Chapter 3

Inverse Scattering Source Problems

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Abstract

The purpose of this chapter is to discuss some of the highlights of the mathematical theory of direct and inverse scattering and inverse source scattering problem for acoustic, elastic and electromagnetic waves. We also briefly explain the uniqueness of the external source for acoustic, elastic and electromagnetic waves equation. However, we must first issue a caveat to the reader. We will also present the recent results for inverse source problems. The resents results including a logarithmic estimate consists of two parts: the Lipschitz part data discrepancy and the high frequency tail of the source function. In general, it is known that due to the existence of non-radiation source, there is no uniqueness for the inverse source problems at a fixed frequency.

Keywords: scattering theory, inverse scattering theory, Helmholtz equation, Bessel functions

1. Introduction

This chapter tries to provide some results and materials on inverse scattering, direct scattering theory and inverse source scattering problems. There have been many scientists who have contributed to the different components of this field, such as linearity or non-linearity of the inverse source problem, computational and numerical solution to the inverse source problem and analytical aspects of the problem, which have their own interests. We obviously cannot give a complete account of inverse scattering here from all angles. Hence, instead of attempting the impossible, we have chosen to present inverse scattering theory from the of our own interests and research program. Particularly, we will focus on inverse source problems for acoustic, elastics and electromagnetic waves. In other words, certain areas of inverse scattering theory are either ignored.

Scattering theory has played a central role in twentieth century mathematical physics and applied mathematics. Indeed, from Rayleigh's explanation of why the sky is blue, to Rutherford's discovery of the atomic nucleus, through the modern medical and clinical applications of computerized tomography, scattering phenomena have attracted scientists and mathematicians for over a hundred years. Broadly speaking, scattering theory is concerned with the effect an inhomogeneous medium has on an incident particle or wave. In particular, if the total field is viewed as the sum of an incident field u^i and a scattered field u^s then the direct scattering problem is to determine u^s form a knowledge of u^i and the differential equation governing the wave motion. There are even more in the inverse scattering problem of determining the nature of the inhomogeneity from a knowledge of the asymptotic behavior of u^s, i.e., to reconstruct the differential equation or its domain or source functions of definition from the behavior of solutions of the direct problems. In this chapter, we are following this notation; C denote generic constants depending on the domain Ω or domain D, which is different in different results, and $\|u\|_{(l)}(\Omega)$ denotes the standard norm in Sobolev space $H^l(\Omega)$.

2. The direct and inverse scattering problem

The stationary incoming wave u of frequency k is a solution to the perturbed Helmholtz equation (scattering by medium)

$$Au - k^2 u = 0\, in\, \mathbb{R}^3 \tag{1}$$

(A is the elliptic operator $A = -\nabla(a\nabla) + b.\nabla + c$ with $\mathfrak{Re}b = 0, \nabla b = 0$, and $\mathfrak{Im}c \leq 0$, which coincides with the Laplace operator outside a ball B and which possesses the uniqueness of continuation property) or to the Helmholtz equation (scattering by an obstacle D for acoustic waves)

$$\Delta u + k^2 u = 0\, in\, \mathbb{R}^3 \backslash \overline{D}, \tag{2}$$

with the Dirichlet boundary data

$$u = 0\, on\, \partial D \quad (soft\ obsacle\, D). \tag{3}$$

or the Neumann boundary data

$$\partial_\nu u = 0\, on\, \partial D \quad (hardo\ bsacle\, D). \tag{4}$$

The function u is assumed to be the sum of the so-called incident plane wave $u^i(x) = \exp(ik\xi \cdot x)$ and a scattered wave u^s satisfying the Sommerfeld radiation condition

$$\lim_{r \to \infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{5}$$

where $\xi \in \mathbb{R}^3, |\xi| = 1$, is the so-called incident direction and

$$u(x) = \exp(ik\xi \cdot x) + u^s(x). \tag{6}$$

The electromagnetic scattering problem corresponding to the electric field E and magnetic field H such as

$$curlE - ikH = 0, \quad curlH + ikE = 0\, in\, \mathbb{R}^3 \backslash \overline{D}, \tag{7}$$

$$E(x) = \frac{i}{k} curl\, curl \exp(ik\xi \cdot x) + E^s(x), \tag{8}$$

$$H(x) = curl \exp(ik\xi \cdot x) + H^s(x), \tag{9}$$

$$\nu \times E = 0 \quad on \quad \partial D, \tag{10}$$

with the Silver- Muller radiation condition;

