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# Modeling Inflation Dynamics with Fractional Brownian Motions and Lévy Processes

*Bodo Herzog*

## Abstract

The article studies a novel approach of inflation modeling in economics. We utilize a stochastic differential equation (SDE) of the form  $dX_t = a(X, t)dt + b(X, t)dB_t^H$ , where  $dB_t^H$  is a fractional Brownian motion in order to model inflationary dynamics. Standard economic models do not capture the stochastic nature of inflation in the Eurozone. Thus, we develop a new stochastic approach and take into consideration fractional Brownian motions as well as Lévy processes. The benefits of those stochastic processes are the modeling of interdependence and jumps, which is equally confirmed by empirical inflation data. The article defines and introduces the rules for stochastic and fractional processes and elucidates the stochastic simulation output.

**Keywords:** inflation, dynamics, modeling, stochastic differential equation, fractional Brownian motion, Lévy process, jump-diffusion

## 1. Introduction

Modeling inflation dynamics is a tricky topic, particularly in the Eurozone. The determinants of inflation are multifaced, including interest rates, GDP growth, supply and demand of goods and services, exchange rates, etc. Moreover, inflation is somehow persistent over time, such as the low inflation rates in the recent years. In order to model the empirical pattern of inflation, we need a stochastic model with a mean-reversion property as well as time-dependent increments. Both features are mathematically difficult to design because all basic stochastic processes, such as a standard Brownian motion have time-independent increments and it is not mean-reverting.

We propose a novel approach by utilizing a fractional Brownian motion (fBm) and a Lévy process. Both stochastic concepts are relatively new in economic applications. Yet, recent discoveries about fBm's in mathematics already unravel striking insights to economics and finance, such as the modeling of inflation dynamics. We model inflation dynamics by a stochastic process,  $X_t$ . Before discussing the mathematical details, we provide a brief summary of the relationship across the different stochastic processes (**Figure 1**).

Each of the three stochastic processes have special properties. Interestingly, the overlap of the three stochastic processes gives a subset of new processes with highly interesting and uncommon properties. In this article, we study the subset of a



derivative to non-integer order, in particular  $d^{1/2}$ , is already in the correspondence of Leibniz with Bernoulli and L'Hospital. Laplace, Fourier, Abel and recently up to Riesz Feller and Mishura [4] contributed to the development of fractional calculus as it is of today.

The interest to fractional calculus has to do with its relationship to dynamics and stochastic processes in general. In the past decade, the field of fractional calculus is growing anew due to new discoveries in mathematics and theoretical physics. The first book on fraction calculus is by [5]. Considerable interest in fractional calculus has been stimulated by the many applications in different fields of sciences, such as physics, biology, engineering, economics and finance.

Now, let us compute a fractional derivative of a concrete example: What is the semi-derivative of  $\frac{d^{1/2}x^0}{dx^{1/2}} = \frac{d^{1/2}1}{dx^{1/2}}$ ? This example is a semi-derivative or half-derivative of a constant. From standard calculus, we know that the derivative of a constant is zero. Yet, the half-derivative is not zero as we will see soon. In general, you can compute fractional derivatives by the following formula:

$$D^m x^p = \frac{\Gamma(1+p)}{\Gamma(1+p-m)} x^{p-m}, \quad (2)$$

where  $\Gamma(x)$  is the Gamma function. Similarly, you can compute the fractional derivatives and fractional integrals by the Riemann-Liouville formula. For simplification, we do not introduce the Riemann-Liouville calculus here. The interested reader is referred to [5]. For  $m = \frac{1}{2}$  and  $p = 0$ , we obtain from Eq. (2)

$$D^{\frac{1}{2}}x^0 = \frac{\Gamma(1+0)}{\Gamma(1+0-\frac{1}{2})} x^{0-\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}. \quad (3)$$

The result is perhaps the most remarkable result in this brief discussion of fractional calculus. It cannot be embraced too much and deserves a special place in the hall of fame in fractional calculus. Note, the semi-derivative of a constant is surprisingly dependent on  $\pi$  and on the variable  $x$ . Indeed, this result is utilized repeatedly in fractional calculus in order to simplify solutions.

