Chapter

Derived Tensor Products and Their Applications

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Abstract

In this research we studied the tensor product on derived categories of Étale sheaves with transfers considering as fundamental, the tensor product of categories $X \otimes Y = X \times Y$, on the category Cor_k, (finite correspondences category) by understanding it to be the product of the underlying schemes on k. Although, to this is required to build a total tensor product on the category $\mathrm{PST}(k)$, where this construction will be useful to obtain generalizations on derived categories using pre-sheaves and contravariant and covariant functors on additive categories to define the exactness of infinite sequences and resolution of spectral sequences. Some concrete applications are given through a result on field equations solution.

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1. Introduction

This study is focused on the derived tensor product whose functors have images as cohomology groups that are representations of integrals of sheaves represented for its pre-sheaves in an order modulo k. This study is remounted to the K-theory on the sheaves cohomologies constructed through pre-sheaves defined by the tensor product on commutative rings. The intention of this study is to establish a methodology through commutative rings and their construction of a total tensor product $\otimes^L$, [1]on the category $\mathrm{PST}(k)$, considering extensions of the tensor products $\otimes_{R(\mathcal{A})}$, to obtain resolution in the projective sense of infinite sequences of modules of Étale sheaves. These sheaves are pre-sheaves of Abelian groups on the category of smooth separated schemes restricted to scheme X.

Likewise, the immediate application of the derived tensor products will be the determination of the tensor triangulated category $\mathrm{DM}^{-}_{\mathrm{ett}}(k, \mathbb{Z}/m)$, of Étale motives to be equivalent to the derived category of discrete $\mathbb{Z}/m$- modules over the Galois

[1] $\mathbb{L}$, is a Lefschetz motive $\mathbb{Z}(1)$, [1].

group $G = \mathrm{Gal}(k_{sep}/k)$, which says on the equivalence of functors of tensor triangulated categories[2].

Then the mean result of derived tensor products will be in tensor triangulated category $\mathrm{DM}_{Nis}^{\mathrm{eff},-}(k, R)$, of effective motives and their subcategory of effective geometric motives $\mathrm{DM}_{gm}^{\mathrm{eff},-}(k, R)$. Likewise, the motive $M(\mathrm{X})$, of a scheme X, is an object of $\mathrm{DM}_{Nis}^{\mathrm{eff},-}(k, R)$, and belongs to $\mathrm{DM}_{gm}^{\mathrm{eff},-}(k, R)$, if X, is smooth. However, this requires the use of cohomological properties of sheaves associated with homotopy invariant pre-sheaves with transfers for Zariski topology, Nisnevich and cdh topologies.

Finally, all this treatment goes in-walked to develop a motivic cohomology to establish a resolution in the field theory incorporating singularities in the complex Riemannian manifolds where singularities can be studied with deformation theory through operads, motives, and deformation quantization.

2. Fundaments of derived tensor products

We consider the Abelian category Ab, which is conformed by all functor images that are contravariant additive functors $F : \mathcal{A} \to \mathrm{Ab}$, on small category of $\mathbb{Z}(\mathcal{A})$. Likewise, $\mathbb{Z}(\mathcal{A})$, is the category of all additive pre-sheaves on $\mathcal{A}$. Likewise, we can define this category as of points space:

$$\mathbb{Z}(\mathcal{A}) = \{F | F : \mathcal{A} \to \mathrm{Ab}\}, \tag{1}$$

Likewise, we have the Yoneda embedding as the mapping[3]:

$$h : \mathcal{A} \to \mathbb{Z}(\mathcal{A}), \tag{2}$$

which has correspondence rule

$$\mathrm{X} \mapsto \oplus \mathrm{X}_i, \tag{3}$$

or

$$h_{\mathrm{X}} = \oplus h_{\mathrm{X}_i}, \tag{4}$$

We need a generalization of the before categories and functors, therefore we give a ring R, originating the ring structure $\mathcal{A}(R)$, to be the Abelian category of the additive functors

$$F : \mathcal{A} \to R - \mathrm{mod}, \tag{5}$$

being R-mod, the category of the modules that originate the ring structure. Then h_X, is the functor

[2] **Theorem.** If $1/\mathrm{m} \in k$, the space $(\mathcal{L}, \otimes_{\mathcal{L}})$, is a tensor triangulated category and the functors

$$\mathrm{D}^-(G, \mathbb{Z}/m) \overset{\pi *}{\to} \mathcal{L} \to \mathrm{D}(W_A^{-1}) = \mathrm{DM}_{Et}^{\mathrm{eff},-}(k, \mathbb{Z}/m),$$

until the category $\mathrm{D}^-(Sh_{et}(Cor_k, \mathbb{Z}/m))$.

[3] The obtained image by the Yoneda embedding has the pre-sheaf $\mathcal{A}^{\oplus} \subset \mathbb{Z}(\mathcal{A})$.

$$h_X : A \mapsto R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A, X), \tag{6}$$

which is representable of the R-mod.

Likewise, the following lemma introduces the representable pre-sheaves and functors and their role to construct pre-sheaves $\otimes_{R(\mathcal{A})}$, that can be extended to pre-sheaves $\otimes^L$, first using the projective objects of $R(\mathcal{A})$, and define the projective resolution to infinite complexes sequence.

Lemma 1.1. Every representable pre-sheaf h_X, is a projective object of $R(\mathcal{A})$, every projective object of $R(\mathcal{A})$ is a direct summand of a direct sum of representable functors, and every F, in $R(\mathcal{A})$, has a projective resolution.