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0 \tag{11}$$

where (7) are the time-harmonic Maxwell equations and ν is again the unit outward normal to ∂D. As in previous case more general boundary condition can also be considered, for example the impedance boundary condition

$$\nu \times curlE - i\lambda(\nu \times E) \times \nu = 0, \tag{12}$$

where λ is a positive constant. The mathematical technique used to investigate the direct scattering problems for and electromagnetic waves depends heavily on the frequency of the wave motion. The first question about direct scattering is about uniqueness of a solution. The basic tools used to prove the uniqueness are Green's theorems and the unique continuation property of solutions to elliptic equations. Since Eqs. (2)–(5) for exterior problem have constant coefficients, the uniqueness question is much easier to handle. Similar argument can be applied on the Maxwell equations. The first result being given by Sommerfeld in 1912 for the case of acoustic case [1]. His work was generalized by Rellich [2] and Vekua [3], all under the assumption $\Im\mathfrak{m}k \geq 0$. The uniqueness of a solution to the exterior scattering problems for acoustic and electromagnetic is more difficult since use must now be made of the unique continuation principle for elliptic equations with non-analytic coefficients. After uniqueness, the most important questions would be the existence and numerical approximation of the solution. The most common technique to existence has been through the method of integral equation. For example for Eqs. (2)–(5), it is easy to see that for all positive values of wave number k the field u is the unique solution of the Lippman-Schwinger equation

$$u(x) = u^i + u^s \tag{13}$$

$$= exp\,(ik\xi \cdot x) + \frac{1}{r}\,exp\,(ikr)\mathcal{A}\left(\frac{x}{r}, \xi, k\right) + O\left(\frac{1}{r^2}\right) \tag{14}$$

where $r = |x|$ and the function $\mathcal{A}$ is called the scattering amplitude (or the scattering pattern or far field pattern).

The representation (14) follows from the fact that any solution u^s to the Helmholtz equation satisfying the radiation condition (5) has the representation by a single layer potential

$$u^i(x) = \int_{\partial B} g(y)K(x-y;k)d\Gamma(y), \tag{15}$$

where $K(x;k) = e^{ik|x|}/(4\pi|x|$ and B is some large ball (i.e., see [4]).

As showed above, the direct scattering problem has been thoroughly investigated and a considerable amount of information is available concerning its solution. In contrast, the inverse scattering problem has only recently progressed. It is worth to mention that the inverse problem is inherently nonlinear. In areas such as radar, sonar, geophysical exploration, medical imaging and nondestructive testing. As in with direct problem, the first question in inverse scattering problem is, how about uniqueness?. The first result in uniqueness brought up to the attention by Schiffer [5] who showed for the problem (2)–(5) the far field pattern $\mathcal{A}\left(\frac{x}{r}, \xi, k\right)$ with fixed wave number k uniquely determines the scattering obstacle D. And result for corresponding exterior problem obtained by Nachman [6], Novikov [7, 8]. Uniqueness theorems for electromagnetic problems were obtained by Colton and Päivärinta [9]. The next step will be the question of existence of the to the inverse scattering problem. The mathematically speaking, the solution of the inverse scattering problem does not exist, but we can speak about stabilization and approximation of the solutions. The earliest efforts in this direction attempted to linearize the problem by reducing it to the problem of solving a linear integral equation of the first kind. The initial attempts to treat the inverse scattering problem without linearizing were investigated by Imbriale and Mittra [10]. Their techniques were based on analytic continuation. In 1980's a number of methods were given to solving the inverse scattering problem which explicitly acknowledged the nonlinear and ill-posed nature of the problem. The two-dimensional case can be used as an approximation for the scattering from finitely long cylinders. In the next sections, we will discuss Helmholtz equation and two and three dimensional inverse source scattering problems for acoustic, elastic and electromagnetic waves. The following lemma is establishing the uniqueness for the direct solution(1) in $\mathbf{R}^3$. The following lemma is stated in [4].

Theorem 1.1. If u solves Eq. (1) in $\mathbb{R}^3$ and satisfies the radiation condition (5), then $u = 0$.