The chapter is organized as follows: Section 2 studies the modeling with fractional Brownian motions. We introduce the concept by defining a fractional Brownian motion in more detail. Section 3 defines a Lévy process and relates it to a Brownian motion. Finally, in Section 4, we start the simulation exercise. We study the stylized facts of inflation rates in the Eurozone from 1997 to 2020. Subsequently, we specify a stochastic differential equation with a fractional Brownian motion and a Lévy process and run several numerical simulations. Section 5 concludes the chapter.

## 2. Inflation modeling with fractional Brownian motion (fBm)

In this section, we define a “fractional Brownian Motion” (fBm). First of all, a fBm is not a (semi-)martingale. Thus, Ito's calculus does not apply anymore. Consequently, the lack of the martingale property has major implications in stochastic calculus. Indeed, one have to develop – similar to Ito's Lemma – completely new stochastic integration and differentiation rules for fractional Brownian motions.

We define an ordinary Brownian motion as a special case of a fractional Brownian motion. Indeed, Mandelbrot and van Ness [6] defined a fractional Brownian motion,  $B_t^H$ , as a Brownian motion together with a Hurst-Index,  $H \in (0, 1)$ ,

in the exponent. The parameter  $H$  is a moving average of the past increments  $dB_t^H$  weighted by the kernel  $(t-s)^{H-1/2}$ . Consequently, fractional Brownian motions have the feature that increments are interdependent. The latter property is known as self-similarity, which displace an invariance of the stochastic process with respect to changes of time scale. Almost all other stochastic processes, such as the ordinary Brownian motion or Lévy process have time-independent increments (at least almost surely). They create the famous class of Markov processes.

Empirically, however, there is evidence that economic and particularly financial time-series have a spectral density with a sharp peak. Additionally, we observe the phenomena of extremely long interdependence of certain trends over time in economics and finance. This presence of interdependence between past increments, directly speaks for the modeling with fractional Brownian motions. A standard Brownian motion is defined by the following properties:

1.  $B_t$  is almost surely continuous;  $B_{t=0} = 0$ ;
2. The increments  $B_t - B_s$  for  $t > s$  have mean zero and variance  $t - s$ ;
3. The increments  $B_t - B_s$  are independent over time and stationary.

Indeed, we know that the variance of the increment is of  $\text{Var}[B_t - B_s] = \mathbb{E}[(B_t - B_s)^2] = \mathbb{E}[dB_t^2]$ . Likewise, the standard deviation is:  $\sigma = \sqrt{\text{Var}[B_t - B_s]} = \sqrt{dB_t^2} \sim dt^{1/2}$ . This is often referred to as the  $t^{1/2}$ -law. Now, we are ready to define a fractional Brownian motion:

**Definition “Fractional Brownian Motion (fBm).”** Let the Hurst-Index,  $H$ , be  $0 < H < 1$ , then we call  $B_t^H$  a fractional Brownian Motion with parameter  $H$ , such as

$$B_{t=0}^H = B_0$$

and;

$$B_t^H - B_0^H = \frac{1}{\Gamma(H + \frac{1}{2})} \left[ \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB_s + \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \right].$$

Part two of the definition is the so-called Weyl fractional integral. Equivalently, you can use the more intuitive Riemann-Liouville fractional integral, defined by

$$B_t^H - B_0^H = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \quad (4)$$

where  $\Gamma(H + \frac{1}{2})$  is the Gamma function. The rules about fractional integration and fractional differentiation are discussed in detail in [5]. It trivially follows that for  $H = 1/2$ , we obtain the ordinary Brownian Motion,  $B_t$ . For other values of  $H$ , such as  $0 < H < 1/2$  and  $1/2 < H < 1$  the fractional Brownian Motion  $B_t^H$  is a fractional derivative or integral. Note, if  $0 < H < 1/2$  we say it has the property of counter persistent or short memory. This is associated with negative correlation. Vice versa for  $1/2 < H < 1$ , we say it is persistent. This is associated with positive correlation. Thus, modeling with fractional Brownian motions display the property of short- and long-term memory, a property very common in economic and financial time-series.

