Chapter

Mode-I and Mode-II Crack Tip Fields in Implicit Gradient Elasticity Based on Laplacians of Stress and Strain. Part I: Governing Equations

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Abstract

Models of implicit gradient elasticity based on Laplacians of stress and strain can be established in analogy to the models of linear viscoelastic solids. The most simple implicit gradient elasticity model including both, the Laplacian of stress and the Laplacian of strain, is the counterpart of the three-parameter viscoelastic solid. The main investigations in Parts *I*, *II*, and *III* concern the "three-parameter gradient elasticity model" and focus on the near-tip fields of Mode-I and Mode-II crack problems. It is proved that, for the boundary and symmetry conditions assumed in the present work, the model does not avoid the well-known singularities of classical elasticity. Nevertheless, there are significant differences in the form of the asymptotic solutions in comparison to the classical elasticity. These differences are discussed in detail on the basis of closed-form analytical solutions. Part *I* provides the governing equations and the required boundary and symmetry conditions for the considered crack problems.

Keywords: implicit gradient elasticity, Laplacians of stress, Laplacians of strain, micromorphic and micro-strain elasticity, plane strain state

1. Introduction

The most simple constitutive law in explicit gradient elasticity is the model with equation:

$$\Sigma_{ij} = \mathbb{C}_{ijmn}\,\varepsilon_{mn} - c_2\mathbb{C}_{ijmn}\,(\Delta\varepsilon)_{mn} \quad (\mathrm{KG-Model}). \tag{1}$$

Here, $\mathbf{\Sigma} = \mathbf{\Sigma}^T$ is the Cauchy stress tensor, ε is the strain tensor, $\mathbb{C}$ is the isotropic elasticity tensor, Δ is the Laplacian operator, and c_2 is a material parameter, with $\sqrt{c_2}$ denoting an internal material length. The components in Eq. (1) are referred to a Cartesian coordinate system. It seems that the constitutive law (1) has been introduced for the first time by Altan and Aifantis [1]. These authors (cf. also Georgiadis [2]) showed that the constitutive Eq. (1) leads to regular strain solutions at the crack tip of Mode-III crack problems. However, the stress field remains singular at the crack tip as in the case of classical elasticity. Moreover, Altan and Aifantis [1], as well as Georgiadis [2], presented an appropriate isotropic energy function for the mechanical model in Eq. (1) in the context of Mindlins gradient elasticity theory (see Mindlin [3] as well as Mindlin and Eshel [4]). An alternative approach to this model has been proposed in Broese et al. [5], where an analogy between gradient elasticity models and linear viscoelastic solids is established. According to this analogy, Eq. (1) is regarded as the gradient elasticity counterpart of the Kelvin viscoelastic solid. The short hand notation "KG-Model" in Eq. (1) stands for "Kelvin-Gradient-Elasticity-Model."

Now, the question arises, if a gradient elasticity model including both, the Laplacian of stress and the Laplacian of strain, could remove both, the singularities of stress and the singularities of strain at the crack tip (cf. Gutkin and Aifantis [6]). The most simple generalization of Eq. (1), including the Laplacians of stress and strain, reads as follows:

$$\Sigma_{ij} - c_1 (\Delta \mathbf{\Sigma})_{ij} = \mathbb{C}_{ijmn} \varepsilon_{mn} - c_2 \mathbb{C}_{ijmn} (\Delta \varepsilon)_{mn} \quad (3 - \mathrm{PG} - \mathrm{Model}), \qquad (2)$$

where the same notation as in Eq. (1) applies and $\sqrt{c_1}$ is a further internal material length. To our knowledge, model (2) has been introduced for the first time by Gutkin and Aifantis [6]. These authors proposed the gradient elasticity law (2) ad hoc in an attempt to eliminate the singularities of stress and strain of defects. Equations of the form (2) are known as models of implicit gradient elasticity (see Askes and Gutiérrez [7]).

Broese et al. [5] proved that Eq. (2) can be derived as a particular case of Mindlins micro-structured elasticity, which arises whenever the micro-deformation of the micromorphic continuum is supposed to be a symmetric tensor. Because the micro-structured elastic continuum of Mindlin and the micromorphic elastic continuum of Eringen (see, e.g., Eringen and Suhubi [8] and Eringen [9]) are essentially equivalent to each other, in the present work we will call both as micromorphic continua. According to Forest and Sievert [10], the resulting micromorphic theory is named micro-strain theory. It is shown in Broese et al. [5] that, in the context of micro-strain elasticity, the 3-PG-Model (2) can be derived as a combination of elasticity constitutive laws and the equilibrium equation for the so-called double stress.

On the other hand, Broese et al. [5] showed that Eq. (2) can be established alternatively by supposing the continuum to be classical, i.e., exhibiting only classical displacement degrees of freedom, but in the framework of the non-conventional thermodynamics proposed in Alber et al. [11]. To be more specific, the micro-deformation variable of the micro-strain approach has to be viewed as an internal state variable analogous to the inelastic strain in linear viscoelasticity. Eq. (2) then turns out to be a constitutive law, which is the counterpart in gradient elasticity of the three-parameter viscoelastic solid. The short hand notation "3-PG-Model" stands for "3-Parameter-Gradient-Elasticity-Model." A general analogy to the constitutive laws describing viscoelastic solids can be established by using a nonstandard spring in gradient elasticity corresponding to the dashpot element in linear viscoelasticity and the Laplacian operator Δ in place of the ordinary time derivative in the evolution laws of dashpot elements. Using virtual power balance arguments, Broese et al. [5] derived the same boundary conditions along the lines of the second approach as in the micro-strain approach. But now the boundary conditions have to be understood as constitutive boundary conditions, analogous to the constitutive initial conditions in viscoelasticity.

Because all resulting governing equations and boundary conditions in the two approaches are equal to each other, we shall proceed further by regarding the

3-PG-Model as a particular case of the micro-strain elasticity. The present work (Parts *I*, *II*, and *III*) is concerned with the near-tip fields predicted by the 3-PG-Model for Mode-I and Mode-II types of crack problems. Unlike statements made somewhere else (see Part *II*), we prove that, for the assumptions made here, the 3-PG-Model does not eliminate the well-known singularities of classical elasticity. Nevertheless, compared with the form of asymptotic solutions in classical elasticity, there are interesting new aspects, which are discussed in detail in Part *II* on the basis of closed-form analytical solutions. Part *I* provides the governing equations and the required boundary and symmetry conditions in order to establish the analytical solutions.

2. Preliminaries: notation

Throughout the paper, we largely use the same notation as in Mindlin [3] and Mindlin and Eshel [4], in order to facilitate the comparison with these works. The deformations are assumed to be small, so we do not distinguish, as usually done, between reference and actual configuration. All indices will have the range of integers $(1, 2, 3)$, while summation over repeated indices is implied. Explicit reference to space and time variables, upon which a function may depend, will be dropped in most part of the paper. Also, we shall not distinguish between functions and their values. However, if necessary, we shall give explicitly the set of variables which the function depends on.

Let $\mathcal{B}$ be a material body which may be identified by the position vectors $\boldsymbol{x} = x_i \boldsymbol{e}_i$, with respect to a Cartesian coordinate system $\{x_i\}$ inducing the orthonormal basis $\{\boldsymbol{e}_i\}$. The body $\mathcal{B}$ occupies the space V in the three-dimensional Euclidean space we deal with. We indicate by $\boldsymbol{n}$ the outward unit normal vector to the surface ∂V bounding the space V. Small Latin indices will be used in conjunction with Cartesian coordinates and related components. If f is a function of the Cartesian coordinates x_i, then we shall use the notations for partial derivatives as follows:

$$\partial_i f \coloneqq \frac{\partial f}{\partial x_i}, \quad \partial_{ij} f \coloneqq \frac{\partial^2 f}{\partial x_i \, \partial x_j}. \tag{3}$$

Let $\nabla = \partial_i \boldsymbol{e}_i$ be the nabla operator and $\mathfrak{a}$ be some vector or higher order tensor. The gradient, the divergence, and the Laplacian of $\mathfrak{a}$ are defined, respectively, by $\operatorname{grad} \mathfrak{a} \equiv \nabla \mathfrak{a} \coloneqq \nabla \otimes \mathfrak{a}$, $\operatorname{div} \mathfrak{a} \coloneqq \nabla \cdot \mathfrak{a}$, and $\Delta \mathfrak{a} = \operatorname{div} \operatorname{grad} \mathfrak{a}$, where $\cdot$ and $\otimes$ are the scalar and the tensorial products between two vectors. It is helpful to use notations for components of the form $(\mathfrak{a})_{ij\dots} = \mathfrak{a}_{ij\dots}$. Thus, if A_{ij} and $\mathcal{A}_{ijk}$ are the Cartesian components of a second-order tensor $\boldsymbol{A}$ and a third-order tensor $\boldsymbol{\mathcal{A}}$, then, with respect to the Cartesian basis $\{\boldsymbol{e}_i\}$, we have the following equations:

$$(\nabla \boldsymbol{A})_{ijk} = \partial_i A_{jk}, \tag{4}$$

$$(\operatorname{div} \boldsymbol{A})_i = \partial_j A_{ji}, \tag{5}$$

$$(\Delta \boldsymbol{A})_{ij} = \partial_{kk} A_{ij}, \tag{6}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{ij} = \partial_k \mathcal{A}_{kij}. \tag{7}$$

