Chapter

A Study of Fuzzy Sequence Spaces

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Abstract

The purpose of this chapter is to introduce and study some new ideal convergence sequence spaces ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$, ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ and ${}_F\mathcal{S}_\infty^{\mathcal{J}_\theta}(\mathcal{T})$ on a fuzzy real number F defined by a compact operator $\mathcal{T}$. We investigate algebraic properties like linearity, solidness and monotinicity with some important examples. Further, we also analyze closedness of the subspace and inclusion relations on the said spaces.

Keywords: Ideal, J{convergence, J{Cauchy, Fuzzy number, Lacunary sequence, Compact operator

1. Introduction

The concepts of fuzzy sets were initiated by Zadeh [1], since then it has become an active area of researchers. Matloka [2] initiated the notion of ordinary convergence of a sequence of fuzzy real numbers and studied convergent and bounded sequences of fuzzy numbers and some of their properties, and proved that every convergent sequence of fuzzy numbers is bounded. Nanda [3] investigated some basic properties for these sequences and showed that the set of all convergent sequences of fuzzy real numbers form a complete metric space. Alaba and Norahun [4] studied fuzzy Ideals and fuzzy filters of pseudocomplemented semilattices Moreover, Nuray and Savas [5] extended the notion of convergence of the sequence of fuzzy real numbers to the notion of statistical convergence.

Fast [6] introduced the theory of statistical convergence. After that, and under different names, statistical convergence has been discussed in the ergodic theory, Fourier analysis and number theory. Furthermore, it was examined from the sequence space point of view and linked with summability theory. Esi and Acikgoz [7] examined almost λ-statistical convergence of fuzzy numbers. Kostyrko et al. [8] introduced ideal $\mathcal{J}-$convergence which is based on the natural density of the subsets of positive integers. Kumar and Kumar [9] extended the theory of ideal convergence to apply to sequences of fuzzy numbers. Khan et al. [10–12] studied the notion of $\mathcal{J}$-convergence in intuitionistic fuzzy normed spaces. Subsequently, Hazarika [3] studied the concept of lacunary ideal convergent sequence of fuzzy real numbers. Where a lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals are determined by θ and defined by $I_r = (k_{r-1}, k_r]$.

We outline the present work as follows. In Section 2, we recall some basic definitions related to the fuzzy number, ideal convergent, monotonic sequence and compact operator. In Section 3, we introduce the spaces of fuzzy valued lacunary ideal convergence of sequence with the help of a compact operator and prove our maim results. In Section 4, we state the conclusion of this chapter.

2. Preliminaries

In this section, we recall some basic notion, definitions and lemma that are required for the following sections.

Definition 2.1. A mapping $F : \mathbb{R} \mapsto \lambda(= [0, 1])$ is a real fuzzy number on the set $\mathbb{R}$ associating real number s with its grade of membership $F(s)$. Let $\mathcal{C}$ denote the set of all closed and bounded intervals $F = [f_1, f_2]$ on the real line $\mathbb{R}$. For $G = [g_1, g_2]$ in $\mathcal{C}$, one define $F \leq G$ if and only if $f_1 \leq f_2$ and $g_1 \leq g_2$. Determine a metric ρ on $\mathcal{C}$ by

$$\rho(F, G) = \max\{|f_1 - g_1|, |f_2 - g_2|\}. \quad (1)$$

It can be easily seen that "$\leq$" is a partial order on $\mathcal{C}$ and $(\mathcal{C}, \rho)$ is a complete metric space. The absolute value $|F|$ of $F \in \mathbb{R}(\lambda)$ is defined by

$$|F|(s) = \begin{cases} \max\{F(s), F(-s)\}, & \text{if } s > 0 \\ 0, & \text{if } s < 0. \end{cases}$$

Suppose $\overline{\rho} : \mathbb{R}(\lambda) \times \mathbb{R}(\lambda) \mapsto \mathbb{R}$ be determined as

$$\overline{\rho}(F, G) = \sup \rho(F, G).$$

Hence, $\overline{\rho}$ defines a metric on $\mathbb{R}(\lambda)$. The multiplicative and additive identity in $\mathbb{R}(\lambda)$ are denoted by $\overline{1}$ and $\overline{0}$, respectively.