There exists an alternative definition of a fractional Brownian motion:

**Proposition.** Let the Hurst-Index,  $H$ , be  $0 < H < 1$ , and  $B_t^H$  be fractional Brownian motion. The covariance of a fractional Brownian motion is

$$\text{Cov}(B_t^H, B_s^H) = \frac{1}{2} [t^{2H} + s^{2H} - (t - s)^{2H}].$$

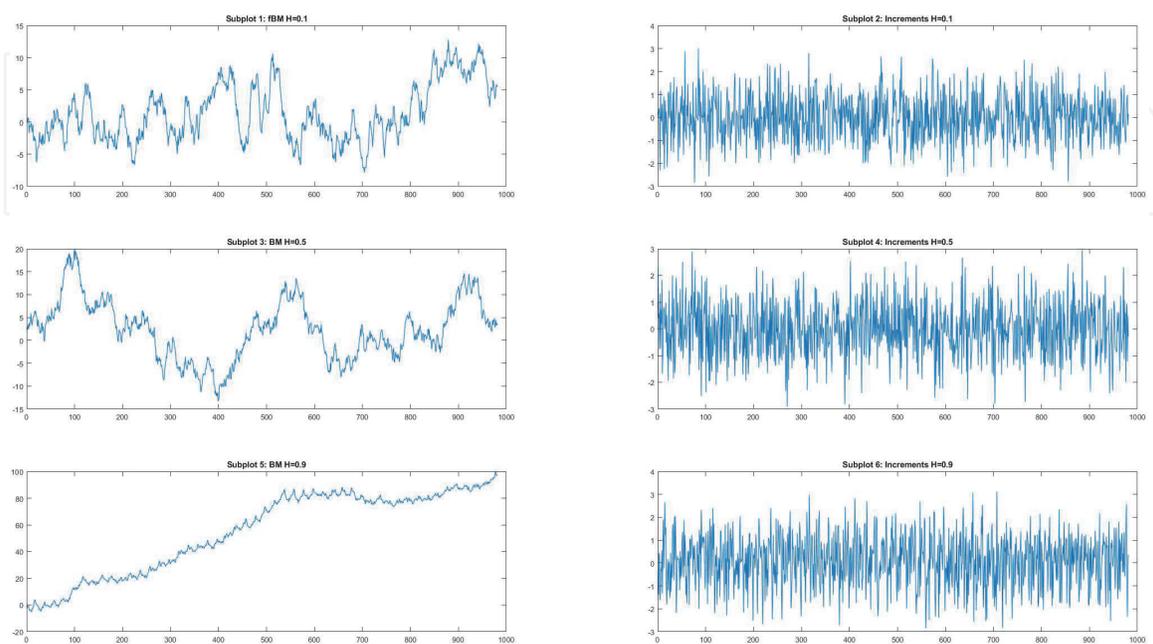
**Proof.** To prove that the covariance for a fractional Brownian motion is correct, we remind the reader that the variance of a fractional Brownian motion is defined as  $\text{Var}[B_t - B_s] = (t - s)^{2H}$ . Note, for  $H = 1/2$  the variance simplifies to the variance of ordinary Brownian motion. Thus, the covariance can be rewritten as

$$\begin{aligned} \text{Cov}(B_t, B_s) &= \mathbb{E}[B_t^H B_s^H] = \frac{1}{2} [\mathbb{E}[(B_t^H)^2] + \mathbb{E}[(B_s^H)^2] - \mathbb{E}[(B_t^H - B_s^H)^2]] \\ &= \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]. \end{aligned}$$

A trivial corollary is that if  $H = 1/2$ , we obtain for the covariance  $\text{Cov}(B_t, B_s) = \min[t, s]$ , the result of a standard Brownian motion. Similarly, by trivial computation, you can show that the increments of a fBm have mean zero and variance of  $|t - s|^{2H}$ . Finally, you can demonstrate that two non-overlapping increments of fractional Brownian motions have the property that they are not independent. In fact, they are interdependent!