Proof. We consider an analogue to (6) in the functor context:

$$\mathrm{Hom}_{R(\mathcal{A})}(h_X, F) \cong F(X), \tag{7}$$

Then each object h_X, is a projective object in $R(\mathcal{A})$. Likewise, each $F \in R(\mathcal{A})$, is a quotient

$$F = h_X/\mathcal{A}^{\oplus}, \tag{8}$$

then there exist a surjection x, such that

$$\mathrm{x} : \oplus h_X \to F, \tag{9}$$

Then from the additive category until functional additive category modulus $\mathcal{A}^{\oplus} \subset \mathbb{Z}(\mathcal{A})$, we have:

$$F = \bigoplus_{X \in \mathcal{A}} \bigoplus_{\substack{\mathrm{x} \in F(X) \\ \mathrm{x} \neq 0}} h_X, \tag{10}$$

which proves the lemma.

■

Now suppose that $\mathcal{A}$, with an additive symmetric monoidal structure $\otimes$, is such that

$$\mathcal{A} = Cor_k, \tag{11}$$

This means that $\otimes$, commutes with direct sum. Let N_α, $\alpha \in A$, and M, be R-modules; then is clear that:

$$M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha \right) \cong \bigoplus_{\alpha \in A} (M \otimes N_\alpha), \tag{12}$$

We extend $\otimes$, on $\mathcal{A}^{\oplus}$, in the same way, and this extends to tensor product of corresponding projectives. Then $\otimes$, can be extended to a tensor product on all of $R(\mathcal{A})$.

Likewise, if $F, G \in R(\mathcal{A})$, then we have a pre-sheaf tensor product in the following way:

$$(F \otimes_R G)(X) = F(X) \otimes_R G(X), \tag{13}$$

However, this does not correspond to $R(\mathcal{A})$, since $F \otimes_R G$, is not additive. However, this could be additive when one component F, or G, is element of $\mathcal{A}^{\oplus}$. But if we want to get a tensor product on $R(\mathcal{A})$, we need a more complicated or specialized construction. For this, we consider $X, Y \in \mathcal{A}$, then $h_X \otimes h_Y$, of their representable pre-sheaves should be represented by $X \otimes Y$. As a first step, we can extend $\otimes$, to a tensor product

$$\otimes : \mathcal{A}^{\oplus} \times \mathcal{A}^{\oplus} \to \mathcal{A}^{\oplus}, \tag{14}$$

commuting with $\oplus$. Thus if $L_1, L_2 \in \mathrm{Ch}^{-}\left(\mathcal{A}^{\oplus}\right)$, of the above co-chain complexes as follows:

$$\dots \to F^n \to 0 \to \dots, \tag{15}$$

the chain complex $L_1 \otimes L_2$, is defined as the total complex of the double complex $L_1^* \otimes L_2^*$.

Then we can define a legitimate tensor product between two categories $F, G \in R(\mathcal{A})$, as follows:

Definition 1.1. Let be $F, G \in R(\mathcal{A})$, choosing projective resolutions

$$P_* \to F, \quad Q_* \to G, \tag{16}$$

we define $F \otimes^L G$, [4]to be $P \otimes Q$, which means that the tensor product is total having that $\mathrm{Tot}(P_* \otimes Q_*)$. Then the tensor product to these pre-sheaves and the Hom, pre-sheaves is defined as:

$$F \otimes G = H_0\left(F \otimes^L G\right), \tag{17}$$

and

$$\mathrm{Hom}(F, G) : X \mapsto \mathrm{Hom}_{R(\mathcal{A})}(F \otimes h_X, G), \tag{18}$$

The relation (17) means the chain homotopy equivalent of the $F \otimes^L G$, is well defined up to chain homotopy equivalence, and analogous for $\mathrm{Hom}(F, G)$.

In particular, given that h_X, and h_Y, are projective, we have

$$\begin{aligned} &h_X \otimes^L h_Y = h_X \otimes h_Y = h_{X \otimes Y}, \\ &\forall X, Y \in \mathcal{A}^{\oplus}. \end{aligned} \tag{19}$$

Likewise, the ring $R(\mathcal{A})$, is an additive symmetric monoidal category.

We consider the following lemma.

Lemma 1.2. The functor $\mathrm{Hom}(F, \bullet)$, is right adjoint to $F \otimes \bullet$. In particular $\mathrm{Hom}(F, \bullet)$, is left exact and $F \otimes \bullet$, is right exact.

Proof. Let be

$$F = \mathrm{Hom}(h_Y, G), \tag{20}$$

Then[5]

$$\mathrm{Hom}_{R(\mathcal{A})}(h_X, F) = \mathrm{Hom}_{R(\mathcal{A})}(h_X, \mathrm{Hom}(h_Y, G)), \tag{21}$$

We consider

$$\mathrm{Hom}(h_Y, G) = \mathrm{Hom}_{\mathrm{R}(\mathcal{A})}(h_X \otimes h_Y, G), \tag{22}$$

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[4] $\otimes^L$, is a total tensor product.

[5] $\mathrm{Hom}(h_Y, G) : Y \to \mathrm{Hom}_{R(\mathcal{A})}(F \otimes h_Y, \mathrm{G})$.

Then in (20) we have:

$$\mathrm{Hom}_{R(\mathcal{A})}\left(h_{\mathrm{X}}, \mathrm{Hom}_{R(\mathcal{A})}(h_{\mathrm{X}} \otimes h_{\mathrm{Y}}, G)\right) = G(\mathrm{X} \otimes \mathrm{Y}), \tag{23}$$

But also,

$$G(\mathrm{X} \otimes \mathrm{Y}) = \mathrm{Hom}_{R(\mathcal{A})}(h_{\mathrm{X}} \otimes h_{\mathrm{Y}}, \mathrm{G}), \tag{24}$$

where the lemma is proved. ■

We consider the following examples.

Example 1.1. We consider the category $\mathcal{A}$, of free R-modules over a commutative ring $R(\mathcal{A})$. This category is equivalent to the category of all R-modules where pre-sheaf associated to M, is $M \otimes_R$, and Hom, and $\otimes$, are the familiar Hom_R, and $\otimes_R$.