Definition 2.2. A family of subsets $\mathcal{J}$ of the power set $P(\mathbb{N})$ of the natural number $\mathbb{N}$ is known as an ideal if and only if the following conditions are satisfied [8]

i. $\varnothing \in \mathcal{J}$,

ii. for every $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{J}$ one obtain $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{J}$,

iii. for every $\mathcal{A}_1 \in \mathcal{J}$ and every $\mathcal{A}_2 \subseteq \mathcal{A}_1$ one obtain $\mathcal{A}_2 \in \mathcal{J}$.

An ideal $\mathcal{J}$ is known as non– trivial if $\mathcal{J} \neq P(\mathbb{N})$ and non– trivial ideal is said to be an admissible if $\mathcal{J} \supseteq \{\{n\} : n \in \mathbb{N}\}$.

Lemma 1. *If ideal $\mathcal{J}$ is maximal, then for every $\mathcal{A} \subset \mathbb{N}$ we have either $\mathcal{A} \in \mathcal{J}$ or $\mathbb{N} \backslash \mathcal{A} \in \mathcal{J}$* [8].

Definition 2.3. A family of subsets $\mathcal{H}$ of the power set $P(\mathbb{N})$ of the natural number $\mathbb{N}$ is known as filter in if and only if following condition are satisfied [8].

i. $\varnothing \notin \mathcal{H}$,

ii. for every $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{H}$ one have $\mathcal{A}_1 \cap \mathcal{A}_1 \in \mathcal{H}$,

iii. for every $\mathcal{A}_1 \in \mathcal{H}$ and $\mathcal{A}_2 \supseteq \mathcal{A}_1$ one have $\mathcal{A}_2 \in \mathcal{H}$.

Remark 1. Filter associated with the ideal $\mathcal{J}$ is defined by the family of sets

$$\mathcal{H}(\mathcal{J}) = \{K \subset \mathbb{N} : \exists \quad \mathcal{A} \in \mathcal{J} : K = \mathbb{N} \backslash \mathcal{A}\}$$

Definition 2.4. A sequence (F_k) of fuzzy real numbers is known as $\mathcal{J}-$ convergent to fuzzy real numbers F_0 if for each $\varepsilon > 0$, the set [9].

$$\{k \in \mathbb{N} : \overline{\rho}(F_k, F_0) \geq \varepsilon\} \in \mathcal{J}.$$

Definition 2.5. A sequence (F_k) is known as $\mathcal{J}-$ null if there exists a fuzzy real numbers $\overline{0}$ such that for each $\varepsilon > 0$ [9],

$$\left\{k \in \mathbb{N} : \overline{\rho}\left(F_k, \overline{0}\right) \geq \varepsilon\right\} \in \mathcal{J}.$$

Definition 2.6. A sequence (F_k) of fuzzy real numbers is known as $\mathcal{J}$-Cauchy if there exists a subsequence (F_{l_ε}) of (F_k) in such a way that for every $\varepsilon > 0$ [13],

$$\left\{k \in \mathbb{N} : \overline{\rho}(F_k, F_{l_\varepsilon}) \geq \varepsilon\right\} \in \mathcal{J}.$$

Definition 2.7. A sequence (F_k) is known as $\mathcal{J}-$ bounded if there exists a fuzzy real numbers $\mathcal{M} > 0$ so that, the set [9].

$$\left\{k \in \mathbb{N} : \overline{\rho}\left(F_k, \overline{0}\right) > \mathcal{M}\right\} \in \mathcal{J}.$$

Definition 2.8. Suppose $K = \{k_i \in \mathbb{N} : k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and $\mathbb{E}$ be a sequence space. A $K-$ step space of E is a sequence space [12].

$$\Lambda_K^{\mathbb{E}} = \left\{(x_{k_i}) \in \omega : (x_k) \in \mathbb{E}\right\}.$$

The canonical pre-image of a sequence $(x_{k_i}) \in \lambda_K^{\mathbb{E}}$ is a sequence $\left(y_k\right) \in \omega$ defined as follows:

$$y_k = \begin{cases} x_k, & \text{if } \; n \in K \\ 0, & \text{otherwise}. \end{cases}$$

y is in canonical pre-image of $\Lambda_K^{\mathbb{E}}$ if y is canonical pre-image of some element $x \in \Lambda_K^{\mathbb{E}}$.