In summary, a fBm has novel properties following empirical observations in economics, yet different to ordinary stochastic processes. Indeed, a fBm has stationary and interdependent increments. Additionally, a fBm is  $H$ -self similar, meaning that  $B_{at}^H = a^H B_t^H$ .

The rules of fractional integration and fractional differentiation are more sophisticated than the Ito-stochastic calculus. Details about those rules are in [4]. In the remaining part of this section, we demonstrate the empirical patterns of a fractional Brownian motion for different Hurst-Indices over time (**Figure 2**).



**Figure 2.** Simulation of fBm for different Hurst-Index.  $H = 0.1$  (top panel),  $H = 0.5$  (middle panel),  $H = 0.9$  (bottom panel). Source: B Herzog (2020).

For  $H = 0.1$ , we obtain in the top-panel a time-series with short-term memory (**Figure 2**). Contrary in the bottom panel ( $H = 0.9$ ), we observe a strong interdependence or a non-stationary stochastic process. This process reflects long-term memory. The middle panel ( $H = 0.5$ ) denotes a standard Brownian motion. It is interesting that a fractional Brownian motion is a generalization of a standard Brownian motion. **Figure 2** summarizes the different empirical patterns in relationship to the H-Index.

### 3. Inflation modeling with Lévy processes

On first encounter, a Poisson process and a Brownian motion seem to be considerably different. Firstly, a Brownian motion has continuous paths whereas a Poisson process does not. Secondly, a Poisson process is a non-decreasing process and thus has paths of bounded variation over finite time horizons, whereas a Brownian motion does not have monotone paths. In fact, the Brownian motion has unbounded variation over finite time horizons.

Yet, both stochastic processes have a lot in common. Both processes are right continuous with left limits (so-called càdlàg). Consequently, we use these common properties to define a general class of stochastic processes, which are so-called Lévy processes. The class of Lévy processes is rather rich, and the Brownian motion or Poisson process are two prominent subcases.

In general, Lévy processes play a major role in several fields of sciences, such as physics, engineering, economics and mathematical finance. Lévy processes are becoming fashionable to describe the observed reality of financial markets more accurately than models based on a Brownian motion alone. Lévy processes result in a more realistic modeling because it captures the empirical reality of jump-diffusions. Indeed, asset prices have jumps and spikes and thus risk managers have to consider Lévy processes in order to hedge the risks appropriately. Similarly, the pattern of implied volatility or incomplete markets is reliant to Lévy processes too.

#### 3.1 Introduction to Lévy processes

The term Lévy process honors the work of the French mathematician Paul Lévy in the 1940s. He pioneered the understanding and characterization of stochastic processes with stationary and independent increments.

**Definition ‘Lévy Process.’** A process  $X = \{X_t : t > 0\}$  defined on a probability space  $(\Omega, F, P)$  is said to be a Lévy process if it possesses the following properties:

$$P(X_0 = 0) = 1.$$

1. The paths of  $X$  are  $P$ -almost surely right continuous with left limits. Mathematically,  $X$  is stochastically continuous for every  $0 < t < T$  and  $\varepsilon > 0$  such as  $\lim_{s \rightarrow t} P(X_t - X_s > \varepsilon) = 0$ .
2. For  $0 < s < t$ , the increments  $X_t - X_s$  are stationary and equal in distribution to  $X_{t-s}$ , i.e. the increment have the same distribution whenever time elapses.
3. For  $0 < s < t$ , the increment  $X_t - X_s$  is independent of  $\{X_u : u > s\}$  or we say the increment is independent of filtration  $F_s$ .

The definition does not immediately make visible the richness of the class of Lévy processes. One simple Lévy process is a Brownian motion with drift. Other examples of Lévy processes are the Poisson process. Or a Brownian motion

combined with a compound Poisson process. The last process is labeled a jump-process because it exhibits random jumps.

In order to identify Lévy processes, we use the property of infinitely divisible distributions. As soon as you can show that a process belongs to the class of infinitely divisible distributions, you immediately say that this process is a Lévy process. Indeed, there is an intimate relationship of Lévy processes to infinitely divisible distributions in general.

