Chapter

Bright, Dark, and Kink Solitary Waves in a Cubic-Quintic-Septic-Nonical Medium

Mati Youssoufa, Ousmanou Dafounansou and Alidou Mohamadou

Abstract

In this chapter, evolution of light beams in a cubic-quintic-septic-nonical medium is investigated. As the model equation, an extended form of the well-known nonlinear Schrödinger (NLS) equation is taken into account. By the use of a special ansatz, exact analytical solutions describing bright/dark and kink solitons are constructed. The existence of the wave solutions is discussed in a parameter regime. Moreover, the stability properties of the obtained solutions are investigated, and by employing Stuart and DiPrima's stability analysis method, an analytical expression for the modulational stability is found.

Keywords: higher-order nonlinear Schrödinger equation, spatial solitons, stability analysis method, modulational instability, optical fibers

1. Introduction

The study of spatial solitons in the field of fiber-optical communication has attracted considerable interest in recent years. In a uniform nonlinear fiber, soliton can propagate over relatively long distance without any considerable attenuation. The formation of optical solitons in optical fibers results from an exact balancing between the diffraction and/or group velocity dispersion (GVD) and the self-phase modulation (SPM). The theorical prediction of a train of soliton pulses from a continuous-wave (CW) light in optical fibers was first suggested by Hasegawa and Tappert [1, 2] and first experimentally demonstrated by Mollenauer et al. [3] in single-mode fibers in the case of negative GVD, in liquid CS_2 by Barthelemy et al. in 1985 [4]. In nonlinear optic, optical solitons are localized electromagnetic waves that transmit in nonlinear Kerr or non-Kerr media with dispersion or (and) diffraction without any change in shapes. In nonlinear media, the dynamics of spatial optical solitons is governed by the well-known nonlinear Schrödinger (NLS) equation. Depending on the signs of GVD, the NLS equation admits two distinct types of soliton, namely, bright and dark solitons. The bright soliton exists in the regime of anomalous GVD, and the dark soliton arises in the regime of normal GVD. The physics governing the soliton differs depending on whether one considers a bright or a dark soliton and accordingly features distinct applications [5–8]. The unique property of optical solitons, either bright or dark, is their particle-like behavior in interaction [9].

In addition to fundamental bright and dark solitons, various other forms and shapes of solitary waves can appear in nonlinear media. Kink solitons, for example, are an important class of solitons which may propagate in nonlinear media exhibiting higher-order effects such as third-order dispersion, self-steepening, higher-order nonlinearity, and intrapulse stimulated Raman scattering. In the setting of nonlinear optics, a kink soliton represents a shock front that propagates undistorted inside the dispersive nonlinear medium [10]. This type of solitons has been studied extensively, both analytically and numerically [11–13]. These spatial soliton solutions can maintain their overall shapes but allow their widths and amplitudes and the pulse center to change according to the management of the system's parameters, such as the dispersion, nonlinearity, gain, and so on [14].

The cubic nonlinear Schrödinger equation (CNLSE) has been widely used to model the propagation of light pulse in material's systems involving third-order susceptibility $\chi^{(3)}$, though, for moderate pulse intensity, the higher-order nonlinearities are related to higher-order nonlinear susceptibilities (nonlinear responses) of a material. For example, the cubic-quintic-nonlinear Schrödinger equation (CQNLSE) models materials with fifth-order susceptibility $\chi^{(5)}$. This kind of nonlinearity (cubic-quintic CQ) is named as *parabolic law nonlinearity* and existing in nonlinear media such as the p-toluene sulfonate (PTS) crystals. The parabolic law can closely describe the nonlinear interaction between the high-frequency Langmuir waves and the ion acoustic waves by ponderomotive forces [15, 16], in a region of reduced plasma density, and the nonlinear interaction between Langmuir waves and electrons. In addition, CQ was experimentally proposed as an empirical description of special semiconductor (e.g., AlGaAs, CdS, etc.) waveguides and semiconductor-doped glasses, particularly for the CdS_xSe_{1-x}-doped glass, which exhibit a significant fifth-order susceptibilities $\chi^{(5)}$ as experimentally reported earlier [17, 18]. Moreover, using high laser intensity, the saturation of nonlinearity has been established experimentally in many materials such as nonlinear organic polymers, semiconductor-doped glasses, and so on, which have the property that their absorption coefficient decreases [19]. More generally, a self-defocusing $\chi^{(5)}$ usually accounts for the saturation of $\chi^{(3)}$.