Here, for any two modules $A, B \in R(\mathcal{A})$, we have:

$$A \otimes_R B : A \otimes B = B \otimes A, \tag{25}$$

Example 2.1. Let $\mathcal{A}$, be the category of R-modules M, such that:

$$K \otimes_R M \cong K \otimes_R (M/M_{\mathrm{tor}}), \tag{26}$$

where K, is a fraction field[6] and M_{tor}, is the torsion submodule of M. Then associated to M, is $1 \otimes_R M$, which is pre-sheaf. Here Hom. and $\otimes$, are Hom_R, and $\otimes_{\mathrm{tor}}$.

Example 3.1. Let R, be a simplicial commutative ring and $R(\mathcal{A}) \to \mathcal{A}$, be a category cofibrant replacement. Here, the pre-sheaf associated to M, which is the Kähler 1-differentials module, is $M \otimes^{\mathrm{L}}_{R(\mathcal{A})}$, and here Hom, and $\otimes$, are Hom_R, and $\otimes^{L}_{R(\mathcal{A})}$. Here the category is of the cotangent complexes of R.

Proposition 1.1. If F_i, and G_i, are in $R(\mathcal{A})$, then there is a natural mapping

$$\mathrm{Hom}(F_1, G_1) \otimes \mathrm{Hom}(F_2, G_2) \to \mathrm{Hom}(F_1 \otimes G_1, F_2 \otimes G_2), \tag{27}$$

compatible with the monoidal pairing

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(U \times A_1, \mathrm{X}_1) \otimes \mathrm{Hom}_{\mathcal{A}}(U \times A_2, \mathrm{X}_2) &\to \mathrm{Hom}_{\mathcal{A}}(U \times U \times A_1 \times A_2, \mathrm{X}_1 \times \mathrm{X}_2) \\ &\to \mathrm{Hom}_{\mathcal{A}}(U \times A_1 \times A_2, \mathrm{X}_1 \times \mathrm{X}_2), \end{aligned} \tag{28}$$

Proof. We have Hom, as defined in (18):

$$\mathrm{Hom}(F_1, G_1) : \mathrm{X}_1 \to \mathrm{Hom}_{R(\mathcal{A})}(F_1 \otimes h_{\mathrm{X}_1}, G_1), \tag{29}$$

and

$$\mathrm{Hom}(F_2, G_2) : \mathrm{X}_2 \to \mathrm{Hom}_{R(\mathcal{A})}(F_2 \otimes h_{\mathrm{X}_2}, G_2), \tag{30}$$

If $F_1, G_1, F_2, G_2 \in R(\mathcal{A}).,$ then

[6] The field of fractions of an integral domain is the smallest field in which this domain can be embedded.

$$\begin{aligned}\mathrm{Hom}(F_1,G_1)\otimes\mathrm{Hom}(F_2,G_2) &= \mathrm{Hom}_{R(\mathcal{A})}(F_1\otimes h_{X_1}\otimes F_2\otimes h_{X_2},G_1\otimes G_2)\\ &= \mathrm{Hom}_{R(\mathcal{A})}(F_1\otimes F_2\otimes h_{X\otimes Y},G_1\otimes G_2)\\ &= \mathrm{Hom}(F_1\otimes F_2,G_1\otimes G_2),\end{aligned} \tag{31}$$

We consider the Universal mapping which is commutative:

$$\begin{array}{ccc} F\times G & \xrightarrow{\lambda} & F\otimes G \\ & \mathrm{f}\searrow & \\ & & S \end{array}$$

Then (31) is compatible with the monoidal pairing. ■

If the (projective) objects h_X, are flat, that is to say, $h_X\otimes\bullet$, is an exact functor then $\otimes$, is called a balanced functor [2]. Here $F\otimes^L G$, agrees with the usual left derived functor $L(F\otimes\bullet)G$. But here we do not know when the h_X, are flat. This is true in Example 1.1. But it is not true in $\mathrm{PST}(k)=\mathbb{Z}(Cor_k)$. Then we need to extend $\otimes^L$, to a total tensor product on the category $\mathrm{Ch}^-R(\mathcal{A})$, of bounded above cochain complexes (15). This would be the usual derived functor if $\otimes$, were balanced [2], and our construction is parallel. Likewise, if C, is a complex in $\mathrm{Ch}^-R(\mathcal{A})$, there is a quasi-isomorphism $P\cong\!\rightarrow \mathrm{C}$, with P, a complex of projective objects. Any such complex P, is called a projective resolution of C, and any other projective resolution of C, is chain homotopic to P [3].

Likewise, if D, is any complex in $\mathrm{Ch}^-R(\mathcal{A})$, and

$$Q\overset{\cong}{\rightarrow}D, \tag{32}$$

is a projective resolution, we define

$$\mathrm{C}\otimes^L D=P\otimes Q, \tag{33}$$

Now, how do we understand the extensions of these tensor products in chain homotopy equivalence?

Since P, and Q, are bounded above, each

$$(P\otimes Q)^n=\bigoplus_{i+j=n}P^i\otimes Q^j, \tag{34}$$

is a finite sum, and $C\otimes^L D$, is bounded above. Then, since P, and Q, are defined up to chain homotopy, the complex $C\otimes^L D$, is independent (up to chain homotopy equivalence) of the choice of P, and Q. Then there exists a mapping

$$C\otimes^L D\rightarrow C\otimes D, \tag{35}$$

which extends the mapping

$$F\otimes^L G\rightarrow F\otimes G, \tag{36}$$

of Definition 1.1.

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We consider the following lemma to obtain in the extension (36) a derived triangulated category that will be useful in the context of derived tensor categories whose pre-sheaves are Étale pre-sheaves.

The importance of a triangulated category together with the additional structure as the given by pre-sheaves $\otimes^L$, lies in obtaining distinguished triangles of categories that generate the long exact sequences of homology that can be described through of short exact sequences of Abelian categories. Likewise, the immediate examples are the derived categories of Abelian category and the stable homotopy category of spectra or more generally, the homotopy category of a stable ∞-category. In both cases is carried a structure of triangulated category.

