Chapter

Some Applications of Clifford Algebra in Geometry

Ying-Qiu Gu

Abstract

In this chapter, we provide some enlightening examples of the application of Clifford algebra in geometry, which show the concise representation, simple calculation, and profound insight of this algebra. The definition of Clifford algebra implies geometric concepts such as vector, length, angle, area, and volume and unifies the calculus of scalar, spinor, vector, and tensor, so that it is able to naturally describe all variables and calculus in geometry and physics. Clifford algebra unifies and generalizes real number, complex, quaternion, and vector algebra and converts complicated relations and operations into intuitive matrix algebra independent of coordinate systems. By localizing the basis or frame of space-time and introducing differential and connection operators, Clifford algebra also contains Riemann geometry. Clifford algebra provides a unified, standard, elegant, and open language and tools for numerous complicated mathematical and physical theories. Clifford algebra calculus is an arithmetic-like operation that can be well understood by everyone. This feature is very useful for teaching purposes, and popularizing Clifford algebra in high schools and universities will greatly improve the efficiency of students to learn fundamental knowledge of mathematics and physics. So, Clifford algebra can be expected to complete a new big synthesis of scientific knowledge.

Keywords: Clifford algebra, geometric algebra, gamma matrix, multi-inner product, connection operator, Keller connection, spin group, cross ratio, conformal geometric algebra

1. A brief historical review

It is well known that a rotational transformation in the complex plane is equivalent to multiplying the complex number by a factor $e^{\theta i}$. How to generalize this simple and elegant operation to three-dimensional space is a difficult problem for many outstanding mathematicians in the early nineteenth century. William Rowan Hamilton (1805–1865) spent much of his later years studying the issue and eventually invented quaternion [1]. This generalization requires four elements $\{1, i, j, k\}$, and the spatial basis should satisfy the multiplying rules $i^2 = j^2 = k^2 = -1$ and $jk = -kj = i$, $ki = -ik = j$, and $ij = -ji = -k$. Although a quaternion is still a vector, it constitutes an associative algebra according to the above rules. However, the commutativity of multiplication is violated. Quaternion can solve the rotational transformation in three-dimensional space very well and simplify the representation of Maxwell equation system of electromagnetic field.

When Hamilton introduced his quaternion algebra, German high school teacher Hermann Gunther Grassmann (1809–1877) was constructing his exterior algebra [2]. He defined the exterior product or outer product $a \wedge b$ of two vectors a and b, which satisfies anti-commutative law $a \wedge b = -b \wedge a$ and associativity $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. The exterior product is a generalization of cross product in three-dimensional Euclidian space. Its geometrical meaning is the oriented volume of a parallel polyhedron. Exterior product is now the basic tool of modern differential geometry, but Grassmann's work was largely neglected in his lifetime.

British mathematician William Kingdon Clifford (1845–1879) was one of the few mathematicians who read and understood Grassmann's work. In 1878, he combined the algebraic rules of Hamilton and Grassmann to define a new algebraic system, which he himself called geometric algebra [3]. In this algebra, both the inner and exterior products of vectors can be uniquely represented by a linear combination of geometric product. In addition, geometric algebra is always isomorphic to some special matrix algebra.

Clifford algebra combines all the advantages of quaternion with the advantages of vector algebra and uniformly and succinctly describes the contents of geometry and physics. However, the vector calculus introduced by Gibbs had also successfully described the mathematical physics problem in three-dimensional space [4]. Clifford died prematurely at the age of 34, so that the theory of geometric algebra was not deeply researched and fully developed, and people still could not see the superiority of this algebra at that time. Thus, the important insights of Grassmann and Clifford were lost in the late nineteenth century papers. Mathematicians abstracted Clifford algebra from its geometric origins, and, for the most part of a century, it languished as a minor subdiscipline of mathematics and became one more algebra among so many others.

With the establishment of relativity, especially the introduction of Pauli and Dirac's matrix algebra for spin and the successful application in quantum theory [5], it was felt that there is an urgent need for a mathematical system to deal with problems in high-dimensional space-time. In the 1920s, Clifford algebra re-entered the field of vision and was paid attention and researched by some of the famous mathematicians and physicists such as R. Lipschitz, T. Vahlen, E. Cartan, E. Witt, C. Chevalley, and M. Riesz [6–8]. When only formal algebra is involved, we usually use the term "Clifford algebra," but more often use the "geometric algebra" named by Clifford himself if applied to geometric problems.

The first person who realized that Clifford algebra is a unified language in geometry and physics should be David Hestenes. By the 1960s, Hestenes began to restore the geometric meaning behind Pauli and Dirac algebra. Although his initial motivation was to gain insight into the nature of quantum mechanics, he quickly realized that Clifford algebra was a unified language and tool for mathematics, physics, and engineering. He published "space-time algebra" in 1966 and has been working on the promotion of Clifford algebra in teaching and research [9–12]. Because representation and algorithm in geometric algebra are seemingly as ordinary as arithmetic, his work has been neglected by the scientific community for more than 20 years. Only with the joint impetus of computer-aided design, computer vision and robotics, protein folding, neural networks, modern differential geometry, mathematical physics [13–17], and especially the Journal "Advances in Applied Clifford algebras" founded by Professor Jaime Keller, geometric algebra began to move towards popularity and prosperity.

As a unified and universal language of natural science, Clifford algebra is developed by many mathematicians, physicists, and engineers according to their different requirements and knowledge background. Such situation leads to "There are a thousand Hamlets in a thousand people's eyes." In this chapter,

by introducing typical application of Clifford algebra in geometry, we show some special feature and elegance of the algebra.

2. Application of Clifford algebra in differential geometry

In Euclidean space, we have several important concepts such as vector, length, angle, area, volume, and tensor. The study of relationship between these concepts constitutes the whole content of Euclidean geometry. The mathematical tools previously used to discuss these contents are vector algebra and geometrical method, which are complex and require much fundamental knowledge. Clifford algebra exactly and faithfully describes the intrinsic properties of vector space by introducing concepts such as inner, exterior, and geometric products of vectors and thus becomes a unified language and standard tool for dealing with geometric and physical problems. Clifford algebra has the characteristics of simple concept, standard operation, completeness in conclusion, and easy understanding.

Definition 1 *For Minkowski space* $\mathbb{M}^n$ *over number field* $\mathbb{F}$*, if the multiplication rule of vectors satisfies*

$$1.\ \text{Antisymmetry},\quad \boldsymbol{x}\wedge\boldsymbol{y}=-\boldsymbol{y}\wedge\boldsymbol{x}; \tag{1}$$

$$2.\ \text{Associativity},\quad (\boldsymbol{x}\wedge\boldsymbol{y})\wedge\boldsymbol{z}=\boldsymbol{x}\wedge(\boldsymbol{y}\wedge\boldsymbol{z}); \tag{2}$$

$$3.\ \text{Distributivity},\quad \boldsymbol{x}\wedge(a\boldsymbol{y}+b\boldsymbol{z})=a\boldsymbol{x}\wedge\boldsymbol{y}+b\boldsymbol{x}\wedge\boldsymbol{z},\quad a,b\in\mathbb{F}, \tag{3}$$

the algebra is called **Grassmann algebra** *and* $\boldsymbol{x}\wedge\boldsymbol{y}$ **exterior product**.

The Grassmann is also called **exterior algebra**. The geometrical meaning of $\mathbf{x}\wedge\mathbf{y}$ is oriented area of a parallelogram constructed by $\mathbf{x}$ and $\mathbf{y}$, and the geometrical meaning of $\mathbf{x}\wedge\mathbf{y}\wedge\cdots\wedge\mathbf{z}$ is the oriented volume of the parallelohedron constructed by the vectors (see **Figure 1**). We call $\mathbf{x}\wedge\mathbf{y}$ two-vector, $\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$ three-vector, and so on. For k-vector $\mathbf{x}\in\Lambda^k$ and l-vector $\mathbf{y}\in\Lambda^l$, we have

$$\mathbf{x}\wedge\mathbf{y}=(-1)^{kl}\mathbf{y}\wedge\mathbf{x}\in\Lambda^{k+l}.$$

By the definition, we can easily check:

Theorem 1 *For exterior algebra defined in* $V=\mathbb{M}^n$*, we have*

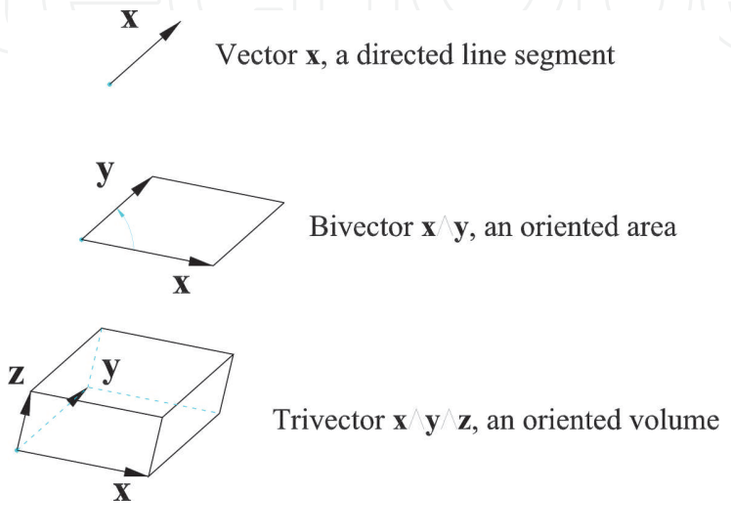

Figure 1.
Geometric meaning of exterior products of vectors.

$$\mathbb{W}^n = \mathbb{F} \oplus V \oplus \Lambda^2(V) \cdots \oplus \Lambda^n(V) = \bigoplus_{r=0}^{n} \Lambda^r(V).$$

The dimension of the algebra is

$$\dim(\mathbb{W}^n) = \sum_{k=0}^{n} C_n^k = 2^n.$$

Under the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$, *the exterior algebra takes the following form*:

$$\boldsymbol{w} = w^0 + w^k \boldsymbol{e}_k + \sum_{k<l} w^{kl} \boldsymbol{e}_{kl} + \sum_{j<k<l} w^{jkl} \boldsymbol{e}_{jkl} + \cdots + w^{12\cdots n} \boldsymbol{e}_{12\cdots n}, \tag{4}$$

in which $\forall w^{jk\cdots l} \in \mathbb{F}$, $\boldsymbol{e}_{jk\cdots l} = \boldsymbol{e}_j \wedge \boldsymbol{e}_k \wedge \cdots \wedge \boldsymbol{e}_l$, *and* $\forall |\boldsymbol{e}_{jk\cdots l}| = 1$.

The exterior product of vectors contains alternating combinations of basis, for example:

$$\begin{aligned} V_n &= \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n = x_1^j x_2^k \cdots x_n^l \mathbf{e}_{jk\cdots l} \\ &= \epsilon_{jk\cdots l} x_1^j x_2^k \cdots x_n^l \mathbf{e}_{12\cdots n} = \det\left(x_j^k\right) \mathbf{e}_{12\cdots n}. \end{aligned} \tag{5}$$

Definition 2 *For any vectors* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{M}^n$, **Clifford product** *of vectors is denoted by*

$$\boldsymbol{xy} = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y}, \tag{6}$$

$$(\boldsymbol{x} \wedge \boldsymbol{y})\boldsymbol{z} = (\boldsymbol{y} \cdot \boldsymbol{z})x - (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = -(\boldsymbol{y} \wedge \boldsymbol{x})\boldsymbol{z}, \tag{7}$$

$$\boldsymbol{z}(\boldsymbol{x} \wedge \boldsymbol{y}) = (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} - (\boldsymbol{y} \cdot \boldsymbol{z})\boldsymbol{x} + \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = -\boldsymbol{z}(\boldsymbol{y} \wedge \boldsymbol{x}), \tag{8}$$

$$(\boldsymbol{xy})\boldsymbol{z} = (\boldsymbol{y} \cdot \boldsymbol{z})\boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} + (\boldsymbol{x} \cdot \boldsymbol{y})\boldsymbol{z} + \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \boldsymbol{x}(\boldsymbol{yz}). \tag{9}$$

Clifford product is also called ***geometric product***.

