Chapter

Singular Boundary Integral Equations of Boundary Value Problems for Hyperbolic Equations of Mathematical Physics

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Abstract

The method of boundary integral equations is developed for solving the nonstationary boundary value problems (BVP) for strictly hyperbolic systems of second-order equations, which are characteristic for description of anisotropic media dynamics. The generalized functions method is used for the construction of their solutions in spaces of generalized vector functions of different dimensions. The Green tensors of these systems and new fundamental tensors, based on it, are obtained to construct the dynamic analogues of Gauss, Kirchhoff, and Green formulas. The generalized solution of BVP has been constructed, including shock waves. Using the properties of integrals kernels, the singular boundary integral equations are constructed which resolve BVP. The uniqueness of BVP solution has been proved.

Keywords: hyperbolic equations, generalized solution, Green tensor, boundary value problem, generalized function method

1. Introduction

Investigation of continuous medium dynamics in areas with difficult geometry with various boundary conditions and perturbations acting on the medium leads to boundary value problems for systems of hyperbolic and mixed types. An effective method to solve such problems is the boundary integral equation method (BIEM), which reduces the original differential problem in a domain to a system of boundary integral equations (BIEs) on its boundary. This allows to lower dimension of the soluble equations, to increase stability of numerical procedures of the solution construction, etc. Note that for hyperbolic systems, BIEM is not sufficiently developed, while for solving boundary value problems (BVPs) for elliptic and parabolic equations and systems, this method is well developed and underlies the proof of their correctness. It is connection with the singularity of solutions to wave equations, which involve characteristic surfaces, i.e., wavefronts, where the solutions and their derivatives can have jump discontinuities. As a result, the fundamental solutions on wavefronts are essentially singular, and the standard methods for constructing BIEs typical for elliptic and parabolic equations cannot be used. Therefore, for the development of the BIEM for hyperbolic equations, the theory of generalized functions [1, 2] is used. At present, BIEM are applied very extensively to solve engineering problems.

Here, the second-order strictly hyperbolic systems in spaces of any dimension are considered. The fundamental solutions of consider systems of equations are constructed and their properties are studied. It is shown that the class of fundamental solutions for our equations in spaces of odd dimensions is described by singular generalized functions with a surface support (e.g. for $R^3 \times t$, this is a single layer on a light cone). The constructed fundamental solutions of consider systems of equations are the kernels of BIEs. For systems of hyperbolic equations, the BIE method is developed. Here, the ideas for solving nonstationary BVPs for the wave equations in multidimensional space [3, 4] are used and the methods were elaborated for boundary value problems of dynamics of elastic bodies [5–8].

2. Generalized solutions and conditions on wave fronts

Consider the second-order system of hyperbolic equations with constant coefficients:

$$L_{ij}(\partial_x, \partial_t)u_j(x,t) + G_i(x,t) = 0, \quad (x,t) \in R^{N+1} \tag{1}$$

$$L_{ij}(\partial_x, \partial_t) = C_{ij}^{ml}\partial_m\partial_l - \delta_{ij}\partial_t^2, \quad i,j = \overline{1,M}, \quad m,l = \overline{1,N} \tag{2}$$

$$C_{ij}^{ml} = C_{ij}^{lm} = C_{ji}^{ml} = C_{ml}^{ij} \tag{3}$$

where $G_i \in L_2(R^{N+1})$ and δ_{ij} are Kronecker symbols; $\partial_x = (\partial_1, \dots \partial_N)$, $\partial_i = \partial/\partial x_i$, and $\partial_t = \partial/\partial t$ are Partial derivatives; and also we will use following notations $u_{i,j} = \partial_j u_i$ and $u_{i,t} = \partial_t u_i$.

The matrix C_{ij}^{ml}, whose indices may be permitted in accordance with above indicated symmetry properties (3), satisfies the following condition of strict hyperbolicity:

$$W(n,v) = C_{ij}^{ml} n_m n_l v^i v^j > 0 \quad \forall n \neq 0, \quad v \neq 0$$

Here everywhere like numbered indices indicate summation in specified limits of their change (so as in tensor convolutions).