3. Derived triangulated categories with structure by pre-sheaves $\otimes^L$, and $\otimes^{tr}_{L,ét}$

We enounce the following proposition.

Proposition 3.1. The derived category $\mathrm{D}^{-}R(\mathcal{A})$, equipped with $\otimes^L$- structure is a tensor-triangulated category.

Proof. We consider a projective object $\mathrm{X} \in \mathcal{P}$, where $\mathcal{P}$, is a projective category defined as the points set

$$\mathcal{P} = \{\mathrm{X} \in R(\mathcal{A}) | \mathcal{A} \text{ is addative with } \otimes_R - \text{structure}\}, \tag{37}$$

We consider the application Λ, defined by the mapping:

$$\Lambda : \mathcal{P} \to \mathrm{K}^{-}(\mathcal{P}), \tag{38}$$

where the objects $\Lambda(\mathrm{X})$, are those that are determined by

$$H_0\left(\Lambda \otimes^L Q\right) = \mathrm{Hom}(h_{\mathrm{X}}, \mathrm{Hom}(\Lambda \otimes h_{\mathrm{Y}}, G)), \tag{39}$$

or

$$\Lambda(\mathrm{X}) = \mathrm{Hom}_{R(\mathcal{A})}(h_{\mathrm{X}}, \Lambda) \in \mathrm{D}^{-}R(\mathcal{A}), \tag{40}$$

Then we have

$$\mathrm{D}^{-}R(\mathcal{A}) \cong \mathrm{K}^{-}(\mathcal{P}), \tag{41}$$

via the chain homotopy. For other side

$$\mathrm{X} \to \mathrm{Hom}(h_{\mathrm{X}}, \mathrm{Hom}(\Lambda \otimes h_{\mathrm{Y}}, G)), \tag{42}$$

which is risked from $\otimes^L-$ structure when $\otimes \cong \otimes^L$, in $\mathcal{P}$, which then is true from the lemma 2.1. ■

Now, for bounded complexes of pre-sheaves we can give the following definitions.

Definition 3.1. Let C, and D, be bounded complexes of pre-sheaves. There is a canonical mapping:

$$C \otimes_R D \to C \otimes D, \tag{43}$$

which was foresee in the Definition 1.1. By right exactness of $\otimes_R$, and $\otimes$, given in Lemma 1.1, it suffices to construct a natural mapping of pre-sheaves

$$\eta : h_{\mathrm{X}} \otimes_R h_{\mathrm{Y}} \to h_{\mathrm{X}\otimes\mathrm{Y}}, \qquad \forall \mathrm{X}, \mathrm{Y} \in \mathcal{A}^{\oplus}. \tag{44}$$

For U, in $\mathcal{A}$, η_U, is the monoidal product in $\mathcal{A}$, followed by the diagonal mapping of triangle:

$$\eta : U \to U \otimes U, \tag{45}$$

that is to say,

$$h_{\mathrm{X}}(U) \otimes_R h_{\mathrm{Y}}(U) \to \mathrm{Hom}_{\mathcal{A}}(U, \mathrm{X}) \otimes_R \mathrm{Hom}_{\mathcal{A}}(U, \mathrm{Y}), \tag{46}$$

satisfies the triangle[7]:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(U,\mathrm{X}) \otimes_R \mathrm{Hom}_{\mathcal{A}}(U,\mathrm{Y}) & \stackrel{\otimes}{\to} & \mathrm{Hom}_{\mathcal{A}}(U \otimes U, \mathrm{X} \otimes \mathrm{Y}) \\ \Delta \searrow & & \downarrow \Delta \\ & & \mathrm{Hom}_{\mathcal{A}}(U, \mathrm{X} \otimes \mathrm{Y}) \end{array} \tag{47}$$

where

$$\mathrm{Hom}_{\mathcal{A}}(U, \mathrm{X} \otimes \mathrm{Y}) = h_{\mathrm{X}\otimes\mathrm{Y}}(U), \tag{48}$$

With all these dispositions and generalities, now we can specialize to the case when[8]

$$\mathcal{A} = Cor_k, \tag{49}$$

and $\otimes$, is the tensor product

$$\mathrm{X} \otimes \mathrm{Y} = \mathrm{X} \times \mathrm{Y}', \tag{50}$$

Then we have the Yoneda embedding:

$$\mathrm{PST}(k) \subset {Cor_k}^{\oplus} \subset Cor_k, \tag{51}$$

We denote as $\otimes^{tr}$, for the tensor product on $\mathrm{PST}(k) = \mathbb{Z}(Cor_k)$, or

$$\mathrm{PST}(k, R) = R(Cor_k), \tag{52}$$

and ${\otimes^{tr}}_L$, for $\otimes^L$. Then there is a natural mapping

$$C \otimes_L^{tr} D \to C \otimes^{tr} D, \tag{53}$$

Here ${\otimes^{tr}}_L$, is the tensor product induced to $\mathbb{Z}(Cor_k)$. But, before we will keep using the product $\otimes^{tr}$, which we can define as:

$$\mathrm{X} \otimes \mathrm{Y} = \mathrm{X} \times \mathrm{Y}. \qquad R_{tr}(\mathrm{X}) \otimes^{tr} R_{tr}(\mathrm{Y}) = R_{tr}(\mathrm{X}, \mathrm{Y}), \tag{54}$$

being $h_{\mathrm{X}} = R_{\mathrm{tr}}(\mathrm{X})$, $\forall h_{\mathrm{X}} \in \mathrm{Hom}$, $\forall \mathrm{X} \in \mathcal{A}^{\oplus}$.

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[7] $\eta_U \circ \Delta = \Delta'$.