**Definition “Infinitely divisible distribution.”** *A real-valued random variable  $X$  has an infinitely divisible distribution if for each  $n = 1, 2, \dots$  there exist a sequence of independent, identical distributed random variables  $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ , such that*

$$X := X_{1,n} + X_{2,n} + \dots + X_{n,n}$$

*the process  $X$  has the same distribution as the processes of  $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ .*

One way to establish whether a given random variable has an infinitely divisible distribution is via the study of the exponent of the characteristic function. This idea is summarized by the rather sophisticated concept of the Lévy-Khintchine formula (e.g. in [7]).

### 3.2 A Brownian motion is a Lévy process

In this subsection, we briefly show that a Brownian motion is a Lévy process. Suppose a Gaussian random variable with distribution  $X \sim N(\mu, \sigma^2)$  and the characteristic function of  $\phi_{X_t}(t) = e^{i\mu t - \frac{1}{2}t^2\sigma^2}$ . We know that the increments of a Brownian motion follow a Gaussian process. By the characteristic function, we show that the increments of the Brownian motion are stationary and independent. Thus it stratifies the Lévy process properties:

$$\phi_{X_t}^n = e^{\left(\frac{i\mu}{n} - \frac{1}{2} \frac{t^2\sigma^2}{n}\right)^n} \quad (5)$$

$$\phi_{X_{t+s}} = \phi_{X_t} * \phi_{X_s}. \quad (6)$$

Eq. (5) demonstrates that the Brownian motion is an infinitely divisible distribution. Eq. (6) shows that the Brownian motion has independent and stationary increments. Thus, we find that the random variable  $X$  is Lévy by computing the sum of  $n$ -random variables  $X = X_1^n + \dots + X_i^n + \dots + X_n^n$  with each  $X_i^n \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$ . Therefore, we obtain  $X \sim N(\mu, \sigma^2)$  and  $X_1^n \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$ . Hence, the Brownian motion is infinitely divisible by  $n$  and it consists of independent, identical distributed (i.i.d) increments. Consequently, a Brownian motion satisfies the properties of a Lévy process.

**Remark.** Markov processes are the best-known family of stochastic processes in mathematical probability theory. Informally, a Markov process has the property that the future behavior of the process depends on the past only. One can show that Lévy processes are related to Markov processes and even simplify the theory significantly. The link between both stochastic processes is so-called random-stopping times. One can show that a random-stopping time on a Lévy process has the Markov property. Consequently, Lévy processes concern many aspects of probability theory and its applications.

## 4. Numerical simulation

In this section, we simulate different fractional Brownian motions and Lévy processes. The simulation reveals different new patterns of inflation dynamics. Our

model is calibrated to the monthly frequency of the past inflation dynamics in the Eurozone from 1997 to 2020.

The simulation follows a mean-reverting stochastic differential equation driven by a fractional Brownian motion and a Lévy process. Suppose  $X_t$  denotes the inflation process over time  $t$ . We model the inflation dynamics by a stochastic differential equation of the form

$$dX_t = (\alpha - \beta * X_t)dt + \sigma dB_t^H + N(\mu, \gamma)dN(\lambda) \quad (7)$$

where  $\alpha$  and  $\beta$  are the mean-reversion trends and  $\sigma$  denotes the volatility coming from the fractional Brownian motion,  $B_t^H$ . The parameter  $H$  reflects the Hurst-Index of the fractional Brownian motion. The last term is a jump-process modelled by a Poisson process,  $N(\mu, \gamma)$ , with parameters  $\mu$  and  $\gamma$ . The jump-frequency is of  $\lambda$ .

The numerical simulation is computed over 1000 time steps and over 1000 different stochastic processes. The Eurozone inflation data are downloaded from the ECB Statistical Data Warehouse. We calibrate the model to the aggregate inflation dynamics of the Eurozone (**Figures 3 and 4**).