Proof. There is a weak solution to Eq. (1) in B with the test function $\phi = u$ we have

$$\int_{\partial B} \partial_\nu u\overline{u} = \int_B \left(a\nabla u \cdot \nabla\overline{u} + b \cdot \nabla u\overline{u} + \left(c - k^2\right)u\overline{u}\right) \tag{16}$$

$$= \int_B \left(a\nabla u \cdot \nabla\overline{u} + \overline{b \cdot \nabla\overline{u}u} + \left(c - k^2\right)u\overline{u}\right) \tag{17}$$

using the condition $\mathfrak{Re} b = 0, \nabla b = 0$ and integration by part over ∂B the internal is a sum of two part; one involving ∇u and another $cu\overline{u}$. The first term coincides with its complex conjugate, so its imaginary part is zero, and the second one has a non-positive imaginary part due to the condition on c, hence

$$\mathfrak{Im}\int_{\partial B} \partial_\nu u\overline{u} \leq 0. \tag{18}$$

since u satisfied the Helmholtz equation and the radiation condition, the known results imply that $u = 0$ outside B. By uniqueness of the continuation for the elliptic operator $A - k^2$ we obtain that $u = 0$ in $\mathbb{R}^3$, so the proof is complete.

2.1 Helmholtz equation

Studying an inverse problem always requires a solid knowledge of the theory for the corresponding direct problem. Therefore in this section is devoted to presenting the foundations of obstacle scattering problems for time harmonic acoustic waves. The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents a time-independent form of the wave equation, results from applying the technique of separation of variables to reduce the complexity of the analysis. Colton and Kress showed that [11] how one can derived the Helmholtz equation from the Euler's equation. Then the domain of the solution is outside a bounded open set $D \in \mathbb{R}^d$, describing the scatterer. The equation is

$$\Delta u + k^2 u = 0 \tag{19}$$

where the wave number k is given by the positive constant $k = \omega/c$, with inhomogeneous boundary conditions on D of Dirichlet or Neumann type:

$$u(x) = g(x) \quad (Dirichlet), \quad \frac{\partial u(x)}{\partial \nu} = h(x), x \in \partial D, \quad (Neumann) \tag{20}$$

and it is a well-posed problem if

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial r} - iku \right) = 0. \tag{21}$$

Let $G(x)$ be the Green's function for Helmholtz in d dimensions, e.g.

$$G(x-y) = \begin{cases} \frac{i}{4} H_0^1(k|x-y|), & d = 2, \quad x \neq y \\ \frac{e^{ik(x-y)}}{4\pi|x-y|}, & d = 3, \quad x \neq y \end{cases}$$

where $H_0^1(z) = \frac{1}{\pi i} \int_{1+i\infty}^{1} e^{izs} (s^2 - 1)^{-1/2} ds$, for $Re z > 0$, is the Hankel function of the first kind [12]. It is also can be defined as

$$H_0^{(1)}(z) = J_0(z) + iY_0(z), \tag{22}$$

where

$$\begin{aligned} J_0(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{2m}}{(m!)^2}, \\ Y_0(z) &= 2\left\{\gamma + \log\left(\frac{1}{2}z\right)\right\} J_0(z) - 2\sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{2m}}{(m!)^2} \left\{1 + \frac{1}{2} + \ldots + \frac{1}{m}\right\}, \end{aligned} \tag{23}$$

and $\gamma = 0.5772157 \ldots$ is the Euler's constant.
Then, we can solve the single layer potential integral equation

$$u(x,k) = \int_{\partial D} G(x-y) \psi(y) dy, \qquad x \in \overline{D}^c. \tag{24}$$

Alternatively, we can solve the double layer potential integral equation

$$-u^i(x,k) = \frac{1}{2} \psi(x) - \int_{\partial D} \frac{\partial G(x-y)}{\partial n} \psi(y) dy, \qquad x \in \partial D. \tag{25}$$

In this case the scattered solution outside D is given by

$$u(x) = -\int_{\partial D} \frac{\partial G(x-y)}{\partial n} \psi(y) dy, \qquad x \in \overline{D}^c. \tag{26}$$