By the virtue of positive definiteness W, the characteristic equation of the system (1)

$$det\left\{C_{ij}^{ml} n_m n_l - c^2\delta_{ij}\right\} = 0, \|n\| = 1 \tag{4}$$

has $2M$ valid roots (with the account of multiplicity):

$$c = \pm c_k(n) : 0 < c_k \leq c_{k+1}, k = \overline{1, M-1}$$

They are *sound velocities* of wave prorogations in physical media which are described by such equations. In a general case, they depend on a wave vector n.

It is known that the solutions of the hyperbolic equations can have characteristic surfaces on which the jumps of derivatives are observed [9]. To receive the conditions on jumps, it is convenient to use the theory of generalized functions.

Denote through $D'_M(R^{N+1})$ the space of generalized vector functions $\hat{f}(x,t) = \left(\hat{f}_1, \dots, \hat{f}_M\right)$ determined on the space $D_M(R^{N+1})$ of finite and indefinitely differentiable vector functions $\varphi(x,t) = (\varphi_1, \dots, \varphi_M)$. For regular $\hat{f}$, this linear function is presented in integral form:

$$\left(\hat{f}(x,t), \varphi(x,t)\right) = \int_{-\infty}^{\infty} d\tau \int_{R^N} f_i(x,\tau)\varphi_i(x,\tau)dV(x), \quad \forall\varphi \in D_M\left(R^{N+1}\right), \quad i = \overline{1,M}$$

$dV = dx_1 \dots dx_N$ (further, we shall say everywhere *generalized function* instead of *generalized vector function*).

Let $u(x,t)$ be the solution of Eq. (1) in R^{N+1}, continuous, twice differentiable almost everywhere, except for characteristic surface F which is motionless in R^{N+1} and mobile in R^N (wave front F_t). On surface, F_t derivatives can have jumps. The equation of F is Eq. (4). We denote $\nu = (n_1, \dots {}_{nN}, n_t) = (n, t_t)$, $n = (n_1, \dots {}_{nN})$, where ν is a normal vector to the characteristic surface F in R^{N+1}, and n is unit wave wave vector in R^N directed in the direction of propagation F_t. It is assumed that the surface F is piecewise smooth with continuous normal on its smooth part.

Let us consider Eq. (1) in the space $D'_M\left(R^{N+1}\right)$ and its solutions in this space are named as *generalized solutions* of Eq. (1) (or *solutions in generalized sense)*.

The solution $u(x,t)$ is considered as a regular generalized function and we denote $\hat{u}(x,t) = u(x,t)$, accordingly $\hat{G}(x,t) = G(x,t)$. Let $\hat{u}(x,t)$ be the solution of Eq. (1) in $D'_M\left(R^{N+1}\right)$.

Theorem 2.1. *If $\hat{u}(x,t)$ is the generalized solution of Eq. (1), then there are next conditions on the jumps of its components and derivatives*:

$$[u_i(x,t)]_{F_t} = 0 \tag{5}$$

$$\left[\sigma_i^m n_m - c u_{i,t}\right]_{F_t} = 0 \tag{6}$$

where $\sigma_i^m = C_{ij}^{ml} u_{j,l}$ *and the velocity* c *of a wave front* F_t *coincides with one of* c_k.

Proof. By the account of differentiation of regular generalized function rules [2], we receive:

$$\begin{aligned} L_{ij}(\partial_x, \partial_t)\hat{u}_j(x,t) + \hat{G}_i(x,t) &= \left[\sigma_i^m \nu_m - \nu_t u_{i,t}\right]_F \delta_F(x,t) + \\ &+ C_{ij}^{ml}\partial_m\left(\left[u_j\right]_F \nu_l \delta_F(x,t)\right) - \left([u_i]_F \nu_t \delta_F(x,t)\right)_{,t} \end{aligned} \tag{7}$$

Here, $\alpha(x,t)\delta_F(x,t)$ is singular generalized function, which is a simple layer on the surface F with specified density $\alpha = (\alpha_1, \dots, \alpha_M)$:

$$(\alpha(x,t)\delta_F(x,t), \varphi(x,t)) = \int_F \alpha_i(x,t)\varphi_i(x,t)dS(x,t), \forall\varphi(x,t) \in D_M\left(R^{N+1}\right)$$

$dS(x,t)$ is the differential of the surface in a point (x,t) and $(\nu,\nu_t) = (\nu_1, \dots, \nu_N, \nu_t)$ is a unit vector, normal to characteristic surface F.