Definition 2.9. A sequence space $\mathbb{E}$ is known as monotone, if it is contains the canonical pre-images of it is step space [12].

That is, if for all infinite $K \subseteq \mathbb{N}$ and $(x_k) \in \mathbb{E}$ the sequence $(\alpha_k x_k)$, where $\alpha_k = 1$ for $k \in K$ and $\alpha_k = 0$ otherwise, belongs to $\mathbb{E}$.

Definition 2.10. [12] A sequence space $\mathbb{E}$ is known as convergent free, if $(x_k) \in \mathbb{E}$ whenever $\left(y_k\right) \in \mathbb{E}$ and $\left(y_k\right) = 0$ implies that $(x_k) = 0$ for all $k \in \mathbb{N}$.

Lemma 2.1. Every solid space is monotone [12].

Definition 2.11. Suppose U and V are normed spaces. An operator $\mathcal{T} : U \to V$ is known as compact linear operator if [12].

1. $\mathcal{T}$ is linear

2. $\mathcal{T}$ maps every bounded sequence (x_k) in U onto a sequence $\mathcal{T}(x_k)$ in V which has a convergent subsequence.

3. Main results

In this section, we introduce the spaces of fuzzy valued lacunary ideal convergence of sequence with the help of a compact operator and investigate some topological and algebraic properties on these spaces. We denote ω^F the class of all sequences of fuzzy real numbers and $\mathcal{J}$ be an admissible ideal of the subset of the natural numbers $\mathbb{N}$.

$${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T}) = \left\{F = (F_k) \in \omega^F : \{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \geq \varepsilon\} \in \mathcal{J} \text{ for some } F_0 \in \mathbb{R}(\lambda)\right\},$$

$${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T}) = \left\{F = (F_k) \in \omega^F : \{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), \overline{0}) \geq \varepsilon\} \in \mathcal{J}\right\},$$

$${}_F\mathcal{S}_\infty^{\mathcal{J}_\theta}(\mathcal{T}) = \left\{F = (F_k) \in \omega^F : \exists \quad \mathcal{M} > 0 : \{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), \overline{0}) \geq \mathcal{M}\} \in \mathcal{J}\right\},$$

$${}_F\mathcal{S}_\infty^{\theta}(\mathcal{T}) = \left\{F = (F_k) \in \omega^F : \sup_r h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), \overline{0}) < \infty\right\}.$$

Theorem 3.1. The sequence spaces ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$, ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ and ${}_F\mathcal{S}_\infty^{\mathcal{J}}(\mathcal{T})$ are linear spaces.

Proof. Suppose α and β be scalars, and assume that $F = (F_k)$, $G = (g_k) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Since $F_k,\ G_k \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Then for a given $\varepsilon > 0$, there exists $F_1, F_2 \in \mathbb{C}$ in such a manner that

$$\left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_1) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{J}.$$

and

$$\left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_2) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{J}.$$

Now, let

$$A_1 = \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_1) < \frac{t}{2|\alpha|}\right\} \in \mathcal{H}(\mathcal{J}).$$

$$A_2 = \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_2) < \frac{t}{2|\beta|}\right\} \in \mathcal{H}(\mathcal{J}).$$

be such that $A_1^c, A_2^c \in \mathcal{J}$. Therefore, the set

$$\begin{aligned} A_3 &= \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(\alpha(F_k) + \beta(G_k)), (\alpha L_1 + \beta L_2)) < \varepsilon\right\} \\ &\supseteq \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), L_1) < \frac{\varepsilon}{2|\alpha|}\right\} \cap \left\{r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), L_1) < \frac{\varepsilon}{2|\beta|}\right\}. \end{aligned} \tag{2}$$

Thus, the sets on right hand side of (2) belong to $\mathcal{H}(\mathcal{J})$. Therefore A_3^c belongs to $\mathcal{J}$. Therefore, $\alpha(F_k) + \beta(G_k) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Hence ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is linear space.

