Chapter

Modular Sumset Labelling of Graphs

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Abstract

Graph labelling is an assignment of labels or weights to the vertices and/or edges of a graph. For a ground set X of integers, a sumset labelling of a graph is an injective map $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X)$ is defined by $f^{+}(uv) = f(u) + f(v)$, for all $uv \in E(G)$, where $f(u) + f(v)$ is the sumset of the set-label, the vertices u and v. In this chapter, we discuss a special type of sumset labelling of a graph, called modular sumset labelling and its variations. We also discuss some interesting characteristics and structural properties of the graphs which admit these new types of graph labellings.

Keywords: sumset labelling, modular sumset labelling, weak modular sumset labelling, strong modular sumset labelling, arithmetic modular sumset labelling

1. Introduction

For terminology and results in graph theory, we refer to [1–5]. For further notions and concepts on graph classes, graph operations, graph products and derived graphs, refer to [3, 6–10]. Unless mentioned otherwise, all graphs mentioned in this chapter are simple, finite, connected and undirected.

1.1 Basics of graph labelling

Labelling of a graph G can broadly be considered as an assignment of labels or weights to the elements (vertices and edges) of G subject to certain pre-defined conditions. The research on graph labelling has flourished in the second half of twentieth century after the introduction of the notion of β-valuations of graphs in [11]. The *β-valuation* of a graph G is an injective map $f : V(G) \to \{1, 2, 3, \dots, |E|\}$ such that the induced function $f^{*}: E(G) \to \{1, 2, 3, \dots, |E|\}$, defined by $f^{*}(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$, is also injective. Later, β-valuation of graphs was popularly known to be the *graceful labelling* of graphs (see [12]). Many variations of number valuations have been defined in the literature since then and most of those studies were based on the number theory and/or number theoretic properties of sets. For concepts and results in number theory, see [13–16].

Analogous to the number valuations of graphs, the notion of set-labelling of graphs has been introduced in [17] as follows: Given a non-empty ground set X, a *set-labelling* or a *set-valuation* of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)$, the power set of X, such that the induced function $f^{\oplus} : V(G) \to \mathcal{P}(X)$ defined by

$f^{\oplus}(uv) = f(u) \oplus f(v)$, for all $uv \in E(G)$, where $\bigoplus$ is the symmetric difference of two sets. A graph which admits a set-labelling is called a *set-labelled graph* or a *set-valued graph*. If the induced function $f^{\oplus}$ is also injective, then the set-labelling f is called a *set-indexer*.

Subsequent to this study, many intensive investigations on set-labelling of graphs and its different types have been taken place. An overview of such studies on set-labelling of graphs can be seen in [17–23]. The study on set-labelling has been extended by replacing the binary operation $\bigoplus$ by some other binary operations of sets. For example, two different types of set-labellings—called disjunctive set-labelling and conjunctive set-labelling—of graphs have been studied in [24]. These set-labellings are defined respectively in terms of the union and the intersection of two sets instead of the symmetric difference of two sets.

1.2 Sumsets and integer additive set-labelled graphs

Integer additive set-labelling or sumset labelling of graphs has been a new addition to the theory of set-labelling of graphs recently. The notion of sumsets of two sets is explained as follows: Let A and B be two sets of numbers. The *sumset* of A and B is denoted by $A + B$ and is defined by $A + B = \{a + b : a \in A, b \in B\}$ (see [25]). Remember that a sumset of two sets can be determined if and only if both of them are number sets. If either A or B is countably infinite, then their sumset $A + B$ is also a countably infinite set and if any one of them is a null set, then the sumset is also a null set. If C is the sumset of two sets A and B, then both A and B are said to be the *summands* of C.

We note that $A + \{0\} = A$ and hence A and $\{0\}$ are called the trivial summands of the set A. Also, note that $A + B$ need not be a subset or a super set of A and/or B. But, $A \subset A + B$ if $0 \in B$. Furthermore, the sumset of two subsets of a set X need not be a subset of the ground set X. These observations are clear deviations from the other common binary operations of sets and thus the study of sumsets becomes more interesting. For the terms, concepts and results on sumsets, we refer to [25–34].

Note that if A and B are two non-empty finite sets of integers, then $|A| + |B| - 1 \leq |A + B| \leq |A \times B| = |A||B|$ (see [25]). The exact cardinality of the sumset $A + B$ always depends on the number as well as the pattern of elements in both the summands A and B. The counting procedure in this case is explained in [35] as follows: Two ordered pairs (a, b) and (c, d) in $A \times B$ is said to be *compatible* if $a + b = c + d$. If (a, b) and (c, d) are compatible, then it is written as $(a, b) \sim (c, d)$. It can easily be verified that this relation is an equivalence relation. A *compatibility class* of an ordered pair (a, b) in $A \times B$ with respect to the integer $k = a + b$ is the subset of $A \times B$ defined by $\{(c, d) \in A \times B : (a, b) \sim (c, d)\}$ and is denoted by $[(a, b)]_k$ or C_k. The cardinality of a compatibility class in $A \times B$ lies between 1 and $\min\{|A|, |B|\}$. Note that the sum of coordinates of all elements in a compatibility class is the same and this sum will be an element of the sumset $A + B$. That is, the cardinality of the sumset of two sets is equal to the number of equivalence classes on the Cartesian product of the two sets generated by the compatibility relation defined on it.