2.2 Inverse source scattering problem

Motivated by the significant applications, the inverse source problems, as an important research subject in inverse scattering theory, have continuously attracted much attention by many researchers. Consequently, a great deal of mathematical and numerical results are available. In general, it is known that there is no uniqueness for the inverse source problem at a fixed frequency due to the existence of non-radiation sources. Hence, additional information is required for the source in order to obtain a unique solution, such as to seek the minimum energy solution. From the numerical and computational point of view, a more challenging issue is lack of stability. A small variation of the data might lead to a huge error in the reconstruction. Recently, it has been realized that the use of multi-frequency data is an effective approach to overcome the difficulties of non- uniqueness and instability which are encountered at a single frequency. An attempt was made in [13] to extend the stability results to the inverse random source of the one-dimensional stochastic Helmholtz equation. The inverse source problem seeks for the right hand side of a partial differential equation from boundary data. The inverse source problems are also considered as a basic mathematical tool for solving many imaging problems including reflection tomography, diffusion-based optical tomography, lidar imaging for chemical and biological threat detection, and fluorescence microscopy. In general, a feature of inverse problems for elliptic equations is a logarithmic type stability estimate which results in a robust recovery of only few parameters describing the source and hence yields very low resolution numerically.

For the Helmholtz equations, the results have shown increasing (getting nearly Lipschitz) stability when the Dirichlet data or Cauchy data are given on the whole boundary and K is getting large. Similar results are obtained for the time periodic solutions of the more complicated dynamical elasticity system. For elastic waves, the inverse source problem is to determine the external force that produces the measured displacement. The inverse source scattering problem for Maxwell equation arises in many scientific areas such as medical imaging. More specifically, Magnetoencephalography (MEG), the imaging modality is a non-invasive neurophysiological technique that measures the electric or magnetic fields generated by neuronal activity of the brain. For electromagnetic waves, the inverse source problem is to reconstruct the electric current density from tangential trace of electric field. As we know in [14], the inverse source problem does not have a unique solution at a single or at finitely many wave numbers. On the other hand, if we use all wave numbers in $(0, K)$ one can regain uniqueness. Another purpose of this chapter is to establish uniqueness for the source from the Cauchy data on any open non empty part of the boundary for arbitrary positive K. For uniqueness, we will show two different techniques. The first technique is to use the stability estimate for the source functions and the second technique is a direct proof.

First increasing stability results were obtained in [15] by using the spatial Fourier transform. In [16, 17] more general and sharp results were obtained in sub-domain of $\mathbb{R}^3$ *and* $\mathbb{R}^2$ in an arbitrary domains with C^2 boundary by the temporal Fourier transform, with a possibility of handling spatially variable coefficients. The recent results showed that the estimate for source functions is a logarithmic type. The right hand-side of the estimate consists of two parts: data discrepancy and the high frequency tail. In the papers [15, 18], Li, Bao and others showed the similar results for disc and ball. For instance, the results by Entekhabi and Isakov are as follows;

Let the radiated wave field $u(x, k)$ solve the scattering problem in $\mathbb{R}^2$ with the source term $-f_1 - ikf_0$ and the radiation condition

$$(\Delta + k^2)u = -f_1 - ikf_0 \text{ in } \mathbb{R}^2, \tag{27}$$

$$\lim r^{1/2}(\partial_r u - iku) = 0 \text{ as } r = |x| \to +\infty. \tag{28}$$

Both $f_0, f_1 \in L^2(\Omega)$ are assumed having $suppf_0, suppf_1 \subset \Omega$ where Ω is a bounded domain with the boundary $\partial\Omega \in C^2$.

The stability of functions f_0, f_1 from the data

$$u = u_0, \partial_\nu u = u_1 \text{ on } \Gamma, \text{ when } K_* < k < K, \tag{29}$$

where Γ is an non empty open subset of $\partial\Omega$ with outer unit normal ν and $0 < K_* < K$, was the following theorem;

Theorem 1.2. Let $\|f_0\|^2_{(4)}(\Omega) + \|f_1\|^2_{(3)}(\Omega) \leq M$, $1 \leq M$, and $\delta < |x - y|, x \in \partial\Omega$, $y \in suppf_0 \cup suppf_1$ for some positive δ.

