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Chapter

# Use of Daubechies Wavelets in the Representation of Analytical Functions

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## Abstract

This chapter aims to use Daubechies' wavelets as basis functions to generate analytical functions, thus being able to rewrite the Taylor series using these wavelets. This makes it possible to analyze functions with a high degree of complexity, in problems that require a high degree of precision in their solution. Wavelet analysis can be applied to practical problems that require a high degree of precision, for example, in the study and analysis of electromagnetic propagation in optical fibers, solutions of differential equations involving engineering problems, in the transmission of WiFi signals, in the treatment and analysis of biomedical images, detection of oil sources through the study of seismic signals.

**Keywords:** wavelets, Daubechies, analytical functions, basis functions, Taylor series

## 1. Introduction

Wavelets [1] were born from the need to generate functions, especially those that present singularities, high gradients, discontinuities both in the time domain and in the frequency domain. Wavelets enable the high-resolution analysis of functions with these characteristics. An example of a problem that occurs when generating functions with a Fourier base is the Gibbs phenomenon. Such a phenomenon occurs because there is no way to represent functions that present discontinuities, even adding more elements in the base that will generate the function. A characteristic of wavelets is that they do not produce such an effect.

Wavelets are widely used in the solution of numerical problems in several areas of knowledge such as image compression, Numerical Harmonic Analysis [2], financial analysis, oil detection, differential Equations [3, 4], biomedical signals, analysis of electromagnetic integral Equations [5], optical fibers [6], among others. Many of these applications use the specific properties of wavelets, such as coefficients that are determined numerically, multi-resolution analysis to decompose a signal, integrals, and derivatives obtained numerically, energy concentrated in its compact and base with orthogonal elements.

## 2. Short introduction to wavelet theory

For the development of topics presented in this chapter, the reader must have as a prerequisite knowledge of functional analysis, linear algebra, measure theory and integration, differential and integral calculus. It is important to note that the wavelet basis is for the wavelet transform as well as the trigonometric basis is for the Fourier transform. Generally, the term wavelet is also used as a wavelet transform. The following subsections present these initial prerequisites to the reader.

### 2.1 Preliminaries on Hilbert spaces

In this subsection, some mathematical concepts necessary for a better formal understanding of the wavelet tool are defined. The definitions, contained in this section, are due to the author [2].

Definition 2.1 The space $\mathcal{H}$ is said to be a Hilbert space, if an inner product $<,>$, associated with a standard $|| = \sqrt{<,>}$ has been defined in it. And a set of vectors $\{v_i\}$, for $i \in \mathbb{N}$ an orthonormal system is said if the internal product $<v_n, v_m> = \delta_{mn}$, for $m, n \in \mathbb{N}$.

Definition 2.2 A set of vectors $\{v_n\}$ is orthonormal, if and only if, for every finite set of complex numbers $x_n$, there is $\|\sum_n a_n x_n\|^2 = \sum |a_n|^2$, for $n \in \mathbb{N}$.

Definition 2.3 In Hilbert's $\mathcal{H}$ space, a set of vectors $\{v_n\}$ is said to be a Riez system, if there are constants $0 \leq c \leq C < \infty$ such that for any finite set of complex numbers $x_n$ if you have:

$$c \sum |a_n|^2 \leq \|\sum_n a_n x_x\|^2 \leq C \sum_n |a_n|^2 \tag{1}$$

Definition 2.4 The space $L^2(\mathbb{R})$ is said to be an integrable square function space, that is,

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \mapsto \mathbb{C} : \int_{\mathbb{R}} |f|(x)|^2 dx < \infty \right\} \tag{2}$$

For $f, g \in L^2(\mathbb{R})$, define the inner product $<f, g> = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$. On what, $\overline{g(x)}$ is the complex conjugate of the function $g(x)$.

In particular $\|f\| = \|f\|_2 = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{\frac{1}{2}}$, and $f$ is said to be an integrable square.

Definition 2.5 Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a function. The support of $f$, denoted by $suppf$, is the closing of the set $\{x \in \mathbb{R} : f(x) \neq 0\}$. A function $f$ is said to have compact support if the $suppf$ set is compact.[1]

Definition 2.6 We say that a function $f$ is generated by the basis functions $\{f_1, \ldots f_n\}$, if coefficients exist $\{c_1, \ldots, c_n\}$ such that:

$$f = \sum_{i=1}^{n} f_i c_i \tag{3}$$

The concepts presented here about orthogonality and support of a $f$ function, are fundamental to formalize the definition of wavelet. The following subsection presents the formal mathematical concept of wavelet.

[1] A set is said to be compact if it is limited and closed.

### 2.2 Definition of wavelet

This subsection aims to define wavelet [2], the main mathematical tool used in the development of this chapter. However, it is necessary to define the expansion and translation of mathematical operations beforehand.