Using the concepts of the sumsets of sets, the notion of integer additive set-labelling of graphs has been introduced in [36] as follows: Let X be a set of non-negative integers and $\mathcal{P}_0(X)$ be the collection of the non-empty subsets of X. Then, an *integer additive set-labelling* or an *integer additive set-valuation* of a graph G is an injective map $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X)$ is defined by $f^{+}(uv) = f(u) + f(v)$, for all $uv \in E(G)$, where $f(u) + f(v)$ is the sumset of the set-label the vertices u and v (see [36, 37]). A graph with an integer additive set-labelling is called an *integer additive set-labelled graph*. It can very easily be verified that every graph G admits an integer additive set-labelling, provided the ground set X is chosen judiciously.

Following to the above path-breaking study, the structural properties and characteristics of different types of integer additive set-labellings of graphs are studied intensively in accordance with the cardinality of the set-label, nature and pattern of elements in the set-label, nature of the collection of set-label, etc. Some interesting and significant studies in this area can be found in [35–44]. Later, the studies in this area have been extended by including the sets of integers (including negative integers also) for labelling the elements of a graph. Some extensive studies in this area, can be seen in [45, 46].

As a specialisation of the sumset labelling of graphs, the notion of modular sumset labelling of graphs and corresponding results are discussed in the following section.

2. Modular sumset labelling of graphs

2.1 Basics of modular sumsets

Recall that $\mathbb{Z}_n$ denotes the set of integers modulo n, where n is a positive integer. The *modular sumset* of two subsets A and B of $\mathbb{Z}_n$ is the set $\{k : a \in A, b \in B, a + b \equiv k \pmod{n}\}$. Unless mentioned otherwise, throughout this chapter, the notation $A + B$ denotes the modular sumset of the sets A and B. Unlike the ordinary sumsets, the modular sumset $A + B \subseteq \mathbb{Z}_n$ if and only if $A, B \subseteq \mathbb{Z}_n$. This fact will ease many restrictions imposed on the vertex set-label of a sumset graph G in order to ensure that the edge set-label are also subsets of the ground set.

If we assign the null set Ø to any vertex as the set-label, the set-label of every edge incident at that vertex will also be a null set. To avoid such an embarrassing situation, we do not consider the null set for labelling any vertex of graphs. Thus, the set of all non-empty subsets of a set X is denoted by $\mathcal{P}_0(X)$. That is, $\mathcal{P}_0(X) = \mathcal{P}(X) \backslash \{0\}$.

2.2 Modular sumset graphs

In view of the facts stated above, the modular sumset labelling of a graph is defined as follows:

Definition 2.1. [47] A modular sumset labelling of a graph G is an injective function $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ such that the induced function $f^+ : E(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ is defined as $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the modular sumset of the set labels of the vertices u and v. A graph which admits a modular sumset labelling is called a *modular sumset graph*.

Definition 2.2. [47] A modular sumset labelling of a graph G is said to be a *uniform modular sumset labelling* of G if the set-label of all its edges have the same cardinality. A modular sumset labelling f of G is said to be a *k-uniform modular sumset labelling* if $f^+(uv) = k, \forall uv \in E(G)$.

Proposition 2.1. [47] *Every graph G admits modular sumset labelling (for a suitable choice of n).*

The proof of the above proposition is immediate from the fact that $f(u) + f(v) \subseteq \mathbb{Z}_n$ if and only if $f(u), f(v) \subseteq \mathbb{Z}_n$.

An immediate question that arises in this context is about the minimum size of the ground set $\mathbb{Z}_n$ (that is, the minimum value of n) required for the existence of a modular sumset labelling of G.

As in the case of sumsets, the cardinality of the modular sumsets also attracted the attention. Hence, we have the bounds for the cardinality of an edge set-label of a modular sumset graph G is as follows:

Theorem 2.2. [47] *Let $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ be a modular sumset labelling of a given graph G. Then, for any edge $uv \in E(G)$, we have*

$$|f(u)| + |f(v)| - 1 \leq |f^+(uv)| = |f(u) + f(v)| \leq |f(u)|\,|f(v)| \leq n. \tag{1}$$

The theorem follows immediately from the theorem on the cardinality of sumsets (see Theorem 2.7, p. 52, [25]).

In this context, it is quite interesting to investigate whether the bounds are sharp. It has also been proved in [25] that the lower bound is sharp when both $f(u)$ and $f(v)$ are arithmetic progressions (we call set an arithmetic progression if its elements are in arithmetic progression) with the same common difference. We shall discuss the different types of modular sumset graphs based on the set-labelling numbers of its vertices and edges, one by one in the coming discussions.

3. Arithmetic modular sumset graphs

As mentioned above, the lower bound of the inequality (1) is sharp if both summand set-label are arithmetic progressions with the same common difference. If the context is clear, the common difference of the set-label (if exists) of an element may be called the *common difference* of that element. The *deterministic ratio* of an edge of G is the ratio, $k \geq 1$ between the common differences of its end vertices. In view of this terminology we have the following definition.

