$ and the operators in (81) act on these states as follows,

$$\begin{aligned} K_z|m;k\rangle &= (m+k)|m;k\rangle, & K^2|m;k\rangle &= k(k-1)|m;k\rangle, \\ K_+|m;k\rangle &= \sqrt{(m+1)(m+2k)}|m+1;k\rangle, & K_-|m;k\rangle &= \sqrt{m(m+2k-1)}|m-1;k\rangle. \end{aligned} \tag{83}$$

It is the case that $K_-|0;m\rangle = 0$ so this is the lowest level state. The $su(1,1)$ Lie algebra can be realized in terms of boson annihilation and creation operators and it is isomorphic to the Lie algebra of the non-compact $SU(1,1)$ group. For the Hamiltonian (80) define operators $K_\pm^{(i)}$ and $K_z^{(i)}$ as

$$K_+^{(i)} = \frac{1}{2} a_i^{\dagger^2}, \quad K_-^{(i)} = \frac{1}{2} a_i^2, \quad K_z^{(i)} = \frac{1}{2} \left(a_i^\dagger a_i + \frac{1}{2} \right), \qquad i = 1, 2, \tag{84}$$

where the Bargmann index k is either $1/4$ for the even parity states while $3/4$ applies to the odd-parity states.

Using these operators, (80) is written in terms of $su(2)$ and $su(1,1)$ operators such that it has the form,

$$\frac{1}{\hbar} H = \sum_{i=1}^{2} \eta_i K_z^{(i)} + \frac{\omega}{2} \sigma_z + \lambda \left(K_+^{(1)} K_+^{(2)} \sigma_- + K_-^{(1)} K_-^{(2)} \sigma_+ \right). \tag{85}$$

The Heisenberg equations of motion obtained from (85) gives

$$i \frac{d}{dt} K_z^{(1)} = \lambda \left(K_+^{(1)} K_+^{(2)} \sigma_- - K_-^{(1)} K_-^{(2)} \sigma_+ \right), \quad i \frac{d}{dt} K_z^{(2)} = \lambda \left(K_+^{(1)} K_+^{(1)} \sigma_- - K_-^{(1)} K_-^{(2)} \sigma_+ \right),$$
$$i \frac{d}{dt} \sigma_z = \lambda \left(K_-^{(1)} K_-^{(2)} \sigma_+ - K_+^{(1)} K_+^{(1)} \sigma_- \right). \tag{86}$$

The following two operators N_1 and N_2 are constants of the motion

$$N_1 = K_z^{(1)} + \sigma_z, \qquad N_2 = K_z^{(2)} + \sigma_z. \tag{87}$$

Hamiltonian (80) can now be put in the equivalent form

$$\frac{1}{\hbar} H = N + C + I, \tag{88}$$

where I is the identity operator and N and C are the operators

$$N = \sum_{i=1}^{2} \eta_i N_i, \qquad C = \Delta \sigma_z + \lambda \left(K_+^{(1)} K_+^{(2)} \sigma_- + K_-^{(1)} K_-^{(2)} \sigma_+ \right). \tag{89}$$

The constant Δ is the detuning parameter defined as

$$\Delta = \frac{\omega}{2} - \eta_1 - \eta_2. \tag{90}$$

As N and C commute, each commutes with the Hamiltonian H so N and C are constants of the motion. The time evolution operator $U(t)$ is given by

$$U(t) = \exp \left(-i \frac{H}{\hbar} t \right) \cdot \exp(-iNt) \cdot \exp(iCt). \tag{91}$$

In the space of the two-level eigenstates

$$e^{-iNt} = \begin{pmatrix} e^{-iW_1 t} & 0 \\ 0 & e^{-iW_2 t} \end{pmatrix}. \tag{92}$$

The operators W_i, $i = 1, 2$ are defined by $W_1 = \eta_1 K_z^{(1)} + \eta_2 K_z^{(2)} + 1$ and $W_2 = \eta_1 K_z^{(1)} + \eta_2 K_z^{(2)} - 1$. The second exponential on the right of (91) takes the form,

$$\exp(-iCt) = \begin{pmatrix} \cos\tau_i t - \frac{i\Delta}{\tau}\sin\tau_1 t & -i\lambda\frac{\sin\tau_1 t}{\tau_1}K_-^{(1)}K_-^{(2)} \\ -i\lambda K_+^{(1)}K_+^{(2)}\frac{\sin\tau_1 t}{\tau_1} & \cos\tau_2 t - \frac{i\Delta}{\tau_2}\sin\tau_2 t \end{pmatrix} \tag{93}$$

where $\tau_j^2 = \Delta^2 + \nu_j, j = 1, 2$ and

$$\tau_1 = \lambda^2 K_-^{(1)}K_+^{(1)}K_-^{(2)}K_+^{(2)}, \qquad \tau_2 = \lambda^2 K_+^{(1)}K_-^{(1)}K_+^{(2)}K_-^{(2)} \tag{94}$$