In recent years, many influential works have devoted to construct exact analytical solutions of CQNLSE, such as the pioneering work of Serkin et al. [20]. In particular, Dai et al. [21–25] obtained exact self-similar solutions (similaritons), their nonlinear tunneling effects of the generalized CQNLSE, and their higher-dimensional forms with spatially inhomogeneous group velocity dispersion, cubic-quintic nonlinearity, and amplification or attenuation.

Since the measurement of third-, fifth-, and seventh-order nonlinearities of silver nanoplatelet colloids using a femtosecond laser [26], an extension of nonlinear Schrödinger equation including the cubic-quintic-septic nonlinearity was used to model the propagation of spatial solitons. In [27], for example, the authors performed numerical calculations based on higher-order nonlinearity parameters including seventh-order susceptibility $\chi^{(7)}$ (a chalcogenide glass is an example). This seeds several motivations to discover new features of solitons with combined effects of higher-order nonlinear parameters. In this regard, Houria et al. [28] constructed dark spatial solitary waves in a cubic-quintic-septic-nonlinear medium, with a profile in a functional form given in terms of "$sech^{2/3}$". They have also investigated chirped solitary pulses for a derivative nonical-NLS equation on a CW background [29]. It is obvious to notice that the contributions of the higher-order nonlinearities can give way to generate stable solitons in homogeneous isotropic media and influence many aspects of filamentation in gases and condensed matters [30–33].

Recently, the study of modulational instability (MI) in non-Kerr media has receiving particular attention. MI is a fundamental and ubiquitous process that appears in most nonlinear systems [6, 9, 34–37]. This instability is referred to as modulation instability because it leads to a spontaneous temporal modulation of the CW beam and transforms it into a pulse train. During this process, small perturbations upon a uniform intensity beam grow exponentially due to the interplay between nonlinearity and dispersion or diffraction. As a result, under specific conditions, a CW light often breaks up into trains of ultrashort solitons like pulses [9]. To date, there has not been any report of MI in the cubic-quintic-septic-nonical-nonlinear Schrödinger equation (CQSNNLSE).

Our study will be focused on the analysis of solitary wave's solutions of systems described by the higher-order NLSE named CQSNNLSE. We will discuss the model with higher-order nonlinearities and explore the dynamics of bright, dark, and kink soliton solutions. Finally, the linear stability analysis of the MI is formulated, and the analytical expression of the gain of MI is obtained. Moreover, the typical outcomes of the nonlinear development of the MI are reported.

2. Model equation

The dynamics of (1 + 1)-dimensional (one spatial and one temporal variables) spatial optical solitons is the well-known nonlinear Schrödinger equation. If we consider the higher-order effects, an extended model is required, and the propagation of optical pulses through the highly nonlinear waveguides can be described by the CQSNNLSE:

$$E_z = i\alpha_1 E_{tt} + i\alpha_2 |E|^2 E + i\alpha_3 |E|^4 E + i\alpha_4 |E|^6 E + i\alpha_5 |E|^8 E, \tag{1}$$

where $E(z, t)$ is the slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivatives in the frame moving with the pulsed solutions, α_1 is the parameter of diffraction or dispersion, and α_2, α_3, α_4, and α_5 are the cubic, quintic, septic, and nonical nonlinear terms, respectively. This model is relevant to some applications in which higher-order nonlinearities are important.

For example, Eq. (1) with $\alpha_1 = \frac{1}{2}$, $\alpha_4 = 1$, and $\alpha_5 = 0$ was used to study numerically the stability conditions of one-dimensional spatial solitons [38]. Recently, Eq. (1) with $\alpha_5 = 0$ was analyzed for systems that are valid for several types of septic nonlinear materials [28]. Here, we consider arbitrary parameters $\alpha_j(\ j = 1, 2, 3, 4, 5)$ for the sake of a general analysis that is valid for several types of nonical media.

To obtain the exact analytic optical solitary-wave solutions of Eq. (1), we can employ the following transformation:

$$E(\ z, t\ ) = \theta(t + \beta z) e^{i(kz - \omega t)} = \theta(\zeta) e^{i(kz - \omega t)}. \tag{2}$$

Here, $\theta(\zeta)$ is a real function and β is a real constant to be determined.