In similar manner, one can easily prove that ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ and ${}_F\mathcal{S}_\infty^{\mathcal{J}}(\mathcal{T})$ are linear spaces. □

Example 3.1. Suppose $\mathcal{J} = \mathcal{J}_\rho = \{\mathcal{B} \subseteq \mathbb{N} : \rho(\mathcal{B}) = 0\}$, where $\rho(\mathcal{B})$ denotes the asymptotic density of $\mathcal{B}$. In this case ${}_F\mathcal{S}^{\mathcal{J}_\rho}(\mathcal{T}) = {}_F\mathcal{S}_\theta(\mathcal{T})$, where

$$_F\mathcal{S}_\theta(T) = \left\{ F_k \in \omega^F : \rho\left(\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \geq \varepsilon \right\}\right) = 0 \ \text{ for some } \ F_0 \in \mathbb{R}(\gamma) \right\}.$$

Example 3.2. Suppose $\mathcal{J} = \mathcal{J}_f = \{\mathcal{B} \subseteq \mathbb{N} : \mathcal{B} \ \text{ is finite}\}$. $\mathcal{J}_f$ is an admissible ideal in $\mathbb{N}$ and $_F\mathcal{S}^{\mathcal{J}_f}(\mathcal{T}) = {_F\mathcal{S}^\theta}(\mathcal{T})$.

Theorem 3.2. The spaces $_F\mathcal{S}^{\mathcal{J}_\theta}(T)$ and $_F\mathcal{S}_0^{\mathcal{J}_\theta}(T)$ are not convergent free.

Proof. For the proof of the theorem, we consider the subsequent example. □

Example 3.3. Suppose $\mathcal{J} = \mathcal{J}_\delta$ and $\mathcal{T}(F_k) = F_k$.

Consider the sequence $F_k \in {_F\mathcal{S}_0^{\mathcal{J}_\theta}}(\mathcal{T}) \subset {_F\mathcal{S}^{\mathcal{J}_\theta}}(\mathcal{T})$ as:

For $k \neq i^2, i \in \mathbb{N}$

$$F_k(s) = \begin{cases} 1, & \text{if } \ 0 \leq s \leq k^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

For $k = i^2, i \in \mathbb{N}, F_k(s) = \overline{0}$. Therefore

$$F_k = \begin{cases} [0, 0], & \text{if } \ k^2 = i \\ \left[0, k^{-1}\right], & k \neq i^2. \end{cases}$$

Hence, $F_k \to \overline{0} \ \ as \ \ k \to \infty$. Thus $F_k \in {_F\mathcal{S}_0^{\mathcal{J}_\theta}}(\mathcal{T}) \subset {_F\mathcal{S}^{\mathcal{J}_\theta}}(\mathcal{T})$. Let G_k be sequence in such a way that, for $k = i^2, i \in \mathbb{N}, F_k(s) = \overline{0}$. Therefore, one obtain

$$G_k(s) = \begin{cases} 1, & \text{if } \ 0 \leq s \leq k \\ 0, & \text{otherwise.} \end{cases}$$

For $k = i^2, i \in \mathbb{N}, F_k(s) = \overline{0}$. Therefore

$$G_k = \begin{cases} [0, 0], & \text{if } \ k^2 = i \\ [0, k], & k \neq i^2. \end{cases}$$

It can be easily seen that $G_k \notin {_F\mathcal{S}_0^{\mathcal{J}_\theta}}(\mathcal{T}) \subset {_F\mathcal{S}^{\mathcal{J}_\theta}}(\mathcal{T})$.

Hence $_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ and $_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ are not convergence free.

Theorem 3.3. The sequence $F = (F_k) \in {_F\mathcal{S}_\infty^\theta}(\mathcal{T})$ is $\mathcal{J}-$ convergent if and only if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ in such a way that

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), \mathcal{T}(F_{N_\varepsilon})) < \varepsilon \right\} \in \mathcal{H}(\mathcal{J}). \tag{3}$$

Proof. Suppose that $F = (F_k) \in {_F\mathcal{S}_\infty^\theta}(\mathcal{T})$ Let $F_0 = \mathcal{J} - \lim F$. Then for every $\varepsilon > 0$, the set

$$B_\varepsilon = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) < \frac{\varepsilon}{2} \right\} \in \mathcal{H}(\mathcal{J}).$$