Definition 3.1. For any vertex v of G, if $f(v)$ is an arithmetic progression, then the modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ is called a *vertex arithmetic modular sumset labelling* of G. In a similar manner, for any edge e of G, if $f(e)$ is an arithmetic progression, then the modular sumset labelling $f : E(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ is called an *edge arithmetic modular sumset labelling* of G.

The difference set of a non-empty set A, denoted by D_A, is the set defined by $D_A = \{|a-b| : a, b \in A\}$. Note that if A is an arithmetic progression, then its difference set D_A is also an arithmetic progression and vice versa. Analogous to the corresponding result of the edge-arithmetic sumset labelling of graphs (see [44, 46]), the following result is a necessary and sufficient condition for a graph G to be edge-arithmetic modular sumset graph in terms of the difference sets the set-label of vertices of G.

Theorem 3.1. *Let f be a modular sumset labelling defined on a graph G. If the set-label of an edge of G is an arithmetic progression if and only if the sumset of the difference sets of set-label of its end vertices is an arithmetic progression.*

Proof. Let $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ be a modular sumset labelling defined on G. Let a_i, a_j be two arbitrary elements in $f(u)$ and let b_r, b_s be two elements in $f(v)$. Then, $|a_i - a_j| \in D_{f(u)}$ and $|a_i - a_j| \in D_{f(u)}$. That is, $D_{f(u)} = \{|a_i - a_j| : a_i, a_j \in f(u)\}$ and $D_{f(v)} = \{|b_r - b_s| : b_r, b_s \in f(v)\}$.

Now, assume that $f^+(e) = f^+(uv)$ is an arithmetic progression for an edge $e = uv \in E(G)$. That is, $A = f(u) + f(v)$ is an arithmetic progression. Then, the difference set $D_A = \{|a-b| : a, b \in A = f(u) + f(v)\}$ is also an arithmetic progression. Since $a, b \in A$, we have $a = a_i + b_r$ and $b = a_j + b_s$, where $a_i, a_j \in f(u)$ and $b_r, b_s \in f(v)$. Then,

$$\begin{aligned} D_A &= \{|a-b| : a, b \in A\} \\ &= \{|a_i + b_r - (a_j + b_s)| : a_i, a_j \in f(u), b_r, b_s \in f(v)\} \\ &= \{|a_i - a_j| + |b_r - b_s| : a_i, a_j \in f(u), b_r, b_s \in f(v)\} \\ &= \{|a_i - a_j| : a_i, a_j \in f(u)\} + \{|b_r - b_s| : b_r, b_s \in f(v)\} \\ &= D_{f(u)} + D_{f(v)} \end{aligned}$$

Hence, $D_{f(u)} + D_{f(v)}$ is an arithmetic progression.

Conversely, assume that $D_{f(u)} + D_{f(v)}$ is an arithmetic progression. Then, by previous step, we have $D_{f(u)} + D_{f(v)} = D_A$, where $A = f(u) + f(v)$. Then, we have D_A is an arithmetic progression. Since the difference set D_A is an arithmetic progression, then by the above remark, we have $A = f(u) + f(v) = f^+(uv)$ is also an arithmetic progression. Hence, the edge $e = uv$ has an arithmetic progression as its set-label. □

In view of the notions mentioned above, we note that there are some graphs, all whose elements have arithmetic progressions as their set-label and there are some graphs, the set-label of whose edges are not arithmetic progressions. Keeping this in mind, we define the following notion.

Definition 3.2. An *arithmetic sumset labelling* of a graph G is a modular sumset labelling f of G, with respect to which the set-label of all vertices and edges of G are arithmetic progressions. A graph that admits an arithmetic modular sumset labelling is called an *arithmetic modular sumset graph*.

Analogous to the condition for an arithmetic sumset graphs (see [44]), a necessary and sufficient condition for a graph to admit an arithmetic modular sumset labelling is discussed in the following theorem.

Theorem 3.2. *A graph G admits an arithmetic modular sumset labelling f if and only if for any two adjacent vertices in G, the deterministic ratio of every edge of G is a positive integer, which is less than or equal to the set-labelling number of its end vertex having smaller common difference.*

Proof. Here, we need to consider the following two cases:

Case 1: First note that if the set-label of two adjacent vertices are arithmetic progressions with the same common difference, say d, then the set-label of the corresponding edge is also an arithmetic progression with the same common difference d. Then, it is clear that a vertex arithmetic modular sumset graph is an arithmetic modular sumset graph if the common differences between any two adjacent vertices of G are the same.

Case 2: Assume that u, v be any two adjacent vertices in G with common differences d_u and d_v respectively such that $d_u \leq d_v$. Also, assume that $f(u) = \{a_r = a + rd_u : 0 \leq r < m\}$ and $f(v) = \{b_s = b + sd_v : 0 \leq s < n\}$. Then, $|f(u)| = m$ and $|f(v)| = n$. Now, arrange the terms of $f(u) + f(v) = f^+(uv)$ in rows and columns as follows. For any $b_s \in f(v)$, $0 \leq s < n$, arrange the terms of $A + b_s$ in $(s+1)$th row in such a way that equal terms of different rows come in the same column of this arrangement. Without loss of generality, assume that $d_v = kd_u$ and $k \leq m$. If $k < m$, then for any $a \in f(u)$ and $b \in f(v)$ we have $a + (b + d_v) = a + b + kd_u < a + b + md_i$. That is, a few final elements of each row of the above arrangement occur as the initial elements of the succeeding row (or rows) and the difference between two successive elements in each row is d_u itself. If $k = m$, then the difference between the final element of each row and the first element of the next row is d_u and the difference between two consecutive elements in each row is d_u. Hence, if $k \leq m$, then $f^+(uv)$ is an arithmetic progression with common difference d_u.