We denote by $\mathbb{C}$ the fourth-order isotropic elasticity tensor with Cartesian components as follows:

$$\mathbb{C}_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu \left(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} \right), \tag{8}$$

where λ and μ are the Lamé constants and δ_{ij} is the Kronecker delta. For some calculations, it will be convenient to use the Young's modulus E and the Poisson ratio ν,

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = 2\mu(1+\nu). \tag{9}$$

Since $\mathbb{C}$ satisfies the symmetry properties

$$\mathbb{C}_{ijmn} = \mathbb{C}_{jimn} = \mathbb{C}_{mnij} = \mathbb{C}_{ijnm}, \tag{10}$$

we have for every second-order tensor $\boldsymbol{A}$, that

$$\mathbb{C}_{ijmn} A_{mn} = A_{mn} \mathbb{C}_{mnij}. \tag{11}$$

For any tensor $\mathfrak{a}$ with components $\mathfrak{a}_{i\,\ldots\,jk\,\ldots\,p}$, we write $\mathfrak{a}_{i\,\ldots\,(jk)\,\ldots\,p}$ for its symmetric part with respect to the indices j and k. Thus, if $\boldsymbol{A}^{(s)}$ is the symmetric part of a second-order tensor $\boldsymbol{A}$, then $A_{(ij)} = A_{ij}^{(s)}$. Corresponding notations apply with regard to components related to curvilinear coordinate systems. Of particular interest for our work are cylindrical coordinates $\{r, \varphi, z\}$. We find it convenient to use Greek indices $\alpha, \beta, \ldots$ to indicate both, physical components with respect to cylindrical coordinates and cylindrical coordinates itself. Thus, e.g., we write $A_{\alpha\beta}$ for the physical components of the second-order tensor $\boldsymbol{A}$ and denote by $[A_{\alpha\beta}]$ the matrix of components,

$$[A_{\alpha\beta}] = \begin{pmatrix} A_{rr} & A_{r\varphi} & A_{rz} \\ A_{\varphi r} & A_{\varphi\varphi} & A_{\varphi z} \\ A_{zr} & A_{z\varphi} & A_{zz} \end{pmatrix}. \tag{12}$$

Similar notations hold for any tensor of arbitrary order. The summation convention applies in analogous manner, e.g., we have $A_{\alpha\alpha} = A_{rr} + A_{\varphi\varphi} + A_{zz}$. Because cylindrical coordinate systems are orthogonal, the algebraic operations between the corresponding physical components of tensors are identical in form to those with respect to Cartesian components. Moreover, the physical components of isotropic tensors are identical to their Cartesian components, e.g., the physical components of the isotropic elasticity tensor $\mathbb{C}$ in Eq. (8) are given as follows:

$$\mathbb{C}_{\alpha\beta\gamma\zeta} = \lambda \delta_{\alpha\beta} \delta_{\gamma\zeta} + \mu \left(\delta_{\alpha\gamma} \delta_{\beta\zeta} + \delta_{\alpha\zeta} \delta_{\beta\gamma} \right). \tag{13}$$

The physical components with respect to cylindrical coordinates of $\nabla \boldsymbol{A}$, $\Delta \boldsymbol{A}$ and $\operatorname{div}\boldsymbol{\mathcal{A}}$ are calculated in A. For partial derivatives of a function f with respect to cylindrical coordinates, we use notations, in analogy to Eq. (3), of the forms

$$\partial_r f := \frac{\partial f}{\partial r}, \quad \partial_\varphi f := \frac{\partial f}{\partial \varphi}, \quad \partial_z f := \frac{\partial f}{\partial z}, \tag{14}$$

$$\partial_{rr} f := \frac{\partial^2 f}{\partial r\, \partial r}, \ \ldots. \tag{15}$$

3. Governing equations for the 3-PG-Model

This section provides a short overview about the 3-PG-Model in a form which is adequate for developing analytical solutions.

3.1 The 3-PG-Model as particular case of micro-strain elasticity

Assume the material body to be a micromorphic continuum. Besides the classical kinematical degrees of freedom, micromorphic continua are characterized by additional degrees of freedom due to the deformations of the micro-continua, which are assumed to be attached at every point of the macro-continuum (see Mindlin [3] and Broese et al. [5]). Therefore, in the micromorphic continuum theory, a nonclassical (double) stress and a nonclassical stress power are introduced in addition to the classical ones, but otherwise the theory is formulated in the framework of classical thermodynamics.

Let $\boldsymbol{\Psi}$ be the micro-deformation tensor of a micromorphic continuum, $\boldsymbol{u}$ be the macro-displacement vector, and ε be the macro-strain tensor,

$$\varepsilon_{ij} \equiv \varepsilon_{(ij)} := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right). \tag{16}$$

All component representations in Section 3 are referred to the Cartesian coordinate system $\{x_i\}$. Assume $\boldsymbol{\Psi}$ and the so-called relative deformation γ to be symmetric,

$$\Psi_{ij} \equiv \Psi_{(ij)}, \quad \gamma_{ij} \equiv \gamma_{(ij)} := \varepsilon_{ij} - \Psi_{ij}. \tag{17}$$

This means that $\boldsymbol{\Psi}$ and γ are strain tensors and that the components of the so-called micro-deformation gradient $\boldsymbol{k}$,

$$k_{ijk} := (\nabla\boldsymbol{\Psi})_{ijk} = \partial_i \Psi_{jk}, \tag{18}$$

exhibit the symmetry property

$$k_{ijk} \equiv k_{i(jk)}. \tag{19}$$

Following Forest and Sievert [10], we denote a micromorphic elasticity theory based on Eqs. (16)–(18) as micro-strain elasticity.

According to Broese et al. [5], the 3-PG-model can be established as a particular case of the micro-strain elasticity by assuming the existence of a free energy (per unit macro-volume) ψ of the form:

$$\begin{aligned}\psi = \psi(\varepsilon, \gamma, \boldsymbol{k}) = &\frac{1}{2}\,\varepsilon_{ij}\,\mathbb{C}_{ijmn}\,\varepsilon_{mn} + \frac{1}{2}\frac{c_2 - c_1}{c_1}\,\gamma_{ij}\,\mathbb{C}_{ijmn}\,\gamma_{mn} \\ &+ \frac{1}{2}\,(c_2 - c_1)\,k_{ijk}\,\mathbb{C}_{jkmn}\,k_{imn}.\end{aligned} \tag{20}$$

The components $\mathbb{C}_{ijmn}$ are defined in Eq. (8) and c_1 as well as c_2 are scalar parameters constrained to $c_2 > c_1 > 0$ with $\sqrt{c_1}$ and $\sqrt{c_2}$ denoting internal material lengths as noted in Section 1. The Cauchy stress tensor $\boldsymbol{\Sigma}$ is then given by (cf. Broese et al. [5]):

$$\Sigma_{ij} \equiv \Sigma_{(ij)} = \tau_{ij} + \sigma_{ij} = \frac{c_2}{c_1}\,\mathbb{C}_{ijmn}\,\varepsilon_{mn} - \frac{c_2 - c_1}{c_1}\,\mathbb{C}_{ijmn}\,\Psi_{mn}, \tag{21}$$

where

$$\tau_{ij} \equiv \tau_{(ij)} = \frac{\partial\psi}{\partial\varepsilon_{ij}} = \mathbb{C}_{ijmn}\,\varepsilon_{mn}, \tag{22}$$

$$\sigma_{ij} \equiv \sigma_{(ij)} = \frac{\partial \psi}{\partial \gamma_{ij}} = \frac{c_2 - c_1}{c_1} \left(\mathbb{C}_{ijmn} \, \varepsilon_{mn} - \mathbb{C}_{ijmn} \, \Psi_{mn} \right). \tag{23}$$

Further, there exists a double stress $\boldsymbol{\mu}$ which satisfies the potential relation

$$\mu_{ijk} \equiv \mu_{i(jk)} = \frac{\partial \psi}{\partial k_{ijk}} = (c_2 - c_1) \, \partial_i \Psi_{mn} \, \mathbb{C}_{mnjk}. \tag{24}$$

For static problems, the classical and nonclassical stresses have to satisfy corresponding equilibrium equations. In the absence of body forces and body double forces, these are (see Mindlin [3] or Broese et al. [5])

$$\partial_j \Sigma_{ji} = 0, \tag{25}$$

$$\partial_k \mu_{kij} + \sigma_{ij} = 0. \tag{26}$$

The concomitant classical and nonclassical boundary conditions are as follows:

$$\text{Either } P_i = n_j \, \Sigma_{ji} \text{ or } u_i \text{ (class.bound.cond.)}, \tag{27}$$

$$\text{and either } T_{ij} = n_k \, \mu_{kij} \text{ or } \Psi_{ij} \text{ (non} - \text{class.bound.cond.)}, \tag{28}$$

have to be prescribed on the boundary ∂V.