Fix an $N_\varepsilon \in B_\varepsilon$. Then, we have

$$\overline{\rho}(\mathcal{T}(F_k),\mathcal{T}(F_{N_\varepsilon})) \leq \overline{\rho}(\mathcal{T}(F_k),F_0)$$
$$+\overline{\rho}(\mathcal{T}(F_{N_\varepsilon}),F_0) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Which holds for all $N_\varepsilon \in B_\varepsilon$. Hence

$$\left\{r\in\mathbb{N}: h_r^{-1}\sum_{k\in J_r}\overline{\rho}(\mathcal{T}(F_k),\mathcal{T}(F_{N_\varepsilon}))<\varepsilon\right\}\in\mathcal{H}(\mathcal{J}).$$

On Contrary, assume that

$$\left\{r\in\mathbb{N}: h_r^{-1}\sum_{k\in J_r}\overline{\rho}(\mathcal{T}(F_k),\mathcal{T}(F_{N_\varepsilon}))<\varepsilon\right\}\in\mathcal{H}(\mathcal{J}).$$

That is

$$\left\{r\in\mathbb{N}: h_r^{-1}\sum_{k\in J_r}\overline{\rho}(\mathcal{T}(F_k),\mathcal{T}(F_{N_\varepsilon}))<\varepsilon\right\}\in\mathcal{H}(\mathcal{J}) \text{ for all } \varepsilon>0.$$

Then, the set

$$C_\varepsilon=\{k\in\mathbb{N}:\mathcal{T}(F_k)\in[\mathcal{T}(F_{N_\varepsilon})-\varepsilon,\mathcal{T}(F_{N_\varepsilon})+\varepsilon]\}\in\mathcal{H}(\mathcal{J}) \text{ for all } \varepsilon>0.$$

Let $I_\varepsilon=[\mathcal{T}(F_{N_\varepsilon})-\varepsilon,\mathcal{T}(F_{N_\varepsilon})+\varepsilon]$. If we fix an $\varepsilon>0$, then we have $C_\varepsilon\in\mathcal{H}(\mathcal{J})$ as well as $C_{\frac{\varepsilon}{2}}\in\mathcal{H}(\mathcal{J})$. Hence $C_{\frac{\varepsilon}{2}}\cap C_\varepsilon\in\mathcal{H}(\mathcal{J})$. This implies that $I=I_\varepsilon\cap I_{\frac{\varepsilon}{2}}\neq\varnothing$. That is

$$\{k\in\mathbb{N}:\mathcal{T}(F_k)\in I\}\in\mathcal{H}(\mathcal{J}).$$

That is

$$\text{diam } I\leq\text{diam } I_\varepsilon.$$

Where the diam of I denotes the length of interval I. Continuing in this way, by induction, we get the sequence of closed intervals.

$$I_\varepsilon=\mathcal{I}_0\supseteq\mathcal{I}_1\supseteq\ \dots\ \supseteq\mathcal{I}_k\supseteq\ \dots$$

With the property that

$$\text{diam } \mathcal{I}_k\leq\frac{1}{2}\text{ diam } \mathcal{I}_{k-1} \text{ for } (k=2,3,4,\dots)$$

and

$$\{k\in\mathbb{N}:\mathcal{T}(F_k)\in\mathcal{I}_k\}\in\mathcal{H}(\mathcal{J}) \text{ for } (k=2,3,4,\dots).$$

Then there exists a $\xi\in\cap\mathcal{I}_k$ where $k\in\mathbb{N}$ in such a way that $\xi=\mathcal{J}-\lim T(F_k)$. Therefore, the result holds. □

Theorem 3.4. *The inclusions* ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})\subset{}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})\subset{}_F\mathcal{S}_\infty^{\mathcal{J}_\theta}(\mathcal{T})$ *hold.*

Proof. Let $F = (F_k) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Then there exists a number $F_0 \in \mathbb{R}$ such that

$$\mathcal{J} - \lim_{k\to\infty} \overline{\rho}(\mathcal{T}(F_k), F_0) = 0.$$

That is, the set

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k\in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \geq \varepsilon \right\} \in \mathcal{J}.$$

Here

$$\begin{aligned} h_r^{-1} \sum_{k\in J_r} \overline{\rho}(\mathcal{T}(F_k), \overline{0}) &= h_r^{-1} \sum_{k\in I_r} \overline{\rho}(\mathcal{T}(F_k), -F_0 + F_0) \\ &\leq h_r^{-1} \sum_{k\in I_r} \overline{\rho}(\mathcal{T}(F_k) - F_0) + h_r^{-1} \sum_{k\in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \end{aligned} \tag{4}$$

On the both sides, taking the supremum over k of the above equation, we obtain $F_k \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Therefore inclusion holds.