In both cases, note that if the given conditions are true, then f is an arithmetic modular sumset labelling of G.

We prove the converse part by contradiction method. For this, assume that f is an arithmetic modular sumset labelling of G. Let us proceed by considering the following two cases.

Case-1: Assume that d_j is not a multiple of d_i (or d_i is not a multiple of d_j). Without loss generality, let $d_i < d_j$. Then, by division algorithm, $d_j = pd_i + q, 0 < q < d_i$. Then, the differences between any two consecutive terms in $f^+(v_iv_j)$ are not equal. Hence, in this case also, f is not an arithmetic modular sumset labelling, contradiction to the hypothesis. Therefore, $d_i | d_j$.

Case 2: Let $d_j = kd_i$ where $k > m$. Then, the difference between two successive elements in each row is d_i, but the difference between the final element of each row and the first element of the next row is td_i, where $t = k - m + 1 \neq 1$. Hence, f is not an arithmetic modular sumset labelling, a contradiction to the hypothesis. Hence, we have $d_j = kd_i; k \leq m$.

Therefore, from the above cases it can be noted that if a vertex arithmetic modular sumset labelling of G is an arithmetic modular sumset labelling of G, then the deterministic ratio of every edge of G is a positive integer, which is greater than or equal to the set-labelling number of its end vertex having smaller common difference. This completes the proof. □

In the following theorem, we establish a relation between the common differences of the elements of an arithmetic modular sumset graph G.

Theorem 3.3. *If G is an arithmetic modular sumset graph, the greatest common divisor of the common differences of vertices of G and the greatest common divisor of the common differences of the edges of G are equal to the smallest among the common differences of the vertices of G.*

Proof. Let f be an arithmetic modular sumset labelling of G. Then, by Theorem 3.2, for any two adjacent vertices v_i and v_j of G with common differences d_i and d_j respectively, either $d_i = d_j$, or if $d_j > d_i, d_j = kd_j$, where k is a positive integer such that $1 < k \leq |f(v_i)|$.

If the common differences of the elements of G are the same, the result is obvious. Hence, assume that for any two adjacent vertices v_i and v_j of G, $d_j = kd_j, k \leq |f(v_i)|$, where d_i is the smallest among the common differences of the vertices of G. If v_r is another vertex that is adjacent to v_j, then it has the common difference d_r which is equal to either d_i or d_j or ld_j. In all the three cases, d_r is a multiple of d_i. Hence, the greatest common divisor of d_i, d_j, d_r is d_i. Proceeding like this, we have the greatest common divisor of the common differences of the vertices of G is d_i.

Also, by Theorem 3.2, the edge u_iv_j has the common difference d_i. The edge v_jv_k has the common difference d_i, if $d_k = d_i$, or d_j in the other two cases. Proceeding like this, we observe that the greatest common divisor of the common differences of the edges of G is also d_i. This completes the proof. □

The study on the set-labelling number of edges of an arithmetic modular sumset graphs arouses much interest. Analogous to the result on set-labelling number of the edges of an arithmetic sumset graph (see [43]), The set-labelling number of an edge of an arithmetic modular sumset graph G, in terms of the set-labelling numbers of its end vertices, is determined in the following theorem.

Theorem 3.4. *Let G be a graph which admits an arithmetic modular sumset labelling, say f and let d_i and d_j be the common differences of two adjacent vertices v_i and v_j in G. If $|f(v_i)| \geq |f(v_j)|$, then for some positive integer $1 \leq k \leq |f(v_i)|$, the edge v_iv_j has the set-labelling number $|f(v_i)| + k(|f(v_j)| - 1)$.*

Proof. Let f be an arithmetic modular sumset labelling defined on G. For any two vertices v_i and v_j of G, let $f(v_i) = \{a_i, a_i + d_i, a_i + 2d_i, a_i + 3d_i, \dots, a_i + (m-1)d_i\}$ and let $f(v_j) = \{a_j, a_j + d_j, a_j + 2d_j, a_j + 3d_j, \dots, a_j + (n-1)d_j\}$. Here $|f(v_i)| = m$ and $|f(v_j)| = n$.

Let d_i and d_j be the common differences of the vertices v_i and v_j respectively, such that $d_i < d_j$. Since f is an arithmetic modular sumset labelling on G, by Theorem 3.2, there exists a positive integer k such that $d_j = k.d_i$, where $1 \le k \le |f(v_i)|$. Then, $f(v_j) = \{a_j, a_j + kd_i, a_j + 2kd_i, a_j + 3kd_i, \dots, a_j + (n-1)kd_i\}$. Therefore, $f^+(v_iv_j) = \{a_i + a_j, a_i + a_j + d_i, a_i + a_j + 2d_i, \dots, a_i + a_j + [(m-1) + k(n-1)]d_i\}$. That is, the set-labelling number of the edge v_iv_j is $m + k(n-1)$. □

4. Strongly modular sumset graphs

The next type of a modular sumset labelling we are going to discuss is the one with the upper bound in Inequality (1) is sharp (that is, $|A + B| = |A||B|$). Thus, we have the following definition.