The coherent atomic state $|\vartheta, \varphi\rangle$ is considered to be the initial state that contains both excited and ground states and has the structure,

$$|\vartheta, \varphi\rangle = \cos\left(\frac{\vartheta}{2}\right)|e\rangle + \sin\left(\frac{\vartheta}{2}\right)e^{-i\varphi}|g\rangle. \tag{95}$$

where ϑ is the coherence angle, φ the relative phase of the two atomic states. The excited state is attained by taking $\vartheta \to 0$, while the ground state of the atom is derived from the limit $\vartheta \to \pi$. The initial state of the system that describes the two $su(1,1)$ Lie algebras is assumed to be prepared in the pair correlated state $|\xi, q\rangle$ defined by

$$K_-^{(1)}K_-^{(2)}|\xi, q\rangle = \xi|\xi, q\rangle, \qquad \left(K_z^{(1)} - K_z^{(1)}\right)|\xi, q\rangle = q|\xi, q\rangle. \tag{96}$$

Since the operators $K_-^{(1)}K_-^{(2)}$ and $\left(K_z^{(1)} - K_z^{(2)}\right)$ commute, $|\xi, q\rangle$ can be introduced which is simultaneously an eigenstate of both operators,

$$|\xi, q\rangle = \sum_{n=0}^{\infty} C_n\,|q + n + k_2 - k_1; k_1; n, k_2\rangle. \tag{97}$$

Then applying $K_-^{(2)}$ and then $K_-^{(1)}$ we obtain,

$$\begin{aligned} &K_-^{(1)}K_-^{(2)}|\xi, q\rangle \\ &= \sum_{n=0}^{\infty} C_n\sqrt{n(n + 2k - 1)(q + n + k_2 - k_1)(q + n + k_2 - k_1 + 2k_1 - 1}|q + n + k_2 - k_1 - 1, k_1; n - 1, k_2\rangle, \\ &= \sum_{n=0}^{\infty} C_{n+1}\sqrt{(n + 1)(n + 2k)(q + n + k_2 - k_1 + 1)((q + n + k_2 - k_1 + 2k_1)}|q + n + k_2 - k_1, k_1; n, k_2\rangle. \end{aligned} \tag{98}$$

This calculation implies that the normalization constant C_n can be obtained by solving

$$\sqrt{(n + 1)(n + 2k)(q + n + k_2 - k_1 + 1)(n + q + k_1 + k_2)}C_{n+1} = \xi C_0. \tag{99}$$

The new state is of the form,

$$|\xi, q\rangle = N_q\sum_{n=0}^{\infty} C_n\,|q + n + k_2 - k_1, k; n, k_2\rangle, \qquad N_q^{-2} = \sum_{n=0}^{\infty} |C_n|^2. \tag{100}$$

If it is assumed that at $t = 0$ the wave function of the system is $|\psi(0)\rangle = |\vartheta, \varphi\rangle \otimes |\xi, q\rangle$, using (91) on $|\psi(0)\rangle$, the state can be calculated for $t > 0$ can be determined

$$|\psi(t)\rangle = e^{-iW_1 t}\left[\left(\cos\tau_1 t\right) - i\frac{\Delta}{\tau_1}\sin\tau_1 t)\cos\left(\frac{\vartheta}{2}\right) - i\frac{\lambda}{\tau_1}\sin(\tau_1 t)K_-^{(1)}K_-^{(2)}e^{-i\varphi}\sin\frac{\vartheta}{2}\right]|e\rangle\otimes|\xi,q\rangle$$
$$+e^{iW_2 t}\left[\left(\cos\tau_2 t + i\frac{\Delta}{\tau_2}\sin\tau_2 t\right)e^{-i\varphi}\sin\frac{\vartheta}{2} - i\frac{\lambda}{\tau_2}\sin\tau_2 t K_+^{(1)}K_+^{(2)}\cos\frac{\vartheta}{2}\right]|g\rangle\otimes|\xi,q\rangle. \tag{101}$$

The reduced density matrix is constructed from this

$$\rho_f(t) = \ \mathrm{Tr}_{atom}\,|\psi(t)\rangle\langle\psi(t)|. \tag{102}$$

6. Summary and conclusions

Explicit structures for quantum phase space have been examined. Quantum phase space provides an inherent geometric structure for an arbitrary quantum system. It is naturally endowed with sympletic and quantum structures. The number of quantum dynamical degrees of freedom has a great effect on determining the quantum phase space. Inherent properties of quantum theory, the Pauli principle, quantum internal degrees of freedom and quantum statistical properties are included. A procedure can be stated for constructing this quantum phase space and canonical coordinates should be derivable for all semi-simple dynamical Lie groups with Cartan decomposition. The coset space G/R provides a way to define coherent states which link physical Hilbert space and quantum phase space. This motivates the study of the algebraic structure of the phase space representation of observables. The algebraic structure of operators is preserved in phase space if the operators are those of the dynamical group G. Through this approach, this property results in an explicit realization of the classical limit of quantum systems. A classical analogy was developed and seen in the examples as well for an arbitrary quantum system independently of the existence of the classical counterpart, so the classical limit of the quantum system can be obtained explicitly if it exists. The classical analogy will contain the first-order quantum correlation. A theorem which pertains to the relationship between dynamical symmetry and integrability has been proved, and is also valid in classical mechanics. It is then possible to construct a way to look for the quantum manifestation of chaos. Finally, it is then consistent with Berry's definition, the study of semi-classical but nonclassical, behavior characteristic of systems whose classical motion exhibits chaos.

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