Similarly, we can define Clifford algebra for many vectors as $\mathbf{xy}\cdots\mathbf{z}$. In (6), $\mathbf{x} \cdot \mathbf{y} = \eta_{ab} x^a y^b$ is the **scalar product** or **inner product** in $\mathbb{M}^n$. By $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$, we find Clifford product is not commutative. By (6), we have

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(\mathbf{xy} + \mathbf{yx}), \qquad \mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx}), \qquad \mathbf{x} \cdot \mathbf{x} = \mathbf{xx} = \mathbf{x}^2. \tag{10}$$

Definition 3 *For Minkowski space* $\mathbb{M}^{p,q}$ *with metric* $\eta_{ab} = diag(I_p, -I_q)$, *if the Clifford product of vectors satisfies*

$$\boldsymbol{e}_k \boldsymbol{e}_l + \boldsymbol{e}_l \boldsymbol{e}_k = 2\eta_{kl}, \text{ or } \boldsymbol{x}^2 = \eta_{kl} x^k x^l,$$

then the algebra

$$\boldsymbol{c} = c^0 + c^k \boldsymbol{e}_k + \sum_{k<l} c^{kl} \boldsymbol{e}_k \boldsymbol{e}_l + \sum_{j<k<l} c^{jkl} \boldsymbol{e}_j \boldsymbol{e}_k \boldsymbol{e}_l + \cdots + c^{12\cdots n} \boldsymbol{e}_1 \boldsymbol{e}_2 \cdots \boldsymbol{e}_n, \tag{11}$$

is called as **Clifford algebra** *or* **geometric algebra**, *which is denoted as* $C\ell_{p,q}$.

There are several definitions for Clifford algebra [18, 19]. The above definition is the original definition of Clifford. Clifford algebra has also 2^n dimensions.

Comparing (11) with (4), we find the two algebras are isomorphic in sense of linear algebra, but their definitions of multiplication rules are different. The Grassmann products have clear geometrical meaning, but the Clifford product is isomorphic to matrix algebra and the multiplication of physical variables is Clifford product. Therefore, representing geometrical and physical variables in the form of (4) will bring great convenience [20, 21]. In this case, the relations among three products such as (6)–(9) are important.

In physics, we often use curvilinear coordinate system or consider problems in curved space-time. In this case, we must discuss problems in n dimensional pseudo Riemann manifold. At each point x in the manifold, the tangent space $T\mathbb{M}(\mathbf{x})$ is a n dimensional Minkowski space-time. The Clifford algebra can be also defined on the tangent space and then smoothly generalized on the whole manifold as follows.

Definition 4 *In $n = p + q$ dimensional manifold $T\mathbb{M}^{p,q}$ over $\mathbb{R}$, the* **element** *is defined by*

$$dx = \gamma_\mu dx^\mu = \gamma^\mu dx_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{12}$$

where γ_a is the local orthogonal frame and γ^a the coframe. The **distance** *$ds = |d\boldsymbol{x}|$ and* **oriented volumes** *dV_k is defined by*

$$d\boldsymbol{x}^2 = \frac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\delta X^a \delta X^b, \tag{13}$$

$$dV_k = d\boldsymbol{x}_1 \wedge d\boldsymbol{x}_2 \wedge \cdots \wedge d\boldsymbol{x}_k = \gamma_{\mu\nu\cdots\omega}dx_1^\mu dx_2^\nu \cdots dx_k^\omega, \quad (1 \le k \le n), \tag{14}$$

in which $(\eta_{ab}) = diag\left(I_p, -I_q\right)$ is **Minkowski metric** *and $g_{\mu\nu}$ is* **Riemann metric**.

$$\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k(T\mathbb{M}^{p,q})$$

is **Grassmann basis**. *The following Clifford-Grassmann number with basis*

$$c = c_0 I + c_\mu \gamma^\mu + c_{\mu\nu}\gamma^{\mu\nu} + \cdots + c_{12\cdots n}\gamma^{12\cdots n}, \quad (\forall c_k(x) \in \mathbb{R}) \tag{15}$$

defines real universal **Clifford algebra** *$C\ell_{p,q}$ on the manifold.*

The definitions and treatments in this chapter make the corresponding subtle and fallible concepts in differential geometry much simpler. For example, in spherical coordinate system of $\mathbb{R}^3$, we have element dx and the area element ds in sphere $dr = 0$ as

$$d\mathbf{x} = \sigma_1 dr + \sigma_2 r d\theta + \sigma_3 r \sin\theta d\varphi,$$

$$d\mathbf{s} = \sigma_2 r d\theta \wedge \sigma_3 r \sin\theta = i\sigma_1 r^2 \sin\theta d\theta d\varphi.$$

We have the total area of the sphere

$$\mathbf{A} = \oint d\mathbf{s} = i\sigma_1 r^2 \oint \sin\theta d\theta d\varphi = i\sigma_1 4\pi r^2.$$

The above definition involves a number of concepts, some more explanations are given in the following:

1. The geometrical meanings of elements $d\mathbf{x}, d\mathbf{y}, d\mathbf{x} \wedge d\mathbf{y}$ are shown in **Figure 2**.
   The relation between metric and vector basis is given by:

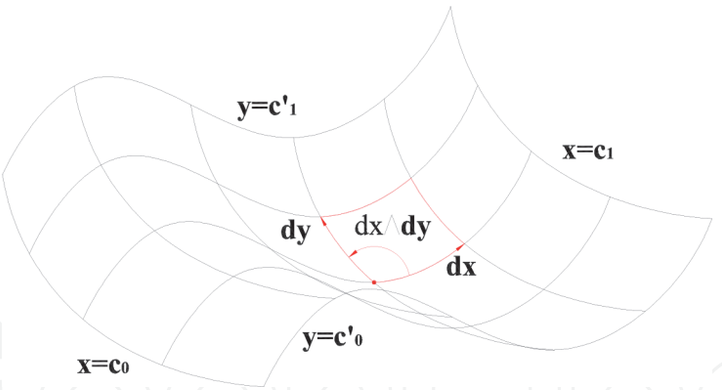

Figure 2.
Geometric meaning of vectors $d\mathbf{x}, d\mathbf{y}$ *and* $d\mathbf{x} \wedge d\mathbf{y}$.

$$g_{\mu\nu} = \frac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) = \gamma_\mu \cdot \gamma_\nu, \tag{16}$$

$$\eta_{ab} = \frac{1}{2}\left(\gamma_a\gamma_b + \gamma_b\gamma_a\right) = \gamma_a \cdot \gamma_b, \tag{17}$$

which is the most important relation in Clifford algebra. Since Clifford algebra is isomorphic to some matrix algebra, by (17) γ_a is equivalent to some special matrices [20]. In practical calculation, we need not distinguish the vector basis from its representation matrix. The relation between the local **frame coefficient** $\left(f_a^\mu, f_\mu^a\right)$ and metric is given by:

$$\gamma^\mu = f_a^\mu\gamma^a, \quad \gamma_\mu = f_\mu^a\gamma_a, \quad f_\mu^a f_b^\mu = \delta_b^a, \quad f_\mu^a f_a^\nu = \delta_\mu^\nu.$$

$$f_a^\mu f_b^\nu \eta^{ab} = g^{\mu\nu}, \quad f_\mu^a f_\nu^b \eta_{ab} = g_{\mu\nu}.$$

2. Assume $\{\gamma_a | a = 1, 2\cdots n\}$ to be the basis of the space-time, then their exterior product is defined by [22]:

$$\gamma_{a_1} \wedge \gamma_{a_2} \cdots \wedge \gamma_{a_k} \equiv \frac{1}{k!}\sum_{\forall\sigma} \sigma_{a_1a_2\cdots a_k}^{b_1b_2\cdots b_k}\gamma_{b_1}\gamma_{b_2}\cdots\gamma_{b_k}, (1 \le k \le n).$$

In which $\sigma_{a_1a_2\cdots a_k}^{b_1b_2\cdots b_k}$ is permutation function, if $b_1b_2\cdots b_k$ is the even permutation of $a_1a_2\cdots a_k$, it equals 1. Otherwise, it equals -1. The above formula is a summation for all permutations, that is, it is antisymmetrization with respect to all indices. The geometric meaning of the exterior product is oriented volume of a higher dimensional parallel polyhedron. Exterior algebra is also called Grassmann algebra, which is associative.

3. By (12) and (13) we find that, using Clifford algebra to deal with the problems on a manifold or in the tangent space, the method is the same. Unless especially mentioned, we always use the Greek alphabet to stand for the index in curved space-time, and the Latin alphabet for the index in tangent space. We use Einstein summation convention.

4. In Eq. (15), each grade-k term is a tensor. For example, $c_0 I \in \Lambda^0$ is a scalar, $c_\mu\gamma^\mu \in \Lambda^1$ is a true vector, and $c_{\mu\nu}\gamma^{\mu\nu} \in \Lambda^2$ is an antisymmetric tensor of rank-2, which is also called a bivector, and so on. In practical calculation, coefficient

and basis should be written together, because they are one entity, such as (12) and (15). In this form, the variables become coordinate free. The coefficient is the value of tensor, which is just a number table, but the geometric meaning and transformation law of the tensor is carried by basis.

The real difficulty in learning modern mathematics is that in order to get a little result, we need a long list of subtle concepts. Mathematicians are used to defining concepts over concepts, but if the chain of concepts breaks down, the subsequent contents will not be understandable. Except for the professionals, the common readers impossibly have so much time to check and understand all concepts carefully. Fortunately, the Clifford algebra can avoid this problem, because Clifford algebra depends only on a few simple concepts, such as numbers, vectors, derivatives, and so on. The only somewhat new concept is the Clifford product of the vector bases, which is isomorphic to some special matrix algebra; and the rules of Clifford algebra are also standardized and suitable for brainless operations, which can be well mastered by high school students.