Definition 4.1. [47] A modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ defined on a given graph G is said to be a *strongly modular sumset labelling* if for the associated function $f^+ : E(G) \to \mathcal{P}_0(\mathbb{Z}_n)$, $|f^+(uv)| = |f(u)||f(v)| \forall uv \in E(G)$. A graph which admits a strongly modular sumset labelling is called a *strongly modular sumset graph*.

Invoking the notion difference set of a set, a necessary and sufficient condition of a modular sumset labelling of a graph G to be a strongly modular sumset labelling is given below:

Theorem 4.1. *A modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ of a given graph G is a strongly modular sumset labelling of G if and only if $D_{f(u)} \cap D_{f(v)} = \emptyset, \forall uv \in E(G)$, where $|f(u)||f(v)| \le n$.*

Proof. Let $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ be a modular sumset labelling on a given graph G. For any vertex $u \in V(G)$, define $D_f(u) = \{a_i - a_j : a_i, a_j \in f(u)\}$.

Let uv be an arbitrary edge in $E(G)$. Assume that f is a strong modular sumset labelling of G. Then, by definition $|f^+(uv)| = |f(u)||f(v)|$. Therefore, for any elements $a_i, a_j \in f(u)$ and $b_r, b_s \in f(v)$, we have $a_i + b_r \neq a_j + b_s$ in $f^+(uv) \forall uv \in E(G)$. That is, $|a_i - a_j| \neq |b_s - b_r|$ for any $a_i, a_j \in f(u)$ and $b_r, b_s \in f(v)$. That is, $D_{f(u)} \cap D_{f(v)} = \emptyset$. Therefore, the difference sets of the set-label of any two adjacent vertices are disjoint.

Conversely, assume that the difference $D_{f(u)} \cap D_{f(v)} = \emptyset$ for any edge uv in G. That is, $|a_i - a_j| \neq |b_s - b_r|$ for any $a_i, a_j \in f(u)$ and $b_r, b_s \in f(v)$. Then, $a_i - a_j \neq b_s - b_r$. That is, $a_i + b_r \neq a_j + b_s$. Therefore, all elements in $f(u) + f(v)$ are distinct. That is, $|f^+(uv)| = |f(u)||f(v)|$ for any edge $uv \in E(G)$. Hence, f is a strongly modular sumset labelling of G.

Also, note that the maximum possible cardinality in the set-label of any element of G is n, the product $|f(u)||f(v)|$ cannot exceed the number n. This completes the proof. □

A necessary and sufficient condition for a modular sumset labelling of a graph G to be a strongly k-uniform modular sumset labelling is given below:

Theorem 4.2. [47] *For a positive integer $k \le n$, a modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ of a given connected graph G is a strongly k-uniform modular sumset labelling of G if and only if either k is a perfect square or G is bipartite.*

Proof. If k is a perfect square, say $k = l^2$, then we can label all the vertices of a graph by distinct l-element sets in such a way that the difference sets of the set-label of every pair of adjacent vertices are disjoint. Hence, assume that k is not a perfect square.

Let G be a bipartite graph with bipartition (X, Y). Let r, s be two divisors of k. Label all vertices of X by distinct r-element sets all of whose difference sets are the same, say D_X. Similarly, label all vertices of Y by distinct s-element sets all of whose difference sets the same, say D_Y, such that $D_X \cap D_Y = \emptyset$. Then, all the edges of G have the set-labelling number $k = rs$. Therefore, G is a strongly k-uniform modular sumset graph.

Conversely, assume that G admits a strongly k-uniform modular sumset labelling, say f. Then, $f^+(uv) = k \forall uv \in E(G)$. Since, f is a strong modular sumset labelling, the set-labelling number of every vertex of G is a divisor of the set-labelling numbers of the edges incident on that vertex. Let v be a vertex of G with the set-labelling number r, where r is a divisor of k, but $r^2 \neq k$. Since f is k-uniform, all the vertices in $N(v)$, must have the set-labelling number s, where $rs = k$. Again, all vertices, which are adjacent to the vertices of $N(v)$, must have the set-labelling number r. Since G is a connected graph, all vertices of G have the set-labelling number r or s. Let X be the set of all vertices of G having the set-labelling number r and Y be the set of all vertices of G having the set-labelling number s. Since $r^2 \neq k$, no two elements in X (and no elements in Y also) can be adjacent to each other. Therefore, G is bipartite. □

The following result is an immediate consequence of the above theorem.

Theorem 4.3. [47] *For a positive non-square integer $k \leq n$, a modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ of an arbitrary graph G is a strongly k-uniform modular sumset labelling of G if and only if either G is bipartite or a disjoint union of bipartite components.*

For a positive integer $k \leq n$, the maximum number of components in a strongly k-uniform modular sumset graph is as follows.