The 3-PG-Model can be obtained from the above equations by first inserting Eq. (24) into Eq. (26), as follows:

$$\begin{aligned} \sigma_{ij} &= -\partial_k \mu_{kij} = -(c_2 - c_1) \, \partial_{kk} \Psi_{mn} \, \mathbb{C}_{mnij} \\ &= -(c_2 - c_1) \, \mathbb{C}_{ijmn} \, (\Delta \boldsymbol{\Psi})_{mn}. \end{aligned} \tag{29}$$

Then take the Laplacian of Eq. (21), as follows:

$$(\Delta \boldsymbol{\Sigma})_{ij} = \frac{c_2}{c_1} \, \mathbb{C}_{ijmn} \, (\Delta \varepsilon)_{mn} - \frac{c_2 - c_1}{c_1} \, \mathbb{C}_{ijmn} \, (\Delta \boldsymbol{\Psi})_{mn}, \tag{30}$$

and use Eq. (21) as well as Eq. (22) in Eq. (29), as follows:

$$-(c_2 - c_1) \, \mathbb{C}_{ijmn} \, (\Delta \boldsymbol{\Psi})_{mn} = \sigma_{ij} = \Sigma_{ij} - \mathbb{C}_{ijmn} \, \varepsilon_{mn}. \tag{31}$$

The latter together with Eq. (30) yield the following equation:

$$\Sigma_{ij} - c_1 \, (\Delta \boldsymbol{\Sigma})_{ij} = \mathbb{C}_{ijmn} \, \varepsilon_{mn} - c_2 \, \mathbb{C}_{ijmn} \, (\Delta \varepsilon)_{mn}, \tag{32}$$

which is nothing but the 3-PG-Model (2).

3.1.1 A useful equation for Ψ

For later reference, we derive a useful equation for the strain $\boldsymbol{\Psi}$. When seeking analytical solutions, there are two possibilities, either to find solutions in terms of $\boldsymbol{\Psi}$ or in terms of $\boldsymbol{\mu}$. In the first case, $\boldsymbol{\mu}$ is eliminated at the cost of a higher order partial differential equation, but no compatibility conditions for $\boldsymbol{\Psi}$ are needed. To be more specific, assuming $\mathbb{C}^{-1}$ to exist, we infer from Eqs. (23) and (29) that

$$\varepsilon_{ij} = \Psi_{ij} - c_1 \, (\Delta \boldsymbol{\Psi})_{ij}. \tag{33}$$

Further, from Eq. (21), we get the following equation:

$$\varepsilon_{ij} = \frac{c_1}{c_2} \left(\mathbb{C}^{-1}\right)_{ijmn} \Sigma_{mn} + \frac{c_2 - c_1}{c_2} \Psi_{ij}. \tag{34}$$

By combining the last two equations, we gain the useful relation as follows:

$$\Psi_{ij} - c_2 (\Delta \boldsymbol{\Psi})_{ij} - \left(\mathbb{C}^{-1}\right)_{ijmn} \Sigma_{mn} = 0. \tag{35}$$

For given $\boldsymbol{\Sigma}$, this is a (Helmholtz) partial differential equation for the components of $\boldsymbol{\Psi}$.

4. Mode-I and mode-II crack problems

In Part *II*, we consider Mode-I and Mode-II loading conditions for a sharp crack in the context of plane strain problems and employ cylindrical coordinates $\{r, \varphi, z\}$ as indicated in **Figure 1**. The aim of this section is to set up all relevant equations which are needed in Part *II*.

4.1 Kinematics

Plane strain state of micro-strain continua in equilibrium is characterized by the assumptions that

$$[u_\alpha] = \begin{pmatrix} u_r \\ u_\varphi \\ 0 \end{pmatrix}, \quad [\varepsilon_{\alpha\beta}] = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\varphi} & 0 \\ \varepsilon_{r\varphi} & \varepsilon_{\varphi\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$[\Psi_{\alpha\beta}] = \begin{pmatrix} \Psi_{rr} & \Psi_{r\varphi} & 0 \\ \Psi_{r\varphi} & \Psi_{\varphi\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{37}$$

and that $\boldsymbol{u}$, ε, and $\boldsymbol{\Psi}$ are independent of z,

$$u_\alpha = u_\alpha(r, \varphi), \quad \varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(r, \varphi), \quad \Psi_{\alpha\beta} = \Psi_{\alpha\beta}(r, \varphi). \tag{38}$$

On the basis of these assumptions, we conclude (see Section A.2) that the physical components $(\nabla \boldsymbol{\Psi})_{\alpha\beta\gamma} \equiv (\nabla \boldsymbol{\Psi})_{\alpha(\beta\gamma)}$ have the explicit form as follows:

$$(\nabla \boldsymbol{\Psi})_{rrr} = \partial_r \Psi_{rr}, \quad (\nabla \boldsymbol{\Psi})_{r\varphi\varphi} = \partial_r \Psi_{\varphi\varphi}, \quad (\nabla \boldsymbol{\Psi})_{rr\varphi} = \partial_r \Psi_{r\varphi}, \tag{39}$$

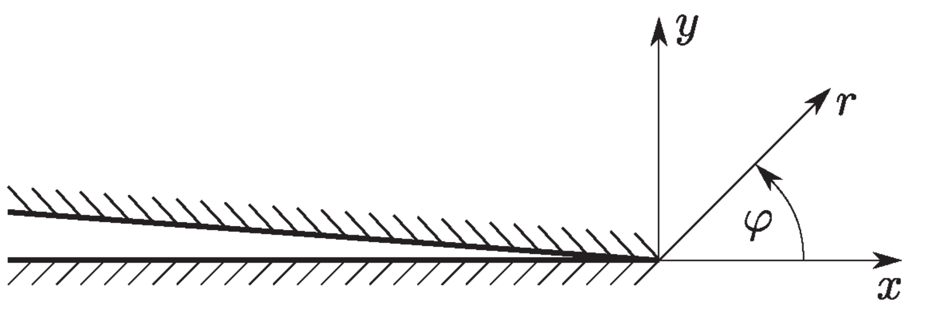

Figure 1.
Coordinate axes ahead of the crack tip.

$$(\nabla \mathbf{\Psi})_{\varphi rr} = \frac{1}{r}\left(\partial_\varphi \Psi_{rr} - 2\Psi_{r\varphi}\right), \tag{40}$$

$$(\nabla \mathbf{\Psi})_{\varphi\varphi\varphi} = \frac{1}{r}\left(\partial_\varphi \Psi_{\varphi\varphi} + 2\Psi_{r\varphi}\right), \tag{41}$$

$$(\nabla \mathbf{\Psi})_{\varphi r\varphi} = \frac{1}{r}\left(\partial_\varphi \Psi_{r\varphi} + \Psi_{rr} - \Psi_{\varphi\varphi}\right), \tag{42}$$

whereas all other components of $\nabla \mathbf{\Psi}$ vanish,

$$(\nabla \mathbf{\Psi})_{\alpha\beta z} = (\nabla \mathbf{\Psi})_{z\alpha\beta} = 0. \tag{43}$$

Similarly, we find (see Section 6.3) for the physical components $(\Delta \mathbf{\Psi})_{\alpha\beta} = (\Delta \mathbf{\Psi})_{(\alpha\beta)}$, that

$$(\Delta \mathbf{\Psi})_{rr} = \partial_{rr}\Psi_{rr} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{rr} + \frac{1}{r}\,\partial_r\Psi_{rr} - \frac{4}{r^2}\,\partial_\varphi\Psi_{r\varphi} - \frac{2}{r^2}\,\Psi_{rr} + \frac{2}{r^2}\,\Psi_{\varphi\varphi}, \tag{44}$$

$$(\Delta \mathbf{\Psi})_{\varphi\varphi} = \partial_{rr}\Psi_{\varphi\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{\varphi\varphi} + \frac{1}{r}\,\partial_r\Psi_{\varphi\varphi} + \frac{4}{r^2}\,\partial_\varphi\Psi_{r\varphi} + \frac{2}{r^2}\,\Psi_{rr} - \frac{2}{r^2}\,\Psi_{\varphi\varphi}, \tag{45}$$

$$(\Delta \mathbf{\Psi})_{r\varphi} = \partial_{rr}\Psi_{r\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{r\varphi} + \frac{1}{r}\,\partial_r\Psi_{r\varphi} + \frac{2}{r^2}\,\partial_\varphi\Psi_{rr} - \frac{2}{r^2}\,\partial_\varphi\Psi_{\varphi\varphi} - \frac{4}{r^2}\,\Psi_{r\varphi}, \tag{46}$$

$$(\Delta \mathbf{\Psi})_{\alpha z} = 0. \tag{47}$$

It is well known (see, e.g., Anderson [12], p. 114) that the nonvanishing physical components of ε are given by the following expressions:

$$\varepsilon_{rr} = \partial_r u_r, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r}\left(u_r + \partial_\varphi u_\varphi\right), \quad \varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\,\partial_\varphi u_r + \partial_r u_\varphi - \frac{1}{r}\,u_\varphi\right). \tag{48}$$

4.2 Cauchy stress: classical equilibrium equations

In view of the assumptions of the last section, we may derive the following results. We conclude from Eqs. (21)–(23), that

$$\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}(r,\varphi), \quad \tau_{\alpha\beta} = \tau_{\alpha\beta}(r,\varphi), \quad \sigma_{\alpha\beta} = \sigma_{\alpha\beta}(r,\varphi), \tag{49}$$

and that

$$[\Sigma_{\alpha\beta}] = \begin{pmatrix} \Sigma_{rr} & \Sigma_{r\varphi} & 0 \\ \Sigma_{r\varphi} & \Sigma_{\varphi\varphi} & 0 \\ 0 & 0 & \Sigma_{zz} \end{pmatrix}, \quad [\tau_{\alpha\beta}] = \begin{pmatrix} \tau_{rr} & \tau_{r\varphi} & 0 \\ \tau_{r\varphi} & \tau_{\varphi\varphi} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}, \tag{50}$$

$$[\sigma_{\alpha\beta}] = \begin{pmatrix} \sigma_{rr} & \sigma_{r\varphi} & 0 \\ \sigma_{r\varphi} & \sigma_{\varphi\varphi} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}. \tag{51}$$

Since $\varepsilon_{zz} = \Psi_{zz} = 0$, we deduce from the elasticity laws (21)–(23) that

$$\Sigma_{zz} = \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right), \quad \tau_{zz} = \nu\left(\tau_{rr} + \tau_{\varphi\varphi}\right), \quad \sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\varphi\varphi}\right), \tag{52}$$

where ν is given by Eq. (9). Thus, as in classical elasticity, once Σ_{rr} and $\Sigma_{\varphi\varphi}$ have been determined, Σ_{zz} is calculated by the first equation of (52).