[8] Def. If X, $\mathrm{Y} \in Cor_k$, their tensor product $\mathrm{X} \otimes \mathrm{Y}$, is defined to be the product underlying schemes over k,

$$\mathrm{X} \otimes \mathrm{Y} = \mathrm{X} \times \mathrm{Y}.$$

The above can be generalized through the following lemma.

Lemma 3.1. The pre-sheaf $\mathbb{Z}_{tr}((X_1, x_1)\wedge\cdots\wedge(X_n, x_n))$, is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. In particular, it is projective object of PST. Likewise, for the following sequence of pre-sheaves with transfers, the exactness is explicit[9]:

$$\begin{aligned} 0 &\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \bigoplus_i \mathbb{Z}_{tr}(X_i) \to \bigoplus_{i,j} \mathbb{Z}_{tr}\left(X_i \times X_j\right) \to \dots \\ &\to \bigoplus_{i,j} \mathbb{Z}_{tr}\left(X_1 \times \dots \times X_j \times \dots \times X_j \times \dots \times X_n\right) \to \bigoplus_{\mathrm{i}} \mathbb{Z}_{tr}\left(X_1 \times \dots \times \hat{X}_j \times \dots \times X_n\right) \\ &\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1\wedge\cdots\wedge X_n) \to 0, \end{aligned} \tag{55}$$

Then, it is sufficient to demonstrate that $\otimes^{tr}_{L,\acute{e}t}$, preserve quasi-isomorphisms.

Definition 3.2. A pre-sheaf with transfers is a contravariant additive functor:

$$F = Cor_k \to \mathrm{Ab}, \tag{56}$$

we write

$$\mathrm{Pre}Sh(Cor_k) \to \mathrm{PST}(k) = \mathrm{PST}, \tag{57}$$

to describe the functor category on the field k, whose objects are pre-sheaves with transfer and whose morphisms are natural transformations.

Likewise, analogously we can define to the tensor product $\otimes^{tr}$, their extension to ${\otimes^{tr}}_{\acute{e}t}$.

Likewise, we have the definition.

Definition 3.3. If F, and G, are pre-sheaves of R-modules with transfers, we write:

$$(F \otimes^{tr} G)_{\acute{e}t} \to F \otimes^{tr}_{\acute{e}t} G, \tag{58}$$

the Étale sheaf associated to $F \otimes^{tr} G$.

If C, and D, are bounded above complexes of pre-sheaves with transfers, we shall write $C \otimes^{tr}_{\acute{e}t} D$, for $(C \otimes^{tr} D)_{\acute{e}t}$, and

$$\left(C \otimes^{tr}_{L} D\right) \cong P \otimes^{tr}_{\acute{e}t} Q, \tag{59}$$

where P, and Q, are complexes of representable sheaves with transfers, $P \cong C$, and $Q \cong D$. Then there is a natural mapping

$$\left(C \otimes^{tr}_{L,\acute{e}t} D\right) \to C \otimes^{tr}_{\acute{e}t} D, \tag{60}$$

induced by

$$\left(C \otimes^{tr}_{L,} D\right) \to C \otimes^{tr} D, \tag{61}$$

Lemma 3.2. If F, and F', are Étale sheaves of R-modules with transfers, and F, is locally constant, the mapping:

$$\begin{aligned} h_X(U) \otimes_R h_Y(U) = \mathrm{Hom}_{\mathcal{A}}(U, X) \otimes \mathrm{Hom}_{\mathcal{A}}(U, Y) &\xrightarrow{\otimes} \mathrm{Hom}_{\mathcal{A}}(U \otimes U, X \otimes Y) \\ &\xrightarrow{\Delta'} \mathrm{Hom}_{\mathcal{A}}(U, X \otimes Y) = h_{X \otimes Y}(U), \end{aligned} \tag{62}$$

[9] $\mathbb{Z}_{tr}(X) \cong \mathbb{Z}_{tr} \oplus \mathbb{Z}_{tr}(X, x)$, $\quad \mathbb{Z}_{tr}(X_1 \times X_2) \cong \mathbb{Z}_{tr} \oplus \mathbb{Z}_{tr}(X_1, x_1) \oplus \mathbb{Z}_{tr}(X_2, x_2) \oplus \mathbb{Z}_{tr}(X_1 \wedge X_2)$.

induces an isomorphism

$$F \otimes_{\mathrm{et}} F' \xrightarrow{\cong} F \otimes^{tr}_{ét} F', \tag{63}$$

Remember that a pre-sheaf is defined as:

Definition 3.4. A pre-sheaf F, of Abelian groups on Sm/k, is an Étale sheaf if it restricts to an Étale sheaf on each X, in Sm/k, , that is if:

i. The sequence

$$0 \to F(\mathrm{X}) \overset{\mathrm{diag}}{\to} F(U) \overset{(+,-)}{\to} F(U \times_{\mathrm{X}} U), \tag{64}$$

is exact for every surjective Étale morphism of smooth schemes,

$$U \to \mathrm{X}, \tag{65}$$

ii. $F(\mathrm{X} \underline{\cup} \mathrm{Y}) = F(\mathrm{X}) \oplus F(\mathrm{Y}), \ \forall \mathrm{X}, \mathrm{Y},$ schemes.

We demonstrate Lemma 3.2.

Proof. We want the tensor product $\otimes^{tr}_{L,ét}$, which induces to tensor triangulated structure on the derived category of Étale sheaves of R-modules with transfers[10] defined in other expositions [4]. Considering Proposition 3.1, we have:

$$\left(C \otimes^{tr}_{L,ét} D\right) \to D \otimes^{tr}_{L,ét} C, \tag{66}$$

Then, it is sufficient to demonstrate that $\otimes^{tr}_{L,ét}$, preserve quasi-isomorphisms. The details can be found in [5].