Upon substituting Eq. (2) into Eq. (1) and separating the real and imaginary parts, one obtains

$$\beta = 2\alpha_1 \omega, \tag{3}$$

$$\theta_{\zeta\zeta} = \frac{k + \alpha_1 \omega^2}{\alpha_1}\theta - \frac{\alpha_2}{\alpha_1}\theta^3 - \frac{\alpha_3}{\alpha_1}\theta^5 - \frac{\alpha_4}{\alpha_1}\theta^7 - \frac{\alpha_5}{\alpha_1}\theta^9, \tag{4}$$

Eq. (4) represents the evolution of an anharmonic oscillator with an effective potential energy V [28] defined by

$$V(\theta) = -\frac{k+\alpha_1\omega^2}{2\alpha_1}\theta^2 + \frac{\alpha_2}{4\alpha_1}\theta^4 + \frac{\alpha_3}{6\alpha_1}\theta^6 + \frac{\alpha_4}{8\alpha_1}\theta^8 + \frac{\alpha_5}{10\alpha_1}\theta^{10}. \tag{5}$$

Integrating Eq. (4) yields

$$(\theta_\zeta)^2 = a_1\theta^2 - a_2\theta^4 - a_3\theta^6 - a_4\theta^8 - a_5\theta^{10} + 2\xi, \tag{6}$$

where

$$a_1 = \frac{k+\alpha_1\omega^2}{2\alpha_1}, a_n = \frac{\alpha_n}{n\alpha_1}(n = 2, 3, 4, 5), \tag{7}$$

and ξ is the constant of integration, which can represent the energy of the anharmonic oscillator [39].

In order to get the exact soliton solutions, we first rewrite Eq. (6) in a simplified form by using transformation:

$$\theta(\zeta) = u^{1/2}(\zeta). \tag{8}$$

By substituting Eq. (8) into Eq. (6), we obtain a new auxiliary equation possessing a sixth-degree nonlinear term:

$$\frac{1}{4}(u_\zeta)^2 = a_1u^2 - a_2u^3 - a_3u^4 - a_4u^5 - a_5u^6 + 2\xi. \tag{9}$$

To solve Eq. (9), we will employ three types of localized solutions named bright, dark, and kink solitons. In the following, we solve Eq. (9) by using appropriate ansatz and obtain alternative types of solitary-wave solutions on a CW background and investigate parameter domains in which these optical spatial solitary waves exist.

3. Exact solitary-wave solutions

In this section, we find bright-, dark-, and kink-solitary-wave localized solutions of Eq. (9), by using a special ansatz:

3.1 Bright solitary-wave solutions

The bright solitary solutions of Eq. (9) have the form:

$$u_b(\zeta) = \frac{A_b}{\sqrt{1 + N_b \cosh(\alpha_b\zeta)}}, \tag{10}$$

where A_b, N_b, and α_b are real constants which represent wave parameters (A_b and α_b related to the amplitude and pulse width of the bright wave profiles, respectively) to be determined by the physical coefficients of the model.

Substituting the ansatz Eq. (10) into Eq. (9), we obtain the unknown parameters A_b, N_b, α_b, and energy ξ:

$$\begin{cases} A_b = \sqrt{\dfrac{2a_1}{a_3}} \\ \alpha_b = \sqrt[4]{a_1} \\ N_b = \sqrt{\dfrac{a_3{}^2 + 4a_1a_5}{a_3{}^2}}, \\ \xi = \dfrac{a_1a_2}{2a_3} \end{cases} \tag{11}$$

with parametric conditions

$$a_2a_3 + 4a_1a_4 = 0, a_1 > 0, a_3 > 0, a_3{}^2 + 4a_1a_5 > 0. \tag{12}$$

Thus, the exact bright solitary-wave solutions on a CW background of Eq. (1) are of the form:

$$E_b(z,t) = \left\{ \frac{\frac{2a_1}{a_3}}{1 + \sqrt{\frac{a_3{}^2+4a_1a_5}{a_3{}^2}} \cosh\left[\sqrt[4]{a_1}(t + 2\alpha_1\omega z)\right]} \right\}^{\frac{1}{4}} \times e^{i(kz-\omega t)}. \tag{13}$$

3.2 Dark solitary-wave solutions

The dark solitary solutions of Eq. (9) take the form [40]:

$$u_d(\zeta) = \frac{A_d \sinh(\alpha_d\zeta)}{\sqrt{1 + N_d \sinh^2(\alpha_d\zeta)}}. \tag{14}$$

Here N_d is a real constant supposed to be positive. Real parameters A_d and α_d are related to the amplitude and pulse width of the dark wave profiles, respectively.