Quite similar to the case of classical elasticity (see, e.g., Anderson [12], p. 114), the matrix of the components of $\boldsymbol{\Sigma}$ in Eq. (50) and the classical equilibrium condition (24), expressed in physical components, lead to the following two equations:

$$\partial_r \Sigma_{rr} + \frac{1}{r}\,\partial_\varphi \Sigma_{r\varphi} + \frac{1}{r}\left(\Sigma_{rr} - \Sigma_{\varphi\varphi}\right) = 0, \tag{53}$$

$$\partial_r \Sigma_{r\varphi} + \frac{1}{r}\,\partial_\varphi \Sigma_{\varphi\varphi} + \frac{2}{r}\,\Sigma_{r\varphi} = 0. \tag{54}$$

4.3 Classical compatibility condition

For the analytical solutions in Part *II*, we need the classical compatibility condition for the components of the strain tensor ε (see, e.g., Malvern [13], p. 669)

$$\partial_{rr}\varepsilon_{\varphi\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\varepsilon_{rr} - \frac{2}{r}\,\partial_{r\varphi}\varepsilon_{r\varphi} - \frac{1}{r}\,\partial_r\varepsilon_{rr} + \frac{2}{r}\,\partial_r\varepsilon_{\varphi\varphi} - \frac{2}{r^2}\,\partial_\varphi\varepsilon_{r\varphi} = 0. \tag{55}$$

The aim is now to rewrite this equation in terms of the physical components of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ by using Eq. (34). There are various equivalent representations for $\mathbb{C}^{-1}$, depending on the chosen set of elasticity constants. We find it convenient here to express $\mathbb{C}^{-1}$ in terms of the elasticity constants μ and ν. Thus, from Eq. (34), expressed in physical components,

$$\varepsilon_{rr} = \frac{c_1}{2\mu c_2}\left[\Sigma_{rr} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_1}\,\Psi_{rr}, \tag{56}$$

$$\varepsilon_{\varphi\varphi} = \frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_1}\,\Psi_{\varphi\varphi}, \tag{57}$$

$$\varepsilon_{r\varphi} = \frac{c_1}{2\mu c_2}\,\Sigma_{r\varphi} + \frac{c_2 - c_1}{c_1}\,\Psi_{r\varphi}. \tag{58}$$

By inserting these equations into Eq. (55), we get

$$\begin{aligned}
&\partial_{rr}\left(\frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_2}\,\Psi_{\varphi\varphi}\right)\\
&-\frac{2}{r}\,\partial_{r\varphi}\left(\frac{c_1}{2\mu c_2}\,\Sigma_{r\varphi} + \frac{c_2 - c_1}{c_2}\,\Psi_{r\varphi}\right)\\
&+\frac{1}{r^2}\,\partial_{\varphi\varphi}\left(\frac{c_1}{2\mu c_2}\left[\Sigma_{rr} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_2}\,\Psi_{rr}\right)\\
&-\frac{1}{r}\,\partial_r\left(\frac{c_1}{2\mu c_2}\left[\Sigma_{rr} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_2}\,\Psi_{rr}\right)\\
&+\frac{2}{r}\,\partial_r\left(\frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_2}\,\Psi_{\varphi\varphi}\right)\\
&-\frac{2}{r^2}\,\partial_\varphi\left(\frac{c_1}{2\mu c_2}\,\Sigma_{r\varphi} + \frac{c_2 - c_1}{c_2}\,\Psi_{r\varphi}\right) = 0.
\end{aligned} \tag{59}$$

This is equivalent to a vanishing sum of two functions of $\Psi_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$, respectively:

$$0 = \chi_1\left(\Psi_{\alpha\beta}\right) + \chi_2\left(\Sigma_{\alpha\beta}\right), \tag{60}$$

with

$$\chi_1(\Psi_{\alpha\beta}) := \frac{c_2 - c_1}{c_2}\left[\partial_{rr}\Psi_{\varphi\varphi} - \frac{2}{r}\,\partial_{r\varphi}\Psi_{r\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{rr}\right.$$
$$\left. - \frac{1}{r}\,\partial_r\Psi_{rr} + \frac{2}{r}\,\partial_r\Psi_{\varphi\varphi} - \frac{2}{r^2}\,\partial_\varphi\Psi_{r\varphi}\right] \tag{61}$$

and

$$\chi_2(\Sigma_{\alpha\beta}) := \frac{c_1}{2\mu c_2}\left[\partial_{rr}\Sigma_{\varphi\varphi} - \nu\,\partial_{rr}(\Sigma_{rr} + \Sigma_{\varphi\varphi}) - \frac{2}{r}\,\partial_{r\varphi}\Sigma_{r\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Sigma_{rr}\right.$$
$$- \frac{\nu}{r^2}\,\partial_{\varphi\varphi}(\Sigma_{rr} + \Sigma_{\varphi\varphi}) - \frac{1}{r}\,\partial_r\Sigma_{rr} \tag{62}$$
$$\left. - \frac{\nu}{r}\,\partial_r(\Sigma_{rr} + \Sigma_{\varphi\varphi}) + \frac{2}{r}\,\partial_r\Sigma_{\varphi\varphi} - \frac{2}{r^2}\,\partial_\varphi\Sigma_{r\varphi}\right],$$

The right hand side of Eq. (62) can be simplified by invoking the equilibrium Eqs. (53) and (54). First recast Eq. (54) to solve for $\Sigma_{r\varphi}$,

$$\Sigma_{r\varphi} = -\frac{r}{2}\,\partial_r\Sigma_{r\varphi} - \frac{1}{2}\,\partial_\varphi\Sigma_{\varphi\varphi}, \tag{63}$$

and then take the derivative with respect to φ,

$$\partial_\varphi\Sigma_{r\varphi} = -\frac{r}{2}\,\partial_{r\varphi}\Sigma_{r\varphi} - \frac{1}{2}\,\partial_{\varphi\varphi}\Sigma_{\varphi\varphi}. \tag{64}$$

On the other hand, from Eq. (53),

$$\partial_\varphi\Sigma_{r\varphi} = -r\,\partial_r\Sigma_{rr} - \Sigma_{rr} + \Sigma_{\varphi\varphi}, \tag{65}$$

and, after differentiation with respect to r,

$$\partial_{r\varphi}\Sigma_{r\varphi} = -r\,\partial_{rr}\Sigma_{rr} - 2\,\partial_r\Sigma_{rr} + \partial_r\Sigma_{\varphi\varphi}. \tag{66}$$

By substituting the latter into Eq. (64),

$$\partial_\varphi\Sigma_{r\varphi} = \frac{r^2}{2}\,\partial_{rr}\Sigma_{rr} - \frac{1}{2}\,\partial_{\varphi\varphi}\Sigma_{\varphi\varphi} + r\,\partial_r\Sigma_{rr} - \frac{r}{2}\,\partial_r\Sigma_{\varphi\varphi}. \tag{67}$$

Finally, by substituting Eqs. (66) and (67) into Eq. (62) and after some rearrangement of terms, we find that

$$\chi_2(\Sigma_{\alpha\beta}) = \frac{(1-\nu)c_1}{2\mu c_2}\left[\partial_{rr}(\Sigma_{rr} + \Sigma_{\varphi\varphi}) + \frac{1}{r^2}\,\partial_{\varphi\varphi}(\Sigma_{rr} + \Sigma_{\varphi\varphi})\right.$$
$$\left. + \frac{1}{r}\,\partial_r(\Sigma_{rr} + \Sigma_{\varphi\varphi})\right]. \tag{68}$$

4.4 Field equations for Ψ

By expressing $\mathbb{C}^{-1}$ in terms of μ and ν and using Eqs. (50), (37) and (44)–(46), we can readily obtain from Eq. (35), expressed in physical components, the following three equations:

$$\begin{aligned}&\partial_{rr}\Psi_{rr} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{rr} + \frac{1}{r}\,\partial_r\Psi_{rr} - \frac{4}{r^2}\,\partial_\varphi\Psi_{r\varphi} - \left(\frac{2}{r^2} + \frac{1}{c_2}\right)\Psi_{rr} + \frac{2}{r^2}\,\Psi_{\varphi\varphi}\\ &+ \frac{1-\nu}{2\mu c_2}\,\Sigma_{rr} - \frac{\nu}{2\mu c_2}\,\Sigma_{\varphi\varphi} = 0,\end{aligned} \tag{69}$$

$$\begin{aligned}&\partial_{rr}\Psi_{\varphi\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{\varphi\varphi} + \frac{1}{r}\,\partial_r\Psi_{\varphi\varphi} + \frac{4}{r^2}\,\partial_\varphi\Psi_{r\varphi} + \frac{2}{r^2}\,\Psi_{rr} - \left(\frac{2}{r^2} + \frac{1}{c_2}\right)\Psi_{\varphi\varphi}\\ &+ \frac{1-\nu}{2\mu c_2}\,\Sigma_{\varphi\varphi} - \frac{\nu}{2\mu c_2}\,\Sigma_{rr} = 0,\end{aligned} \tag{70}$$

$$\begin{aligned}&\partial_{rr}\Psi_{r\varphi} + \frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{r\varphi} + \frac{1}{r}\,\partial_r\Psi_{r\varphi} + \frac{2}{r^2}\,\partial_\varphi\Psi_{rr} - \frac{2}{r^2}\,\partial_\varphi\Psi_{\varphi\varphi} - \left(\frac{4}{r^2} + \frac{1}{c_2}\right)\Psi_{r\varphi}\\ &+ \frac{1}{2\mu c_2}\,\Sigma_{r\varphi} = 0.\end{aligned} \tag{71}$$