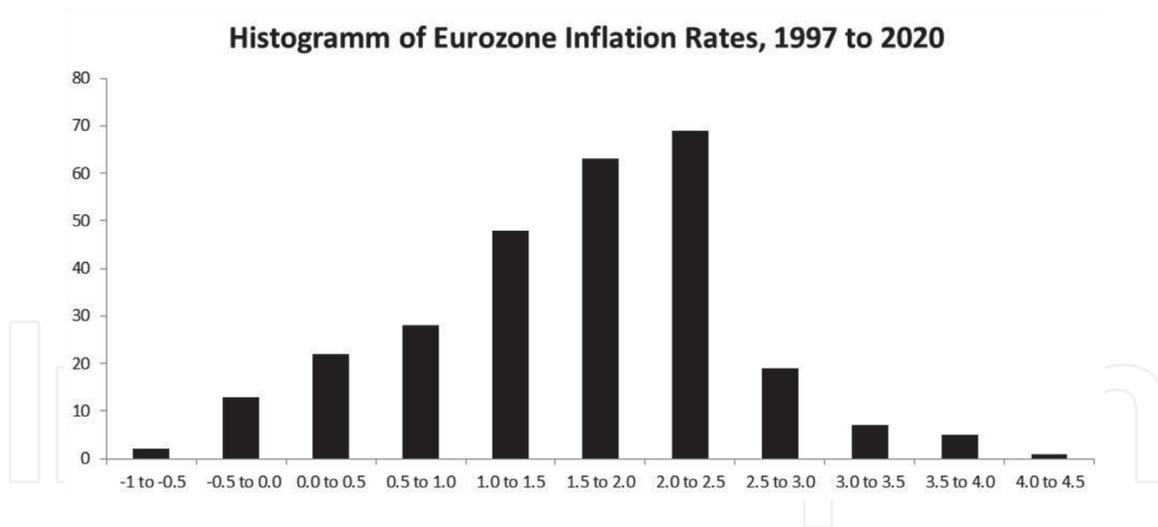
**Figure 3** represents the Harmonized Index of Consumer Price (HICP) of the Eurozone on monthly frequency from 1997 to 2020. One clearly sees the sharp drop in inflation rates during the global financial crisis of 2008–2009. Subsequently inflation rebounded, however, afterwards with low inflation rates, partly deflation, in the years of 2013–2016. In recent years, inflation rates were in the range of 1.0–2.0%. Thus, the inflation rate in the Eurozone is following Article 127 TFEU and the definition of price-stability by the European Central Bank [8]: “... inflation rates below, but close to 2% over the medium term.”

Based on the inflation data, we compute the histogram of Eurozone inflation rates in **Figure 4**. The distribution displays particularly a right-skewedness. Indeed, the mean is of 1.66, the median of 1.80 and the modus is of 2.10. Moreover, the standard deviation is of 0.77, the variance of 0.60, the skewness of  $-0.22$  and the kurtosis of  $-0.06$  is almost zero. These parameters characterize the Eurozone’s inflation rate properties over time.

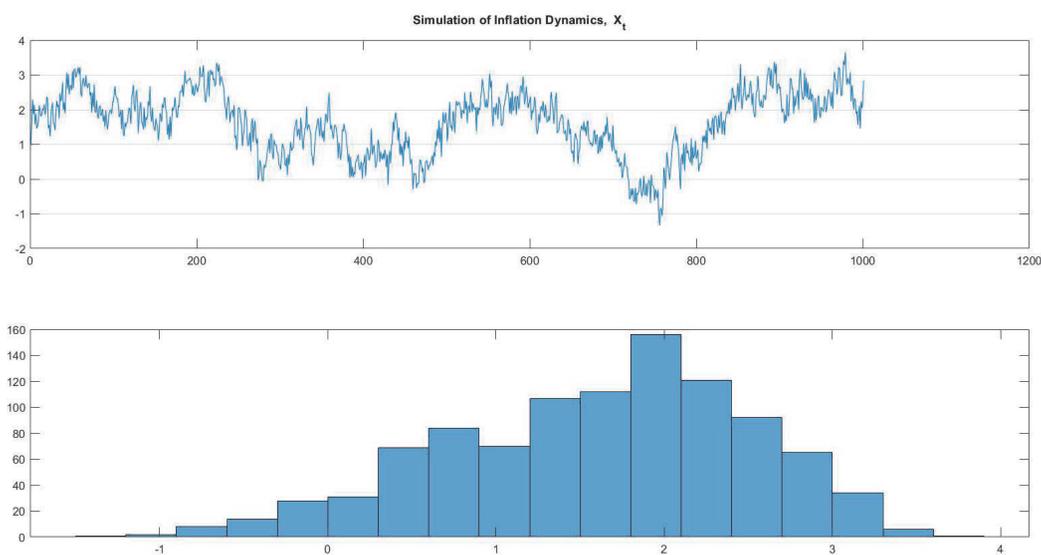
Next, we choose the following parameters in our stochastic differential equation (Eq. (7)):  $\alpha = 1.7$ ,  $\beta = 1.0$ ,  $\sigma = 0.4$ ,  $\mu = -2.0$ ,  $\gamma = 0.5$ ,  $\lambda = 0.01$  and  $H = 0.2$ . We run the simulation model for 1000-time steps. **Figure 5** represents the result of one simulation, where the mean is of 1.60, the median of 1.73, the variance of 0.81 and