Then there exist a constant $C = C(\Omega, \delta)$ such that

$$\|f_1\|^2_{(0)}(\Omega) + \|f_0\|^2_{(1)}(\Omega) \leq C\left(\varepsilon^2 + \frac{M^2}{1 + K^{\frac{2}{3}}E^{\frac{1}{4}}}\right) \tag{30}$$

for all $u \in H^2(\Omega)$ solving (27) and (28) with $1 < K$. Here

$$\epsilon^2 = \int_0^K \left(\omega^2 \|u(,\omega)\|^2_{(0)}(\partial\Omega) + \|\nabla u(,\omega)\|^2_{(0)}(\partial\Omega)\right) d\omega, \quad 0 < E = -\ln \epsilon. \tag{31}$$

While Bao, Li and Lu used Dirichlet to Neumann map to simplify the boundary conditions for two dimensional and three dimensional domains (disks and balls), Isakov, Lu, Chang and Entekhabi used the Fourier transform and observability bound for corresponding hyperbolic initial value boundary problem (wave equation) for two and three dimensional domain with C^2-boundary. In papers [19, 20], authors considered inverse source scattering problems with damping factor for two and three dimensional domains, that is, they considered the following equation:

$$\left(\Delta + k^2 + ikb\right)u = -f_1 - bf_0 + ikf_0 \tag{32}$$

where $b > 0$ is the damping factor. In particular attenuation can have various reasons and in application, one of the fundamental reasons of poor resolution in inverse problems is a spatial decay of the signal due in part to the damping factor. The results was the following theorem:

Theorem 1.3. There exists a generic constant C depending on the domain Ω such that

$$\|f_0\|^2_{(1)}(\Omega) + \|f_1\|^2_{(1)}(\Omega) \leq Ce^{Cb^2}\left(\varepsilon^2 + \frac{(b^2+1)M_3^2}{1 + K^{\frac{2}{3}}E^{\frac{1}{4}} + b}\right) \tag{33}$$

for all $u \in H^2(\Omega)$ solving (1), with $1 < K$ and $M_3 = \max\left\{\|f_0\|_{(4)}(\Omega) + \|f_1\|_{(3)}(\Omega), 1\right\}$. As you can see, the results showed a deterioration of stability with growing attenuation/damping constant b.

In papers [21, 22], authors considered inverse source scattering problems for double layers medium. The results in the papers [23, 24] showed an stability estimate for elastic and electromagnetic waves. Also authors in [23] proved a stability estimate using just Dirichlet data. Increasing stability for the Schrodinger potential from the complete set of the boundary data (the Dirichlet-to Neumann map) was demonstrated in [25, 26]. They showed that the boundary condition for elastic waves they considered the following equation

$$\sigma(\mathbf{u}) + k^2\mathbf{u} = -\mathbf{f}_1 - ik\mathbf{f}_0 \; in \; \mathbb{R}^n, \tag{34}$$

$$\mathbf{u} = \mathbf{u}_0 \text{ in } \Gamma, \tag{35}$$

where $\sigma = (\mu\Delta + (\mu + \lambda)\nabla \cdot \nabla)$, where μ, λ are Lame constants satisfying $\mu > 0$ and $\mu + \lambda > 0$, functions $\mathbf{f_1}, \mathbf{f_0} \in L^2(\Omega)$ are the external force are assumed to be compactly supported in a C^2-boundary domain $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \partial\Omega$ is an open non-void set. By the Helmholtz decomposition, the displacement filed $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \text{ in } \mathbb{R}^n \backslash \overline{\Omega}, \tag{36}$$

where $\mathbf{u}_p$ and the sheer part $\mathbf{u}_s$ which satisfy Sommerfeld radiation conditions

$$\lim_{r \to \infty} r\left(\partial_r \mathbf{u}_p - \mathrm{i}k_p \mathbf{u}_p\right) = 0, \quad \lim_{r \to \infty} r(\partial_r \mathbf{u}_s - \mathrm{i}k_s \mathbf{u}_s) = 0, \quad r = |\mathbf{x}|, \tag{37}$$

To achieve the result, authors used Helmholtz decomposition. The decomposition was allowed them to break the Navier-Lame equation to two elliptic equations.

The results for discrete data for inverse source problem which was obtained in [27] are as follows:

Theorem 1.4. Let $\mathbf{u}$ be the solution of the following scattering problem corresponding source $\mathbf{f} \in F_M(\mathbb{B}_{\mathbb{R}})$,

$$(\mu\Delta + (\mu + \lambda)\nabla \cdot \nabla)(\mathbf{u}) + \omega^2 \mathbf{u} = \mathbf{f} \; in \; \mathbb{R}^n, \tag{38}$$

with radiation condition (37),

Then

$$\|\mathbf{f}\|^2_{(0)}(B_R) \le C \left(\epsilon_2^2 + \frac{M^2}{\left(\frac{N^{\frac{5}{8}} |\, ln\, \epsilon_3|^{\frac{1}{9}}}{(6m-3n+3)^3}\right)^{2m-n+1}} \right) \tag{39}$$