4.5 Double stress

4.5.1 Elasticity law for double stress

With respect to physical components, the elasticity law (24) becomes

$$\begin{aligned}\mu_{\alpha\beta\gamma} \equiv \mu_{\alpha(\beta\gamma)} &= (c_2 - c_1)\,(\nabla\boldsymbol{\Psi})_{\alpha\rho\zeta}\,\mathbb{C}_{\rho\zeta\beta\gamma}\\ &= (c_2 - c_1)\Big\{2\mu\,(\nabla\boldsymbol{\Psi})_{\alpha\beta\gamma} + \lambda\,(\nabla\boldsymbol{\Psi})_{\alpha\zeta\zeta}\,\delta_{\beta\gamma}\Big\}.\end{aligned} \tag{72}$$

Keeping in mind that the physical components of $\nabla\boldsymbol{\Psi}$ for a plane strain state are given by Eqs. (39)–(41), it is not difficult to derive the following results:

$$\mu_{rrr} = (c_2 - c_1)\left[(\lambda + 2\mu)\,\partial_r\Psi_{rr} + \lambda\,\partial_r\Psi_{\varphi\varphi}\right], \tag{73}$$

$$\mu_{r\varphi\varphi} = (c_2 - c_1)\left[(\lambda + 2\mu)\,\partial_r\Psi_{\varphi\varphi} + \lambda\,\partial_r\Psi_{rr}\right], \tag{74}$$

$$\mu_{rzz} = (c_2 - c_1)\,\lambda\,\partial_r\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right) = \nu\left(\mu_{rrr} + \mu_{r\varphi\varphi}\right), \tag{75}$$

$$\mu_{rr\varphi} = (c_2 - c_1)\,2\mu\,\partial_r\Psi_{r\varphi}, \tag{76}$$

$$\mu_{\varphi rr} = \frac{c_2 - c_1}{r}\left[2\mu\left(\partial_\varphi\Psi_{rr} - 2\Psi_{r\varphi}\right) + \lambda\,\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right)\right], \tag{77}$$

$$\mu_{\varphi\varphi\varphi} = \frac{c_2 - c_1}{r}\left[2\mu\left(\partial_\varphi\Psi_{\varphi\varphi} + 2\Psi_{r\varphi}\right) + \lambda\,\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right)\right], \tag{78}$$

$$\mu_{\varphi zz} = \frac{c_2 - c_1}{r}\,\lambda\,\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right) = \nu\left(\mu_{\varphi rr} + \mu_{\varphi\varphi\varphi}\right), \tag{79}$$

$$\mu_{\varphi r\varphi} = \frac{c_2 - c_1}{r}\,2\mu\left(\partial_\varphi\Psi_{r\varphi} + \Psi_{rr} - \Psi_{\varphi\varphi}\right), \tag{80}$$

$$\mu_{rrz} = \mu_{r\varphi z} = \mu_{\varphi rz} = \mu_{\varphi\varphi z} = 0, \tag{81}$$

$$\mu_{z\alpha\beta} = 0. \tag{82}$$

4.5.2 Non-classical equilibrium conditions

The physical components of the non-classical equilibrium condition (26) are

$$(\operatorname{div} \boldsymbol{\mu})_{\alpha\beta} + \sigma_{\alpha\beta} = 0. \tag{83}$$

With the aid of the physical components of $\boldsymbol{\mu}$ given by Eqs. (73)–(81) and the physical components of σ stated by Eq. (51), we can verify that the equilibrium conditions (83) furnish only four nontrivial equations:

$$(\operatorname{div} \boldsymbol{\mu})_{rr} + \sigma_{rr} = 0, \tag{84}$$

$$(\operatorname{div} \boldsymbol{\mu})_{\varphi\varphi} + \sigma_{\varphi\varphi} = 0, \tag{85}$$

$$(\operatorname{div} \boldsymbol{\mu})_{r\varphi} + \sigma_{r\varphi} = 0, \tag{86}$$

$$(\operatorname{div} \boldsymbol{\mu})_{zz} + \sigma_{zz} = 0, \tag{87}$$

or equivalently (cf. Section A.4)

$$\partial_r \mu_{rrr} + \frac{1}{r}\,\partial_\varphi \mu_{\varphi rr} + \frac{1}{r}\left(\mu_{rrr} - 2\mu_{\varphi r\varphi}\right) + \sigma_{rr} = 0, \tag{88}$$

$$\partial_r \mu_{r\varphi\varphi} + \frac{1}{r}\,\partial_\varphi \mu_{\varphi\varphi\varphi} + \frac{1}{r}\left(\mu_{r\varphi\varphi} + 2\mu_{\varphi r\varphi}\right) + \sigma_{\varphi\varphi} = 0, \tag{89}$$

$$\partial_r \mu_{rr\varphi} + \frac{1}{r}\,\partial_\varphi \mu_{\varphi r\varphi} + \frac{1}{r}\left(\mu_{rr\varphi} - \mu_{\varphi\varphi\varphi} + \mu_{\varphi rr}\right) + \sigma_{r\varphi} = 0, \tag{90}$$

$$\partial_r \mu_{rzz} + \frac{1}{r}\,\partial_\varphi \mu_{\varphi zz} + \frac{1}{r}\,\mu_{rzz} + \sigma_{zz} = 0. \tag{91}$$

4.6 Nonclassical compatibility conditions

Besides the classical compatibility condition for the strain ε in Eq. (55), further compatibility conditions for the micro-strain $\boldsymbol{\Psi}$ can be established by considering the following identities:

$$\partial_{r\varphi}\Psi_{rr} - \partial_{\varphi r}\Psi_{rr} = 0, \tag{92}$$

$$\partial_{r\varphi}\Psi_{\varphi\varphi} - \partial_{\varphi r}\Psi_{\varphi\varphi} = 0, \tag{93}$$

$$\partial_{r\varphi}\Psi_{r\varphi} - \partial_{\varphi r}\Psi_{r\varphi} = 0. \tag{94}$$

From these, we obtain useful relations by involving the physical components $\mu_{\alpha\beta\gamma}$ with the aid of the elasticity law (24). To illustrate, we recall from Eqs. (39)–(43) that

$$\begin{aligned} \partial_{r\varphi}\Psi_{rr} &= \partial_r\left(\partial_\varphi \Psi_{rr}\right) = \partial_r\Big(r\,(\nabla\boldsymbol{\Psi})_{\varphi rr} + 2\Psi_{r\varphi}\Big) \\ &= (\nabla\boldsymbol{\Psi})_{\varphi rr} + r\,\partial_r(\nabla\boldsymbol{\Psi})_{\varphi rr} + 2\,(\nabla\boldsymbol{\Psi})_{rr\varphi} \end{aligned} \tag{95}$$

and that

$$\partial_{\varphi r}\Psi_{rr} = \partial_\varphi(\partial_r \Psi_{rr}) = \partial_\varphi(\nabla\boldsymbol{\Psi})_{rrr}. \tag{96}$$

By inserting these into Eq. (92), we obtain the following equation:

$$\partial_\varphi(\nabla\boldsymbol{\Psi})_{rrr} - (\nabla\boldsymbol{\Psi})_{\varphi rr} - r\,\partial_r(\nabla\boldsymbol{\Psi})_{\varphi rr} - 2\,(\nabla\boldsymbol{\Psi})_{rr\varphi} = 0. \tag{97}$$

In a similar way, we conclude from Eqs. (93) and (94) that

$$\partial_\varphi (\nabla \boldsymbol{\Psi})_{r\varphi\varphi} - (\nabla \boldsymbol{\Psi})_{\varphi\varphi\varphi} - r\,\partial_r (\nabla \boldsymbol{\Psi})_{\varphi\varphi\varphi} + 2(\nabla \boldsymbol{\Psi})_{rr\varphi} = 0, \tag{98}$$

$$\partial_\varphi (\nabla \boldsymbol{\Psi})_{rr\varphi} - (\nabla \boldsymbol{\Psi})_{\varphi r\varphi} - r\,\partial_r (\nabla \boldsymbol{\Psi})_{\varphi r\varphi} + (\nabla \boldsymbol{\Psi})_{rrr} - (\nabla \boldsymbol{\Psi})_{r\varphi\varphi} = 0. \tag{99}$$