Definition 2.7 *Given $a > 0$, the expansion operator, $D_a$, defined over a $f(x)$ function in $L^1$ or $L^2$ over $\mathbb{R}$, is given by, $D_a f(x) = a^{\frac{1}{2}} f(x)$.*

Definition 2.8 *Given $b \in R$, the translation operator, $T_b$, defined over a function $f(x)$, in $L^1$ or $L^2$ over $\mathbb{R}$, is given by, $T_b f(x) = f(xb)$.*

Thus, using the expansion and translation operations defined above, a family of functions $\psi_{j,k}(x)$ was built: $L^2 \to \mathbb{R}$, base orthogonal to $L^2(\mathbb{R})$.

$$\left\{\psi_{j,k}(x)\right\}_{j,k \in \mathbb{Z}} = \left\{2^{\frac{j}{2}}\psi\left(2^j x - k\right)\right\}_{j,k \in \mathbb{Z}} = \left\{D_{2^j} T_k \psi(x)\right\}_{j,k \in \mathbb{Z}} \tag{4}$$

The Definition 2.9, uses the family of functions $\left\{\psi_{j,k}(x)\right\}_{j,k \in \mathbb{Z}}$, to define the term mathematically wavelet.

Definition 2.9 A function $\psi(x)$ is called wavelet if the collection $\left\{\psi_{j,k}(x)\right\}_{j,k \in \mathbb{Z}}$ is an orthogonal basis on $L^2(\mathbb{R})$. Where $j$ and $k$ are the resolution and translation of wavelet respectively.

By varying the values of $j$ and/or $k$, it is possible to analyze with greater precision, for example, the behavior of functions that present abrupt changes in values and discontinuity. This type of analysis makes the wavelet a tool as or more efficient than the basic Fourier functions.

The definition 2.10 is another way used to define a wavelet.

Definition 2.10 A wavelet[2] is a short duration wave, which has an average value equal to zero.

Due to the definition 2.10, wavelets resemble Fourier sine and cosine basis functions. Analogously to what is done in the Fourier transform, which has sine and cosine functions as base functions, in wavelet analysis, a function is decomposed into a base of wavelet functions.

The Fourier transform $F(\omega)$ expression of a $f(t)$ function is given by (5):

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \tag{5}$$

The expression (5) means that the Fourier transform is the sum of every $f(t)$ sign multiplied by a complex exponential, which can be separated into cosine and sinusoidal components in the real and complex parts, respectively.

Similarly, the expression of the wavelet transform $W_{j,k}(f)$ of a function $f(t)$, is given by (6):

$$W_{j,k}(f) = \int_{-\infty}^{+\infty} f(t)\psi_{j,k}(t) dt \tag{6}$$

Similarly, the expression of the wavelet transform (6) is the internal product of the signal to be transformed by a wavelet function.

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[2] Anglophone term to designate a small wave, in the sense of having a fast duration.

In the following subsection, among the most varied types of wavelets, the Daubechies wavelets are highlighted, which are the basis for the development of this chapter.

## 2.3 Daubechies wavelet properties

At 1988, a family of compact support wavelets [7] is built by Ingrid Daubechies. This family of wavelets has highly well-located elements. Each member wavelet is governed by a set of $N$ integer coefficients and $k = \{0.1, \ldots, N-1\}$ coefficients through scale relations (7) and (8). The $a_k$ and $a_{1-k}$ coefficients, which appear in the (7) and (8), are called filter coefficients and verify the following relations:

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k) \tag{7}$$

$$\psi(x) = \sum_{k=2-N}^{1} (-1)^k a_{1-k} \phi(2x - k) \tag{8}$$

In the **Figures 1** and **2** below, we have the graphical representation of the Daubechies wavelet functions $\phi$ and $\psi$ of kind 4.

The functions $\phi$ in (7) and $\psi$ in (8) are called the scale function $\phi$ and wavelet function $\psi$, respectively. The fundamental support of the scale function[3] is the interval $[0, N-1]$ as the fundamental support of wavelet function $\psi(x)$ is the interval $\left[1 - \frac{N}{2}, \frac{N}{2}\right]$. In the case of $N = 4$, we have the graphs of the **Figures 1** and **2**.