Proposition 4.4. [47] *Let f be a strongly k-uniform modular sumset labelling of a graph G with respect to the ground set $\mathbb{Z}_n$. Then, the maximum number of components in G is the number of distinct pairs of divisors r and s of k such that $rs = k$.*

The following theorem discusses the condition for an arithmetic modular sumset labelling of a graph G to be a strongly modular sumset labelling of the graph.

Theorem 4.5. *Let G be a graph which admits an arithmetic modular sumset labelling, say f. Then, f is a strongly modular sumset labelling of G if and only if the deterministic ratio of every edge of G is equal to the set-labelling number of its end vertex having smaller common difference.*

Proof. Let f be an arithmetic modular sumset labelling of G. Let v_i and v_j are two adjacent vertices in G and d_i and d_j be their common differences under f. Without loss of generality, let $d_i < d_j$. Then, by Theorem 3.4, the set-labelling number of the edge v_iv_j is $|f(v_i)| + k(|f(v_j)| - 1)$.

Assume that f is a strongly modular sumset labelling. Therefore, $f^+(v_iv_j) = mn$. Then,

$$\begin{aligned} |f(v_i)| + k(|f(v_j)| - 1) &= |f(v_i)||f(v_j)| \\ \Rightarrow k(|f(v_j)| - 1) &= |f(v_i)|(|f(v_j)| - 1) \\ \Rightarrow k &= |f(v_i)|. \end{aligned}$$

Conversely, assume that the common differences d_i and d_j of two adjacent vertices v_i and v_j respectively in G, where $d_i < d_j$ such that $d_j = |f(v_i)|.d_i$. Assume that $f(v_i) = \{a_r = a + rd_i : 0 \leq r < |f(v_i)|\}$ and $f(v_j) = \{b_s = b + skd_i : 0 \leq s < |f(v_j)|\}$, where $k \leq |f(v_i)|$. Now, arrange the terms of $f^+(v_iv_j) = f(v_i) + f(v_j)$ in rows and columns as follows. For $b_s \in f(v_j), 0 \leq s < |f(v_j)|$, arrange the terms of $f(v_i) + b_s$ in $(s+1)$-th row in such a way that equal terms of different rows come in the same column of this arrangement. Then, the common difference between consecutive elements in each row is d_i. Since $k = |f(v_i)|$, the difference between the final element of any row (other than the last row) and first element of its succeeding row is also d_i. That is, no column in this arrangement contains more than one element. Hence, all elements in this arrangement are distinct. Therefore, total number of elements in $f(v_i) + f(v_j)$ is $|f(v_i)||f(v_j)|$. Hence, f is a strongly modular sumset labelling. □

5. Supreme modular sumset labelling of G

In both types of modular sumset labelling discussed above, it is observed that the cardinality of the edge set-label cannot exceed the value n. This fact creates much interest in investigating the case where all the edge set-label have the cardinality n.

Definition 5.1. [47] A modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ of a given graph G is said to be a *supreme modular sumset labelling* or *maximal modular sumset labelling* of G if and only if $f^+(E(G)) = \{\mathbb{Z}_n\}$.

Put in a different way, a modular sumset labelling $f : V(G) \to \mathcal{P}(\mathbb{Z}_n)$ of a given graph G is a supreme modular sumset labelling of G if the set-label of every edge of G is the ground set $\mathbb{Z}_n$ itself.

A necessary and sufficient condition for a modular sumset labelling of a graph G to be its supreme modular sumset labelling is discussed in the theorem given below:

Theorem 5.1. [47] *The modular sumset labelling $f : V(G) \to \mathcal{P}(\mathbb{Z}_n)$ of a given graph G is a supreme modular sumset labelling of G if and only if for every pair of adjacent vertices u and v of G some or all of the following conditions hold.*

i. $|f(u)| + |f(v)| \geq n$ if $D_{f(u)} \cap D_{f(v)} \neq \emptyset$. The strict inequality hold when $D_{f(u)}$ and $D_{f(v)}$ are arithmetic progressions containing the same elements.

ii. $|f(u)||f(v)| \geq n$ if $D_{f(u)} \cap D_{f(v)} = \emptyset$.

Proof. For two adjacent vertices u and v in G, let $D_{f(u)} = D_{f(v)} = \{d\}$ are arithmetic progressions containing the same elements. Then, the elements in $f(u)$ and $f(v)$ are also in arithmetic progression, with the same common difference d. Then, by Theorem 3.4, $n = |f(u) + f(v)| = |f(u)| + |f(v)| - 1$. Therefore, the set-labelling number of the edge uv is n if and only if $|f(u)| + |f(v)| > n$.

Now, let $D_{f(u)} \cap D_{f(v)} \neq \emptyset$ such that $D_{f(u)} \neq D_{f(v)}$. Then, clearly $|f(u) + f(v)| \geq |f(u)| + |f(v)|$. Therefore, we have $|f^+(uv)| = n$ if and only if $|f(u)| + |f(v)| \geq n$. □

Next assume that $D_{f(u)} \cap D_{f(v)} = \emptyset$. Then, $|f(u) + f(v)| = |f(u)||f(v)|$. Therefore, we have $|f^+(uv)| = n$ if and only if $|f(u)||f(v)| \geq n$.

A necessary and sufficient condition for a strong modular sumset labelling of a graph G to be a maximal modular sumset labelling of G.