In order to involve the components of $\boldsymbol{\mu}$, we invert Eq. (24) to obtain

$$(\nabla \boldsymbol{\Psi})_{\alpha\beta\gamma} = \frac{1}{(c_2 - c_1)E} \left((1+\nu)\,\mu_{\alpha\beta\gamma} - \nu\,\mu_{\alpha\zeta\zeta}\,\delta_{\beta\gamma} \right), \tag{100}$$

where E is given by Eq. (9). Explicitly, we get

$$(\nabla \boldsymbol{\Psi})_{rrr} = \frac{1}{(c_2 - c_1)E} \left((1-\nu^2)\,\mu_{rrr} - \nu(1+\nu)\,\mu_{r\varphi\varphi} \right), \tag{101}$$

$$(\nabla \boldsymbol{\Psi})_{r\varphi\varphi} = \frac{1}{(c_2 - c_1)E} \left((1-\nu^2)\,\mu_{r\varphi\varphi} - \nu(1+\nu)\,\mu_{rrr} \right), \tag{102}$$

$$(\nabla \boldsymbol{\Psi})_{rr\varphi} = \frac{1+\nu}{(c_2 - c_1)E}\,\mu_{rr\varphi}, \tag{103}$$

$$(\nabla \boldsymbol{\Psi})_{\varphi rr} = \frac{1}{(c_2 - c_1)E} \left((1-\nu^2)\,\mu_{\varphi rr} - \nu(1+\nu)\,\mu_{\varphi\varphi\varphi} \right), \tag{104}$$

$$(\nabla \boldsymbol{\Psi})_{\varphi\varphi\varphi} = \frac{1}{(c_2 - c_1)E} \left((1-\nu^2)\,\mu_{\varphi\varphi\varphi} - \nu(1+\nu)\,\mu_{\varphi rr} \right), \tag{105}$$

$$(\nabla \boldsymbol{\Psi})_{\varphi r\varphi} = \frac{1+\nu}{(c_2 - c_1)E}\,\mu_{\varphi r\varphi}, \tag{106}$$

where in addition use has been made of Eqs. (75) and (80). By inserting these components into Eqs. (97)–(99), we can verify that

$$\begin{aligned} &-(1-\nu^2)\,r\,\partial_r\mu_{\varphi rr} + \nu(1+\nu)\,r\,\partial_r\mu_{\varphi\varphi\varphi} + (1-\nu^2)\,\partial_\varphi\mu_{rrr} - \nu(1+\nu)\,\partial_\varphi\mu_{r\varphi\varphi} \\ &-(1-\nu^2)\,\mu_{\varphi rr} + \nu(1+\nu)\,\mu_{\varphi\varphi\varphi} - 2(1+\nu)\,\mu_{rr\varphi} = 0, \end{aligned} \tag{107}$$

$$\begin{aligned} &-(1-\nu^2)\,r\,\partial_r\mu_{\varphi\varphi\varphi} + \nu(1+\nu)\,r\,\partial_r\mu_{\varphi rr} + (1-\nu^2)\,\partial_\varphi\mu_{r\varphi\varphi} - \nu(1+\nu)\,\partial_\varphi\mu_{rrr} \\ &-(1-\nu^2)\,\mu_{\varphi\varphi\varphi} + \nu(1+\nu)\,\mu_{\varphi rr} + 2(1+\nu)\,\mu_{rr\varphi} = 0, \end{aligned} \tag{108}$$

$$-r\,\partial_r\mu_{\varphi r\varphi} + \partial_\varphi\mu_{rr\varphi} - \mu_{\varphi r\varphi} + \mu_{rrr} - \mu_{r\varphi\varphi} = 0. \tag{109}$$

The last equation is independent of material parameters. In order to rewrite Eqs. (107) and (108) also in a form independent of material parameters, we add and subtract them from each other to obtain, respectively,

$$\partial_\varphi\mu_{r\varphi\varphi} + \partial_\varphi\mu_{rrr} - \mu_{\varphi\varphi\varphi} - \mu_{\varphi rr} - r\,\partial_r\mu_{\varphi\varphi\varphi} - r\,\partial_r\mu_{\varphi rr} = 0, \tag{110}$$

$$\partial_\varphi\mu_{r\varphi\varphi} - \partial_\varphi\mu_{rrr} - \mu_{\varphi\varphi\varphi} + \mu_{\varphi rr} - r\,\partial_r\mu_{\varphi\varphi\varphi} + r\,\partial_r\mu_{\varphi rr} + 4\,\mu_{rr\varphi} = 0. \tag{111}$$

4.7 Boundary conditions

As usually, near-tip field solutions rely upon boundary conditions, which are imposed only on the crack faces. Especially, we assume the classical traction $\boldsymbol{P}$ (see Eq. (27)) and the double force $\boldsymbol{T}$ (see Eq. (28)) to vanish on the crack faces. With regard to **Figure 1** this implies that

$$[P_\alpha]_{\varphi=\pm\pi} = 0, \quad [T_{\alpha\beta}]_{\varphi=\pm\pi} = 0. \tag{112}$$

Now, we have from Eq. (27), expressed in physical components, that $P_\alpha = n_\beta \Sigma_{\beta\alpha}$, where on the crack faces $[\boldsymbol{n}]_{\varphi=\pm\pi} = \mp \boldsymbol{e}_\varphi$. Therefore, and by virtue of Eq. (50), the nontrivial classical boundary conditions implied by Eq. (112) are as follows:

$$[\Sigma_{r\varphi}]_{\varphi=\pm\pi} = 0, \tag{113}$$

$$[\Sigma_{\varphi\varphi}]_{\varphi=\pm\pi} = 0. \tag{114}$$

Similarly, we get from Eq. (28), expressed in physical components, that $T_{\alpha\beta} = n_\gamma \mu_{\gamma\alpha\beta}$, so that the nonclassical part of Eq. (112) implies

$$[\mu_{\varphi\alpha\beta}]_{\varphi=\pm\pi} = (c_2 - c_1)\left[(\nabla\boldsymbol{\Psi})_{\varphi\gamma\zeta}\right]_{\varphi=\pm\pi} \mathbb{C}_{\gamma\zeta\alpha\beta} = 0, \tag{115}$$

where use has been made of the elasticity law (24). As the isotropic elasticity tensor $\mathbb{C}$ has been assumed to be invertible, we infer from Eq. (115), that

$$0 = \left[(\nabla\boldsymbol{\Psi})_{\varphi\alpha\beta}\right]_{\varphi=\pm\pi}. \tag{116}$$

Keeping in mind Eqs. (39)–(43), the only nontrivial conditions implied are as follows:

$$\left[(\nabla\boldsymbol{\Psi})_{\varphi rr}\right]_{\varphi=\pm\pi} = \left[(\nabla\boldsymbol{\Psi})_{\varphi\varphi\varphi}\right]_{\varphi=\pm\pi} = \left[(\nabla\boldsymbol{\Psi})_{\varphi r\varphi}\right]_{\varphi=\pm\pi} = 0, \tag{117}$$

or equivalently

$$[\partial_\varphi \Psi_{rr} - 2\Psi_{r\varphi}]_{\varphi=\pm\pi} = 0, \tag{118}$$

$$[\partial_\varphi \Psi_{\varphi\varphi} + 2\Psi_{r\varphi}]_{\varphi=\pm\pi} = 0, \tag{119}$$

$$[\partial_\varphi \Psi_{r\varphi} + \Psi_{rr} - \Psi_{\varphi\varphi}]_{\varphi=\pm\pi} = 0. \tag{120}$$

4.8 Symmetry conditions

Symmetry conditions are important to classify the near-tip field solutions into types according to Mode-I and Mode-II crack problems. Each type of loading condition is characterized by the following symmetry conditions.

4.8.1 Mode-I

As in classical elasticity (see, e.g., Hellan [14], p. 10), we suppose for the macro-displacement the following symmetry conditions:

$$u_r(r,\varphi) = u_r(r,-\varphi), \quad u_\varphi(r,\varphi) = -u_\varphi(r,-\varphi), \tag{121}$$

i.e., u_r is an even function of φ, whereas u_φ is an odd function of φ. It follows from Eq. (48) that the physical components of the macro-strain ε exhibit the following properties:

$$\varepsilon_{rr}(r,\varphi) = \varepsilon_{rr}(r,-\varphi), \quad \varepsilon_{\varphi\varphi}(r,\varphi) = \varepsilon_{\varphi\varphi}(r,-\varphi), \tag{122}$$

$$\varepsilon_{r\varphi}(r,\varphi) = -\varepsilon_{r\varphi}(r,-\varphi), \tag{123}$$

implying that ε_{rr} and $\varepsilon_{\varphi\varphi}$ are even functions of φ, whereas $\varepsilon_{r\varphi}$ is an odd function of φ. Since $\boldsymbol{\Psi}$ is also a strain tensor, we assume for its components, in analogy to Eqs. (122) and (123), that

$$\Psi_{rr}(r,\varphi) = \Psi_{rr}(r,-\varphi), \quad \Psi_{\varphi\varphi}(r,\varphi) = \Psi_{\varphi\varphi}(r,-\varphi), \tag{124}$$

$$\Psi_{r\varphi}(r,\varphi) = -\Psi_{r\varphi}(r,-\varphi). \tag{125}$$

Then, it can be verified, with the help of the elasticity law (21), expressed in physical components, that Eqs. (122)–(125) engender the following conditions for the components of $\boldsymbol{\Sigma}$:

$$\Sigma_{rr}(r,\varphi) = \Sigma_{rr}(r,-\varphi), \quad \Sigma_{\varphi\varphi}(r,\varphi) = \Sigma_{\varphi\varphi}(r,-\varphi), \tag{126}$$