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k\in J_r} \overline{\rho}\left(\mathcal{T}\left(F_k^p\right), a_p\right) \geq \varepsilon \right\} \in \mathcal{J}.$$

Then, it proves that ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T}) \subset {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Hence ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T}) \subset {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T}) \subset {}_F\mathcal{S}_\infty^{\mathcal{J}_\theta}(\mathcal{T})$. □

Theorem 3.5. The space ${}_F\mathcal{S}^{\mathcal{J}_\theta}(T)$ is neither normal nor monotone if $\mathcal{J}$ is not maximal ideal.

Example 3.4. Suppose fuzzy number

$$G_k(s) = \begin{cases} \dfrac{1+s}{2}, & \text{if } \; -1 \leq s \leq 1 \\ \dfrac{3-s}{2}, & \text{if } \; 1 \leq s \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Then $G_k(s) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Applying Lemma 1, there exists a subset K of $\mathbb{N}$ in such a way that $K \notin \mathcal{J}$ and $\mathbb{N} \backslash K \notin \mathcal{J}$. Determine a sequence $G = (G_k)$ as

$$G_k = \begin{cases} F_k, & k \in K \\ 0, & \text{otherwise.} \end{cases}$$

Therefore (G_k) belong to the canonical pre- image of the K- step spaces of ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. But $(G_k) \notin {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Hence ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is not monotone. Hence, by Lemma (2.1) ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is not normal.

Theorem 3.6. The sequence space ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ is solid and monotone.

Proof. Suppose $F = (F_k) \in {}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$, then for $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k\in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \geq \varepsilon \right\} \in \mathcal{J}. \tag{5}$$

Suppose that sequence of scalars (α_k) with the property $|\alpha_k| \leq 1 \;\; \forall \;\; k \in \mathbb{N}$. Therefore

$$\begin{aligned} &\overline{\rho}(\mathcal{T}(\alpha_k F_k), F_0) = \overline{\rho}(\alpha_k \mathcal{T}(F_k), F_0) \\ &\leq |\alpha_k| \overline{\rho}(\mathcal{T}(F_k), F_0) \;\; \text{for all} \;\; k \in \mathbb{N}. \end{aligned} \tag{6}$$

Hence, from the Eq. (5) and above inequality, one obtain

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(\alpha_k F_k), F_0) \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F_k), F_0) \geq \varepsilon \right\} \in \mathcal{J}.$$

Implies

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(\alpha_k F_k), F_0) \geq \varepsilon \right\} \in \mathcal{J}.$$

Therefore, $(\alpha_k F_k) \in {}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$. Then ${}_F\mathcal{S}_0^{\mathcal{J}_\theta}(\mathcal{T})$ is solid and monotone by lemma 2.1. □

Theorem 3.7. The sequence space ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is closed subspace of ${}_F\mathcal{S}_\infty^\theta(\mathcal{T})$.

Proof. Suppose $\left(F_k^q\right)$ be a Cauchy sequence in ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$. Then, $F_k^q \to F$ in ${}_F\mathcal{S}_\infty(\mathcal{T})$ as $q \to \infty$. Since $\left(F_k^q\right) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$, then for each $\varepsilon > 0$ there exists a_q such that converges to a.

$$\mathcal{J} - \lim F = a.$$

Since $\left(F_k^q\right)$ be a Cauchy sequence in ${}_F\mathcal{S}^{I_\theta}(\mathcal{T})$. Then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$$\sup_k \overline{\rho}\left(\mathcal{T}\left(F_k^q\right), \mathcal{T}\left(F_k^s\right)\right) < \frac{\varepsilon}{3} \;\; \text{for all} \;\; q, s \geq n_0.$$

For a given $\varepsilon > 0$,

$$B_{q,s} = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}\left(\mathcal{T}\left(F_k^q\right), \mathcal{T}\left(F_k^s\right)\right) < \frac{\varepsilon}{3} \right\}.$$

Now,

$$B_q = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}\left(\mathcal{T}\left(F_k^p\right), a_q\right) < \frac{\varepsilon}{3} \right\}$$

and

$$B_s = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \overline{\rho}\left(\mathcal{T}\left(F_k^s\right), a_s\right) < \frac{\varepsilon}{3} \right\}.$$