If $F(x,t) = 0$ is an equation of wave front, then

$$(\nu, \nu_t) = (gradF, F_{,t})/\|(gradF, F_{,t})\|.$$

If the right part of expression (7) is equal to zero, then the function $\hat{u}(x,t)$ will satisfy to the Eq. (1) in a generalized sense. The natural requirement of the continuity of the solutions at transition through wave front F

$$[u_i(x,t)]_F = 0 \tag{8}$$

vanishes only two last composed right parts of Eq. (7). Hence, it is necessary that

$$\left[\sigma_i^m \nu_m - \nu_t u_{i,\,t}\right]_F = 0 \tag{9}$$

These conditions on the appropriate mobile wave front F_t we can write down with the account Eq. (4). By virtue of continuity of function $u(x,t)$ for $(x,t) \in F_t$, we have

$$\begin{aligned}[f(x,t)]_F &= \lim_{\varepsilon\to+0} (f(x+\varepsilon\nu, t+\varepsilon\nu_t) - f(x-\varepsilon\nu, t-\varepsilon\nu_t)) \\ &= \lim_{\varepsilon\to+0} (f(x+\varepsilon n, t) - f(x-\varepsilon n, t)) = [f(x,t)]_{F_t};\end{aligned}$$

therefore the condition (5) is equivalent to (8).
If $(x,t) \in F_{t,}$ then $(x + cn\Delta t, t + \Delta t) \in F_{t+\Delta t}$. Therefore,

$$F(x + cn\Delta t, t + \Delta t) - F(x,t) = \left(c\left(F,_j, n_j\right) + F,_t\right)\Delta t = 0$$

From here, we have

$$c = -F,_t/\left(F,_j, n_j\right) = -\nu_t/\sqrt{\nu_i\nu_i}$$

By virtue of it, the condition (9) will be transformed to the kind (6), where c, for each front, coincides with one of c_k. The theorem has been proved.

Corollary. On the wave fronts

$$[n_l u_{i,\,t} + c u_{i,\,l}]_{F_t} = 0, \quad i = \overline{1,M}, \quad l = \overline{1,N} \tag{10}$$

The proof follows from the condition of continuity (5). The expression (10) is the condition of the continuity of tangent derivative on the wave front.

In the physical problems of solid and media, the corresponding condition (6) is a condition for conservation of an impulse at fronts. This condition connects a jump of velocity at a wave fronts with stresses jump. By this cause, such surfaces are named as *shock wave fronts*.

Definition 1. The solution of Eq. (1), $u(x,t)$, is named as classical one if it is continuous on R^{N+1}, twice differentiable almost everywhere on R^{N+1}, and has limited number of piecewise smooth wave fronts on which conditions jumps (5) and (6) are carried out.

3. Fundamental matrices

3.1 The Green's matrix of second-order system of hyperbolic equations

Let us construct fundamental solutions of Eq. (1) on $D'_M\left(R^{N+1}\right)$.

Definition 2. $U_{jk}(x,t)$ is the Green's matrix of Eq. (1) if it satisfies to equations

$$L_{ij}(\partial_x, \partial_t) U_{jk}(x,t) + \delta_{ik}\delta(x)\delta(t) = 0, \quad i,j,k = \overline{1,M} \tag{11}$$

and next conditions:

$$U_{jk}(x,t) = 0 \quad for \quad t < 0, \forall x, \tag{12}$$

$$U_{jk}(x,0) = 0 \quad for \quad x \neq 0 \tag{13}$$

Here, by definition,

$$(\delta_{ik}\delta(x,t), \varphi_i(x,t)) = \varphi_k(0,0) \quad \forall \varphi \in D'_M(R^{N+1})$$

For construction of Green's matrix, it is comfortable to use Fourier transformation, which brings Eq. (11) to the system of linear algebraic equations of the kind

$$L_{jk}(-i\xi, -i\omega)\overline{U}_{kl}(\xi,\omega) + \delta_{jl} = 0, \quad j,k,l = \overline{1,M}$$

Here, $(\xi,\omega) = (\xi_{1,\dots,}\xi_{N,}\omega)$ is the Fourier variables appropriate to (x,t).