Definition 5 *For vector* $\boldsymbol{x} = \gamma_\mu x^\mu \in \Lambda^1$ *and multivector* $\boldsymbol{m} = \gamma_{\theta_1\theta_2\cdots\theta_k} m^{\theta_1\theta_2\cdots\theta_k} \in \Lambda^k$, *their inner product is defined as*

$$\boldsymbol{x} \odot \boldsymbol{m} = \left(\gamma_\mu \odot \gamma_{\theta_1\theta_2\cdots\theta_k}\right) x^\mu m^{\theta_1\theta_2\cdots\theta_k}, \qquad \boldsymbol{m} \odot \boldsymbol{x} = \left(\gamma_{\theta_1\theta_2\cdots\theta_k} \odot \gamma_\mu\right) x^\mu m^{\theta_1\theta_2\cdots\theta_k}, \tag{18}$$

in which

$$\gamma^\mu \odot \gamma^{\theta_1\theta_2\cdots\theta_k} \equiv g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} - g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + (-1)^{k+1} g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}, \tag{19}$$

$$\gamma^{\theta_1\theta_2\cdots\theta_k} \odot \gamma^\mu \equiv (-1)^{k+1} g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} + (-1)^k g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}. \tag{20}$$

Theorem 2 *For basis of Clifford algebra, we have the following relations*

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} = \gamma^\mu \odot \gamma^{\theta_1\theta_2\cdots\theta_k} + \gamma^{\mu\theta_1\cdots\theta_k}, \tag{21}$$

$$\gamma^{\theta_1\theta_2\cdots\theta_k}\gamma^\mu = \gamma^{\theta_1\theta_2\cdots\theta_k} \odot \gamma^\mu + \gamma^{\theta_1\cdots\theta_k\mu}. \tag{22}$$

$$\gamma_{a_1a_2\cdots a_{n-1}} = \epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_n}, \tag{23}$$

$$\gamma_{a_1a_2\cdots a_{n-2}} = \frac{1}{2!}\epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_{n-1}a_n}, \tag{24}$$

$$\gamma_{a_1a_2\cdots a_{n-k}} = \frac{1}{k!}\epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_{n-k+1}\cdots a_n}. \tag{25}$$

Proof. Clearly $\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} \in \Lambda^{k-1} \cup \Lambda^{k+1}$, so we have

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} = a_1 g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} + a_2 g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + a_k g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}} + A\gamma^{\mu\theta_1\cdots\theta_k}. \tag{26}$$

Permuting the indices θ_1 and θ_2, we find $a_2 = -a_1$. Let $\mu = \theta_1$, we get $a_1 = 1$. Check the monomial in exterior product, we get $A = 1$. Thus, we prove (21). In like manner, we prove (22). For orthonormal basis γ_a, by (22) we have:

$$\gamma_{a_1a_2\cdots a_{n-1}}\gamma_{a_n} = \epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}. \tag{27}$$

Again by $\gamma_{a_n}\gamma^{a_n} = 1$ (not summation), we prove (23). Other equations can be proved by antisymmetrization of indices. The proof is finished.

Likewise, we can define **multi-inner product** $\mathbf{A}\odot^k\mathbf{B}$ between multivectors as follows:

$$\gamma^{\mu\nu}\odot\gamma^{\alpha\beta}=g^{\mu\beta}\gamma^{\nu\alpha}-g^{\mu\alpha}\gamma^{\nu\beta}+g^{\nu\alpha}\gamma^{\mu\beta}-g^{\nu\beta}\gamma^{\mu\alpha}, \tag{28}$$

$$\gamma^{\mu\nu}\odot^2\gamma^{\alpha\beta}=g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta},\quad \gamma^{\mu\nu}\odot^k\gamma^{\alpha\beta}=0,(k>2).\quad\cdots \tag{29}$$

We use $\mathbf{A}\odot^k\mathbf{B}$ rather $\mathbf{A}\cdot^k\mathbf{B}$, because the symbol "$\cdot$" is too small to express exponential power. Then for the case $\gamma^{\mu_1\mu_2\cdots\mu_f}\gamma^{\theta_1\theta_2\cdots\theta_k}$, we have similar results. For example, we have

$$\gamma^{\mu\nu}\gamma^{\alpha\beta}=\gamma^{\mu\nu}\odot^2\gamma^{\alpha\beta}+\gamma^{\mu\nu}\odot\gamma^{\alpha\beta}+\gamma^{\mu\nu\alpha\beta}. \tag{30}$$

In $C\ell_{1,3}$, denote the Pauli matrices by

$$\sigma^a\equiv\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}, \tag{31}$$

$$\sigma^0=\tilde{\sigma}^0=I,\qquad \tilde{\sigma}^k=-\sigma^k,\ \ (k=1,2,3). \tag{32}$$

We use k,f,j standing for spatial indices. Define Dirac $\gamma-$ matrix by:

$$\gamma^a=\begin{pmatrix}0&\tilde{\sigma}^a\\\sigma^a&0\end{pmatrix},\qquad \gamma^5=diag(I,-I). \tag{33}$$

γ^a forms the grade-1 basis of Clifford algebra $C\ell_{1,3}$. In equivalent sense, the representation (33) is unique. By γ-matrix (33), we have the complete bases of $C\ell_{1,3}$ as follows [21]:

$$I,\quad \gamma^a,\quad \gamma^{ab}=\frac{i}{2}\epsilon^{abcd}\gamma_{cd}\gamma^5,\quad \gamma^{abc}=i\epsilon^{abcd}\gamma_d\gamma^5,\quad \gamma^{0123}=-i\gamma^5. \tag{34}$$

Based on the above preliminaries, we can display some enlightening examples of application, which show how geometric algebra works efficiently. For a skew-symmetrical torsion $\mathcal{T}_{\mu\nu\omega}\equiv g_{\mu\beta}\mathcal{T}^{\beta}_{\nu\omega}$ in $\mathbb{M}^{1,3}$, by Clifford calculus, we have:

$$\mathcal{T}=\mathcal{T}_{\mu\nu\omega}\gamma^{\mu\nu\omega}=\mathcal{T}_{abc}\gamma^{abc}=\mathcal{T}_{abc}\epsilon^{abcd}\gamma_d\left(i\gamma^5\right)\equiv i\gamma_d\gamma^5\mathcal{T}^d=i\gamma_\alpha\gamma^5\mathcal{T}^\alpha, \tag{35}$$

and then

$$\mathcal{T}^\alpha=f^\alpha_d\mathcal{T}_{abc}\epsilon^{abcd}=\mathcal{T}_{\mu\nu\omega}f^\mu_a f^\nu_b f^\omega_c f^\alpha_d\epsilon^{abcd}=\frac{1}{\sqrt{g}}\epsilon^{\mu\nu\omega\alpha}\mathcal{T}_{\mu\nu\omega}, \tag{36}$$

where $g=|\det\left(g_{\mu\nu}\right)|$. So we get:

$$\mathcal{T}_{\mu\nu\omega}=\sqrt{g}\epsilon_{\mu\nu\omega\alpha}\mathcal{T}^\alpha,\qquad \mathcal{T}_{\mu\nu\omega}\mathcal{T}^\omega=0,\qquad \mathcal{T}^\alpha_{\mu\nu}\mathcal{T}^\nu=0. \tag{37}$$

So, the skew-symmetrical torsion is equivalent to a pseudo vector in $\mathbb{M}^{1,3}$. This example shows the advantages to combine variable with basis together.

The following example discusses the absolute differential of tensors. The definition of vector, tensor, and spinor in differential geometry involving a number of refined concepts such as vector bundle and dual bundle, which are too complicated for readers in other specialty. Here, we inherit the traditional definitions based on

the bases γ^a and γ^μ. In physics, basis of tensors is defined by direct products of grade-1 bases γ^μ. For metric, we have [23]:

$$\begin{aligned}\mathbf{g} &= g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu = g^{\mu\nu}\gamma_\mu \otimes \gamma_\nu = \delta^\nu_\mu \gamma^\mu \otimes \gamma_\nu \\ &= \eta_{ab}\gamma^a \otimes \gamma^b = \eta^{ab}\gamma_a \otimes \gamma_b = \delta^b_a \gamma^a \otimes \gamma_b.\end{aligned} \tag{38}$$

For simplicity, we denote tensor basis by:

$$\otimes \gamma^{\mu_1\mu_2\cdots\mu_n} = \gamma^{\mu_1} \otimes \gamma^{\mu_2} \otimes \cdots \gamma^{\mu_n}, \quad \otimes \gamma_{\mu_1}{}^{\mu_2\mu_3\cdots\mu_n} = \gamma_{\mu_1} \otimes \gamma^{\mu_2} \otimes \cdots \gamma^{\mu_n}, \quad \cdots \tag{39}$$

In general, a tensor of rank n is given by:

$$\mathbf{T} = T_{\mu_1\mu_2\cdots\mu_n} \otimes \gamma^{\mu_1\mu_2\cdots\mu_n} = T^{\mu_1}{}_{\mu_2\cdots\mu_n} \otimes \gamma_{\mu_1}{}^{\mu_2\mu_3\cdots\mu_n} = \cdots \tag{40}$$

The geometrical information of the tensor such as transformation law and differential connection are all recorded by basis γ^μ, and all representations of rank (r,s) tensor denote the same one practical entity $\mathbf{T}(\mathbf{x})$. $T^{\nu\cdots}_{\mu\cdots}$ is just a quantity table similar to $c_{\mu\nu}$ in (15), but the physical and geometrical meanings of the tensor $\mathbf{T}$ are represented by basis γ^μ. Clifford algebra is a special kind of tensor with exterior product. Its algebraic calculus exactly reflects the intrinsic property of space-time and makes physical calculation simple and clear.

For the absolute differential of vector field $\mathbf{A} = \gamma_\mu A^\mu$, we have

$$\begin{aligned}dA &\equiv \lim_{\Delta x \to 0}[A(x+\Delta x) - A(x)] \\ &= \left(\partial_\alpha A^\mu \gamma_\mu + A^\mu d_\alpha \gamma_\mu\right)dx^\alpha = \left(\partial_\alpha A_\mu \gamma^\mu + A_\mu d_\alpha \gamma^\mu\right)dx^\alpha.\end{aligned} \tag{41}$$

We call $\mathfrak{d}_\alpha$ **connection operator** [23]. According to its geometrical meanings, connection operator should satisfy the following conditions:

1. It is a real linear transformation of basis γ^μ,

2. It satisfies metric consistent condition $d\mathbf{g} = 0$.

Thus, the differential connection can be generally expressed as:

$$\mathfrak{d}_\alpha \gamma^\mu = -\left(\Pi^\mu_{\alpha\beta} + \mathcal{T}^\mu_{\alpha\beta}\right)\gamma^\beta, \qquad \Pi^\mu_{\alpha\beta} = \Pi^\mu_{\beta\alpha}, \qquad \mathcal{T}^\mu_{\alpha\beta} = -\mathcal{T}^\mu_{\beta\alpha}. \tag{42}$$

For metric $\mathbf{g} = g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu$, by metric consistent condition we have:

$$\begin{aligned}0 = d\mathbf{g} &= d\left(g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu\right) \\ &= \left[\left(\partial_\alpha g_{\mu\nu}\right)\gamma^\mu \otimes \gamma^\nu + g_{\mu\nu}(\mathfrak{d}_\alpha \gamma^\mu) \otimes \gamma^\nu + g_{\mu\nu}\gamma^\mu \otimes \mathfrak{d}_\alpha \gamma^\nu\right]dx^\alpha \\ &= \left[\left(\partial_\alpha g_{\mu\nu} - g_{\nu\beta}\Pi^\beta_{\alpha\mu} - g_{\mu\beta}\Pi^\beta_{\alpha\nu}\right)dx^\alpha - \left(g_{\nu\beta}\mathcal{T}^\beta_{\alpha\mu} + g_{\mu\beta}\mathcal{T}^\beta_{\alpha\nu}\right)dx^\alpha\right]\gamma^\mu \otimes \gamma^\nu.\end{aligned} \tag{43}$$

By (43), we have:

$$\left(\partial_\alpha g_{\mu\nu} - g_{\nu\beta}\Pi^\beta_{\alpha\mu} - g_{\mu\beta}\Pi^\beta_{\alpha\nu}\right)dx^\alpha - \left(g_{\nu\beta}\mathcal{T}^\beta_{\alpha\mu} + g_{\mu\beta}\mathcal{T}^\beta_{\alpha\nu}\right)dx^\alpha = 0. \tag{44}$$

Since $dx^\alpha \leftrightarrow \delta X^a$ is an arbitrary vector in tangent space, (44) is equivalent to:

$$\partial_\alpha g_{\mu\nu} - g_{\nu\beta}\Pi^\beta_{\alpha\mu} - g_{\mu\beta}\Pi^\beta_{\alpha\nu} = g_{\nu\beta}T^\beta_{\alpha\mu} + g_{\mu\beta}T^\beta_{\alpha\nu}. \tag{45}$$

(45) is a linear nonhomogeneous algebraic equation of $\left(\Pi^\mu_{\alpha\beta}, T^\mu_{\alpha\beta}\right)$.