Then $B^c_{q,s}, B^c_q, B^c_s \in \mathcal{J}$. Let $B^c = B^c_{q,s} \cup B^c_q \cup B^c_s$. Where

$$B = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}\left(a_q, a_s\right) < \varepsilon \right\} \text{ then } B^c \in \mathcal{J}.$$

Assume $n_0 \in B^c$. Then, for every $q, s \geq n_0$ it has

$$\begin{aligned} \overline{\rho}\left(a_q, a_s\right) &\leq \overline{\rho}\left(\mathcal{T}\left(F_k^q\right), a_q\right) + \overline{\rho}\left(\mathcal{T}\left(F_k^s\right), a_s\right) + \overline{\rho}\left(\mathcal{T}\left(F_k^q\right), \mathcal{T}\left(F_k^s\right)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, a_q is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists scalar $a \in \mathbb{C}$ in such a way that $\left(a_q\right) \to a$ as $q \to \infty$. For this step, let $0 < \alpha < 1$ be given. Therefore it proved that whenever

$$U = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F), a) < \alpha \right\} \text{ then } U^c \in \mathcal{J}. \tag{7}$$

Since $\left(F_k^q\right) \to F$, there exists $q_0 \in \mathbb{N}$ so that

$$P = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F^{q_0}), \mathcal{T}(F)) < \frac{\alpha}{3} \right\}.$$

implies that $P^c \in \mathcal{J}$. The number, q_o can be chosen together with Eq. (7), it have

$$Q = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}\left(a_{q_0}, a\right) < \frac{\alpha}{3} \right\}.$$

Which implies that $Q^c \in \mathcal{J}$. Since

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}\left(F_k^{q_0}\right), a_{q_0}) \geq \frac{\alpha}{3} \right\} \in \mathcal{J}.$$

Then it has a subset S such that $S^c \in \mathcal{J}$, where

$$S = \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}\left(F_k^{q_0}\right), a_{q_0}) < \frac{\alpha}{3} \right\}.$$

Suppose that $U^c = P^c \cup Q^c \cup S^c$. Then, for every $k \in U^c$ it has

$$\begin{aligned} \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}(F), a) < \alpha \right\} &\supseteq \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} \overline{\rho}(\mathcal{T}(F^{q_0}), (F)) < \frac{\alpha}{3} \right\} \cap \\ &\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}\left(a_{q_0}, a\right) < \frac{\alpha}{3} \right\} \cap \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in J_r} \overline{\rho}(\mathcal{T}\left(F_k^{q_0}\right), a_{q_0}) < \frac{\alpha}{3} \right\}. \end{aligned} \tag{8}$$

The right hand side of Eq. (8) belongs to $\mathcal{H}(\mathcal{J})$. Hence, the sets on the left hand side of Eq. (8) belong to $\mathcal{H}(\mathcal{J})$. Therefore its complement belongs to $\mathcal{J}$. Thus, $\mathcal{J} - \lim \overline{\rho}(a_{q_0}, a) = 0$. □

In the following example to prove that ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is closed subspace of ${}_F\mathcal{S}^{\theta}_{\infty}(\mathcal{T})$.

Example 3.5. Suppose that sequence of fuzzy number determine by

$$F_k(s) = \begin{cases} 2^{-1}(1+s), & \text{for } s \in [-1,1] \\ 2^{-1}(3-s), & \text{for } s \in [1,3] \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\overline{\rho}(F_k, \overline{0}) = \sup \rho(F_k, \overline{0})$. Therefore $(F_k) \in {}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ and $L = \overline{0}$. So, it can be easily seen that ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is closed subspace of ${}_F\mathcal{S}^{\theta}_{\infty}(\mathcal{T})$.

4. Conclusion

The spaces of fuzzy valued lacunary ideal convergence of sequence with the help of a compact operator and investigate algebraic and topological properties together with some examples on the given spaces. We proved that the new introduced sequence spaces are linear. Some spaces are convergent free and we also proved that space ${}_F\mathcal{S}^{\mathcal{J}_\theta}(\mathcal{T})$ is closed subspace of ${}_F\mathcal{S}^{\theta}_{\infty}(\mathcal{T})$. These new spaces and results provide new tools to help the authors for further research and to solve the engineering problems.

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