By substituting the ansatz Eq. (14) into Eq. (9), we get the unknown parameters A_d, α_d, and energy ξ:

$$\begin{cases} A_d = \sqrt{\dfrac{2a_1N_d}{a_3}} \\ \alpha_d = \sqrt[2]{a_1} \\ \xi = \dfrac{a_1}{2a_3} \end{cases}, \tag{15}$$

subject to the parametric conditions

$$2(a_2a_3 + 2a_1a_4) - a_3 = 0, a_1 > 0, a_3 > 0, a_3{}^2 + 4a_1a_5 > 0. \tag{16}$$

The exact dark solitary-wave solutions on a CW background of Eq. (1) are of the form:

$$E_d(z,t) = \left\{ \frac{2a_1N_d}{a_3} \frac{\sinh^2\left[\sqrt[2]{a_1}(t + \beta z)\right]}{1 + N_d \sinh^2\left[\sqrt[2]{a_1}(t + 2\alpha_1\omega z)\right]} \right\}^{\frac{1}{4}} \times e^{i(kz-\omega t)}. \tag{17}$$

3.3 Kink solitary-wave solutions

The kink solitary solutions of Eq. (9) are in the following form:

$$u_k(\zeta) = A_k\sqrt{1 + tanh\,(\alpha_k\zeta)}, \tag{18}$$

where A_k and α_k are real parameters related to the amplitude and pulse width of the kink wave profiles, respectively.

Substituting Eq. (18) into Eq. (9), we get

$$\begin{cases} A_k = \sqrt{-\dfrac{a_3}{4a_5}} \\ \alpha_k = \sqrt{-\dfrac{{a_3}^2}{4a_5}}, \\ \xi = -\dfrac{{a_2}^2}{4a_4} \end{cases} \tag{19}$$

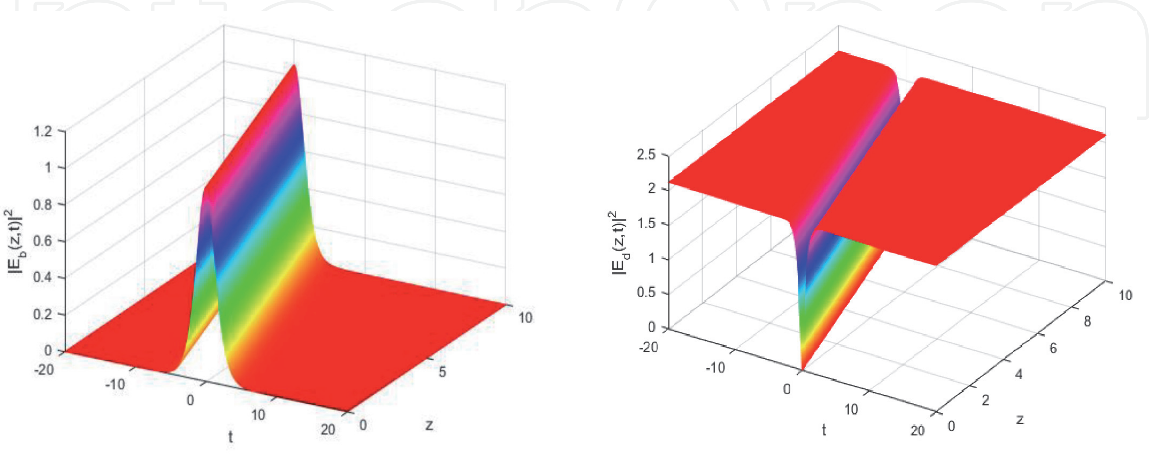
under the parametric conditions

$$2a_2a_5 - a_3a_4 = 0, {a_3}^2 + 4a_1a_5 = 0, a_3 > 0, a_5 < 0. \tag{20}$$

Thus, the exact bright solitary-wave solutions on a CW background of Eq. (1) are of the form:

$$E_k(z,t) = \left\{-\frac{a_3}{4a_5}\left[1 + tanh\left(\sqrt{-\frac{{a_3}^2}{4a_5}}(t + 2\alpha_1\omega z)\right)\right]\right\}^{\frac{1}{4}} \times e^{i(kz-\omega t)}. \tag{21}$$