Solving (45), we get the symmetrical particular solution "Christoffel symbols" as follows;

$$\Pi^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) + \pi^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \pi^\alpha_{\mu\nu}, \tag{46}$$

in which $\Gamma^\alpha_{\mu\nu}$ is called Levi-Civita connection determined by metric, $\pi^\alpha_{\mu\nu} = \pi^\alpha_{\nu\mu}$ is a symmetrical post-metric part of connection. In this chapter, the "**post-metric connection**" means the parts of connection cannot be determined by metric, i.e., the components $\pi^\alpha_{\mu\nu}$ and $T^\alpha_{\mu\nu}$ different from Levi-Civita connection $\Gamma^\alpha_{\mu\nu}$. Denote

$$\mathcal{T}_{\mu|\nu\alpha} = g_{\mu\beta}\mathcal{T}^\beta_{\nu\alpha}, \quad \pi_{\mu|\nu\alpha} = g_{\mu\beta}\pi^\beta_{\nu\alpha}, \quad K_{\mu\nu\alpha} = \pi_{\mu|\nu\alpha} + \mathcal{T}_{\mu|\nu\alpha}, \tag{47}$$

where $K_{\mu\nu\alpha}$ is called contortion with total n^3 components [24]. Substituting (46) and (47) into metric compatible condition (45), we get $\frac{1}{2}(n+1)n^2$ constraints for $K_{\mu\nu\alpha}$,

$$K_{\mu\nu\alpha} + K_{\nu\mu\alpha} = 0 = \left(\pi_{\mu|\nu\alpha} + \pi_{\nu|\mu\alpha}\right) + \left(\mathcal{T}_{\mu|\nu\alpha} + \mathcal{T}_{\nu|\mu\alpha}\right). \tag{48}$$

By (48), $K_{\mu\nu\alpha}$ has only $\frac{1}{2}(n-1)n^2$ independent components. Noticing torsion $\mathcal{T}_{\mu|\nu\alpha}$ has just $\frac{1}{2}(n-1)n^2$ independent components, so $K_{\mu\nu\alpha}$ or $\pi_{\mu|\nu\alpha}$ can be represented by $\mathcal{T}_{\mu|\nu\alpha}$.

Theorem 3 *For post-metric connections we have the following relations*

$$\pi_{\mu|\nu\alpha} = \mathcal{T}_{\nu|\alpha\mu} + \mathcal{T}_{\alpha|\nu\mu}, \tag{49}$$

$$K_{\mu\nu\alpha} = \mathcal{T}_{\nu|\alpha\mu} + \mathcal{T}_{\alpha|\nu\mu} + \mathcal{T}_{\mu|\nu\alpha}, \tag{50}$$

$$\mathcal{T}_{\mu|\nu\alpha} = \frac{1}{3}\left(\pi_{\alpha|\mu\nu} - \pi_{\nu|\mu\alpha}\right) + \tilde{\mathcal{T}}_{\mu\nu\alpha}, \tag{51}$$

and consistent condition

$$\pi_{\mu|\nu\alpha} + \pi_{\alpha|\mu\nu} + \pi_{\nu|\alpha\mu} = 0. \tag{52}$$

$\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_{\mu\nu\omega}\gamma^{\mu\nu\omega} \in \Lambda^3$ *is an arbitrary skew-symmetrical tensor*.

Proof If we represent $\pi_{\mu|\nu\alpha}$ by $\mathcal{T}_{\mu|\nu\alpha}$, by (48) and symmetry we have solution as (49). By (49), we get consistent condition (52). By (49) and (47), we get (50).

If we represent $\mathcal{T}_{\mu|\nu\alpha}$ by $\pi_{\mu|\nu\alpha}$, we generally have linear relation

$$\mathcal{T}_{\mu|\nu\alpha} = k\left(\pi_{\nu|\mu\alpha} - \pi_{\alpha|\mu\nu}\right) + \tilde{\mathcal{T}}_{\mu\nu\alpha}, \tag{53}$$

in which k is a constant to be determined, $\tilde{\mathcal{T}}_{\mu\nu\alpha}$ is particular solution as $\pi_{\mu|\nu\alpha} \equiv 0$. $\tilde{\mathcal{T}}_{\mu\nu\alpha}$ satisfies

$$\tilde{\mathcal{T}}_{\mu\nu\alpha} = \tilde{\mathcal{T}}_{\alpha\mu\nu} = \tilde{\mathcal{T}}_{\nu\alpha\mu} = -\tilde{\mathcal{T}}_{\mu\alpha\nu} = -\tilde{\mathcal{T}}_{\nu\mu\alpha} = -\tilde{\mathcal{T}}_{\alpha\nu\mu}. \tag{54}$$

So this part of torsion is a skew-symmetrical tensor $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_{\mu\nu\omega}\gamma^{\mu\nu\omega} \in \Lambda^3$, which has $C^3_n = \frac{1}{6}(n-2)(n-1)n$ independent components. Substituting (53) into (48), we get

$$(k-1)\left(\pi_{\mu|\nu\alpha}+\pi_{\nu|\mu\alpha}\right)=2k\pi_{\alpha|\mu\nu}. \tag{55}$$

Calculating the summation of (55) for circulation of $\{\mu,\nu,\alpha\}$, we also get consistent condition (52). Substituting (52) into (55) we get $k=\frac{1}{3}$. Again by (53), we get solution (51). It is easy to check, (49) and (51) are the inverse representation under condition (52). The proof is finished.

Substituting (42) into

$$0=d\mathbf{g}=\delta^{\mu}_{\nu}\left[\left(\mathfrak{d}_{\alpha}\gamma_{\mu}\right)\otimes\gamma^{\nu}+\gamma_{\mu}\otimes\mathfrak{d}_{\alpha}\gamma^{\nu}\right]dx^{\alpha}, \tag{56}$$

we get

$$\mathfrak{d}_{\alpha}\gamma_{\mu}=\left(\Gamma^{\nu}_{\alpha\mu}+\pi^{\nu}_{\alpha\mu}+\mathcal{T}^{\nu}_{\alpha\mu}\right)\gamma_{\nu}. \tag{57}$$

To understand the meaning of $\pi^{\alpha}_{\mu\nu}$ and $\mathcal{T}^{\alpha}_{\mu\nu}$, we examine the influence on geodesic.

$$\begin{aligned}\frac{d\mathbf{v}}{ds}\equiv\frac{dv^{\alpha}}{ds}\gamma_{\alpha}+v^{\alpha}\mathfrak{d}_{\mu}\gamma_{\alpha}v^{\mu}&=\left(\frac{dv^{\alpha}}{ds}+\left(\Gamma^{\alpha}_{\mu\nu}+\pi^{\alpha}_{\mu\nu}+\mathcal{T}^{\alpha}_{\mu\nu}\right)v^{\mu}v^{\nu}\right)\gamma_{\alpha},\\&=\left(\frac{d}{ds}v^{\alpha}+\Gamma^{\alpha}_{\mu\nu}v^{\mu}v^{\nu}\right)\gamma_{\alpha}+\pi^{\alpha}_{\mu\nu}v^{\mu}v^{\nu}\gamma_{\alpha}.\end{aligned} \tag{58}$$

The term $\mathcal{T}^{\alpha}_{\mu\nu}v^{\mu}v^{\nu}=0$ due to $\mathcal{T}^{\alpha}_{\mu\nu}=-\mathcal{T}^{\alpha}_{\nu\mu}$. So the symmetrical part $\pi^{\alpha}_{\mu\nu}$ influences the geodesic, but the antisymmetrical part $\mathcal{T}^{\alpha}_{\mu\nu}$ only influences spin of a particle. This means $\pi^{\alpha}_{\mu\nu}\neq 0$ violates Einstein's equivalent principle. In what follows, we take $\pi^{\alpha}_{\mu\nu}=0$.

By (42) and (57), we get:

Theorem 4 *In the case* $\pi^{\alpha}_{\mu\nu}\equiv 0$, *the absolute differential of vector* $\boldsymbol{A}$ *is given by*

$$d\boldsymbol{A}=\nabla_{\alpha}A^{\mu}\gamma_{\mu}dx^{\alpha}=\nabla_{\alpha}A_{\mu}\gamma^{\mu}dx^{\alpha}, \tag{59}$$

in which ∇_{α} *denotes the absolute derivatives of vector defined as follows*:

$$\nabla_{\alpha}A^{\mu}=A^{\mu}_{;\,\alpha}+\mathcal{T}^{\mu}_{\alpha\beta}A^{\beta},\quad A^{\mu}_{;\,\alpha}=\partial_{\alpha}A^{\mu}+\Gamma^{\mu}_{\alpha\nu}A^{\nu}, \tag{60}$$

$$\nabla_{\alpha}A_{\mu}=A_{\mu;\,\alpha}-\mathcal{T}^{\beta}_{\alpha\mu}A_{\beta},\quad A_{\mu;\,\alpha}=\partial_{\alpha}A_{\mu}-\Gamma^{\nu}_{\alpha\mu}A_{\nu}, \tag{61}$$

where $A^{\mu}_{;\,\alpha}$ and $A_{\mu;\,\alpha}$ *are usual covariant derivatives of vector without torsion. Torsion* $\mathcal{T}_{\mu\nu\omega}\in\Lambda^{3}$ *is an antisymmetrical tensor of* C^{3}_{n} *independent components*.

Similarly, we can calculate the absolute differential for any tensor. The example also shows the advantages to combine variable with basis.

Now we take spinor connection as example to show the power of Clifford algebra. For Dirac equation in curved space-time without torsion, we have [23, 25, 26]:

$$\gamma^{\mu}i\left(\partial_{\mu}+\Gamma_{\mu}\right)\phi=m\phi,\qquad\Gamma_{\mu}=\frac{1}{4}\gamma_{\nu}\left(\partial_{\mu}\gamma^{\nu}+\Gamma^{\nu}_{\mu\alpha}\gamma^{\alpha}\right). \tag{62}$$

Γ_{μ} is called spinor connection. Representing $\gamma^{\mu}\Gamma_{\mu}\in\Lambda^{1}\cup\Lambda^{3}$ in the form of (15), we get:

$$\alpha^{\mu}\hat{p}_{\mu}\phi-s_{\mu}\Omega^{\mu}\phi=m\gamma^{0}\phi, \tag{63}$$

where α^μ is current operator, $\hat{p}_\mu$ is momentum operator, and s_μ spin operator. They are defined respectively by:

$$\alpha^\mu = \mathrm{diag}(\sigma^\mu, \tilde{\sigma}^\mu), \quad \hat{p}_\mu = i\left(\partial_\mu + \Upsilon_\mu\right) - eA_\mu, \quad s^\mu = \frac{1}{2}\mathrm{diag}(\sigma^\mu, -\tilde{\sigma}^\mu), \tag{64}$$

where $\sigma^\mu = f_a^\mu \sigma^a$ and $\tilde{\sigma}^\mu = f_a^\mu \tilde{\sigma}^a$ are the Pauli matrices in curved space-time. $\Upsilon_\mu \in \Lambda^1$ is called Keller connection, and $\Omega_\mu \in \Lambda^3$ is called Gu-Nester potential, which is a pseudo vector [23, 26, 27]. They are calculated by:

$$\Upsilon_\mu = \frac{1}{2} f_a^\nu \left(\partial_\mu f_\nu^a - \partial_\nu f_\mu^a\right), \quad \Omega^\alpha = \frac{1}{2}\epsilon^{abcd} f_d^\alpha f_a^\mu f_b^\nu \partial_\mu f_\nu^e \eta_{ce} = \frac{1}{4}\epsilon^{dabc} f_d^\alpha f_a^\beta S_{bc}^{\mu\nu} \partial_\beta g_{\mu\nu}, \tag{65}$$

where $S_{ab}^{\mu\nu} \equiv f_a^{\{\mu} f_b^{\nu\}} \mathrm{sign}(a-b)$ for LU decomposition of metric. In the Hamiltonian of a spinor, we get a spin-gravity coupling potential $s_\mu \Omega^\mu$. If the metric of the space-time can be orthogonalized, we have $\Omega_\mu \equiv 0$.