By permitting the system, we receive transformation of Green's matrix which by virtue of differential polynomials uniformity looks like:

$$\overline{U}_{jk}(\xi,\omega) = Q_{jk}(\xi,\omega)Q^{-1}(\xi,\omega) \tag{14}$$

where Q_{jk} are the cofactors of the element with index (k,j) of the matrix $\{L(-i\xi, -i\omega)\}$; and Q is the symbol of operator L:

$$Q(\xi,\omega) = \det\{L_{kj}(-i\xi, -i\omega)\}$$

There are the following relations of symmetry and homogeneous:

$$Q_{jk}(\xi,\omega) = Q_{jk}(-\xi,\omega) = Q_{jk}(\xi,-\omega), Q(\xi,\omega) = Q(-\xi,\omega) = Q(\xi,-\omega) \tag{15}$$

$$Q_{jk}(\lambda\xi,\lambda\omega) = \lambda^{2M-2}Q_{jk}(\xi,\omega), Q(\lambda\xi,\lambda\omega) = \lambda^{2M}Q(\xi,\omega) \tag{16}$$

By virtue of strong hyperbolicity characteristic equation,

$$Q(\xi,\omega) = 0$$

has $2M$ roots. It is a singular matrix. There is not a classic inverse Fourier transformation of it. It defines the Fourier transformation of the full class of fundamental matrices which are defined with accuracy of solutions of homogeneous system (1). Components of this matrix are not a generalized function. To calculate the inverse transformation, it is necessary to construct regularisation of this matrix in virtue of properties (12) and (13) of Green tensor. The following theorems has been proved [10]:

Theorem 3.1. *If* $c_q \ (q = \overline{1,M})$ *are unitary roots of Eq. (4), then the Green's matrix of system (1) has form*

$$U_{jk}(x,t) = \sigma_N H(t) \sum_{q=1}^{M} \int_{\|e\|=1} A_{jk}(e, c_q)$$
$$\times \left\{ \left((e,x) + c_q(e)t - i0\right)^{1-N} - \left((e,x) - c_q(e)t - i0\right)^{1-N} \right\} dS(e)$$

where $\sigma_N = (2\pi i)^{-N}(N-2)!$, $A_{jk}(e,c_q) = Q_{jk}(e,c_q)/2(c_q Q_{mm}(e,c_q))$, and $H(t)$ *is Heaviside's function.*

Theorem 3.2. *If* $c_q \ (q = \overline{1,M})$ *are roots of Eq. (4) with multiplicity* m_q, *then the Green's matrix of system (1) has form*

$$U_{jk}(x,t) = \sigma_N H(t) \sum_q m_q \int_{R^N} Q_{jk,\omega}^{(m_q-1)}(e,c_q) \left(Q,_{\omega}^{(m_q)}(e,c_q) \right)^{-1}$$
$$\times \left\{ \left((e,x) + c_q(e)t - i0\right)^{1-N} - \left((e,x) - c_q(e)t - i0\right)^{1-N} \right\} dS(e)$$

Here, the top index in brackets designate the order of derivative on ω.

So, the construction of a Green's matrix is reduced to the calculation of integrals on unit sphere. For odd N, these theorems allow to build the Green's matrix ε-approach only. For even N and for $\varepsilon-$approach, it is required to integrate multidimensional surface integral over unit sphere. However, in a number of cases, this procedure can be simplified.

We notice that if the original of Q^{-1} is known, i.e.

$$J(x,t) = F^{-1}\left[Q^{-1}(\xi,\omega)\right],$$

which is built in view of conditions (12), then it is easy to restore the Green's matrix

$$U_{jk}(x,t) = Q_{jk}(i\partial_x, i\partial_t)J(x,t) \tag{17}$$

In the case of invariance of Eq. (1) relative to group of orthogonal transformations, a symbol of the operator L_{ij} is a function of only two variables $\|\xi\|, \omega$ and can be presented in the form:

$$Q(\xi,\omega) = (i\omega)^{2M} q\left(\|\xi\|\omega^{-1}\right). \tag{18}$$

It essentially simplifies the construction of the original using the Green's functions of classical wave equations. For this purpose, it is necessary to spread out $Q^{-1}(\xi,\omega)$ on simple fractions. In the case of simple roots,

$$Q(\xi,\omega) = \prod_{k=1}^{M} \left(\|\xi\|^2 - \omega^2/c_k^2 \right)$$

$$Q^{-1}(\xi,\omega) = (-i\omega)^{-2M+2} \sum_{k=1}^{M} A_k \left(\|\xi\|^2 - \omega^2/c_k^2 \right)^{-1} \tag{19}$$

where A_k is the decomposition constant. It is easy to see that summand in round brackets under summation sign is the symbol of the classical wave operator

$$D_k = c_k^{-2}\partial_t^2 - \Delta_N.$$

Here, Δ_N is the Laplacian for which the Green's function $U_N(x,t)$ has been investigated well [11].