The previous three exact solitary-wave solutions (13), (17), and (21) exist for the governing nonical-NLS model due to a balance among diffraction (or dispersion) and competing cubic-quintic-septic-nonical nonlinearities. For better insight, we plot in **Figure 1** the intensity profile on top of the related first two exact solution solitons named bright and dark, corresponding to the CQNLS models



(a) *(b)*

Figure 1.
Intensity $|E_j(z, t)|^2$ distribution of the (a) bright and (b) dark solitons given by Eqs. (14) and (17), respectively, with the parameter values corresponding to CQNLS models as $\alpha_1 = 0.5$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = 0$, $\alpha_5 = 0$, $k = 1$, and $\omega = 1$.

(with $\alpha_4 = 0, \alpha_5 = 0$) that is available in the current literature. As we can see from Eq. (21), the kink solitons exist only if $a_5 \neq 0$, consequently $\alpha_5 \neq 0$; thus, we cannot plot the corresponding CQNLS kink solution.

4. Modulational instability of the CW background

One of the essential aspects of solitary waves is their stability on propagation, in particular their ability to propagate in a perturbed environment over an appreciable distance [41]. Unlike the conventional pulses of different forms, the solitons are relatively stable, even in an environment subjected to external perturbations.

The previous three exact solitary-wave solutions given by the expressions (13), (17), and (21) are sitting on a CW background, which may be subject to MI. If this phenomenon occurs, then the CW background will be quickly destroyed, which will inevitably cause the destruction of the soliton. It is therefore of paramount importance to verify whether the condition of the existence of the soliton can be compatible with the condition of the stability of the CW background. Since MI properties can be used to understand the different excitation patterns on a CW in nonlinear systems, in this section, we perform the standard linear stability analysis [9, 34] on a generic CW:

$$E_0(z,t) = \sqrt{P_0}e^{i\phi_{nl}}, \tag{22}$$

in the system modeled by Eq. (1), where $\phi_{nl} = P_0\left(\alpha_2 + \alpha_3 P_0 + \alpha_4 {P_0}^2 + \alpha_5 {P_0}^3\right)z$ is the nonlinear phase shift induced by self-phase modulation and non-Kerr quintic-septic-nonical nonlinear terms, P_0 being the initial power inside a medium exhibiting optical nonlinearities up to the ninth order. A perturbed nonlinear background plane-wave field for the CQSNNLSE (Eq. (1)) can be written as

$$E(z,t) = \left[\sqrt{P_0} + a(z,t)\right]e^{i\phi_{nl}}, \tag{23}$$

where $a(z,t)$ is a small perturbation field which is given by collecting the Fourier modes as

$$a(z,t) = a_+ e^{i(Kz-\Omega t)} + a_- e^{-i(Kz-\Omega t)}, \tag{24}$$

a_+ and a_- are much less than the background amplitude P_0, and Ω represents the perturbed frequency. Here, the complex field $|a(z,t)| \ll P_0$. Thus, if the perturbed field grows exponentially, the steady state (CW) becomes unstable. Inserting the expression of a perturbed nonlinear background Eq. (23) into Eq. (1), with respect to Eq. (24), we obtain after linearization the following dispersion relation:

$$K = |\alpha_1||\Omega|\sqrt{\Omega^2 - sgn\,(\alpha_1){\Omega_c}^2}, \tag{25}$$

where $\Omega_c = \frac{1}{\sqrt{\alpha_1}}\sqrt{2\alpha_2 P_0 + 4\alpha_3 {P_0}^2 + 6\alpha_4 {P_0}^3 + 8\alpha_5 {P_0}^4}$ and $sgn\,(\alpha_1) = \pm 1$ depending on the sign of α_1 [$sgn\,(\alpha_1) = +1$, for $\alpha_1 > 0$, and $sgn\,(\alpha_1) = -1$, for $\alpha_1 < 0$]. The dispersion relation (25) shows that the steady-state stability depends critically on whether the light experiences normal or anomalous GVD inside the fiber. In the case of normal GVD ($\alpha_1 < 0$), the wave number K is real for all Ω, and the steady state is stable against small perturbations. By contrast, in the case of anomalous