where

$$\epsilon_2 = \left(\sum_{n=1}^{N} \|\mathbf{u}(., \omega_{p,n})\|_0^2(\Gamma_R) + \|\mathbf{u}(., \omega_{s,n})\|_0^2(\Gamma_R) \right)^{\frac{1}{2}},$$

$$\epsilon_3 = \sup_{\omega \in \left(0, \frac{\pi}{c_p R},\right]\right)} \|\mathbf{u}(., \omega)\|_0^2(\Gamma_R),$$

and

$$F_M(\mathbb{B}_{\mathbb{R}}) = \left\{ \mathbf{f} \in H^{m+1}(B_R) : \|\mathbf{f}\|_{(m+1)}(B_R) \le M \right\}.$$

The stability increases as N increases, i.e., the inverse problem is more stable when higher frequency data is used.

2.2.1 Uniqueness of source function

To achieve the uniqueness, we introduced two different approaches. The first approach is using the estimate for the source function. Letting the norm of the boundary data goes to zero, then the proof is complete. For instance, consider Theorem 1.2 and let $\epsilon \to 0$. The second approach is the result has proved by Isakov, Chang and Lu. They used classical result of the hyperbolic initial value boundary problem indirectly. The following theorem is the result of [16]. In the following theorem $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ and $\Gamma \subset \partial\Omega$.

Theorem 1.5. Let u be a solution to the scattering problem (27) and (28) with $f_0 \in H^1(\Omega), f_1 \in L^2(\Omega)$. If the Cauchy data $u_0 = u_1 = 0$ on Γ when $k \in (K_*, K)$, then $f_0 = f_1 = 0$ in Ω.

Proof. Denote by U_0 the solution to the following hyperbolic problem

$$\begin{aligned} &\partial_t^2 U_0 - \Delta U_0 = 0 \quad \text{on} \quad \Omega \times (0, \infty), \\ &U_0 = -f_0, \partial_t U_0 = f_1 \quad \text{on} \quad \Omega \times \{0\}, U_0 = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty). \end{aligned} \tag{40}$$

Under their assumptions, there is a unique solution to the problem (40) with $\|U_0(,t)\|_{(1)}(\Omega) + \|\partial_t U_0(,t)\|_{(0)}(\Omega) \leq C\Big(\|f_0\|_{(1)}(\Omega) + \|f_1\|_{(0)}(\Omega)\Big)$.

Now let

$$u^*(x,k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty U_0(x,t) e^{ikt} dt$$

Due to the properties of U_0, in particular to the conservation of the energy, the function $u^*(x,k)$ is well defined and analytic with respect to $k = k_1 + ik_2, k_2 > 0$. Applying the integration by parts and using standard properties of the Fourier-Laplace transform we conclude that

$$\left(\Delta + k^2\right) u^* = -f_1 - ikf_0 \quad \text{in} \quad \Omega, \quad u^* = 0 \quad \text{on} \quad \partial\Omega. \tag{41}$$

Due to the assumption, the function u solves the same Dirichlet problem for $\Delta + k^2$ when $0 < k_1, 0 < k_2$. Indeed, u solves the homogeneous Helmholtz equation in $\mathbb{R}^n \backslash \overline{\Omega}$ and has zero Cauchy data on Γ. By the uniqueness in the Cauchy problem for elliptic equations, $u = 0$ on $\mathbb{R}^n \backslash \overline{\Omega}$ and hence on $\partial\Omega$ provided $K_* < k < K$. As follows from the integral representation of solution (27), the function $u(;k)$ is (complex) analytic when $0 < \mathfrak{Re} k$, hence $u(;k) = 0$ on $\partial\Omega$ provided $0 < \mathfrak{Re} k$. Since $k_2 > 0$, the solution of (41) is unique, hence $u = u^*$ on Ω (see Section 4). Consequently, we obtain $u^* = u = 0$, $\partial_\nu u^* = \partial_\nu u = 0$ on Γ. Since u^* is an analytic function, we can conclude that $u^* = 0, \partial_\nu u^* = 0$ on Γ for all $k = k_1 + ik_2$ with $k_2 > 0$. Due to the uniqueness of the inversion of the Fourier-Laplace transform we will obtain