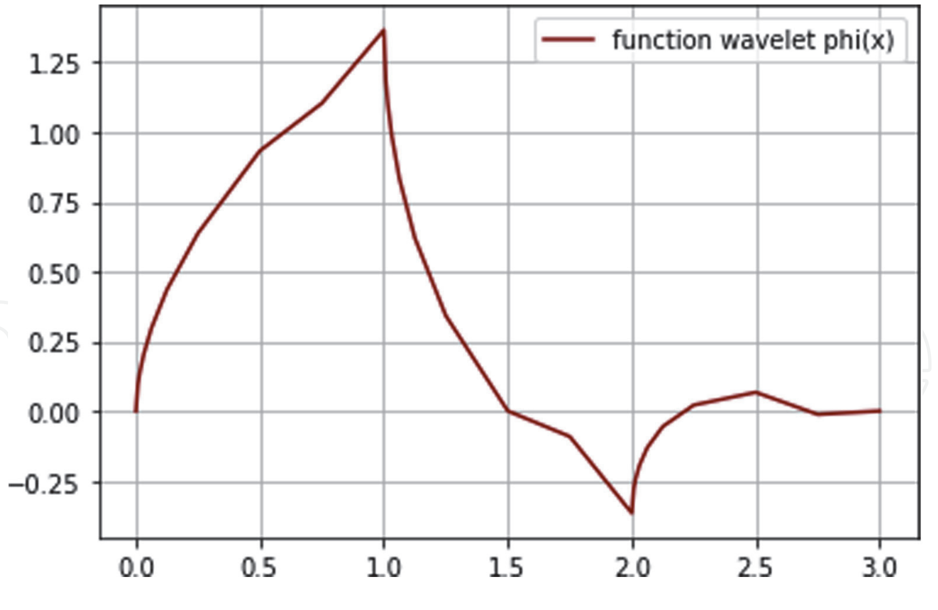


**Figure 1.**
*Daubechies wavelets $\phi$. Source: This figure was generated by the author using the python programming language.*

[3] We emphasize that the scale function has energy concentrated in its support that is determined by the genus of the wavelet, that is, $supp(\phi) = [0, N-1]$, and that the total energy of the scale function is unitary, that is, $\int_{-\infty}^{+\infty} \phi dx = 1$.

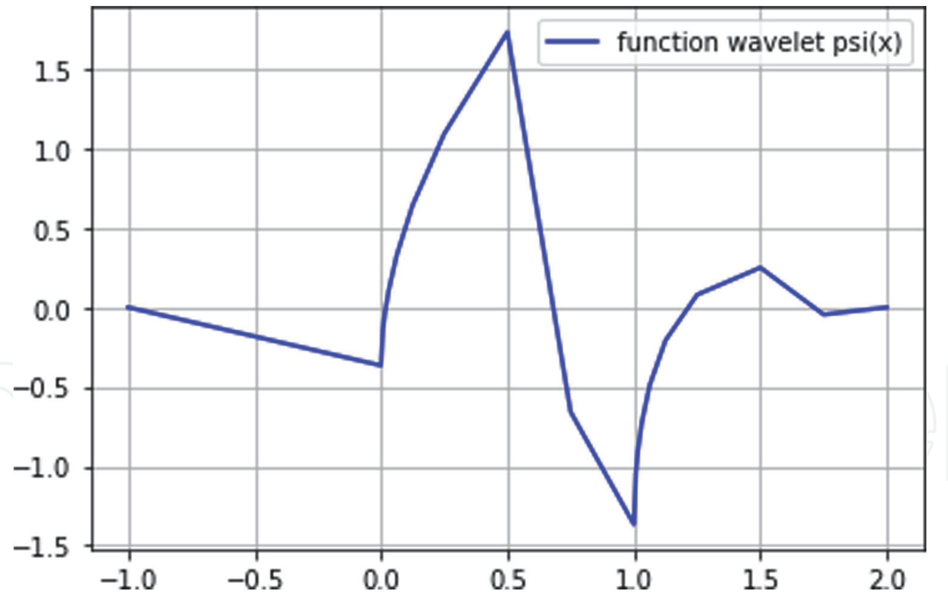


**Figure 2.**
*Daubechies wavelet $\psi$. Source: This figure was generated by the author using the python programming language.*

To determine the filter coefficients $a_k$ and $a_{1-k}$, which appear in the (7) and (8), we use the relations (9)–(12) below.

$$\sum_{k=0}^{N-1} a_k = 2 \tag{9}$$

$$\sum_{k=0}^{N-1} a_k a_{k-m} = \delta_{0,m} \tag{10}$$

$$\sum_{k=0}^{N-1} (-1)^k a_{1-k} a_{k-2m} = 0 \tag{11}$$

$$\sum_{k=0}^{N-1} (-1)^k k^m a_k = 0, \quad m = 0.1, \ldots, \frac{N}{2} - 1, \tag{12}$$

where $\delta_{k,m}$ is the Kronecker Delta function.

## 3. Generating an analytical function of the type $x^k$ using wavelets

Analytical functions are those that can be locally around a point $x_0$ expanded in a Taylor series, according to the following expression.