GVD ($\alpha_1 > 0$), K becomes imaginary for $\Omega < \Omega_c$, and the perturbation $a(z, t)$ grows exponentially with z. As a result, the CW solution E_0 is inherently unstable for $\alpha_1 > 0$. This instability is referred to as modulation instability because it leads to a spontaneous temporal modulation of the CW beam and transforms it into a pulse train. Similar instabilities occur in many other nonlinear systems and are often called self-pulsing instabilities [9, 42–45]. Then, by setting $sgn\,(\alpha_1) = 1$, one can obtain the MI gain $G = 2\,Im(K)$, where the factor 2 convert G to power gain. The gain exists only if for $|\Omega| < \Omega_c$ and is given by

$$G(\Omega) = 2|\alpha_1\Omega|\sqrt{\Omega_c{}^2 - \Omega^2}. \tag{26}$$

The gain attains its peak values when the modulated frequency reaches its optimum value, i.e., its optimum modulation frequency (OMF). The OMF corresponding to the gain spectrum (26) is given by

$$\Omega_{op} = \pm\frac{\Omega_c}{\sqrt{2}}, \tag{27}$$

and the peak value given by

$$G_{op} = G(\Omega_{op}) = |\alpha_1\Omega_c{}^2|. \tag{28}$$

In **Figure 2**, we have shown the variation of OMF, computed from Eq. (27) as a function of the GVD parameter (α_1). The parameter values we have used are given as [34]

$$\begin{aligned} &P_0 = 15W, \alpha_2 = 2736W^{-1}/km, \alpha_3 = 2.63W^{-2}/km, \\ &\alpha_4 = -9.12\times10^{-4}W^{-3}/km, \alpha_5 = 0.5W^{-4}/km. \end{aligned} \tag{29}$$

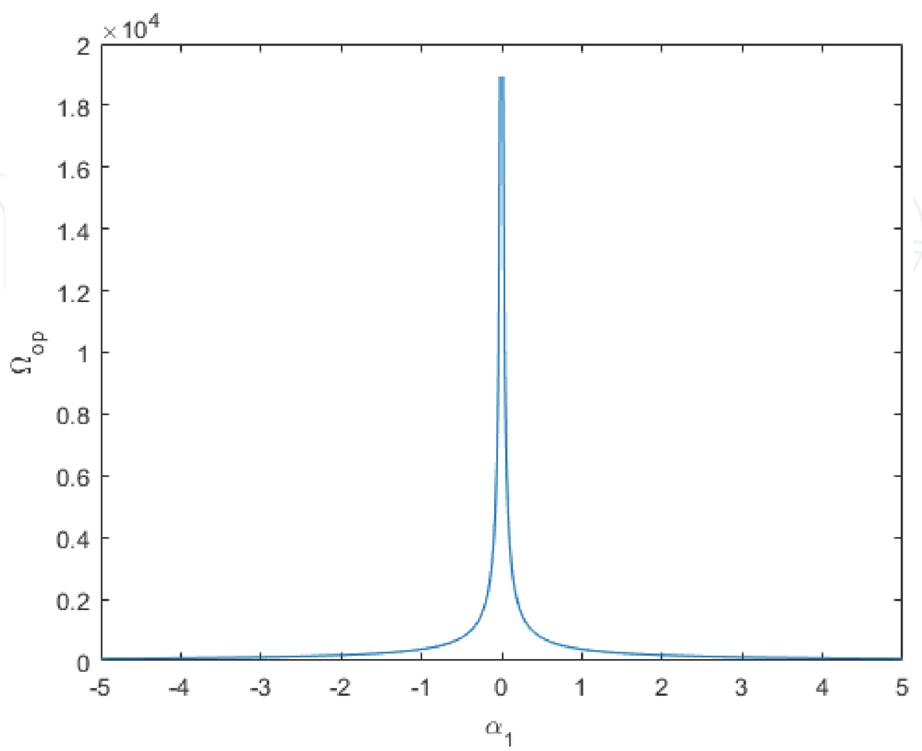


Figure 2.
Variation of optimum modulation frequency Ω_{op} as a function of second-order dispersion α_1.

We can observe that the OMF increases (respectively decreases) with the increasing $\alpha_1 < 0$ (respectively with the increasing $\alpha_1 > 0$).