From Theorem 3.1 follows the support $U_N(x,t,c)$ is:

$$K_c^+ = \{(x,t) : \|x\| \le ct, t > 0\}$$

in R^{N+1} for even N and it is sound cone

$$K_c = \{(x,t) : \|x\| = ct, t > 0\}$$

for odd N.

For example, U_3 is the simple layer on a cone [10] and it is the singular generalized function. In this case, $J(x,t)$ is convolution over t Green's function with $H(t)$:

$$J(x,t) = \sum_{k=1}^{M} A_k (H(t) *_t \ldots (H(t) *_t U_N(x,t,c_k))). \tag{20}$$

Here, the convolution over t undertakes $(2M-2)$ time, which exists, by virtue of, on semi-infinite at the left of supports of functions [11]. It is easy to check up that the boundary conditions (12) and (13) are carried out as $U_N(x,t,c)$ which satisfies them. We formulate this result as:

Theorem 3.3. *If the symbol of the operator L is presented in form* (18) *and c_k are simple roots of Eq.* (4), *then $U_{jk}(x,t)$ is defined by the formula* (17), *where $J(x,t)$ looks like* (20).

If c_k have multiplicity m_k in decomposition as (20), degrees $\left(\|\xi\|^2 - \omega^2/c_k^2\right)^{-m}$ $\left(m = \overline{1, m_k}\right)$ can appear. Using the property of convolution transformation, we receive their original in kind of complete convolution over (x,t):

$$F^{-1}\left[\left(\|\xi\|^2 - \omega^2/c_k^2\right)^{-m}\right] = (U_N(x,t,c) * \ldots_m * U_N(x,t,c))$$

Then, the procedure of construction of a Green's matrix is similar to the described one.

We notice that as follows from (20) in a case of N = 1, 2, the convolution operation is reduced to calculate regular integrals of simple kind:

$$U_N(x,t) *_t H(t) = \int_0^t U_N(x,t-\tau)d\tau$$

$$U_N(x,t) *_t U_N(x,t) = \int_{R^N} dV(y) \int_0^t U_N(x-y,t-\tau)U_N(y,\tau)d\tau$$

But already for N = 3 and more, the construction of convolutions is non-trivial, and for their determination, its definition in a class of generalized functions should be used.

For any regular function $\hat{G} \in D'_M\left(R^{N+1}\right) : \sup_t \ \hat{G} \in (0,\infty)$, the appropriate solution of Eq. (1) looks like the convolution

$$\hat{u}_i = U_{ik} * \hat{G}_k.$$

For regular functions, it has integral representation in form of retarded potential:

$$\hat{u}_i(x,t) = H(t) \int_0^\infty d\tau \int_{R^N} U_{ik}(x-y,\tau)G_k(y,t-\tau)dV(y)$$

If Eqs. (1) are invariant, concerning the group of orthogonal transformations, then c_k do not depend on n. In physical problems, the isotropy of medium is reduced to the specified property.

3.2 The Green's tensor of elastic medium

For isotropic elastic medium constants, the matrix is equal to

$$C_{ij}^{ml} = \rho\left\{\lambda\delta_l^m\delta_i^j + \mu\left(\delta_i^m\delta_j^l + \delta_j^m\delta_i^l\right)\right\}.$$

The coefficients of Eq. (1) depend only on two sound velocities

$$c_1 = \sqrt{(\lambda + 2\mu)/\rho},\ c_2 = \sqrt{\mu/\rho},$$

where ρ is the density of medium, and λ and μ are elastic Lame parameters. These two speeds are velocities of propagation of dilatational and shearing waves. Wave fronts for Green's tensor are two spheres expanding with these velocities.

In the case of plane deformation N = M = 2, an appropriate Green's tensor was constructed in [5, 6]. For the space deformation N = M = 3, the expression of a Green's tensor was represented in [6].