Then the tensor product $\otimes^{tr}_{ét}$, as pre-sheaf to Étale sheaves can have a homology space of zero dimension that vanishes in certain component right exact functor $\Phi(F) = R_{tr}(\mathrm{Y}) \otimes^{tr}_{ét} F$, from the category $\mathrm{PST}(k, R)$, of pre-sheaves of R-modules with transfers to the category of the Étale sheaves of R-modules and transfers. Then every derived functor $L_n\Phi$, vanishes on $H_0\left(\tilde{C}\right)$, to certain complex of Étale.

Then, all right exact functors $R_{tr}(\mathrm{Y}) \otimes^{tr}_{ét} F$, are acyclic. This is the machinery to demonstrate the functor exactness and resolution in modules through of induce from $\otimes^{tr}_{L,ét}$, a tensor-triangulated structure to a derived category more general that $\mathrm{D}^{-}R(\mathcal{A})$.

Also we have:

Lemma 3.3. Fix Y, and set $\Phi = R_{tr}(Y) \otimes^{tr}_{ét}$. If F, is a pre-sheaf of R-modules with transfers such that $F_{ét} = 0$, then $L_n\Phi(F) = 0, \ \forall n$.

4. Some considerations to mathematical physics

Remember that in the derived geometry we work with structures that must support R-modules with characterizations that should be most general to the case of singularities, where it is necessary to use irregular connections, if it is the case, for example in field theory in mathematical physics when studying the quantum field equations on a complex Riemann manifold with singularities.

[10] **Definition.** A pre-sheaf with transfers is a contravariant additive functor from the category Cor_k, to the category of abelian groups Ab.

Through the characterization of connections for derived tensor products, we search precisely generalize the connections through pre-sheaves with certain special properties, as can be the Étale sheaves.

Remember we want to generalize the field theory on spaces that admit decomposing into components that can be manageable in the complex manifolds whose complex varieties can be part of those components called motives, creating a decomposition in the derived category of its spectrum considering the functor Spec, and where solutions of the field equations are defined in a hypercohomology.[11] Likewise, this goes focused to obtain a good integrals theory (solutions) in the hypercohomology context considering the knowledge of spectral theory of the cycle sequences in motivic theory that searches the solution of the field equations even with singularities of the complex Riemann manifold.

We can demonstrate that $\otimes^{tr}_{L,\acute{e}t}$, induces a tensor-triangulated structure to a derived category more general than $D^{-}R(\mathcal{A})$, as for example, $DM^{\mathrm{eff},-}_{\acute{e}t}(k, \mathbb{Z}/m)$, which is our objective. In this case, we want geometrical motives, where this last category $DM^{\mathrm{eff},-}_{\acute{e}t}(k, \mathbb{Z}/m)$, can be identified for the derived category $DM^{-}_{\mathrm{gm}}(k, R)$.

We consider and fix Y, and the right exact functor $\Phi(F) = R_{tr}(\mathrm{Y}) \otimes^{tr}_{\acute{e}t} F$, from the category $\mathrm{PST}(k, R)$, of pre-sheaves of R-modules with transfers to the category of the Étale sheaves of R-modules and transfers. Likewise, their left functors $L_P\Phi(F)$, are the homology sheaves of the total left derived functor $\Phi(F) = R_{tr}(\mathrm{Y}) \otimes^{tr}_{L,\acute{e}t} F$. Considering a chain complex C, the hypercohomology spectral sequence is:

$$\mathrm{E}^2_{p,q} = L_p\Phi\left(H_{\mathrm{q}}C\right), \tag{67}$$

then

$$L_{p+q}\Phi(C) = 0, \tag{68}$$

Then the corresponding infinite sequence is exact.

We consider A, and $B \in \mathcal{A}$, where $\mathcal{A}$, is a category as has been defined before.

We have the following proposition.

Proposition 4.1. There is equivalence between categories $\mathrm{Ab}\left(\mathrm{CRing}_{A//B}\right) \cong \mathrm{Mod}_B$.

Then a hypercohomology as given to $d\mathrm{da} = 0$, can be obtained through double functor work $A \to B \to B$, through an inclusion of a category Mod_B, in $\mathrm{CRing}_{A//B}$. Then is had the result.

Theorem 4.1. The left adjoint to the inclusion functor $\mathrm{Mod}_B, \mathrm{CRing}_{A//B}$. is defined by $\mathrm{X} \mapsto \Omega_{\mathrm{X}/A} \otimes_{\mathrm{X}} B$. In particular, the image of $A \to B \to B$, under this functor is $B \mapsto \Omega_{\mathrm{X}/A}$.

The derived tensor product is a regular tensor product.

Theorem 4.2. The character for an adjoint lifts for a homotopically meaningful adjunction complies:

$$\mathrm{Ch}(B)_{\geq B} \leftrightarrow s\mathrm{CRing}_{A//B}, \tag{69}$$

---

[11] **Definition.** A hyperhomology or hypercohomology of a complex of objects of an abelian category is an extension of the usual homology of an object to complexes. The mechanism to give a hypercohomology is suppose that $\mathcal{A}$, is an abelian category with enough injectives and Φ, a left exact functor to another abelian category $\mathcal{B}$. If C, is a complex of objects of $\mathcal{A}$, bounded on the left, the hypercohomology $H^i(C)$, of C, (for an integer i) is calculated as follows: take a quasi-isomorphism $\psi : C \to I$, where I, is a complex of injective elements of $\mathcal{A}$. The hypercohomology $H^i(C)$, of C, is then the cohomology $H^i(\Phi(I))$, of the complex $\Phi(I)$.

Meaning that, it is an adjunction of categories, which induces an adjunction to level of homotopy categories.

We define the cotangent complex required in derived geometry and QFT.