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tag{13}$$

In general according to the author [8], any $f(x)$ function can be represented in terms of a wavelet base, as follows:

$$f(x) = \sum_{m=-\infty}^{+\infty} c_k \phi(x - m) = \sum_{m=-\infty}^{+\infty} c_m \phi_m(x). \tag{14}$$

The $c_k$ coefficients are called moments of the scale functions. In particular, for $f(x) = x^k$, we have the expression (15), below:

$$x^k = \sum_{m=-\infty}^{+\infty} \frac{M_m^k}{2^{jk}} \phi\left(2^j x - m\right), \tag{15}$$

Since $M_m^k$ the moment of the wavelet scales concerning the $x^k$ monomial, where $k$ is the degree of the polynomial, $m$ and $j$ are the translation and resolution of the $\phi$ wavelet. The justification for the construction of the equation is found in the work of [8–10], in which the author concludes that the $c_m^j$ coefficients for approximating a monomial of the $x^k$ form, using a Daubechies wavelet base $\phi$, looks like this:

$$c_m^j = \frac{M_m^k}{2^{jk}} \tag{16}$$

The justification used in the approximation (15) of a polynomial function of type $f(x) = x^k$ derives from the number of null moments,

$$\int_{-\infty}^{+\infty} x^k \psi(x) dx = 0, \quad k = 0.1, \dots, \frac{N}{2} - 1 \tag{17}$$

According to the Eq. (17), the $N$ Daubechies Wavelet has $\frac{N}{2}$ vanish moments, being possible to represent a polynomial of degree at most $\frac{N}{2} - 1$, using the $\phi(x)$ scale function. The polynomial approximation using the scale function is formalized in the following definition.

Definition 3.1 A wavelet has $p$ vanish moments (18), if and only if, the wavelet scale function $\phi$ can generate polynomials of degree up to $p - 1$ [Eq. (19)]. That is, the scale function alone can be used to represent these polynomials. The fact that it has more null moments means that the scale function can represent more complex functions.

$$\int_{-\infty}^{+\infty} x^m \psi(x) dx = 0; \quad m = 0, 1, \dots, \frac{N}{2} - 1 \tag{18}$$

$$f(x) = p_1 + p_2 x + \dots + p_{k-1} x^k, \quad k \leq \frac{N}{2} - 1 \tag{19}$$

In general, a Daubechies wavelet of kind $N$, properly translated and adjusted to the appropriate resolution level, generates a polynomial of degree $k$, with the relation between $N$ and $k$ given by $N = 2k + 2$. For example, to generate a polynomial of degree 1 a wavelet of Daubechies of kind 4 is necessary.

To generate a polynomial with $n + 1$ terms, in the function of Daubechies wavelets of genres $4, 6, 8, \dots, N - 1$, we use the momentum equation and the polynomial expansion as a function of wavelets.

$$p(x) = \sum_{k=0}^{n} a_k x^k, \tag{20}$$

where $x^k$, takes the form

$$x^k = \sum_{m=-\infty}^{+\infty} \frac{M_m^k}{2^{jk}} \phi\left(2^j x - m\right) \tag{21}$$

Substituting the Eq. (21) in (20), we have:

$$p(x) = \sum_{k=0}^{n} \frac{a_k}{2^{jk}} \sum_{m=-\infty}^{+\infty} \phi\left(2^j x - m\right) M_m^k \tag{22}$$

where $k$ is the degree of the polynomial $j$ and $m$ are the resolution and translation of the wavelet respectively.

In the next subsection, the calculation of the moment generating function, which appears in the expression 21 as a coefficient of $x^k$, is shown in detail.

## 4. Moment generating function

The calculation of the moment generating function according to the author [11] is of fundamental importance to approximate the functions by wavelets. The deduction of the moment-generating function now begins. For this, the mathematical expression is used

$$M_m^k = \int_{-\infty}^{+\infty} x^k \phi(x-m)dx \tag{23}$$

which refers to the moment of the wavelet scale $\phi$ in relation to the monomial $x^k$. For $m = k = 0$, in (23), we have:

$$M_0^0 = \int_{-\infty}^{+\infty} \phi(x)dx = 1. \tag{24}$$

Substituting $m = 0$ in the Eq. (23), we have:

$$M_0^k = \int_{-\infty}^{+\infty} x^k \phi(x)dx = \int_{-\infty}^{+\infty} x^k \sum_{s=0}^{N-1} a_s \phi(2x-s)dx \tag{25}$$

$$M_0^k = \sum_{s=0}^{N-1} a_s \int_{-\infty}^{+\infty} x^k \phi(2x-s)dx. \tag{26}$$

Note that the variable $s$, in the Eq. (26), also represents a translation.