For anisotropic medium in a plane case (N = M = 2), the Green's tensor was constructed in [12, 13]. For such medium, the wave propagation velocities depend on direction n and the form of wave fronts essentially depends on coefficients of Eq. (1). Anisotropic mediums with weak and strong anisotropy of elastic properties in the case of plane deformation were considered in [12–15]. In the first case, the topological type of wave fronts is similar to extending spheres. In the second case, the complex wave fronts and *lacunas* appear [16]. *Lacunas* are the mobile unperturbed areas limited by wave fronts and extended with current of time. Such medium has sharply waveguide properties in the direction of vector of maximal speeds. The wave fronts and the components of Green's tensor for weak and strong anisotropy are presented in [15]. The calculations are carried out for crystals of aragonite, topaz and calli pentaborat.

3.3 The fundamental matrices $\hat{V}, \hat{T}, \hat{W}, \hat{U}^{(s)}, \hat{T}^{(s)}$

For solution of BVP using Green's matrix $\hat{U}$, we introduce the fundamental matrices $\hat{S}$ and $\hat{T}$ with elements given by

$$\hat{S}_{ik}^{m}(x,t) = C_{ij}^{ml}\partial_l\hat{U}_j^k, \qquad \Gamma_i^k(x,t,n) = \hat{S}_{ik}^{m}n_m, \tag{21}$$

$$\hat{T}_k^i(x,t,n) = -\Gamma_i^k(x,t,n) = -C_{ij}^{ml}n_m\partial_l\hat{U}_j^k, \tag{22}$$

$$i,j,k = \overline{1,M}, \qquad m,l = \overline{1,N}.$$

Then, the equation for $\hat{U}$ can be written as

$$\hat{S}_{ik\,,\,l}^{l} - \hat{U}_{i\,,\,tt}^{k} + \delta_i^k\delta(x)\delta(x) = 0.$$

From the invariance of the equations for $\hat{U}$ under the symmetry transformations $y = -x$, some symmetry properties of introduced matrices follows:

$$\hat{U}_i^k(x,t) = \hat{U}_i^k(-x,t), \quad \hat{U}_i^k(x,t) = \hat{U}_k^i(x,t), \quad \hat{S}_{ik}^m(x,t) = -\hat{S}_{ik}^m(-x,t), \tag{23}$$

$$\hat{T}_i^k(x,t,n) = -\hat{T}_i^k(-x,t,n) = -\hat{T}_i^k(x,t,-n). \tag{24}$$

Is easy to prove [17].

Theorem 3.4. *For fixed k and n, the vector* $\hat{T}_i^k(x,t,n)$ *is the fundamental solution of system (1) corresponding to*

$$G_i = C_{ik}^{ml} n_m \delta,_l(x)\delta(t).$$

The matrix $\hat{T}$ is called a *multipole matrix*, since it describes the fundamental solutions of system (1) generated by concentrated multipole sources (see [18]).

Primitives of the matrix. The primitive of the multipole matrix is introduced as convolution over time:

$$\hat{W}_j^k(x,t,n) = \hat{T}_j^k(x,t,n) *_t H(t),$$

which is the primitive of the corresponding matrices with respect to t:

$$\partial_t \hat{V}_i^k = \hat{U}_i^k(x,t), \quad \partial_t \hat{W}_i^k = \hat{T}_i^k(x,t,n).$$

It is easy to see that $\hat{V}_i^k$ and $\hat{W}_i^k$ are fundamental solutions to system (1) of the form

$$L_{ij}(\partial_x, \partial_t)\hat{V}_j^k + \delta_i^k \delta(x) H(t) = 0, \tag{25}$$

$$L_{ij}(\partial_x, \partial_t)\hat{W}_j^k + n_m C_{ki}^{ml} \delta,_l(x) H(t) = 0.$$

Relation (23) implies the following symmetry properties of the above matrices:

$$\hat{V}_i^k(x,t) = \hat{V}_i^k(-x,t), \quad \hat{V}_i^k(x,t) = \hat{V}_k^i(x,t),$$

$$\hat{W}_i^k(x,t,n) = -\hat{W}_i^k(-x,t,n) = -\hat{W}_i^k(x,t,-n). \tag{26}$$

The Green's matrix of the static equations for $\hat{U}_i^{k(s)}(x)$ (when the t-derivatives in (1) are zero) is defined by

$$L_{ij}(\partial_x, 0)\hat{U}_j^{k(s)}(x) + \delta_i^k \delta(x) = 0, \tag{27}$$

$$\hat{U}_i^{k(s)}(x) \to 0, \quad \|x\| \to \infty. \tag{28}$$

By analogy with (22), we define the matrix

$$\hat{T}_i^{k(s)}(x,n) = -C_{kj}^{ml} n_m \partial_l \hat{U}_j^{i(s)}.$$

Obviously, we have the symmetry relations

$$\hat{T}_i^{k(s)}(x,n) = -\hat{T}_i^{k(s)}(-x,n) = -\hat{T}_i^{k(s)}(x,-n). \tag{29}$$

Theorem 3.4 implies the following result.