**Figure 3.** Eurozone HICP-Inflation Rate. Data from ECB Data Warehouse. Source: B Herzog (2020).



**Figure 4.** Histogram of Eurozone Inflation Rates. Data from ECB Data Warehouse. Source: B Herzog (2020).



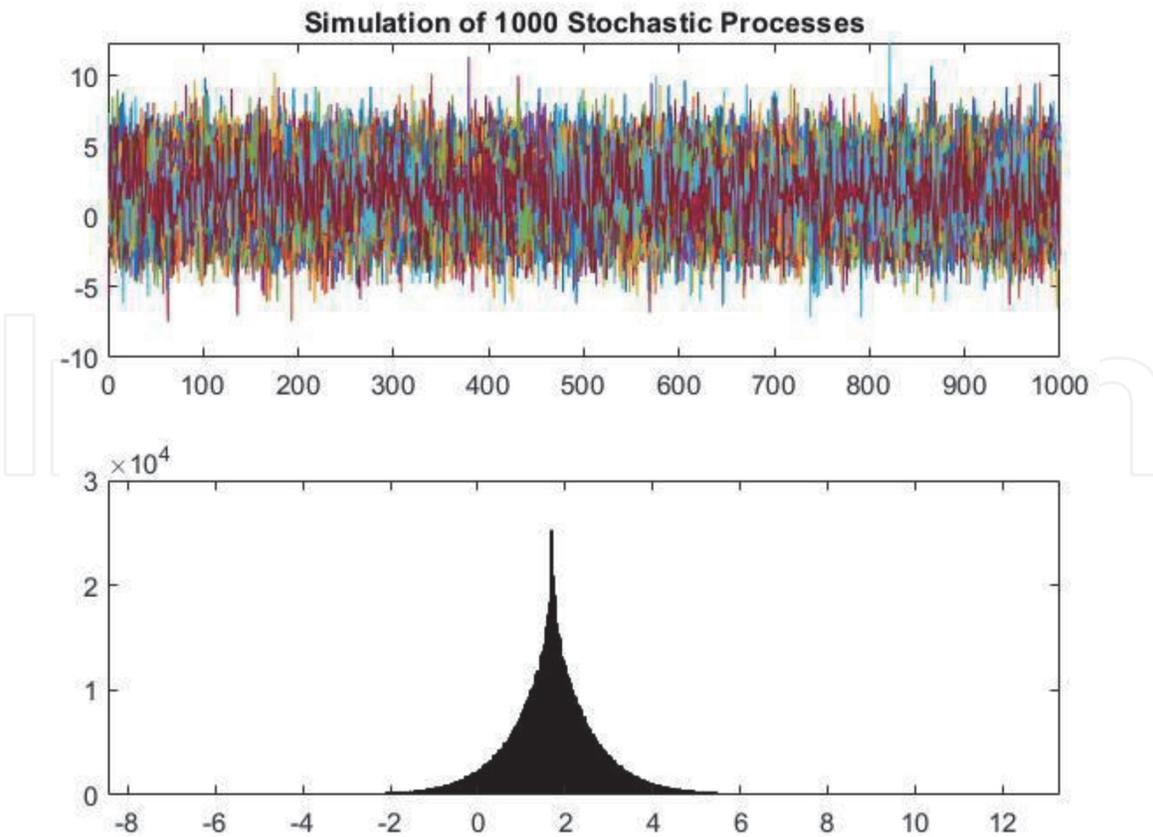
**Figure 5.** Simulation of Inflation Dynamics according to equation (7). Top panel denotes the inflation rate and bottom panel the histogram. Source: B Herzog (2020).

the skewness of  $-0.42$ . This demonstrates that the simulation is following the distribution properties of inflation data, particularly the right-skewedness.