$$\partial_\nu U_0 = 0 \quad \text{on} \quad \Gamma \times (0, \infty).$$

Due to the uniqueness in the lateral Cauchy problem for the wave equation (40) with the Cauchy data on $\Gamma \times (0, +\infty)$ [Holmgren-John theorem ([28], Section 3.4)], we can conclude that $U_0 = 0$ on $\Omega \times (T, +\infty)$ for some positive T. Hence from the uniqueness in the backward initial boundary value problem for the hyperbolic equation (39) in $\Omega \times (0, T)$ with zero boundary data on $\partial\Omega \times (0, T)$ and initial data at $\Omega \times \{T\}$ we conclude that $U_0 = 0$ on $\Omega \times (0, T)$. So $-U_0(,0) = f_0 = 0, \partial_t U(,0) = f_1 = 0$ on Ω which finishes the proof of uniqueness.

3. Conclusions

In this section, the scattering and inverse scattering theory, inverse source scattering problem were considered briefly. The recent results such stability estimates for external source and electric current density from boundary measurements of radiated wave field and uniqueness for source function for Helmholtz equation, Elasticity and Maxwell system have showed. We also show some result for discrete data. In addition, we also showed some results of using just Dirichlet data for improving stability which was a big improvement. There are still many challenges remain in this field. For instance, studying the stability in the inverse source problems for inhomogeneous media where the analytical Green tensors are not available and the present method may not be directly applicable. Another interesting topic in stability of the external source is to consider the governing equation in the time domain. The non-linear case is also is a very challenging problem. The direct and inverse scattering problems when both the source and the linear load are random is also an open problem. Another challenging problem is to study the random source scattering problem for three dimensional elastic wave equation. As I mentioned before, there are many scientist and researcher have been working on inverse scattering and more specifically on inverse source problems. To expand your knowledge and further mathematical development in this field of research, please see the result authors in [29–41], which were discussed different aspects of the problems.

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References

[1] Sommerfeld A. Die Greensche Funktion der Schwingungsgleichung. Jahresbericht der Deutschen Mathematiker-Vereinigung. 1912;**21**: 309-353

[2] Rellich F. Uber des asymptotiche Verhalten der Losungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten. Deutsche Mathematiker-Vereinigung. 1943;**5**: 57-65

[3] Vekua IN. Metaharmonic functions. Trudy Belinskogo Matematicheskoe Instituta. 1943;**12**:105-174

[4] Isakov V. Inverse Problems for Partial Differential Equations. 2nd ed. New York: Springer International Publishing, Verlag; 2017. pp. 173-177. DOI: 10.1007/978-3-319-51658-5

[5] Lax PD, Phillips RS. Scattering Theory. New York: Academic Press; 1967. ISBN 10: 0124400507

[6] Nachman A. Reconstructions from boundary measurements. Annals of Mathematics. 1988;**128**:531-576

[7] Novikov R. Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. Functional Analysis and Its Applications. 1988;**22**: 263-272

[8] Ramm AG. Recovery of the potential from fixed energy scattering data. Inverse Problems. 1988;**4**:877-886

[9] Colton D, Päivärinta L. The uniqueness of a solution to an inverse scattering problem for electromagnetic waves. Archive for Rational Mechanics and Analysis. 1992;**119**:59-70

[10] Imbriale WA, Mittra R. The two-dimensional inverse scattering problem. IEEE Transactions on Antennas and Propagation. 1970;**18**: 633-642

[11] Colton D, Kress R. Inverse Acoustic and Electromagnetic Scattering Theory. New York: Springer Verlag; 2013. ISBN-10: 1461449413

[12] Watson GN. A Treatise on the Theory of Bessel Functions. USA: Cambridge University Press; 1922. ISBN: 9780521483919

[13] Li P, Bao G, Chen C. Inverse random source scattering for elastic waves. SIAM Journal on Numerical Analysis. 2017;**55**:2616-2643

[14] Eller M, Valdivia N. Acoustic source identification using multiple frequency information. Inverse Problems. 2009;**25**: 115005

[15] Bao G, Lin J, Triki F. A multi-frequency inverse source problem. Journal of Difference Equations. 2010; **249**:3443-3465

[16] Cheng J, Isakov V, Lu S. Increasing stability in the inverse source problem with many frequencies. Journal of Difference Equations. 2016;**260**: 4786-4804