If the gravitational field is generated by a rotating ball, the corresponding metric, like the Kerr one, cannot be diagonalized. In this case, the spin-gravity coupling term has nonzero coupling effect. In asymptotically flat space-time, we have the line element in quasi-spherical coordinate system [28]:

$$dx = \gamma_0 \sqrt{U}(dt + W d\varphi) + \sqrt{V}(\gamma_1 dr + \gamma_2 r d\theta) + \gamma_3 \sqrt{U^{-1}} r \sin\theta d\varphi, \tag{66}$$

$$dx^2 = U(dt + W d\varphi)^2 - V\left(dr^2 + r^2 d\theta^2\right) - U^{-1} r^2 \sin^2\theta d\varphi^2, \tag{67}$$

in which (U, V, W) is just functions of (r, θ). As $r \to \infty$ we have:

$$U \to 1 - \frac{2m}{r}, \quad W \to \frac{4L}{r}\sin^2\theta, \quad V \to 1 + \frac{2m}{r}, \tag{68}$$

where (m, L) are mass and angular momentum of the star, respectively. For common stars and planets, we always have $r \gg m \gg L$. For example, we have $m \doteq 3$ km for the sun. The nonzero tetrad coefficients of metric (66) are given by:

$$\begin{cases} f_t^0 = \sqrt{U}, f_r^1 = \sqrt{V}, f_\theta^2 = r\sqrt{V}, f_\varphi^3 = \dfrac{r\sin\theta}{\sqrt{U}}, f_\varphi^0 = \sqrt{U}W, \\ f_0^t = \dfrac{1}{\sqrt{U}}, f_1^r = \dfrac{1}{\sqrt{V}}, f_2^\theta = \dfrac{1}{r\sqrt{V}}, f_3^\varphi = \dfrac{\sqrt{U}}{r\sin\theta}, f_3^t = \dfrac{-\sqrt{U}W}{r\sin\theta}. \end{cases} \tag{69}$$

Substituting it into (65) we get

$$\begin{aligned} \Omega^\alpha &= f_0^t f_1^r f_2^\theta f_3^\varphi \left(0, \partial_\theta g_{t\varphi}, -\partial_r g_{t\varphi}, 0\right) \\ &= \left(Vr^2 \sin\theta\right)^{-1} (0, \partial_\theta(UW), -\partial_r(UW), 0) \\ &\to \frac{4L}{r^4}(0, 2r\cos\theta, \sin\theta, 0). \end{aligned} \tag{70}$$

By (70), we find that the intensity of Ω^α is proportional to the angular momentum of the star, and its force line is given by:

$$\frac{dx^{\mu}}{ds} = \Omega^{\mu} \Rightarrow \frac{dr}{d\theta} = \frac{2r\cos\theta}{\sin\theta} \Leftrightarrow r = R\sin^2\theta. \tag{71}$$

(71) shows that the force lines of Ω^{α} is just the magnetic lines of a magnetic dipole. According to the above results, we know that the spin-gravity coupling potential of charged particles will certainly induce a macroscopic dipolar magnetic field for a star, and it should be approximately in accordance with the Schuster-Wilson-Blackett relation [29–31].

3. Representation of Clifford algebra

The matrix representation of Clifford algebra is an old problem with a long history. As early as in 1908, Cartan got the following periodicity of 8 [18, 19].

Theorem 5 *For real universal Clifford algebra $C\ell_{p,q}$, we have the following isomorphism*

$$C\ell_{p,q} \cong \begin{cases} \mathrm{Mat}\left(2^{\frac{n}{2}}, \mathbb{R}\right), & \text{if } \mathrm{mod}(p-q,8) = 0,2 \\ \mathrm{Mat}\left(2^{\frac{n-1}{2}}, \mathbb{R}\right) \oplus \mathrm{Mat}\left(2^{\frac{n-1}{2}}, \mathbb{R}\right), & \text{if } \mathrm{mod}(p-q,8) = 1 \\ \mathrm{Mat}\left(2^{\frac{n-1}{2}}, \mathbb{C}\right), & \text{if } \mathrm{mod}(p-q,8) = 3,7 \\ \mathrm{Mat}\left(2^{\frac{n-2}{2}}, \mathbb{H}\right), & \text{if } \mathrm{mod}(p-q,8) = 4,6 \\ \mathrm{Mat}\left(2^{\frac{n-3}{2}}, \mathbb{H}\right) \oplus \mathrm{Mat}\left(2^{\frac{n-3}{2}}, \mathbb{H}\right), & \text{if } \mathrm{mod}(p-q,8) = 5. \end{cases} \tag{72}$$

For $C\ell_{0,2}$, we have $C = tI + x\gamma_1 + y\gamma_2 + z\gamma_{12}$ with

$$\gamma_1^2 = \gamma_2^2 = \gamma_{12}^2 = -1, \gamma_1\gamma_2 = -\gamma_2\gamma_1 = \gamma_{12}, \gamma_2\gamma_{12} = -\gamma_{12}\gamma_2 = \gamma_1, \gamma_{12}\gamma_1 = -\gamma_1\gamma_{12} = \gamma_2. \tag{73}$$

By (73), we find C is equivalent to a quaternion, that is, we have isomorphic relation $C\ell_{0,2} \cong \mathbb{H}$.

Similarly, for $C\ell_{2,0}$, we have $C = tI + x\gamma_1 + y\gamma_2 + z\gamma_{12}$ with

$$\gamma_1^2 = \gamma_2^2 = \gamma_{12}^2 = 1, \gamma_1\gamma_2 = -\gamma_2\gamma_1 = \gamma_{12}, \gamma_2\gamma_{12} = -\gamma_{12}\gamma_2 = -\gamma_1, \gamma_{12}\gamma_1 = -\gamma_1\gamma_{12} = -\gamma_2. \tag{74}$$

By (74), the basis is equivalent to

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{75}$$

Thus, (75) means $C\ell_{2,0} \cong \mathrm{Mat}(2, \mathbb{R})$.

In geometry and physics, the matrix representation of generators of Clifford algebra is more important and fundamental than the representation of whole algebra. Define γ^{μ} by

$$\begin{aligned} \gamma^{\mu} &= \begin{pmatrix} 0 & \tilde{\vartheta}^{\mu} \\ \vartheta^{\mu} & 0 \end{pmatrix} \equiv \Gamma^{\mu}(m), \quad \vartheta_{\mu} = \mathrm{diag}\left(\overbrace{\sigma_{\mu}, \sigma_{\mu}, \cdots, \sigma_{\mu}}^{m}\right), \quad \tilde{\vartheta}_{\mu} \\ &= \mathrm{diag}\left(\overbrace{\tilde{\sigma}_{\mu}, \tilde{\sigma}_{\mu}, \cdots, \tilde{\sigma}_{\mu}}^{m}\right). \end{aligned} \tag{76}$$

which forms the generator or grade-1 basis of Clifford algebra $C\ell_{1,3}$. To denote γ^μ by $\Gamma^\mu(m)$ is for the convenience of representation of high dimensional Clifford algebra. For any matrices C^μ satisfying $C\ell_{1,3}$ Clifford algebra, we have [20, 32]:

Theorem 6 *Assuming the matrices C^μ satisfy anti-commutative relation of $C\ell_{1,3}$*

$$C^\mu C^\nu + C^\nu C^\mu = 2\eta^{\mu\nu}, \tag{77}$$

then there is a natural number m and an invertible matrix K, such that $K^{-1}C^\mu K = \Gamma^\mu(m)$.

This means in equivalent sense, we have unique representation (76) for generator of $C\ell_{1,3}$. In [20], we derived complex representation of generators of $C\ell_{p,q}$ based on Theorem 6 and real representations according to the complex representations as follows.

Theorem 7 *Let*

$$\gamma^5 = i\,diag(E, -E), \quad E \equiv diag(I_{2k}, -I_{2l}), \quad k + l = n. \tag{78}$$

Other γ^μ, $(\mu \le 3)$ are given by (76). *Then the generators of Clifford algebra $C\ell_{1,4}$ are equivalent to $\forall\gamma^\mu$, $(\mu = 0, 1, 2, 3, 5)$.*

In order to express the general representation of generators, we introduce some simple notations. I_m stands for $m \times m$ unit matrix. For any matrix $A = (A_{ab})$, denote block matrix

$$A \otimes I_m = (A_{ab}I_m), \qquad [A, B, C, \cdots] = \mathrm{diag}(A, B, C, \cdots). \tag{79}$$

in which the direct product of matrix is Kronecker product. Obviously, we have $I_2 \otimes I_2 = I_4$, $I_2 \otimes I_2 \otimes I_2 = I_8$, and so on. In what follows, we use $\Gamma^\mu(m)$ defined in (76). For $\mu \in \{0, 1, 2, 3\}$, $\Gamma^\mu(m)$ is $4m \times 4m$ matrix, which constitute the generator of $C\ell_{1,3}$. Similar to the above proofs, we can check the following theorem by method of induction.

Theorem 8

1. *In equivalent sense, for $C\ell_{4m}$, the matrix representation of generators is uniquely given by*

$$\begin{Bmatrix} \Gamma^\mu(n), \ \left[\Gamma^\mu\left(\frac{n}{2^2}\right), -\Gamma^\mu\left(\frac{n}{2^2}\right)\right] \otimes I_2, \\ \left[\left[\Gamma^\mu\left(\frac{n}{2^4}\right), -\Gamma^\mu\left(\frac{n}{2^4}\right)\right], -\left[\Gamma^\mu\left(\frac{n}{2^4}\right), -\Gamma^\mu\left(\frac{n}{2^4}\right)\right]\right] \otimes I_{2^2}, \\ \left[\Gamma^\mu\left(\frac{n}{2^6}\right), -\Gamma^\mu\left(\frac{n}{2^6}\right), -\Gamma^\mu\left(\frac{n}{2^6}\right), \Gamma^\mu\left(\frac{n}{2^6}\right), -\Gamma^\mu\left(\frac{n}{2^6}\right), \Gamma^\mu\left(\frac{n}{2^6}\right), \Gamma^\mu\left(\frac{n}{2^6}\right), -\Gamma^\mu\left(\frac{n}{2^6}\right)\right] \otimes I_{2^3}, \cdots \end{Bmatrix}. \tag{80}$$

in which $n = 2^{m-1}N$, where N is any given positive integer. All matrices are $2^{m+1}N \times 2^{m+1}N$ type.

2. *For $C\ell_{4m+1}$, besides* (80) *we have another real generator*

$$\gamma^{4m+1} = [E, -E, -E, E, -E, E, E, -E\cdots], \qquad E = [I_{2k}, -I_{2l}]. \tag{81}$$

If and only if $k = l$, this representation can be uniquely expanded as generators of $C\ell_{4m+4}$.