It turns out that the simulation replicates the distributional properties quite well, except for the kurtosis. Nonetheless, we clearly see in the bottom panel of **Figure 5** that the distribution is right-skewed with more tail events on the left-hand side.

If we run the same model with the Gaussian assumption, by using a standard Brownian motion,  $H = 0.5$ , we obtain a somewhat different result. The mean is of  $0.94$ , the median of  $0.85$ , the variance of  $1.23$ , the skewness of  $0.45$  and the kurtosis of  $2.64$ . This distribution is not right-skewed and has higher variance than the stylized facts. Hence, we conclude that a fractional Brownian motion with a Lévy process provide a better approach in order to model the inflation dynamics of the Eurozone.

Finally, we discuss the results of the simulation exercise with 1000 runs. In this simulation, we have specified our stochastic differential equation (Eq. (7)) as follows:  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\sigma = 0.3$ ,  $\mu = -2.0$ ,  $\gamma = 0.1$ ,  $\lambda = 0.00$  and  $H = 0.2$ . **Figure 6** represents in the top-panel the stochastic paths of all stochastic processes and in the bottom-panel the respective histogram. The numerical simulation yields a mean and median of approximately  $1.7$ , a variance of  $1.4$  and negative skewness of  $-1.71$ .



**Figure 6.** Simulation of equation (7) with calibrated parameters. Top panel denotes all inflation processes and bottom panel the histogram. Source: B Herzog (2020).

Last but not least, by running several simulations we find that inflation dynamics is with high likelihood in a range of  $[-2, 5]$  in the Eurozone. Hence, even with severe positive or negative shocks the inflationary process is stable and anchored around the target level of 2%. Finally, in a scenario analysis, we set the mean-reverting level to the target rate of 4% as proposed by Blanchard et al. [9]. We find inflation dynamics is more volatile and still face deflationary levels during severe negative shocks. In that regard, a higher inflation target does not eliminate deflation events as with the target level of 2% today. Of course, the buffer towards deflation is greater if the inflation target is 4%. But economically, we proclaim that a higher inflation target creates a higher volatility and de-anchor inflation expectations subsequently. Consequently, increasing the inflation target is not free of any risk due to growing uncertainty about inflation expectations and price-stability in general.

## 5. Conclusion

This article models the inflation dynamics of the Eurozone with a novel approach. We utilize a stochastic differential equation driven by fractional Brownian motions and a Lévy process. Empirical inflation data show that the distribution is right-skewed. Thus, any standard approach using the normality assumption in econometrics fails. Therefore, we propose the use of fractional Brownian motions and Lévy processes in order to model time-dependence and jumps. Those processes cover short- and long-term phenomena, which is a prerequisite for empirical distributions.

We find that our modeling and numerical simulation provide good results to the calibrated inflation data. Inflation dynamics of the Eurozone is according to 1000

runs of our simulation stable and strongly anchored at the 2.0% inflation target. Even in the worst negative or positive shock, inflation numbers do not reach levels persistently below 0 or above 4%.

That said, the stable and low inflation rates of the Eurozone are highly contingent of the inflation target defined by the European Central Bank. Currently, inflation expectations are well anchored below the 2% level. Yet, our model simulation demonstrates that proposals to increase the inflation target, such as by Blanchard et al. [9], are highly risky because it leads to a de-anchoring of inflation. In the end, you might have higher volatility and the risk of de-anchored inflation expectations. The latter can create a strong upward bias in inflation rates out of the control of a central bank.

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## Conflict of interest

There is no external funding and no conflict of interest for this research article.

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