$$\Sigma_{r\varphi}(r,\varphi) = -\Sigma_{r\varphi}(r,-\varphi). \tag{127}$$

Further, it can be seen from Eqs. (124) and (125), that

$$\partial_r\Psi_{rr}(r,\varphi) = \partial_r\Psi_{rr}(r,-\varphi), \;\; \partial_\varphi\Psi_{rr}(r,\varphi) = -\partial_\varphi\Psi_{rr}(r,-\varphi), \tag{128}$$

$$\partial_r\Psi_{\varphi\varphi}(r,\varphi) = \partial_r\Psi_{\varphi\varphi}(r,-\varphi), \;\; \partial_\varphi\Psi_{\varphi\varphi}(r,\varphi) = -\partial_\varphi\Psi_{\varphi\varphi}(r,-\varphi), \tag{129}$$

$$\partial_r\Psi_{r\varphi}(r,\varphi) = \partial_r\Psi_{r\varphi}(r,-\varphi), \;\; \partial_\varphi\Psi_{r\varphi}(r,\varphi) = -\partial_\varphi\Psi_{r\varphi}(r,-\varphi), \tag{130}$$

and from the elasticity laws (73)–(82), that

$$\mu_{rrr}(r,\varphi) = \mu_{rrr}(r,-\varphi), \quad \mu_{\varphi rr}(r,\varphi) = -\mu_{\varphi rr}(r,-\varphi), \tag{131}$$

$$\mu_{r\varphi\varphi}(r,\varphi) = \mu_{r\varphi\varphi}(r,-\varphi), \quad \mu_{\varphi\varphi\varphi}(r,\varphi) = -\mu_{\varphi\varphi\varphi}(r,-\varphi), \tag{132}$$

$$\mu_{rzz}(r,\varphi) = \mu_{rzz}(r,-\varphi), \quad \mu_{\varphi zz}(r,\varphi) = -\mu_{\varphi zz}(r,-\varphi), \tag{133}$$

$$\mu_{rr\varphi}(r,\varphi) = \mu_{rr\varphi}(r,-\varphi), \quad \mu_{\varphi r\varphi}(r,\varphi) = -\mu_{\varphi r\varphi}(r,-\varphi). \tag{134}$$

4.8.2 Mode-II

We know from classical elasticity (see, e.g., Hellan [14], p. 10), that the radial component of the displacement vector is an odd function of φ, whereas the circumferential component is an even function of φ. We assume these symmetry properties to also apply for the macro-displacement here, i.e.,

$$u_r(r,\varphi) = -u_r(r,-\varphi), \quad u_\varphi(r,\varphi) = u_\varphi(r,-\varphi). \tag{135}$$

It follows for the macro-strain ε, that

$$\varepsilon_{rr}(r,\varphi) = -\varepsilon_{rr}(r,-\varphi), \quad \varepsilon_{\varphi\varphi}(r,\varphi) = -\varepsilon_{\varphi\varphi}(r,-\varphi), \tag{136}$$

$$\varepsilon_{r\varphi}(r,\varphi) = \varepsilon_{r\varphi}(r,-\varphi), \tag{137}$$

which suggest to assume the following symmetries for $\boldsymbol{\Psi}$:

$$\Psi_{rr}(r,\varphi) = -\Psi_{rr}(r,-\varphi), \quad \Psi_{\varphi\varphi}(r,\varphi) = -\Psi_{\varphi\varphi}(r,-\varphi), \tag{138}$$

$$\Psi_{r\varphi}(r,\varphi) = \Psi_{r\varphi}(r,-\varphi). \tag{139}$$

It can be proved, in a similar fashion to Mode-I, that

$$\Sigma_{rr}(r,\varphi) = -\Sigma_{rr}(r,-\varphi), \quad \Sigma_{\varphi\varphi}(r,\varphi) = -\Sigma_{\varphi\varphi}(r,-\varphi), \tag{140}$$

$$\Sigma_{r\varphi}(r,\varphi) = \Sigma_{r\varphi}(r,-\varphi), \tag{141}$$

and that

$$\mu_{rrr}(r,\varphi) = -\mu_{rrr}(r,-\varphi), \quad \mu_{\varphi rr}(r,\varphi) = \mu_{\varphi rr}(r,-\varphi), \tag{142}$$

$$\mu_{r\varphi\varphi}(r,\varphi) = -\mu_{r\varphi\varphi}(r,-\varphi), \quad \mu_{\varphi\varphi\varphi}(r,\varphi) = \mu_{\varphi\varphi\varphi}(r,-\varphi), \tag{143}$$

$$\mu_{rzz}(r,\varphi) = -\mu_{rzz}(r,-\varphi), \quad \mu_{\varphi zz}(r,\varphi) = \mu_{\varphi zz}(r,-\varphi), \tag{144}$$

$$\mu_{rr\varphi}(r,\varphi) = -\mu_{rr\varphi}(r,-\varphi), \quad \mu_{\varphi r\varphi}(r,\varphi) = \mu_{\varphi r\varphi}(r,-\varphi). \tag{145}$$

Before closing this section, we notice here, that the numerical simulations on the basis of the finite element method in Part *III* confirm the assumed symmetry conditions.

5. Concluding remarks

If the implicit gradient elasticity model in Eq. (2), named the 3-PG-Model, is recognized as a particular case of micromorphic (micro-strain) elasticity, a free energy and associated response functions and boundary conditions can be assigned. Part *I* adopts this conceptual point of view for the 3-PG-Model and provides the reduced form of the governing equations and boundary conditions for plane strain problems. This includes, among others, elasticity laws for classical and nonclassical stresses as well as classical and nonclassical equilibrium equations and compatibility conditions. It also supplies the required symmetry conditions for asymptotic solutions of Mode-I and Mode-II crack problems. A detailed discussion of such analytical solutions is given in Part *II*.

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A. Appendix

This section provides the component representations with respect to cylindrical coordinates of some space derivatives of a second-order tensor $\boldsymbol{A}$ and a third-order tensor $\boldsymbol{\mathcal{A}}$. It is easy to find component representations for $\nabla\boldsymbol{A}$ and $\operatorname{div}\boldsymbol{A}$ in textbooks, whereas it may be harder to find such representations for $\Delta\boldsymbol{A}$ and $\operatorname{div}\boldsymbol{\mathcal{A}}$. But one can calculate them with the help of the relations given below.

A.1 Cylindrical coordinates

We denote by $\{\theta^i\}$ the cylindrical coordinate system with $\theta^1 = r, \theta^2 = \varphi$ and $\theta^3 = z$. The covariant basis induced by $\{\theta^i\}$ is denoted by $\{\boldsymbol{g}_i\}$ where $\boldsymbol{g}_1 = (\cos\varphi)\boldsymbol{e}_1 + (\sin\varphi)\boldsymbol{e}_2, \boldsymbol{g}_2 = -r(\sin\varphi)\boldsymbol{e}_1 + r(\cos\varphi)\boldsymbol{e}_2$ and $\boldsymbol{g}_3 = \boldsymbol{e}_3$. The contravariant basis is denoted by $\{\boldsymbol{g}^i\}, \boldsymbol{g}^i = g^{ij}\boldsymbol{g}_j$, where g^{ij} are the contravariant metric coefficients. The corresponding covariant metric coefficients are g_{ij}. All values of g_{ij} and g^{ij} are vanishing except for the values $g_{11} = g_{33} = g^{11} = g^{33} = 1, g_{22} = r^2$ and $g^{22} = \frac{1}{r^2}$. Moreover, all values of the related Christoffel symbols Γ^k_{ij} also vanish except for the values $\Gamma^1_{22} = -r$ and $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$. Physical components are referred to the orthonormal basis $\{\boldsymbol{e}_{<i>}\}$ where $\boldsymbol{e}_{<1>} \equiv \boldsymbol{e}_r = \boldsymbol{g}_1, \boldsymbol{e}_{<2>} \equiv \boldsymbol{e}_\varphi = \frac{1}{r}\boldsymbol{g}_2$ and $\boldsymbol{e}_{<3>} \equiv \boldsymbol{e}_z = \boldsymbol{g}_3$. According to Section 2, the physical components of a second-order tensor $\boldsymbol{A}$ are denoted by $A_{\alpha\beta}$ and notations of the form $A_{<11>} \equiv A_{rr}, A_{<12>} \equiv A_{r\varphi}$, … apply. Similar notations hold also for any tensor, especially for the third-order tensor $\boldsymbol{\mathcal{A}}$. Finally, the nabla operator ∇ obeys the representation $\nabla = \frac{\partial}{\partial\theta^i}\boldsymbol{g}^i$.