Making the substitution $z = 2x$, $\frac{dz}{dx} = 2$, $dx = \frac{dz}{2}$, we have:

$$M_0^k = \sum_{s=0}^{N-1} a_s \int_{-\infty}^{+\infty} x^k \phi(2x-s)dx \tag{27}$$

$$M_0^k = \frac{1}{2^{k+1}} \sum_{s=0}^{N-1} a_s M_s^k \tag{28}$$

Using the substitution $x - m = t$, $\frac{dx}{dt} = 1$, $dx = dt$, in (23), we have:

$$M_m^k = \int_{-\infty}^{+\infty} x^k \phi(x-m)dx = \sum_{r=0}^{k} \binom{k}{r} m^{k-r} M_0^r \tag{29}$$

Now consider the equations:

$$M_0^k = \frac{1}{2^{k+1}} \sum_{s=0}^{N-1} a_s M_s^k \tag{30}$$

$$M_s^k = \sum_{r=0}^{k} \binom{k}{r} s^{k-r} M_0^r \tag{31}$$

Substituting $M_s^k$ in $M_0^k$, we have: (note that $m = s$)

$$M_0^k = \frac{1}{2^{k+1}} \sum_{s=0}^{N-1} a_s \sum_{r=0}^{k} \binom{k}{r} s^{k-r} M_0^r \tag{32}$$

Now separate the last term of the sum (32), $(r = k)$, to place the term on the left side of the equation:

$$M_0^k = \frac{1}{2^{k+1}} \sum_{s=0}^{N-1} a_s \sum_{r=0}^{k-1} \binom{k}{r} s^{k-r} M_0^r + \frac{1}{2^{k+1}} \sum_{s=0}^{N-1} a_s \binom{k}{r} s^{k-k} M_0^k \tag{33}$$

Using the fact that $\sum_{s=0}^{N-1} a_s = 2$, we have:

$$M_0^k = \frac{1}{2(2^k - 1)} \sum_{r=0}^{k-1} \binom{k}{r} M_0^r \sum_{s=0}^{N-1} a_s s^{k-r} \tag{34}$$

Thus, the equations are obtained:

$$M_m^k = \sum_{r=0}^{k} \binom{k}{r} m^{k-r} M_0^r \tag{35}$$

$$M_0^k = \frac{1}{2(2^k - 1)} \sum_{r=0}^{k-1} \binom{k}{r} \sum_{s=0}^{N-1} a_s s^{k-r} M_0^r \tag{36}$$

From (35), (34), and (24), we get the moment generating function $M_m^k : \mathcal{W} \to \mathbb{R}$, where $\mathcal{W}$ is wavelet space, $m$ is the translation of the scale function and $k$ is the degree of the polynomial to be approximated.

$$M_m^k = \begin{cases} \dfrac{1}{2(2^k - 1)} \displaystyle\sum_{r=0}^{k-1} \binom{k}{r} \sum_{s=0}^{N-1} a_s s^{k-r} M_0^r, & \text{se} \quad m = 0; k \neq 0 \\ \displaystyle\sum_{r=0}^{k} \binom{k}{r} m^{k-r} M_0^r, & \text{se} \quad m \neq 0; k \neq 0 \\ 1, & \text{se} \quad m = k = 0, \end{cases} \tag{37}$$

The analytical expression for $M_m^k$ was developed during the author's research [11] and to validate the results found, a comparative study was made with other numerical results [12, 13] of the scientific literature.

Similar to what was done with the calculation of the moments for the function $\phi$, there is also the calculation of the moments for the function $\psi$. This is given by integral (38)

$$\int_{-\infty}^{+\infty} x^m \psi(x) dx = 0; \quad m = 0, 1, \ldots, \frac{N}{2} - 1. \tag{38}$$

The following is an example of the calculation of the moments for the case of Daubechies wavelets of a kind $N = 4$.

Example 4.1 In this example, the Daubechies wavelet of kind 4 is used to generate the analytical polynomial function $f(x) = x$. According to the definition 3.1, the scale function of Daubechies of genus $N = 4$, generates a line (polynomial of degree 1). To represent a 1 monomial with a 4 Daubechies wavelet in the $[0, 1]$ range, the translations $\phi(x), \phi(x+1), \phi(x+2)$, whose supports are $[0.3], [-1.2], [-2.1]$, that is:

$$x = \sum_{m=-2}^{0} \frac{M_m^1}{2^{jk}} \phi(2^j x - m) = \frac{M_{-2}^1}{2^j} \phi(x+2) + \frac{M_{-1}^1}{2^j} \phi(x+1) + \frac{M_0^1}{2^j} \phi(x) \tag{39}$$

The support of the linear combination (39), represented in **Figure 3**, is obtained by the intersection of the supports of the translations of the function $\phi(x)$. This intersection results in the interval $I = [0.1]$. This fact defines well the function to be integrated in the $I$ range. In **Figure 3**, the number of translations of the function $\phi(x)$ to generate $f(x) = x$ is illustrated.

**Figure 4** shows the graph of translated functions $\phi(x), \phi(x+1)$ and $\phi(x+2)$ respectively, that form a base to generate the function $f(x) = x$.

The calculation using the moment generating function depends on the Daubechies wavelet coefficients of kind 4. These coefficients are obtained by the Eqs. (9)–(12), which gives rise to the following non-linear system.