Definition 4.1. The cotangent complex $L_{A/B}$, is the image of functor $A \to B \to B$, under the left functor of the Kahler differentials module $M \otimes^{L}_{R(\mathcal{A})}$, . Likewise, if $P_{\bullet} \to B$, be a free resolution then

$$L_{A/B} = \Omega_{P_{\bullet}/A} \otimes_{P_{\bullet}} B, \tag{70}$$

The cotangent complex as defined in (69) lives in the derived category Mod_B. We observe that choosing the particular resolution of B, then $\Omega_{P_{\bullet}/A}$, is a co-fibrant object in the derived category $\mathrm{Mod}_{P_{\bullet}}$, which no exist distinction between the derived tensor product and the usual tensor product. Then to any representation automorphic of $G(A)$, the $G(F)/G(A)$, can be decomposed as the tensor product $\otimes_{i=1}^{n} \pi_I$. This last fall in the geometrical Langlands ramifications.

Example 4.1. (66) in the context of solution of field equations as $d\mathrm{da} = 0$, has solution in the hypercohomology of a spectral sequence of $\mathrm{D}^{-}R(\mathcal{A})$, (established on the infinite sequence $\ldots \to F^n \to 0 \to \ldots$. [6]) when its functors whose image $\Omega_{B/A}$, have as its cotangent complex the image under of the functor $L_{A/B}$, which is the functor image $A \to B \to B$, under the left derived functor of Kahler differentials.

To demonstrate this, it is necessary to define an equivalence between derived categories in the level of derived categories $D\left({}^{L}\mathrm{Bun}, \mathcal{D}\right)$, and $D\left({}^{L}Loc, \mathcal{O}\right)$, where geometrical motives can be risked with the corresponding moduli stack to holomorphic bundles. The integrals are those whose functors image will be in $\mathrm{Spec}_H \mathrm{SymT}(\mathrm{OP}_{L_G}(D))$, which is the variety of opers on the formal disk D, or neighborhood of all points in a surface Σ, in a complex Riemannian manifold [6].

5. Applications

As was shown, the geometrical motives required in our research are a result of embedding the derived category $\mathrm{DM}^{-}_{\mathrm{gm}}(k, R)$, (geometrical motives category) in the $\mathrm{DM}^{eff,-}_{\acute{e}t}(k, \mathbb{Z}/m)$, considering the category of smooth schemes on the field k.

We consider the following functors. For each $F \in \mathrm{D}^{-}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right)$, there is $L^{A^1}F \in \mathrm{D}^{\mathrm{eff}}_{-}(k)$, the resulting functor is:

$$L^{A^1} : \mathrm{D}^{-}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right) \to \mathrm{D}^{\mathrm{eff}}_{-}(k), \tag{71}$$

which is exact and left-adjoint to the inclusion

$$\mathrm{D}^{\mathrm{eff}}_{-}(k) \to \mathrm{D}^{-}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right), \tag{72}$$

Also the functor (70) descends to an equivalence of triangulated categories. This is very useful to make $\mathrm{D}^{\mathrm{eff}}_{-}(k)$, into a tensor category as follows. We consider the Nisnevich sheaf $\mathbb{Z}_{tr}(k)$, with transfer $tr : \mathrm{Y} \to c(\mathrm{Y}, \mathrm{X})$. We define

$$\mathbb{Z}_{tr}(k) \otimes \mathbb{Z}_{tr}(k) := \mathbb{Z}_{tr}(\mathrm{X} \times_k \mathrm{Y}), \tag{73}$$

Then it can be demonstrated that the operation realized in (70) can be extended to give $\mathrm{D}^{-}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right)$, with the structure of a triangulated tensor category.

Then the functor L_{A^1}, induces a tensor operation on $\mathrm{D}^-_{A^1}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right)$, making that $\mathrm{D}^-_{A^1}\left(\mathrm{Sh}^{\mathrm{Nis}}(Cor(k))\right)$ also a triangulated tensor category. Likewise, explicitly in $\mathrm{DM}^{\mathrm{eff}}_-(k)$, this gives us the functor

$$m : Sm_k \to \mathrm{DM}^{\mathrm{eff}}_-(k), \tag{74}$$

defined by

$$m(\mathrm{X}) \coloneqq C^{\mathrm{Sus}}(\mathrm{Z}_{tr}(\mathrm{X})), \tag{75}$$

where we have the formula

$$m(\mathrm{X}\times_k\mathrm{Y}) = m(\mathrm{X}) \otimes m(\mathrm{Y}), \tag{76}$$

If we consider the embedding theorem, then we can establish the following triangulated scheme

$$\begin{array}{ccc} Sm_k & \to & \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \\ & m\searrow & \updownarrow \mathrm{Id} \\ & & \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \end{array} \tag{77}$$

which has implications in the geometrical motives applied to bundle of geometrical stacks in mathematical physics.

Theorem 5.1 (F. Bulnes). Suppose that M, is a complex Riemannian manifold with singularities. Let X, and Y, be smooth projective varieties in M [12]. We know that solutions of the field equations $d\mathrm{da} = 0$, are given in a category $\mathrm{Spec}(Sm_{\mathrm{k}})$, (see Example 4). Context Solutions of the quantum field equations for $d\mathrm{da} = 0$, are defined in hyper-cohomology on $\mathbb{Q}$- coefficients from the category Sm_k, defined on a numerical field k, considering the derived tensor product $\otimes^{tr}_{\acute{e}t}$, of pre-sheaves. Then the following triangulated tensor category scheme is true and commutative:

$$\begin{array}{ccc} & \mathrm{DQFT} & \\ i\swarrow & & \searrow F \\ \mathrm{MD}_{\mathrm{gm}}\ (\mathbb{Q}) & \leftrightarrow & \mathrm{MD}(\mathcal{O}_{\mathrm{Y}}) \end{array} \tag{78}$$

The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, R)$, has a tensor structure and the tensor product of its motives is as defined in (75) $m(\mathrm{X}) \otimes m(\mathrm{Y}) = m(\mathrm{X} \times \mathrm{Y})$.