A.2 The gradient of a symmetric second-order tensor

In the case of a second-order tensor $\boldsymbol{A} = A^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$, we have (cf. Section 2)

$$\operatorname{grad}\boldsymbol{A} \equiv \nabla\boldsymbol{A} := \nabla \otimes \boldsymbol{A} = \boldsymbol{g}^i \otimes \partial_{\theta^i}\boldsymbol{A} = (\nabla\boldsymbol{A})_i{}^{jk}\,\boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k, \tag{A1}$$

where $(\nabla\boldsymbol{A})_i{}^{jk}$ is the covariant derivative of the components A^{jk},

$$(\nabla\boldsymbol{A})_i{}^{jk} = A^{jk}\Big|_i = \partial_{\theta^i}A^{jk} + \Gamma^j_{im}A^{mk} + \Gamma^k_{im}A^{jm}. \tag{A2}$$

When $\boldsymbol{A}$ is symmetric, $\boldsymbol{A} = \boldsymbol{A}^{(s)}$, we conclude from Eq. (A2), that

$$(\nabla\boldsymbol{A})_i{}^{jk} = (\nabla\boldsymbol{A})_i{}^{(jk)}. \tag{A3}$$

This symmetry also applies with respect to physical components,

$$(\nabla\boldsymbol{A})_{\alpha\beta\gamma} = (\nabla\boldsymbol{A})_{\alpha(\beta\gamma)}. \tag{A4}$$

It can be seen that Eq. (A2) furnishes the following physical components of $\nabla\boldsymbol{A}$:

$$(\nabla\boldsymbol{A})_{rrr} = \partial_r A_{rr}, \quad (\nabla\boldsymbol{A})_{r\varphi\varphi} = \partial_r A_{\varphi\varphi}, \quad (\nabla\boldsymbol{A})_{rzz} = \partial_r A_{zz}, \tag{A5}$$

$$(\nabla\boldsymbol{A})_{rr\varphi} = \partial_r A_{r\varphi}, \quad (\nabla\boldsymbol{A})_{rrz} = \partial_r A_{rz}, \quad (\nabla\boldsymbol{A})_{r\varphi z} = \partial_r A_{\varphi z}, \tag{A6}$$

$$(\nabla\boldsymbol{A})_{\varphi rr} = \frac{1}{r}\left(\partial_\varphi A_{rr} - 2A_{\varphi r}\right), \tag{A7}$$

$$(\nabla\boldsymbol{A})_{\varphi\varphi\varphi} = \frac{1}{r}\left(\partial_\varphi A_{\varphi\varphi} + 2A_{\varphi r}\right), \tag{A8}$$

$$(\nabla\boldsymbol{A})_{\varphi zz} = \frac{1}{r}\,\partial_\varphi A_{zz}, \tag{A9}$$

$$(\nabla\boldsymbol{A})_{\varphi r\varphi} = \frac{1}{r}\left(\partial_\varphi A_{r\varphi} + A_{rr} - A_{\varphi\varphi}\right), \tag{A10}$$

$$(\nabla\boldsymbol{A})_{\varphi\varphi z} = \frac{1}{r}\left(\partial_\varphi A_{\varphi z} - A_{rz}\right), \tag{A11}$$

$$(\nabla\boldsymbol{A})_{\varphi rz} = \frac{1}{r}\left(\partial_\varphi A_{rz} - A_{\varphi z}\right), \tag{A12}$$

$$(\nabla\boldsymbol{A})_{z\alpha\beta} = 0. \tag{A13}$$

A.3 The Laplacian of a symmetric second-order tensor

The Laplacian of a second-order tensor $\boldsymbol{A}$ is given by (cf. Section 2)

$$\begin{aligned}\Delta\boldsymbol{A} &:= \operatorname{div}\operatorname{grad}\boldsymbol{A} = \nabla\cdot(\nabla\boldsymbol{A}) = \boldsymbol{g}^m\cdot\partial_{\theta^m}(\nabla\boldsymbol{A}) \\ &= \boldsymbol{g}^m\cdot\partial_{\theta^m}\left(A^{jk}\big|_i\boldsymbol{g}^i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}_k\right).\end{aligned} \tag{A14}$$

This may be written as

$$\Delta\boldsymbol{A} = (\Delta\boldsymbol{A})^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j, \tag{A15}$$

where

$$(\Delta\boldsymbol{A})^{ij} = A^{ij}\Big|_{km}g^{km}, \tag{A16}$$

and $A^{ij}\big|_{km}$ is the second covariant derivative of the components A^{ij},

$$A^{ij}\big|_{km} = \left(A^{ij}{}_k\right)\big|_m = \partial_{\theta^m}A^{ij}\big|_k + \Gamma^i_{ml}A^{lj}\big|_k + \Gamma^j_{ml}A^{il}\big|_k - \Gamma^l_{km}A^{ij}\big|_l. \tag{A17}$$

We can calculate the physical components $(\Delta\boldsymbol{A})_{<ij>}$ from Eqs. (A16) and (A17). For the case that $\boldsymbol{A}$ is symmetric, $\boldsymbol{A} = \boldsymbol{A}^{(s)}$, we can derive, after lengthy algebraic manipulations, that

$$\begin{aligned}(\Delta\boldsymbol{A})_{rr} = {}& \partial_{rr}A_{rr} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{rr} + \partial_{zz}A_{rr} + \frac{1}{r}\partial_r A_{rr} - \frac{4}{r^2}\partial_\varphi A_{r\varphi} \\ & - \frac{2}{r^2}A_{rr} + \frac{2}{r^2}A_{\varphi\varphi},\end{aligned} \tag{A18}$$

$$\begin{aligned}(\Delta\boldsymbol{A})_{\varphi\varphi} = {}& \partial_{rr}A_{\varphi\varphi} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{\varphi\varphi} + \partial_{zz}A_{\varphi\varphi} + \frac{1}{r}\partial_r A_{\varphi\varphi} + \frac{4}{r^2}\partial_\varphi A_{r\varphi} \\ & + \frac{2}{r^2}A_{rr} - \frac{2}{r^2}A_{\varphi\varphi},\end{aligned} \tag{A19}$$

$$(\Delta\boldsymbol{A})_{zz} = \partial_{rr}A_{zz} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{zz} + \partial_{zz}A_{zz} + \frac{1}{r}\partial_r A_{zz}, \tag{A20}$$

$$\begin{aligned}(\Delta\boldsymbol{A})_{r\varphi} = {}& \partial_{rr}A_{r\varphi} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{r\varphi} + \partial_{zz}A_{r\varphi} + \frac{1}{r}\partial_r A_{r\varphi} + \frac{2}{r^2}\partial_\varphi A_{rr} \\ & - \frac{2}{r^2}\partial_\varphi A_{\varphi\varphi} - \frac{4}{r^2}A_{r\varphi},\end{aligned} \tag{A21}$$

$$(\Delta\boldsymbol{A})_{rz} = \partial_{rr}A_{rz} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{rz} + \partial_{zz}A_{rz} + \frac{1}{r}\partial_r A_{rz} - \frac{2}{r^2}\partial_\varphi A_{\varphi z} - \frac{1}{r^2}A_{rz}, \tag{A22}$$

$$(\Delta\boldsymbol{A})_{\varphi z} = \partial_{rr}A_{\varphi z} + \frac{1}{r^2}\partial_{\varphi\varphi}A_{\varphi z} + \partial_{zz}A_{\varphi z} + \frac{1}{r}\partial_r A_{\varphi z} + \frac{2}{r^2}\partial_\varphi A_{rz} - \frac{1}{r^2}A_{\varphi z}. \tag{A23}$$

A.4. The divergence of a third-order tensor

Let $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}^{ijk}\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}_k$ be a third-order tensor. Then (cf. Section 2)

$$\operatorname{div}\boldsymbol{\mathcal{A}} := \nabla\cdot\boldsymbol{\mathcal{A}} = \boldsymbol{g}^i\cdot\partial_{\theta^i}\boldsymbol{\mathcal{A}} = (\operatorname{div}\boldsymbol{\mathcal{A}})^{jk}\boldsymbol{g}_j\otimes\boldsymbol{g}_k, \tag{A24}$$

where

$$(\operatorname{div} \boldsymbol{\mathcal{A}})^{jk} = \mathcal{A}^{ijk}\Big|_i, \tag{A25}$$

and $\mathcal{A}^{ijk}\Big|_m$ is the covariant derivative of the components $\mathcal{A}^{ijk}$. If the symmetry condition $\mathcal{A}^{ijk} \equiv \mathcal{A}^{i(jk)}$ holds, we can establish, after lengthy algebraic manipulations, the following results for the physical components of $\operatorname{div} \boldsymbol{\mathcal{A}}$:

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{rr} = \partial_r A_{rrr} + \frac{1}{r}\,\partial_\varphi A_{\varphi rr} + \partial_z A_{zrr} + \frac{1}{r}\,(A_{rrr} - 2A_{\varphi r\varphi}), \tag{A26}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{\varphi\varphi} = \partial_r A_{r\varphi\varphi} + \frac{1}{r}\,\partial_\varphi A_{\varphi\varphi\varphi} + \partial_z A_{z\varphi\varphi} + \frac{1}{r}\,(A_{r\varphi\varphi} + 2A_{\varphi r\varphi}), \tag{A27}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{zz} = \partial_r A_{rzz} + \frac{1}{r}\,\partial_\varphi A_{\varphi zz} + \partial_z A_{zzz} + \frac{1}{r}\,A_{rzz}, \tag{A28}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{r\varphi} = \partial_r A_{rr\varphi} + \frac{1}{r}\,\partial_\varphi A_{\varphi r\varphi} + \partial_z A_{zr\varphi} + \frac{1}{r}\,(A_{rr\varphi} - A_{\varphi\varphi\varphi} + A_{\varphi rr}), \tag{A29}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{rz} = \partial_r A_{rrz} + \frac{1}{r}\,\partial_\varphi A_{\varphi rz} + \partial_z A_{zrz} + \frac{1}{r}\,(A_{rrz} - A_{\varphi\varphi z}), \tag{A30}$$

$$(\operatorname{div} \boldsymbol{\mathcal{A}})_{\varphi z} = \partial_r A_{r\varphi z} + \frac{1}{r}\,\partial_\varphi A_{\varphi\varphi z} + \partial_z A_{z\varphi z} + \frac{1}{r}\,(A_{r\varphi z} + A_{\varphi rz}). \tag{A31}$$

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