Theorem 5.2. [47] *Let f be a strong sumset-labelling of a given graph G. Then, f is a maximal sumset-labelling of G if and only if n is a perfect square or G is bipartite or a disjoint union of bipartite components.*

Proof. The proof is an immediate consequence of Theorem 4.2, when $k = n$. □

6. Weakly modular sumset graphs

Another interesting question we address in the beginning of this section is whether the lower bound and the upper bound of the sumset can be equal. Suppose that A and B be two non-empty subsets of $\mathbb{Z}_n$ such that the bounds of their sumset are equal. Then, we have

$$\begin{aligned}&|A|+|B|-1=|A||B|\\&|A|+|B|-1-|A||B|=0\\&|A|(1-|B|)+|B|-1=0\\&(|A|-1)(|B|-1)=0\end{aligned}$$

which is possible only when $|A|=1$ or $|B|=1$ (or both). Also, note that in this case the cardinality of the sumset is equal to equal to that of one of the summands. This interesting phenomenon leads us to a new type of a modular sumset labelling called weakly modular sumset labelling. This type of labelling is investigated in the following section.

6.1 Weakly modular sumset labelling of graphs

Definition 6.1. A modular sumset labelling f of a graph G is said to be a *weakly modular sumset labelling* of G if the cardinality of the set-label of every edge of G is equal to the cardinality of the set-label of at least one of its end vertices. A graph which admits a weakly modular sumset labelling is called a *weakly modular sumset graph*.

From the above definition, it can be observed that for any edge uv in weakly modular sumset graph G, $|f^+(uv)|=|f(u)|$ or $|f^+(uv)|=|f(v)|$. Putting it in a different way, the set-labelling number of at least one end vertex of every edge of a weakly modular sumset graph is a singleton. An element (a vertex or an edge) of modular sumset graph G with set-labelling number 1 is called a *sparing element* or a *monocardinal elements* of G. Hence, analogous to the condition for a sumset graph to be a weak sumset graph (see [39]), we have.

Theorem 6.1. *A graph G admits a weak modular sumset labelling if and only if G is bipartite or contains sparing edges.*

Proof. Note the fact that at least one end vertex of every edge of G is a sparing vertex. Also, we note that no two vertices with non-singleton set-label in weakly modular sumset graph can be adjacent to each other. Thus, if every of edge of G has exactly one end vertex with singleton set-label, then we can partition the vertex set of G into two subsets X with all sparing vertices and Y with all non-sparing vertices. Here, no two vertices in the same partition are adjacent and hence G is a bipartite graph. If G is not a bipartite graph, then obviously G should have at least one sparing edge, completing the proof.

Invoking Theorem 6.1, the following two results are immediate.

Corollary 6.2. *Every graph G admits a weakly modular sumset labelling.*

Corollary 6.3. *A graph G admits a weakly uniform modular sumset labelling if and only if G is bipartite.*

The above results are similar to the corresponding result of integer additive set-labelled graphs (see [39]) and hence the notion of *sparing number* of graphs defined and studied in [48–57] can be extended to our current discussion also. The notion of the sparing number of graphs is defined as follows:

Definition 6.2. Let G be a weakly modular sumset labelled graph. Then, the *sparing number* of G is the number of sparing edges in G.

A set of vertices X of a graph G is said to have maximal incidence if the maximum number of edges of incidence at the elements of X. Then, analogous to the corresponding result of integer additive set-valued graphs (see [40]), we have.

Theorem 6.4. *Let G be a weakly modular sumset labelled graph and I be I be the largest independent set of G with maximum incidence. Then, the sparing number of G is* $|E(G)| - \sum_{v_i \in I} d(v_i)$.

Proof. Recall that the degree of a vertex v, denoted by $d(v)$, is equal to the number of edges incident on a vertex. Note that any vertex $v_i \in I$ can have a non-singleton set-label which gives non-singleton set labels to $d(v_i)$ edges incident on it. Since I is an independent set, the edges incident at the vertices in I assumes non-singleton set-label. Therefore, the number of edges having non-singleton set-label incident at the vertices in I is $\sum_{v_i \in I} d(v_i)$. Since I is a maximal independent set of that kind, the above expression counts the maximal non-sparing edges in G. Hence, the number of sparing edges in G is $|E(G)| - \sum_{v_i \in I} d(v_i)$. □

6.2 Weakly modular sumset number of graphs

As a special case of the modular sumset number, the notion of weakly modular sumset number is introduced in [47] as follows:

Definition 6.3. The *weakly modular sumset number* of a graph G, denoted by σ_w is defined to be the minimum value of n such that a modular sumset labelling $f : V(G) \to \mathcal{P}_0(\mathbb{Z}_n)$ is a weakly modular sumset labelling of G.

The following theorem discussed the weak sumset number of an arbitrary graph G in terms of its covering and independence numbers.

Theorem 6.5. [47] *Let G be a modular sumset graph and α and β be the covering number and independence number of G respectively. Then, the weak modular sumset number of G is* $\max\{\alpha, r\}$, *where r is the smallest positive integer such that* $2^r - r - 1 \geq \beta$.