Figure 3 shows the variation of MI gain as a function of the nonic nonlinearity α_5. The MI gain increases with the decreasing nonic nonlinearity. In **Figure 4**, as the input power increases, the maximum gain also increases.

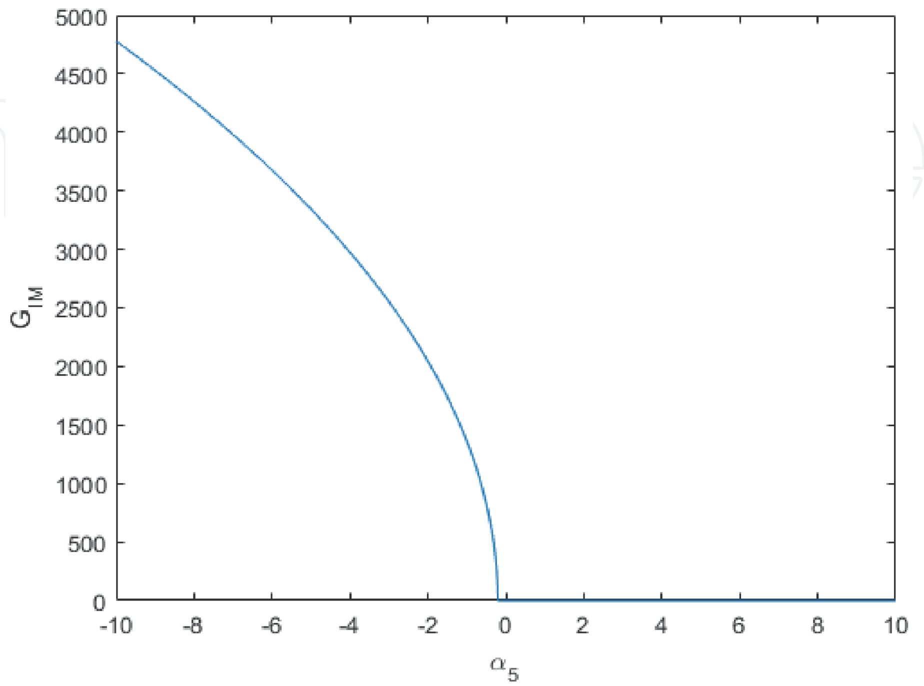


Figure 3.
Variation of the MI gain G as a function of the nonic nonlinearity α_5, with the same parameter values as in ***Figure 2****.*