3. *For any* $C\ell_{p,q}$, $\{p,q|p+q\leq 4m, \mathrm{mod}(p+q,4)\neq 1\}$, *the combination of* $p+q$ *linear independent generators* $\{\gamma^{\mu}, i\gamma^{\nu}\}$ *taking from* (80) *constitutes the complete set of generators. In the case* $\{p,q|p+q\leq 4m, \mathrm{mod}(p+q,4)=1\}$, *besides the combination of* $\{\gamma^{\mu}, i\gamma^{\nu}\}$, *we have another normal representation of generator taking the form* (81) *with* $k\neq l$.

4. *For* $C\ell_m$, $(m<4)$, *we have another* 2×2 *Pauli matrix representation for its generators* $\left\{\sigma^1,\sigma^2,\sigma^3\right\}$.

Then, we get all complex matrix representations for generators of real $C\ell_{p,q}$ explicitly.

The real representation of $C\ell_{p,q}$ can be easily constructed from the above complex representation. In order to get the real representation, we should classify the generators derived above. Let $\mathbf{G}_c(n)$ stand for any one set of all complex generators of $C\ell_n$ given in Theorem 8, and set the coefficients before all σ^{μ} and $\tilde{\sigma}^{\mu}$ as 1 or i. Denote $\mathbf{G}_{c+}$ stands for the set of complex generators of $C\ell_{n,0}$ and $\mathbf{G}_{c-}$ for the set of complex generators of $C\ell_{0,n}$. Then, we have:

$$\mathbf{G}_c=\mathbf{G}_{c+}\cup\mathbf{G}_{c-},\qquad \mathbf{G}_{c-}\cong i\mathbf{G}_{c+}. \tag{82}$$

By the construction of generators, we have only two kinds of γ^{μ} matrices. One is the matrix with real nonzero elements and the other is that with imaginary nonzero elements. This is because all nonzero elements of σ^2 are imaginary but all other $\sigma^{\mu}(\forall\mu\neq 2)$ are real. Again assume

$$\mathbf{G}_{c+}=\mathbf{G}_r\cup\mathbf{G}_i,\qquad \mathbf{G}_r=\left\{\gamma_r^{\mu}|\gamma_r^{\mu}\text{ is real}\right\},\qquad \mathbf{G}_i=\left\{\gamma_i^{\mu}|\gamma_i^{\mu}\text{ is imaginary}\right\}. \tag{83}$$

Denote $J_2=i\sigma^2$, we have $J_2^2=-I_2$. J_2 becomes the real matrix representation for imaginary unit i. Using the direct products of complex generators with (I_2,J_2), we can easily construct the real representation of all generators for $C\ell_{p,q}$ from $\mathbf{G}_{c+}$ as follows.

Theorem 9

1. *For* $C\ell_{n,0}$, *we have real matrix representation of generators as*

$$\boldsymbol{G}_{r+}=\{\gamma^{\mu}\otimes I_2\ (\text{if }\gamma^{\mu}\in\boldsymbol{G}_r);\ \ i\gamma^{\nu}\otimes J_2\ (\text{if }\gamma^{\nu}\in\boldsymbol{G}_i)\}. \tag{84}$$

2. *For* $C\ell_{0,n}$, *we have real matrix representation of generators as*

$$\boldsymbol{G}_{r-}=\{\gamma^{\mu}\otimes J_2|\gamma^{\mu}\in\boldsymbol{G}_{r+}\}. \tag{85}$$

3. *For* $C\ell_{p,q}$, *we have real matrix representation of generators as*

$$\boldsymbol{G}_r=\left\{\Gamma_+^{\mu_a},\Gamma_-^{\nu_b}\left|\begin{array}{l}\Gamma_+^{\mu_a}=\gamma^{\mu_a}\in\boldsymbol{G}_{r+},(a=1,2,\cdots,p)\\ \\ \Gamma_-^{\nu_b}=\gamma^{\nu_b}\in\boldsymbol{G}_{r-},(b=1,2,\cdots,q)\end{array}\right.\right\}. \tag{86}$$

Obviously we have $C_n^pC_n^q=\left(C_n^p\right)^2$ *choices for the real generators of* $C\ell_{p,q}$ *from each complex representation*.

Proof. By calculating rules of block matrix, it is easy to check the following relations:

$$(\gamma^{\mu}\otimes I_2)(\gamma^{\nu}\otimes J_2)+(\gamma^{\nu}\otimes J_2)(\gamma^{\mu}\otimes I_2)=(\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu})\otimes J_2, \tag{87}$$

$$(\gamma^\mu \otimes J_2)(\gamma^\nu \otimes J_2) + (\gamma^\nu \otimes J_2)(\gamma^\mu \otimes J_2) = -(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \otimes I_2. \tag{88}$$

By these relations, Theorem 9 becomes a direct result of Theorem 8.

For example, we have 4×4 real matrix representation for generators of $C\ell_{0,3}$ as follows:

$$i\{\sigma^1, \sigma^2, \sigma^3\} \cong \{\sigma^1 \otimes J_2, i\sigma^2 \otimes I_2, \sigma^3 \otimes J_2\} \equiv \{\Sigma^1, \Sigma^2, \Sigma^3\} =$$
$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}. \tag{89}$$

It is easy to check

$$\Sigma^k\Sigma^l + \Sigma^l\Sigma^k = -2\delta^{kl}, \qquad \Sigma^k\Sigma^l - \Sigma^l\Sigma^k = 2\epsilon^{klm}\Sigma_m. \tag{90}$$

4. Transformation of Clifford algebra

Assume V is the base vector space of $C\ell_{p,q}$, then Clifford algebra has the following global properties [22, 33, 34]:

$$C\ell_{p,q} = \bigoplus_{k=0}^{n} \Lambda^k V = C\ell^+_{p,q} \oplus C\ell^-_{p,q}, \tag{91}$$

$$C\ell^+_{p,q} \equiv \bigoplus_{k=\text{even}} \Lambda^k V, \quad C\ell^-_{p,q} \equiv \bigoplus_{k=\text{odd}} \Lambda^k V, \tag{92}$$

$$C\ell_{p,q} \cong C\ell^+_{p,q+1}. \tag{93}$$

$C\ell_{p,q}$ is a $\mathbb{Z}_2$-graded superalgebra, and $C\ell^+_{p,q}$ is a subalgebra of $C\ell_{p,q}$. We have:

$$C\ell^+C\ell^+ = C\ell^-C\ell^- = C\ell^+, \quad C\ell^+C\ell^- = C\ell^-C\ell^+ = C\ell^-. \tag{94}$$

Definition 6 *The* **conjugation** *of element in $C\ell_{p,q}$ is defined by*

$$\left(\gamma_{k_1k_2\cdots k_m}\right)^* = (-1)^m\gamma_{k_m\cdots k_2k_1} = (-1)^{\frac{1}{2}m(m+1)}\gamma_{k_1k_2\cdots k_m}, (0 \le m \le n). \tag{95}$$

The **main involution** *of element is defined by*

$$\alpha\left(\gamma_{k_1k_2\cdots k_m}\right) = (-1)^m\gamma_{k_1k_2\cdots k_m}, (0 \le m \le n). \tag{96}$$

The **norm** *and* **inverse** *of element are defined by*

$$\mathcal{N}(\boldsymbol{X}) \equiv \boldsymbol{X}\boldsymbol{X}^*, \quad \boldsymbol{X}^{-1} = \boldsymbol{X}^*/N(\boldsymbol{X}) \text{ if } \mathcal{N}(\boldsymbol{X}) \neq 0. \tag{97}$$

By the definition, it is easy to check

$$\gamma_k^* = -\gamma_k, \quad \gamma_{ab}^* = -\gamma_{ab}, \quad \gamma_{abc}^* = \gamma_{abc}, \quad \cdots \tag{98}$$

$$\alpha(\mathbf{x}^*) = \alpha(\mathbf{x})^*, \quad \alpha(\gamma_k) = -\gamma_k, \quad \alpha(\gamma_{ab}) = \gamma_{ab}, \quad \cdots \tag{99}$$

$$\mathbf{g}^{-1} = \mathbf{g}^*, \quad \{\mathbf{g} = \mathbf{g}_1\mathbf{g}_2\cdots\mathbf{g}_m | \forall \mathbf{g}_k \in \Lambda^1, N(\mathbf{g}_k) = 1\}. \tag{100}$$

Definition 7 *The* **Pin group** *and* **Spin group** *of* $C\ell_{p,q}$ *are defined by*

$$Pin_{p,q} = \left\{ \boldsymbol{g} \in C\ell_{p,q} | \mathcal{N}(\boldsymbol{g}) = \pm 1, \alpha(\boldsymbol{g}) \boldsymbol{x} \boldsymbol{g}^* \in V \forall \boldsymbol{x} \in V \right\}, \tag{101}$$

$$Spin_{p,q} = \left\{ \boldsymbol{g} \in C\ell^+_{p,q} | \mathcal{N}(\boldsymbol{g}) = \pm 1, \boldsymbol{g} \boldsymbol{x} \boldsymbol{g}^* \in V \forall \boldsymbol{x} \in V \right\} = Pin \cap C\ell^+. \tag{102}$$

The transformation $\mathbf{x} \mapsto \alpha(\mathbf{g}) \mathbf{x} \mathbf{g}^*$ is called sandwich operator. Pin or Spin group consists of two connected components with $\mathcal{N}(\mathbf{g}) = 1$ or $\mathcal{N}(\mathbf{g}) = -1$,

$$\mathrm{Spin}^+_{p,q} = \left\{ \mathbf{g} \in C\ell^+_{p,q} | \mathcal{N}(\mathbf{g}) = +1, \mathbf{g} \mathbf{x} \mathbf{g}^* \in V \forall \mathbf{x} \in V \right\}, \tag{103}$$

$$\mathrm{Spin}^-_{p,q} = \left\{ \mathbf{g} \in C\ell^+_{p,q} | \mathcal{N}(\mathbf{g}) = -1, \mathbf{g} \mathbf{x} \mathbf{g}^* \in V \forall \mathbf{x} \in V \right\}. \tag{104}$$

For $\forall \mathbf{g} \in \mathrm{Pin}_{p,q}, \mathbf{x} \in V$, the sandwich operator is a linear transformation for vector in V,

$$\mathbf{x}' = \alpha(\mathbf{g}) \mathbf{x} \mathbf{g}^* \Rightarrow X' = KX, X = \left(x^1, x^2 \cdots x^n\right)^T. \tag{105}$$

In all transformations of vector, the reflection and rotation transformations are important in geometry. Here, we discuss the transformation in detail. Let $\mathbf{m} \in \Lambda^1$ be a unit vector in V, then the **reflection transformation** of vector $\mathbf{X} \in \Lambda^1$ with respect to $n-1$ dimensional mirror perpendicular to $\mathbf{m}$ is defined by [35]:

$$\mathbf{X}' = \mathbf{m} \mathbf{X} \mathbf{m}^* = -\mathbf{m} \mathbf{X} \mathbf{m}. \tag{106}$$

Let $\mathbf{m} = \gamma_a m^a, \mathbf{X} = \gamma_a X^a$, substituting it into (106) and using (21), we have:

$$\begin{aligned} \mathbf{X}' &= -\left(\mathbf{m} \odot \mathbf{X} + m^a X^b \gamma_{ab}\right) \mathbf{m} = -(\mathbf{m} \odot \mathbf{X}) \mathbf{m} - m^a X^b m^c (\gamma_{ab} \gamma_c) \\ &= -(\mathbf{m} \odot \mathbf{X}) \mathbf{m} - m^a X^b m^c \left(g_{bc} \gamma_a - g_{ac} \gamma_b + \gamma_{abc}\right) \\ &= -2(\mathbf{m} \odot \mathbf{X}) \mathbf{m} + \mathbf{X} = \mathbf{X}_\perp - \mathbf{X}_\parallel. \end{aligned} \tag{107}$$

Eq. (107) clearly shows the geometrical meaning of reflection. By (106), we learn reflection transformation belongs to $\mathrm{Pin}_{p,q}$ group (**Figure 3**).