Triangulated category of geometrical motives $\mathrm{DM}_{\mathrm{gm}}(k, R)$, or written simply as $\mathrm{DM}_{\mathrm{gm}}(k)$, is defined formally inverting the functor of the Tate objects[13] (are objects of a motivic category called Tannakian category) $\mathbb{Z}\ (1)$, to be image of the complex

[12] Singular projective varieties useful in quantization process of the complex Riemannian manifold. The quantization condition compact quantizable Käehler manifolds can be embedded into projective space.

[13] Let $\mathrm{MT}(\mathbb{Z})$, denote the category of mixed Tate motives unramified over $\mathbb{Z}$. It is a Tannakian category with Galois group $\mathrm{Gal}_{\mathrm{MT}}$.
The inverting of the objects $-\otimes\mathbb{Z}.(1)$.
Remember that a scheme is a mathematical structure that enlarges the concept of algebraic variety in several forms, such as taking account of multiplicities. The schemes can to be of a same algebraic variety different and allowing "varieties" defined over any commutative ring. In many cases, the family of all varieties of a type can be viewed as a variety or scheme, known as a moduli space.

$[\mathbb{P}^1] \rightarrow [\mathrm{Spec}(k)]$, where the motive in degree $p = 2, 3$, will be $m(p) = m \otimes \mathbb{Z}(1)^{\otimes p}$, or to any motive $m \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k), \forall p \in \mathbb{N}$.

Likewise, the important fact is that the canonical functor $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k), \rightarrow \mathrm{DM}_{\mathrm{gm}}(k)$, is full embedding [7]. Therefore we work in the category $\mathrm{DM}_{\mathrm{gm}}(k)$.

Likewise, for X, and Y, smooth projective varieties and for any integer i, there exists an isomorphism:

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)}(m(\mathrm{X}), m(\mathrm{Y})(i)[2i]) \cong A^{m+i}(\mathrm{X} \times \mathrm{Y}), \qquad m = \dim \mathrm{Y}, \tag{79}$$

We demonstrate the Theorem 5.1.

Proof. $\forall\ \mathrm{X} \in Sm_k$, the category Sm_k, extends to a pseudo-tensor equivalence of cohomological categories over motives on k, that is to say, $\mathrm{MM}(k)$, is the image of functors

$$\mathrm{DM}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{\mathrm{gm}}(k), \tag{80}$$

which is an equivalence of the underlying triangulated tensor categories.

On the other hand, the category DQFT can be defined for the motives in a hypercohomology from the category Sm_k, defined as:

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)}(m(\mathrm{X}), \mathbb{Q}(q)[\mathrm{p}]) \cong H^{\bullet}_{Nis}(\mathrm{X}, \mathbb{Q}, (q)) = \mathbb{H}^{p,q}(\mathrm{X}_{Nis}, \mathbb{Q}, (q)), \tag{81}$$

which comes from the hypercohomology

$$H_L^{p,q}(\mathrm{X}, \mathbb{Q}) = \mathbb{H}^{p,q}(\mathrm{X}, \mathbb{Q}), \tag{82}$$

We observe that if a Zariski sheaf of $\mathbb{Q}$-modules with transfers F, is such that $F = H^q C$, for all C, a complex defined on $\mathbb{Q}(q)$-modules (being a special case when $C = \mathbb{Q}(q)$), where the cohomology groups of the isomorphism $H^p_{\acute{e}t}(\mathrm{X}, F_{\acute{e}t}) \cong H^p_{Nis}(\mathrm{X}, F_{Nis})$, can be vanished for $p > \dim(\mathrm{Y})$.

Then survives a hypercohomology $\mathbb{H}^q(\mathrm{X}, \mathbb{Q})$. If we consider $\mathrm{Spec}(Sm_{\mathrm{k}})$, we can to have the quantum version of this hyper-cohomology with an additional work on moduli stacks of the category Mod_B, in a study on equivalence between derived categories in the level of derived categories $D(^L\mathrm{Bun}, \mathcal{D})$, and $D(^L Loc, \mathcal{O})$, where geometrical motives can be risked with the corresponding moduli stack to holomorphic bundles[14].

For other way, with other detailed work of quasi-coherent sheaves [6] we can to obtain the category $\mathrm{MO}_{\mathcal{O}}(\mathrm{Y})$. The functors are constructed using the Mukai-Fourier transforms. ■

[14] We consider the functor F, defined as:
where $\mathcal{K}(F^r)$, the kernel space of the functor F^r, is the functor that induces the equivalence $\mathrm{Mod}_T(D(\mathrm{X} \times_{\mathrm{Y}} \mathrm{X})) \cong {}^{\perp}\mathcal{K}(F^r)$, and the operator $T = F^r \circ F$, acting on category $D(\mathrm{X} \times_{\mathrm{Y}} \mathrm{X})$.

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References

[1] Adams JF. Stable homotopy and generalised homology. In: Chicago Lectures in Mathematics, MR 53 #6534. Chicago, IL: University of Chicago Press; 1974

[2] Mazza C, Voevodsky V, Weibel C, editors. Lecture Notes on Motivic Cohomology. Vol. 2. Cambridge, MA, USA: AMS Clay Mathematics Institute; 2006

[3] Aubry M. Homotopy Theory and Models. Boston, MA: Birkhäuser; 1995

[4] Milne JS. Étale Cohomology. Princeton, NJ: Princeton University Press; 1980. ISBN: 0-691-08238-3

[5] Bosch S. Algebraic Geometry and Commutative Algebra. NY, USA: Springer; 2013

[6] Bulnes F. Extended d-cohomology and integral transforms in derived geometry to QFT-equations solutions using Langlands correspondences. Theoretical Mathematics and Applications. 2017;7(2):51-62

[7] Voevodsky V. Triangulated categories of motives over a field. In: Cycles, Transfers, and Motivic Homology Theories, Vol. 143 of Annals of Mathematics Studies. USA: Princeton University Press; 2000. pp. 188-238