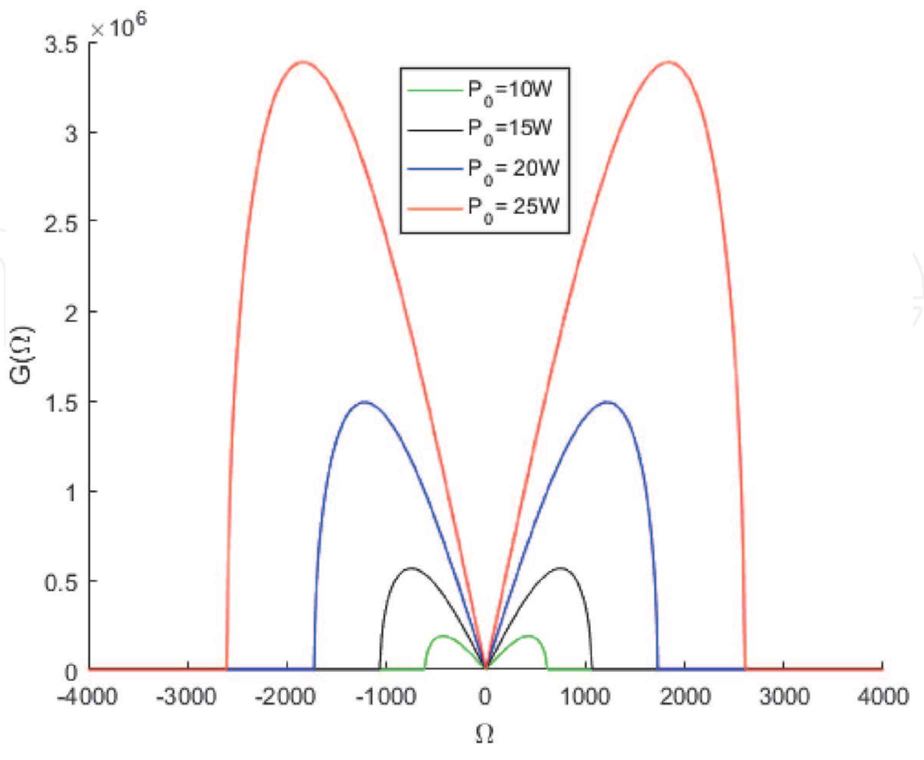


Figure 4.
Variation of the MI gain G (km^{-1}) as a function of frequency Ω (Hz), at a four-power level P_0 for an optical fiber. The other parameters are $\alpha_5 = 0.5ps^2/km, \alpha_2 = 2736W^{-1}/km, \alpha_3 = 2.63W^{-2}/km$, $\alpha_4 = -9.12 \times 10^{-4}W^{-3}/km, \alpha_5 = 0.5W^{-4}/km$.

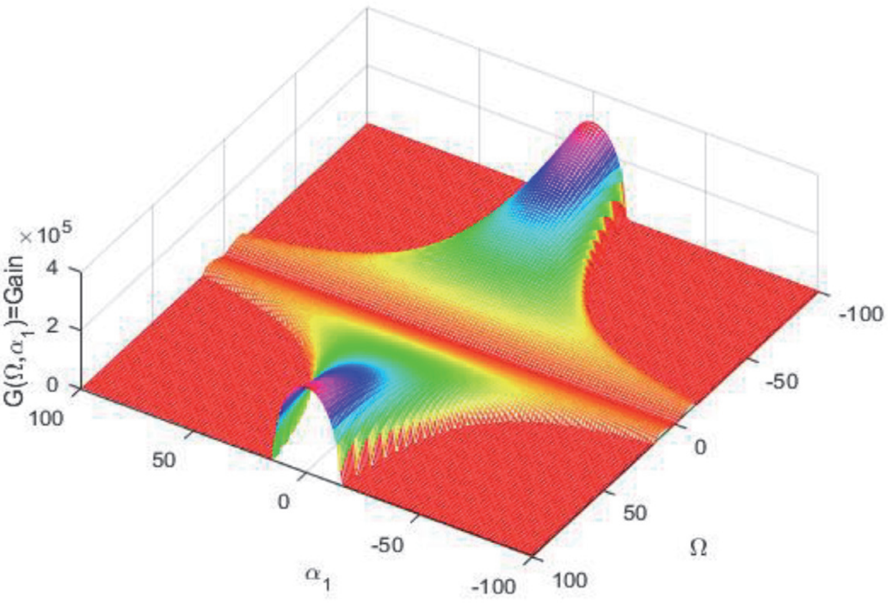


Figure 5.
Variation of the MI gain $G(km^{-1})$ *as a function of frequency* Ω *and the GVD* α_1*. The other parameters are* $P_0 = 15W, \alpha_5 = -0.5ps^2/km, \alpha_2 = 2736W^{-1}/km, \alpha_3 = 2.63W^{-2}/km, \alpha_4 = -9.12 \times 10^{-4}W^{-3}/km,$ $\alpha_5 = 5W^{-4}/km.$

The MI gain spectrum in **Figure 5** is a constitutive of two symmetrical sidebands which stand symmetrically along the line $\Omega = 0$. The maximum gain is nil at the zero perturbation frequency $\Omega = 0$; thus, there is no instability at the zero perturbation frequency.

5. Conclusion

In this chapter, we have investigated the higher-order nonlinear Schrödinger equation involving nonlinearity up to the ninth order. We have constructed exact solutions of this equation by means of a special ansatz. We showed the existence of a family of solitonic solutions: bright, dark, and kink solitons. The conditions on the physical parameters for the existence of this propagating envelope have also been reported. These conditions show a subtle balance among the diffraction or dispersion, Kerr nonlinearity, and quintic-septic-nonical non-Kerr nonlinearities, which has a profound implication to control the wave dynamics. Moreover, by employing Stuart and DiPrima's stability analysis method, an analytical expression for the MI gain has been obtained. The outcomes of the instability development depend on the nonlinearity and dispersion (or diffraction) parameters. Results may find straightforward applications in nonlinear optics, particularly in fiber-optical communication.

Conflict of interest

The authors declare no conflict of interest.

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