The **rotation transformation** $\mathbf{R} \in \mathrm{Spin}_{p,q}$,

$$\mathbf{X}' = \mathbf{R} \mathbf{X} \mathbf{R}^{-1}. \tag{108}$$

The group elements of elementary transformation in Λ^2 are given by [22, 36]:

$$\left(\cosh \frac{\upsilon_{ab}}{2} + \gamma_{ab} \sinh \frac{\upsilon_{ab}}{2}\right)^{-1} = \left(\cosh \frac{\upsilon_{ab}}{2} - \gamma_{ab} \sinh \frac{\upsilon_{ab}}{2}\right), \upsilon_{ab} \in \mathbb{R}, \tag{109}$$

$$\left(\cos \frac{\theta_{ab}}{2} + \gamma_{ab} \sin \frac{\theta_{ab}}{2}\right)^{-1} = \left(\cos \frac{\theta_{ab}}{2} - \gamma_{ab} \sin \frac{\theta_{ab}}{2}\right), \theta_{ab} \in [-\pi, \pi). \tag{110}$$

The total transformation can be expressed as multiplication of elementary transformations as follows:

$$\mathbf{R} = \prod_{\{\eta_{aa}\eta_{bb}=-1\}} \left(\cosh \frac{\upsilon_{ab}}{2} + \gamma_{ab} \sinh \frac{\upsilon_{ab}}{2}\right) \prod_{\{\eta_{aa}\eta_{bb}=1\}} \left(\cos \frac{\theta_{ab}}{2} + \gamma_{ab} \sin \frac{\theta_{ab}}{2}\right). \tag{111}$$

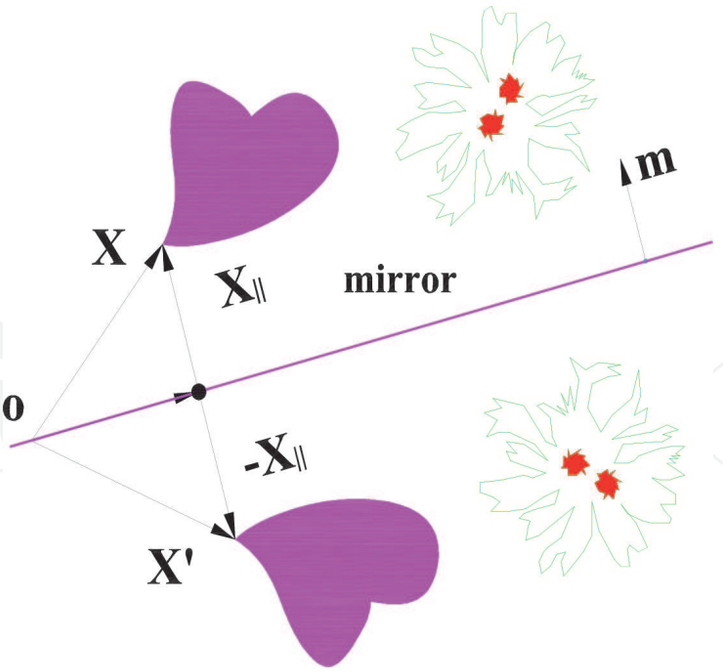

Figure 3.
Reflection transformation $\mathbf{X}' = \mathbf{X}_\perp - \mathbf{X}_\parallel$.

(111) has $\frac{1}{2}(n-1)n$ generating elements like $SO(n)$. In (111), we have commutative relation as follows:

$$\left[\cosh\frac{\upsilon_{ab}}{2}+\gamma_{ab}\sinh\frac{\upsilon_{ab}}{2},\ \cos\frac{\theta_{cd}}{2}+\gamma_{cd}\sin\frac{\theta_{cd}}{2}\right]=2\sinh\frac{\upsilon_{ab}}{2}\sin\frac{\theta_{cd}}{2}\gamma_{ab}\odot\gamma_{cd}, \tag{112}$$

$$\left[\cos\frac{\theta_{ab}}{2}+\gamma_{ab}\sin\frac{\theta_{ab}}{2},\ \cos\frac{\theta_{cd}}{2}+\gamma_{cd}\sin\frac{\theta_{cd}}{2}\right]=2\sin\frac{\theta_{ab}}{2}\sin\frac{\theta_{cd}}{2}\gamma_{ab}\odot\gamma_{cd}, \tag{113}$$

in which

$$\gamma_{ab}\odot\gamma_{cd}=\eta_{bc}\gamma_{ad}-\eta_{ac}\gamma_{bd}+\eta_{ad}\gamma_{bc}-\eta_{bd}\gamma_{ac}\in\Lambda^2. \tag{114}$$

If $a\neq b\neq c\neq d$, the right hand terms vanish, and then two elementary transformations commute with each other.

R forms a Lie Group of $\frac{1}{2}(n-1)n$ paraments. In the case $C\ell_{n,0}$ or $C\ell_{0,n}$, **R** is compact group isomorphic to $SO(n)$. Otherwise, **R** is noncompact one similar to Lorentz transformation. The infinitesimal generators of the corresponding Lie group is γ_{ab}, and the Lie algebra is given by:

$$\mathcal{R}=\varepsilon^{ab}\gamma_{ab},\quad [\gamma_{ab},\gamma_{cd}]=2\gamma_{ab}\odot\gamma_{cd}\in\Lambda^2,\quad \forall\varepsilon^{ab}\in\mathbb{R}. \tag{115}$$

Thus, $\Lambda^2(\mathbb{M}^{p,q})$ is just the Lie algebra of proper Lorentz transformation of the space-time $\mathbb{M}^{p,q}$.

5. Application in classical geometry

Suppose the basic space of projective geometry is n-dimensional Euclidean space π (see **Figure 4**), and the basis is $\{\gamma_a|a=1,2,\cdots,n\}$. The coordinate of point **x** is

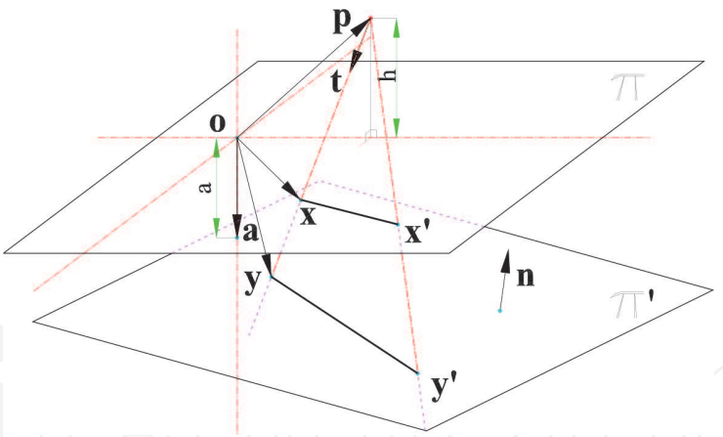

Figure 4.
Diagram of parameter setting for projective geometry.

given by $\mathbf{x} = \gamma_a x^a$. The projective polar is P, and its height from the basic space π is h. The total projective space is $n+1$ dimensional, and an auxiliary basis $\gamma_{n+1} = \gamma_p$ is introduced. The coordinate of the polar P is $\mathbf{p} = \gamma_\mu p^\mu$. In this section, we use Greek characters for $n+1$ indices. Assume the unit directional vector of the projective ray is $\mathbf{t} = \gamma_\mu t^\mu$, the unit normal vector of the image space π' is $\mathbf{n} = \gamma_\mu n^\mu$, coordinate in π' is $\mathbf{y} = \gamma_\mu y^\mu$, and the intercept of π' with the $n+1$ coordinate axis is a. Then, we have:

$$(\mathbf{y} - \mathbf{a}) \odot \mathbf{n} = 0, \quad \text{or} \quad \mathbf{y} \odot \mathbf{n} = n_\mu y^\mu = a n_p. \tag{116}$$

The equation of projective ray is given by:

$$\mathbf{s} = \mathbf{p} + \lambda \mathbf{t}, \tag{117}$$

where λ is parameter coordinate of the line. In the basic space π, we have $s^{n+1} = 0$ and $\lambda = -h/t^p$, so the coordinate of the line in π reads

$$\mathbf{x} = \mathbf{p} - \frac{h}{t^p}\mathbf{t}. \tag{118}$$

Let $\mathbf{s} = \mathbf{y}$ and substitute (117) into (116) we get image equation as follows:

$$\mathbf{y} = \mathbf{p} + \frac{a n_p - \mathbf{p} \odot \mathbf{n}}{\mathbf{t} \odot \mathbf{n}}\mathbf{t}, \qquad \lambda = \frac{a n_p - \mathbf{p} \odot \mathbf{n}}{\mathbf{t} \odot \mathbf{n}}. \tag{119}$$

In the above equation $\mathbf{t} \odot \mathbf{n} \neq 0$, which means $\mathbf{t}$ cannot be perpendicular to $\mathbf{n}$; otherwise, the projection cannot be realized. Eliminating coordinate $\mathbf{t}$ in (118) and (119), we find the projective transformation $\mathbf{y} \leftrightarrow \mathbf{x}$ is nonlinear. In (119), only the parameters $(a, \mathbf{n})$ are related to image space π'; so, all geometric variables independent of two parameters $(a, \mathbf{n})$ are projective invariants. In what follows we prove the fundamental theorems of projective geometry by Clifford algebra.

Theorem 10 *For 4 different points* $\{y_1, y_2, y_3, y_4\}$ *on a straight line L, the following cross ratio is a projective invariant*

$$(12;34) \equiv \frac{|y_1 - y_3|}{|y_2 - y_3|} \cdot \frac{|y_2 - y_4|}{|y_1 - y_4|}. \tag{120}$$

Proof Substituting (119) into (120) we get

$$(12;34) = \frac{|(\mathbf{t}_3 \odot \mathbf{n})\mathbf{t}_1 - (\mathbf{t}_1 \odot \mathbf{n})\mathbf{t}_3|}{|(\mathbf{t}_3 \odot \mathbf{n})t_2 - (\mathbf{t}_2 \odot \mathbf{n})\mathbf{t}_3|} \cdot \frac{|(\mathbf{t}_4 \odot \mathbf{n})\mathbf{t}_2 - (\mathbf{t}_2 \odot \mathbf{n})\mathbf{t}_4|}{|(\mathbf{t}_4 \odot \mathbf{n})\mathbf{t}_1 - (\mathbf{t}_1 \odot \mathbf{n})\mathbf{t}_4|}. \tag{121}$$

By (19) and (20), we get

$$(\mathbf{t}_b \odot \mathbf{n})\mathbf{t}_a - (\mathbf{t}_a \odot n)\mathbf{t}_b = (\mathbf{t}_a \wedge \mathbf{t}_b) \odot \mathbf{n} = \pm|\mathbf{t}_a \wedge \mathbf{t}_b|\mathbf{m} \odot \mathbf{n}, \tag{122}$$

where $\mathbf{m}$ is the unit normal vector of the plane spanned by $(\mathbf{t}_a, \mathbf{t}_b)$, which is independent of the image space π'. Substituting it into (121), we get

$$(12;34) = \frac{|\mathbf{t}_3 \wedge \mathbf{t}_1|}{|\mathbf{t}_3 \wedge \mathbf{t}_2|} \cdot \frac{|\mathbf{t}_4 \wedge \mathbf{t}_2|}{|\mathbf{t}_4 \wedge \mathbf{t}_1|}. \tag{123}$$

(123) is independent of $(a, \mathbf{n})$; so, it is a projective invariant. Likewise, $(13;24)$ and $(14;23)$ are also projective invariants. The proof is finished.