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 2 \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 2 \\ a_0 a_2 + a_1 a_3 = 0 \\ -a_1 + 2a_2 - 3a_3 = 0 \end{cases} \tag{40}$$

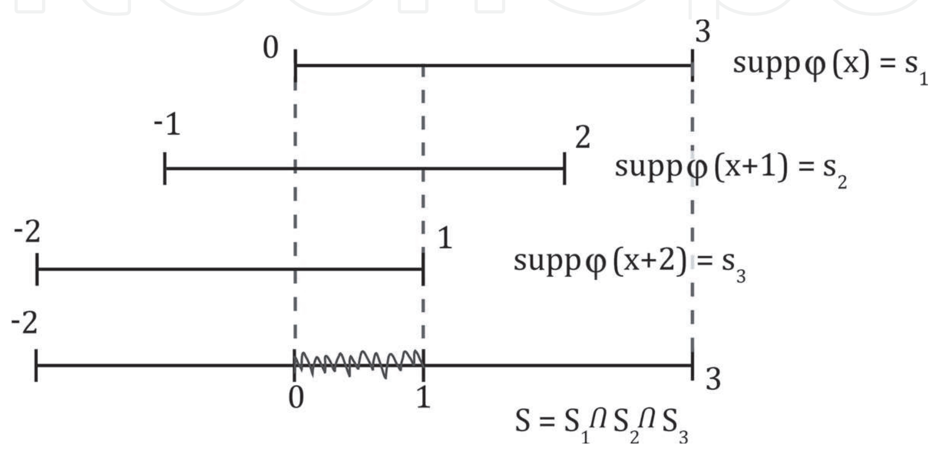


**Figure 3.**
*Translations required to represent the analytical function $f(x) = x$ using Daubechies wavelets of kind 4. Source: Own authorship.*

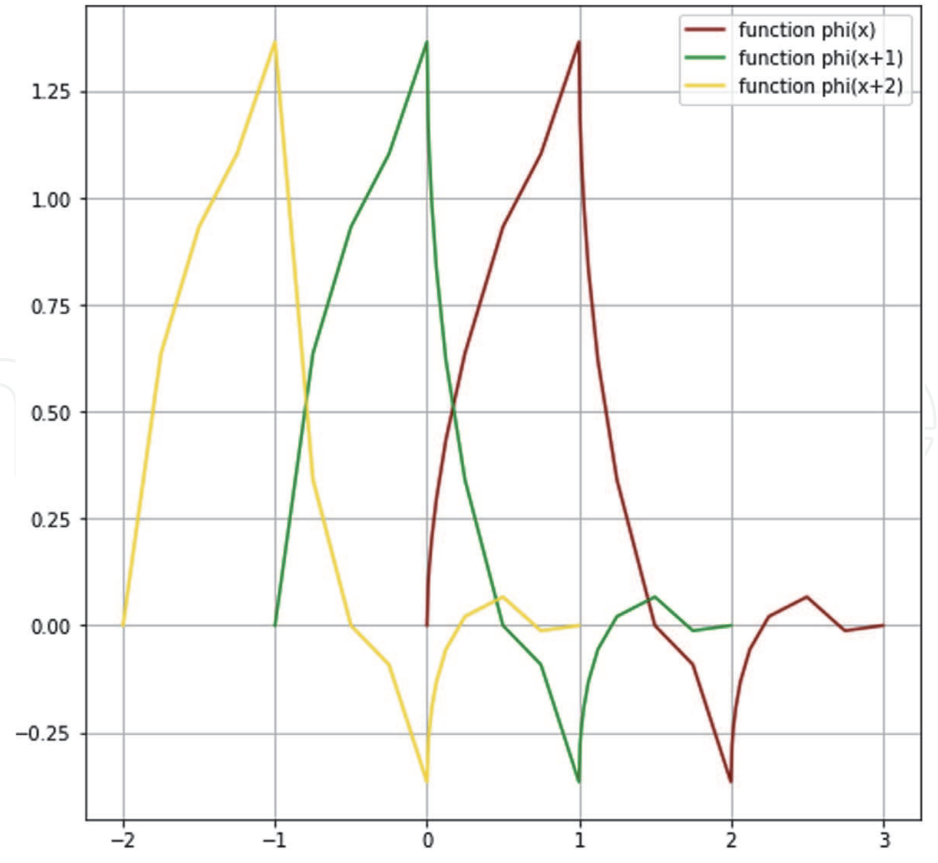


**Figure 4.**
*Translations required to represent the analytical function $f(x) = x$ using Daubechies wavelets of kind 4. Source: This figure was generated by the author using the python programming language.*

The solution of this system is the irrational numbers $a_0, a_1, a_2, a_3$, given by:

$$\begin{cases} a_0 = 0,683012701892219 \\ a_1 = 1.183012701892219 \\ a_2 = 0,316987298107781 \\ a_3 = -0,183012701892219 \end{cases} \tag{41}$$