Proof. Recall that $\alpha(G) + \beta(G) = |V(G)|$ (see [4]). Since G is a modular sumset graph, no two adjacent vertices can have non-singleton set-label simultaneously. Therefore, the maximum number of vertices that have non-singleton set-label is β. Let V' be the set of these independent vertices in G. Therefore, the minimum number of sparing vertices is $|V(G)| - \beta = \alpha$. Since all these vertices in $V - V'$ must have distinct singleton set-label, the ground set must have at least α elements.

Also, the number non-empty, non-singleton subsets of the ground set must be greater than or equal to α. Otherwise, all the vertices in V' cannot be labelled by non-singleton subsets of this ground set. We know that the number of non-empty, non-singleton subsets of a set A is $2^{|A|} - |A| - 1$, where $A \subseteq \mathbb{Z}_n$ is the ground set used for labelling.

Therefore, the weak modular sumset number G is α if $2^\alpha - \alpha - 1 \geq \beta$. Otherwise, the ground set must have at least r elements such that $2^r - r - 1 \geq \beta$. Therefore, in this case, the weak modular sumset number of G is r, where r is the smallest positive integer such that $2^r - r - 1 \geq \beta$. Hence, $\sigma^*(G) = \max\{\alpha, r\}$. This completes the proof. □

The weakly modular sumset number some fundamental graph classes are given in **Table 1**.

The following theorem discusses the minimum cardinality of the ground set when the given graph G admits a weakly uniform modular sumset labelling.

Theorem 6.6. [47] *Let G be a weakly k-uniform modular sumset graph with covering number α and independence number β, where $k < \alpha$. Then, the minimum cardinality of the ground set* $\mathbb{Z}_n$ *is* $\max\{\alpha, r\}$, *where r is the smallest positive integer such that* $\binom{r}{k \geq \beta}$.

Graph class	$\sigma^*(G)$
Path, P_p	2 if $p \leq 2$; $\lfloor \frac{p}{2} \rfloor$ if $p > 2$
Cycle, C_p	$p-1$ if $p = 3, 4$; $\lfloor \frac{p}{2} \rfloor$ if $p > 4$
Wheel graph, $W_{1,p}$	$1 + \lfloor \frac{p}{2} \rfloor$
Helm graph, H_p	p
Ladder graph, L_p	p
Complete graph, K_p	$p-1$

Table 1.
Weakly modular sumset number of some graph classes.

Proof. Let a weakly k-uniform modular sumset labelling be defined on a graph G over the ground set $A \subset \mathbb{Z}_n$. Then, by Corollary 6.3, G is bipartite. Let X, Y be the bipartition of the vertex set $V(G)$. Without loss of generality, let $|X| \leq |Y|$. Then, $\alpha = |X|$ and $\beta = |Y|$. Then, distinct elements of X must have distinct singleton set-label. Therefore, $n \geq \alpha$.

On the other hand, since f is k-uniform, all the elements in Y must have distinct k-element set-label. The number of k-element subsets of a set A (obviously, with more than k elements) is $\binom{|A|}{k}$. The ground set A has α elements only if $\binom{\alpha}{k \geq \beta}$. Otherwise, the ground set A must contain at least r elements, where $r > \alpha$ is the smallest positive integer such that $\binom{r}{k \geq \beta}$. Therefore, $n = \max\{\alpha, r\}$. □

In view of the above theorem, the following result is immediate.

Corollary 6.7. *Let G be a weakly k-uniform modular sumset graph, where $k \geq \alpha$, where α is the covering number of G. Then, the minimum cardinality of the ground set $\mathbb{Z}_n$ is the smallest positive integer n such that $\binom{n}{k \geq \beta}$, where β is the independence number of G.*

The following result explains a necessary and sufficient condition for a weak modular sumset labelling of a given graph G to be a maximal modular sumset labelling of G.

Proposition 6.8. [47] *A weakly modular sumset labelling of a graph G is a supreme modular sumset labelling of G if and only if G is a star graph.*

Proof. Let f be a weak modular sumset labelling of given graph G. First, assume that f is a maximal modular sumset labelling of G. Then, the set-labelling number of one end vertex of every edge of G is 1 and the set-labelling number of the other end vertex is n. Therefore, $\mathbb{Z}_n$ be the set-label of one end vertex of every edge of G, which is possible only if G is a star graph with the central vertex has the set-label $\mathbb{Z}_n$ and the pendant vertices of G have distinct singleton set-label.

Conversely, assume that G is a star graph. Label the central vertex of G by the ground set $\mathbb{Z}_n$ and label other vertices of G by distinct singleton subsets of $\mathbb{Z}_n$. Then, all the edges of G has the set-labelling number n. That is, this labelling is a supreme modular sumset labelling of G. □

7. Conclusion

In this chapter, we have discussed certain types of modular sumset graphs and their structural properties and characterisations. These studies are based on the cardinality of the set-label of the elements of the graphs concerned and the patterns of the elements in these set-label. It is to be noted that several other possibilities can be investigated in this regard. For example, analogous to the topological set-valuations of graphs, the case when the collection of set-label of vertices and/or edges of a graph G forms a topology of the ground set $\mathbb{Z}_n$ can be studied in detail. Another possibility for future investigation is to extend the graceful and sequential concepts of set-labelling of graphs to modular sumset labelling also. All these points highlight the wide scope for further studies in this area.

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