Now we examine affine transformation. In this case, the polar P at infinity and the directional vector $\mathbf{t}$ of rays becomes constant vector. The equation of rays is given by $\mathbf{y} = \mathbf{x} + \lambda\mathbf{t}$. Substituting it into (116), we get the coordinate transformation from basic space π to image space π',

$$\mathbf{y} = \mathbf{x} + \frac{an_p - \mathbf{n} \odot \mathbf{x}}{\mathbf{t} \odot \mathbf{n}}\mathrm{t}, \qquad \lambda = \frac{an_p - \mathbf{n} \odot \mathbf{x}}{\mathbf{t} \odot \mathbf{n}}. \tag{124}$$

Since $\mathbf{t}$ and $\mathbf{n}$ are constant vectors for all rays, the affine transformation $\mathbf{y} \leftrightarrow \mathbf{x}$ is linear. A variable independent of $(a, \mathbf{n})$ is an affine invariant.

Theorem 11 *Assume* $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ *are 3 points on a straight line L in basic space π, and* $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ *are respectively their projective images on line L′ in π′. Then the simple ratio*

$$(12, 13) \equiv \frac{|\boldsymbol{y}_2 - \boldsymbol{y}_1|}{|\boldsymbol{y}_3 - \boldsymbol{y}_1|} \tag{125}$$

is an affine invariant.

Proof By equation of transformation (124) we get

$$\mathbf{y}_k = \mathbf{x}_k + \frac{an_p - \mathbf{n} \odot \mathbf{x}_k}{\mathbf{t} \odot \mathbf{n}}\mathbf{t}. \tag{126}$$

In (126), only the parameters $(a, \mathbf{n})$ are related to image space π'. Substituting (126) into (125), we have:

$$(12, 13) = \frac{|(\mathbf{t} \odot \mathbf{n})(\mathbf{x}_2 - \mathbf{x}_1) - \mathbf{n} \odot (\mathbf{x}_2 - \mathbf{x}_1)\mathbf{t}|}{|(\mathbf{t} \odot \mathbf{n})(\mathbf{x}_3 - \mathbf{x}_1) - \mathbf{n} \odot (\mathbf{x}_3 - \mathbf{x}_1)\mathbf{t}|} = \frac{|((\mathbf{x}_2 - \mathbf{x}_1) \wedge \mathbf{t}) \odot \mathbf{n}|}{|((\mathbf{x}_3 - \mathbf{x}_1) \wedge \mathbf{t}) \odot \mathbf{n}|}. \tag{127}$$

Denote the unit directional vector of line L by $\mathbf{k}$, then we have

$$\mathbf{x}_2 - \mathbf{x}_1 = \pm|\mathbf{x}_2 - \mathbf{x}_1|\mathbf{k}, \qquad \mathbf{x}_3 - \mathbf{x}_1 = \pm|\mathbf{x}_3 - \mathbf{x}_1|\mathbf{k}. \tag{128}$$

Substituting them into (127) we get:

$$(12, 13) = \frac{|\mathbf{y}_2 - \mathbf{y}_1|}{|\mathbf{y}_3 - \mathbf{y}_1|} = \frac{|\mathbf{x}_2 - \mathbf{x}_1|}{|\mathbf{x}_3 - \mathbf{x}_1|}. \tag{129}$$

This proves the simple ratio $(12, 13)$ is an affine invariant. Likewise, we can prove $(12, 23)$ and $(13, 23)$ are also affine invariants. The proof is finished.

The treatment of image information by computer requires concise and general algebraic representation for geometric modeling as well as fast and robust algebraic algorithm for geometric calculation. Conformal geometry algebra was introduced in this context. By establishing unified covariant algebra representation of classical geometry, the efficient calculation of invariant algebra is realized [13–15]. It provides a unified and concise homogeneous algebraic framework for classical geometry and algorithms, which can thus be used for complicated symbolic geometric calculations. This technology is currently widely applied in high-tech fields such as computer graphics, vision calculation, geometric design, and robots.

The algebraic representation of a geometric object is homogeneous, which means that any two algebraic expressions representing this object differ by only one nonzero factor and any such algebraic expressions with different nonzero multiple represent the same geometric object. The embedding space provided by conformal geometric algebra for n dimensional Euclidean space is $n + 2$ dimensional Minkowski space. Since the orthonormal transformation group of the embedding space is exactly double coverage of the conformal transformation group of the Euclidean space, this model is also called the conformal model. The following is a brief introduction to the basic concepts and representation for geometric objects of conformal geometric algebra. The materials mainly come from literature [13].

In conformal geometry algebra, an additional Minkowski plane $\mathbb{M}^{1,1}$ is attached to n dimensional Euclidean space $\mathbb{R}^n$, $\mathbb{M}^{1,1}$ has an orthonormal basis $\{e_+, e_-\}$, which has the following properties:

$$e_+^2 = 1, \qquad e_-^2 = -1, \qquad e_+ \odot e_- = 0. \tag{130}$$

In practical application, $\{e_+, e_-\}$ is replaced by null basis $\{e_0, e\}$

$$e_0 = \frac{1}{2}(e_- - e_+), \qquad e = e_- + e_+. \tag{131}$$

They satisfy

$$e_0^2 = e^2 = 0, \qquad e \odot e_0 = -1. \tag{132}$$

A unit pseudo-scalar E for $\mathbb{M}^{1,1}$ is defined by:

$$E = e \wedge e_0 = e_+ \wedge e_- = e_+ e_-. \tag{133}$$

In conformal geometric algebra, we work with $\mathbb{M}^{n+1,1} = \mathbb{R}^n \oplus \mathbb{M}^{1,1}$.
Define the **horosphere** of $\mathbb{R}^n$ by:

$$\mathcal{N}_e^n = \left\{x \in \mathbb{M}^{n+1,1} | x^2 = 0, x \odot e = -1\right\}. \tag{134}$$

$\mathcal{N}_e^n$ is a homogeneous model of $\mathbb{R}^n$. The powerful applications of conformal geometry come from this model. By calculation, for $\forall \mathbf{x} \in \mathbb{R}^n$ we have:

$$x = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e + e_0, \tag{135}$$

which is a bijective mapping $\mathbf{x} \in \mathbb{R}^n \leftrightarrow x \in \mathcal{N}_e^n$, we have $\mathcal{N}_e^n \cong \mathbb{R}^n$. x is referred to as the **homogeneous point** of $\mathbf{x}$. Clearly, $0 \in \mathbb{R}^n \leftrightarrow e_0 \in \mathcal{N}_e^n$ and $\infty \in \mathbb{R}^n \leftrightarrow e \in \mathcal{N}_e^n$ are in homogeneous coordinate.

Now we examine how conformal geometric algebra represents geometric objects. For a line passing through points $\mathbf{a}$ and $\mathbf{b}$, we have

$$e \wedge a \wedge b = e\mathbf{a} \wedge \mathbf{b} + (\mathbf{b} - \mathbf{a})E. \tag{136}$$

Since $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge (\mathbf{b} - \mathbf{a})$ is the moment for a line through point $\mathbf{a}$ with tangent $\mathbf{a} - \mathbf{b}$, $e \wedge \mathbf{a} \wedge \mathbf{b}$ characterizes the line completely.

Again by using (135) and (136), we get

$$e \wedge a \wedge b \wedge c = e\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} + (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})E. \tag{137}$$

We recognize $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the moment of a plane with tangent $(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})$. Thus, $e \wedge a \wedge b \wedge c$ represents a plane through points $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, or, more specifically, the triangle (2-simplex) with these points as vertices.

For a sphere with radius ρ and center $\mathbf{p} \in \mathbb{R}^n$, we have $(\mathbf{x} - \mathbf{p})^2 = \rho^2$. By (135), the equation in terms of homogeneous points becomes

$$x \odot p = -\frac{1}{2}\rho^2. \tag{138}$$

Using $x \odot e = -1$, we get:

$$x \odot s = 0, \qquad s = p - \frac{1}{2}\rho^2 e = \mathbf{p} + e_0 + \frac{1}{2}(\mathbf{p}^2 - \rho^2)e, \tag{139}$$

where

$$s^2 = \rho^2, \qquad e \odot s = -1. \tag{140}$$

From these properties, the form (139) and center $\mathbf{p}$ can be recovered. Therefore, every sphere in $\mathbb{R}^n$ is completely characterized by a unique vector $s \in \mathbb{M}^{n+1,1}$. According to (140), s lies outside the null cone. Analysis shows that every such vector determines a sphere.

6. Discussion and conclusion

The examples given above are only applications of Clifford algebra in geometry, but we have seen the power of Clifford algebra in solving geometrical problems. In fact, Clifford algebra is more widely used in physics. Why does Clifford algebra work so well? As have been seen from the above examples, the power of Clifford algebra comes from the following features:

1. In the geometry of flat space, the basic concepts are only length, angle, area, and volume, which are already implicitly included in the definition of Clifford algebra. So, Clifford algebra summarizes these contents of classical geometry and algebraize them all. By introducing the concepts of inner, exterior, and direct products of vector, Clifford algebra summarizes the operations of scalars, vectors, and tensors and then can represent all the physical variables in classical physics, because only these variables are included in classical physics.

2. By localizing the basis or frame of space-time, Clifford algebra is naturally suitable for the tangent space in a manifold. If the differential ∂_μ and connection operator $\mathfrak{d}_\mu\gamma_\nu$ are introduced, Clifford algebra can be used for the whole manifold, so it contains Riemann geometry. Furthermore, Clifford algebra can express all contents of classical physics, including physical variables, differential equations, and algebraic operations. Clifford algebra transforms complicated theories and relations into a unified and standard calculus with no more or less contents, and all representations are neat and elegant [23, 36].

3. If the above contents seem to be very natural, Clifford algebra still has another unusual advantage, that is, it includes the theory of spinor. So, Clifford algebra also contains quantum theory and spinor connection. These things are far beyond the human intuition and have some surprising properties.

4. There are many reasons to make Clifford algebra become a unified and efficient language and tool for mathematics, physics, and engineering, such as Clifford algebra generalizes real number, complex number, quaternion, and vector algebra; Clifford algebra is isomorphic to matrix algebra; the derivative operator $\gamma^\mu\nabla_\mu$ contains grad, div, curl, etc. However, the most important feature of Clifford algebra should be taking the physical variable and the basis as one entity, such as $\mathbf{g} = g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu$ and $\mathcal{T} = \mathcal{T}_{\mu\nu\omega}\gamma^{\mu\nu\omega}$. In this representation, the basis is an operator without ambiguity. Clifford algebra calculus is an arithmetic-like operation which can be well understood by everyone.

   "But, if geometric algebra is so good, why is it not more widely used?" As Hestenes replied in [11]: "Its time will come!" The published geometric algebra literature is more than sufficient to support instruction with geometric algebra at intermediate and advanced levels in physics, mathematics, engineering, and computer science. Though few faculty are conversant with geometric algebra now, most could easily learn what they need while teaching. At the introductory level, geometric algebra textbooks and teacher training will be necessary before geometric algebra can be widely taught in the schools. There is steady progress in this direction, but funding is needed to accelerate it. Malcolm Gladwell has discussed social conditions for a "tipping point" when the spread of an idea suddenly goes viral. Place your bets now on a Tipping Point for Geometric Algebra!

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Author details

Ying-Qiu Gu
School of Mathematical Science, Fudan University, Shanghai, China

*Address all correspondence to: yqgu@fudan.edu.cn

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