Using the moment generating function for the case where $m = 0$, we have:

$$M_0^1 = \frac{1}{2}\sum_{r=0}^{0}\binom{1}{r}M_0^r\sum_{s=0}^{3}a_s s^{1-r} = \frac{1}{2}M_0^0(a_1 + 2a_2 + 3a_3) = \frac{a_1 + 2a_2 + 3a_3}{2} \\ = 0,633974600 \tag{42}$$

Proceeding with the calculations, we obtain:

$$M_m^k = \sum_{r=0}^{1}\binom{1}{r}m^{1-r}M_0^r = m + M_0^1 = m + \frac{a_1 + 2a_2 + 3a_3}{2} = m + 0,633974600 \tag{43}$$

Replacing the value of $m$ by $m = -1$, $m = -2$ and $k = 1$, we obtain:

$$M_{-1}^1 = -0,366025400 \tag{44}$$

$$M_{-2}^1 = -1,366025400 \tag{45}$$

So, the representation for the $x$ polynomial (for a resolution $j = 0$) is:

$$x = 0,634\phi(x) - 0,366\phi(x+1) - 1.366\phi(x+2) \tag{46}$$

In **Figure 5**, we have the graphical representation of the function obtained of the expression (46). Here the function $f(x) = x$ is generated by linear combination of wavelets $\phi(x)$, $\phi(x+1)$ and $\phi(x+2)$.

The representation for the expression (46) using the summation is given by,

$$x = \sum_{m_1=-2}^{0} \frac{M^1_{m_1}}{2^{jk}} \phi\left(2^j x - m_1\right) \tag{47}$$

The expression for writing polynomials of degrees $k = 2$ and $k = 3$ in terms of Daubechies wavelets is given by

$$x^2 = \sum_{m_2=-4}^{2} \frac{M^2_{m_2}}{2^{jk}} \phi\left(2^j x - m_2\right) \tag{48}$$

$$x^3 = \sum_{m_3=-6}^{0} \frac{M^3_{m_3}}{2^{jk}} \phi\left(2^j x - m_3\right) \tag{49}$$

See that to generate the polynomials (48), (49) is necessary to use Daubechies wavelets of kind 6 and 8, according with the definition 3.1.

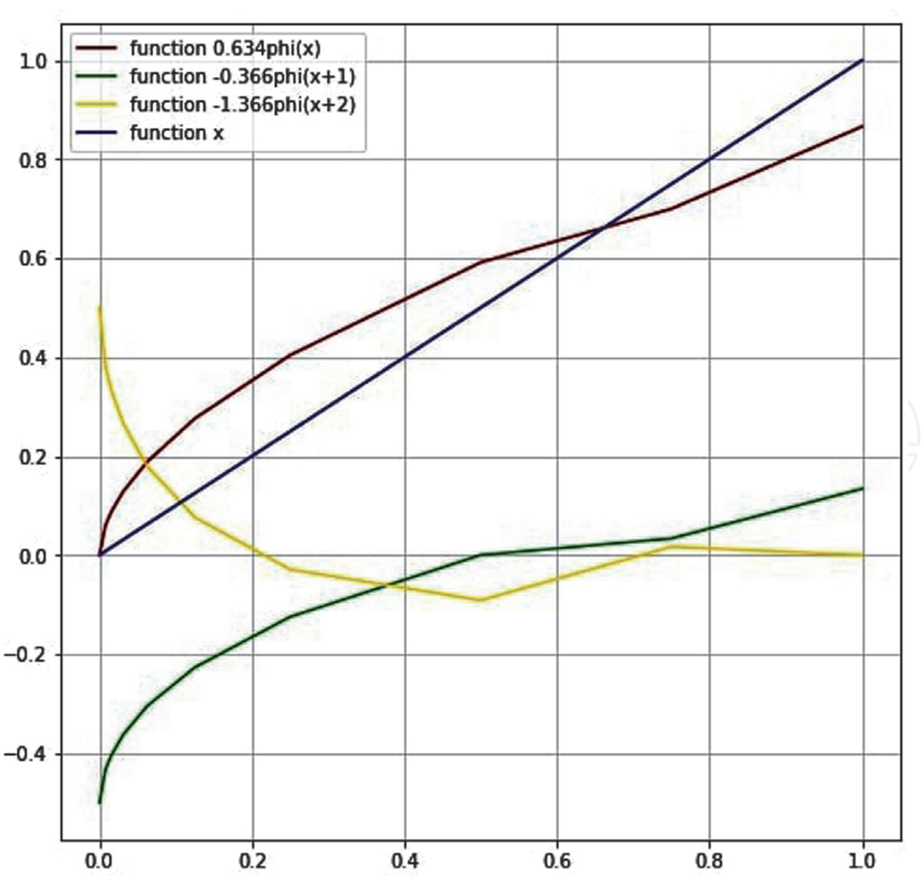


**Figure 5.**
*Function $f(x) = x$ using Daubechies wavelets of kind 4. Source: This figure was generated by the author using the python programming language.*