[17] Entekhabi MN, Isakov V. On increasing stability in the two dimensional inverse source scattering problem with many frequencies. Inverse Problems. 2017;**34**:055005

[18] Bao G, Li P. Inverse medium scattering problems in near-field optics. Journal of Computational Mathematics. 2007;**25**(3):252-265

[19] Isakov V, Lu S. Increasing stability in the inverse source problem with attenuation and many frequencies. SIAM Journal on Applied Mathematics. 2018;**18**:1-18

[20] Entekhabi MN. Increasing stability in the two dimensional inverse source

scattering problem with attenuation and many frequencies. Inverse Problems. 2018;**34**:115001

[21] Entekhabi MN, Gunaratne A. A Logarithmic Estimate for Inverse Source Scattering Problem with Attenuation in a Two-Layered Medium, to be Appeared in Journal of Inverse and Ill-Posed Problems. 2019;. Available from: https://arxiv.org/pdf/1903.03475.pdf

[22] Zhao Y, Li P. Stability on the one-dimensional inverse source scattering problem in a two-layered medium. Applicable Analysis. 2017;**98**(4): 682-692. DOI: 10.1080/00036811.2017.1399365

[23] Entekhabi MN, Isakov V. Increasing stability in acoustic and elastic inverse source problems, SIAM Journal on Mathematical Analysis. 2018. Available from: https://arxiv.org/abs/1808.10528

[24] Li P, Helin T. Inverse Random Source Problems for Time-Harmonic Acoustic and Elastic Waves, Submitted. 2018

[25] Isakov V, Lai R-Y, Wang J-N. Increasing stability for the attenuation and conductivity coefficients. SIAM Journal on Mathematical Analysis. 2016; **48**:1-18

[26] Isakov V. Increasing stability for the Schrodinger potential from the Dirichlet-to Neumann map. Discrete and Continuous Dynamical Systems. 2011;**4**:631-641

[27] Bao G, Li P, Zhao Y. Stability for the inverse source problems in elastic and electromagnetic waves. Journal de Mathématiques Pures et Appliquées. 2018

[28] John F. Partial Differential Equations, Applied Mathematical Sciences. New York/Berlin: Springer-Verlag; 1982

[29] Isakov V, Lu S. Inverse source problems without (pseudo) convexity assumptions. Inverse Problems & Imaging. 2018;**12**:955-970

[30] Isakov V, Lu S, Xu B. Linearized inverse Schrödinger potential problem at a large wave number. SIAM Journal on Applied Mathematics. arXiv: 1812.05011

[31] Isakov V. On increasing stability in the continuation for elliptic equations of second order without (pseudo) convexity assumptions. Inverse Problems & Imaging. 2019;**13**:983-1006

[32] Isakov V, Wang J-N. Increasing stability for determining the potential in the Schrödinger equation with attenuation from the Dirichlet-to Neumann map. Inverse Problems & Imaging. 2014;**8**:1139-1150

[33] Isakov V, Lai R-Y, Wang J-N. Increasing stability for conductivity and attenuation coefficients. SIAM Journal on Mathematical Analysis. 2016;**48**: 569-594

[34] Nakamura G, Saitoh S, Seo JK. Inverse Problems and Related Topics. New York; 2000. ISBN-13: 978-15848819192000

[35] Ivanov VK, Vasin VV, Tanana VP. Theory of Linear Ill-Posed Problems and its Applications. 2002. ISBN: 978-3-11-094482-2

[36] Lavrentiev R, Vasiliev. Inverse problems for second-order elliptic equations. In: Multidimensional Inverse Problems for Differential Equations. Lecture Notes in Mathematics. Vol. 167. Berlin/Heidelberg: Springer; 1970

[37] Tikhonov AN. On the regularization of ill-posed problems. SSSS Doklady Akademii Nauk SSSR. 1963;**153**:49-52

[38] Bao G, Gao J, Li P. Analysis of direct and inverse cavity scattering problems.

Numerical Mathematics: Theory, Methods and Applications. 2011;**4**: 419-442

[39] Bao G, Liu J. Numerical solution of inverse scattering problems with multi-experimental limited aperture data. SIAM Journal on Scientific Computing. 2003;**25**(3):1102-1117

[40] Isakov V. Inverse Source Problems. Vol. 34. USA: American Mathematical Society; 1990. ISBN: 978-0-8218-1532-8

[41] Bukhgeim AL. Introduction to the Theory of Inverse Problems. 2000