### 4.1 Taylor polynomial using Daubechies wavelets

The Taylor polynomial or Taylor series is an expression that allows the calculation of the local value of a function $f$ using your derivatives. For this, the function $f$ must be of class $C$ infinite (represented by $C^{\infty}$) which implies that the $f$ is infinitely derivable in an interval containing a point $x_0$. The expression for the Taylor polynomial for the function $f$ is as follows,

$$f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{50}$$

The expression (50) developed around $x_0 = 0$ is:

$$f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!}(x)^k \tag{51}$$

Making use of the expression (21), we have:

$$f(x) = \sum_{k=0}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{f^{(k)}(0)}{k!} \frac{M_m^k}{2^{jk}} \phi\left(2^j x - m\right) \tag{52}$$

The expression (52) is another way of writing Taylor's polynomial using Daubechies Wavelets.

Example 4.2 Consider the analytical function $f(x) = e^x$, using Daubechies wavelet of kind a $N = 4$ is possible to write this function $f$ in terms of this wavelet. For this, Taylor's series development around the point $x_0 = 0$ of this function is given by:

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \tag{53}$$

Using only two summation terms in the expression (53), we have:

$$e^x \approx \sum_{n=0}^{1} \frac{x^n}{n!} = 1 + x \tag{54}$$

Using the expression (46), we have:

$$e^x \approx 1 + x = 1 + 0,634\phi(x) - 0,366\phi(x + 1) - 1.366\phi(x + 2) \tag{55}$$

The expression (55) allows us to approximate the exponential function using a base of Daubechies wavelets. This type of approximation, although simple for this case, is very useful in the case of representation for functions other types.

In the following example, the expression (46) is used to approximate Taylor's series developments for the functions $s(x) = e^x, f(x) = \cosh(x), g(x) = \sinh(x)$ and $h(x) = \ln(1 + x)$.

Example 4.3 For the functions $f(x) = \cosh(x), g(x) = \cos(x)$ and $h(x) = \sec(x)$. Taylor's series development of these functions around the point $x_0 = 0$ is:

$$\cosh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \tag{56}$$

| Function | Value in $x = 1$, Taylor Series[4] | Value in $x = 1$, Daubechies Wavelets[5] | Error % |
|---|---|---|---|
| $s(x) = e^x$ | 2.716666667 | 2.716735469443329 | 0.0025% |
| $f(x) = \cosh(x)$ | 1.543088161791753 | 1.543058311287478 | 0.0019% |
| $g(x) = \sinh(x)$ | 1.1750199840127897 | 1.1750591521108822 | 0.00376% |
| $h(x) = \ln(1+x)$ | 0.6456349203122008 | 0.6456349190214307 | $1, 9.10^{-7}$% |

[4]*Calculation using Taylor Series.*
[5]*Calculation using Daubechies wavelets of kind 4.*

**Table 1.**
*Comparison of the values obtained by the Taylor series and by Daubechies wavelets.*

$$\sinh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{57}$$

$$\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^n}{n} \tag{58}$$

In order to verify the potentiality of the application of Daubechies wavelets we will calculate the value of the functions in (53), (56), (57) and (58) evaluated at point x = 1. Considering only 7 terms in each summation. For obtain the results using Daubechies wavelets we apply the expression (55) in each summation (53), (56), (57) and (58). In the **Table 1** we have a comparison between the calculation of the values of the functions $s(x) = e^x, f(x) = \cosh(x), g(x) = \sinh(x)$ and $h(x) = \ln(1+x)$ evaluated at point $x = 1$, using the Taylor series and the Daubechies wavelets of kind 4.

**Table 1** appears here only as a way of showing the quality of the approximations using the Daubechies wavelets of kind 4. Obviously if we want more precise values, we must use Daubechies wavelets of the kind greater than 4. This will cause changes in the resolution and translation of each wavelet, but the result will be even better.

## 5. Conclusions

Daubechies wavelets are quite versatile mathematical tools. They can be used to analyze, generate, decompose a function, or even a signal that is represented by an analytical function. This type of application is widely used, for example, in electrical engineering in studies of magnetic fields and electric fields. The theory exposed in this chapter provides tools to carry out these studies. The use of the Taylor series as a way of approximating analytical functions is a very used technique in applied mathematics. Making use of the Taylor series with wavelets is another option to perform an approximation of analytical functions. In future work, we are researching other wavelets, for example Deslauries-Dubuc interpolets, that have an even better approach quality. As Deslauriers-Dubuc interpolets and others in research.

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