Corollary. $\hat{T}_i^{k(s)}$ is a *fundamental solution of the static equations*:

$$L_{ij}(\partial_x, 0)T_j^{k(s)} - n_m C_{ki}^{ml} \delta,_l(x) = 0.$$

It is easy to see that this is an elliptic system.

The following theorem have been proved [17].

Theorem 3.5. *The following representations take place*

$$\hat{V}_i^k(x,t) = U_i^{k(s)}(x)H(t) + V_i^{k(d)}(x,t), \tag{30}$$

$$\hat{W}_i^k(x,t) = T_i^{k(s)}(x)H(t) + W_i^{k(d)}(x,t), \tag{31}$$

where $U_i^{k(s)}(x)H(t)$ and $T_i^{k(s)}(x)H(t)$ are regular functions for $x \neq 0$. As $\|x\| \to 0$,

$$U_i^{k(s)}(x) \sim \ln\|x\| A_{ik}^N(e_x), \quad T_i^{k(s)}(x) \sim \|x\|^{-1} B_{ik}^N(e_x), \quad N = 2,$$

$$U_i^{k(s)}(x) \sim \|x\|^{-N+2} A_{ik}^N(e_x), \quad T_i^{k(s)}(x) \sim \|x\|^{-N+1} B_{ik}^N(e_x), \quad N > 2. \tag{32}$$

Here, $e_x = x/\|x\|$, $A_{ik}^N(e)$, *and* $B_{ik}^N(e)$ *are continuous and bounded functions on the sphere* $\|e\| = 1$, *and* $V_i^{k(d)}$ *and* $W_i^{k(d)}$ *are regular functions that are continuous at* $x = 0$ *and* $t > 0$. *For any N,*

$$V_i^{k(d)}(x,t) = 0 \quad W_i^{k(d)}(x,t) = 0 \quad \textit{for} \ \ \|x\| > \max_{k=\overline{1,M}} \ \max_{\|e\|=1} c_k(e)t,$$

and for odd N, these relations hold for $\|x\| < \min\limits_{k=\overline{1,M}} \ \min\limits_{\|e\|=1} c_k(e)t$..

4. Statement of the initial BVP

Consider the system of strict hyperbolic equations (1). Assume that $x \in S^- \subset R^N$, where S^- is an open bounded set; $(x,t) \in D^-$, $D^- = S^- \times (0,\infty)$, $D_t^- = S^- \times (0,t)$, $t > 0$; $D = S \times (0,\infty)$, and $D_t = S \times (0,t)$.

The boundary S of S^- is a Lyapunov surface with a continuous outward normal $n(x)$ $(\|n\| = 1)$:

$$\|n(x_2) - n(x_1)\| = O\left(\|x_2 - x_1\|^\beta\right), \ \beta > 0, \ x_1 \in S, \ x_2 \in S.$$

It is assumed that G is a locally integrable (regular) vector function.

$$G \to 0 \ \text{as} \ \ t \to +\infty, \quad \forall x \in S^-.$$

Furthermore, $u \in C(D^- + D)$, where u is a twice differentiable vector function almost everywhere on D^-, except for possibly the characteristic surfaces (F) in R^{N+1}, which correspond to the moving wavefronts (F_t) R^N. On them, conditions (5) and (6) are satisfied.

It is assumed that the number of wavefronts is finite and each front is almost everywhere a Lyapunov surface of dimension $N - 1$.

Problem 1. Find a solution of system (1) satisfying conditions (5)–(7) if the boundary values of the following functions are given:

the initial values

$$u_i(x,0) = u_i^0(x), \quad x \in S^- + S \tag{33}$$

$$u_{i,t}(x,0) = u_i^1(x), \quad x \in S^-; \tag{34}$$

the Dirichlet conditions

$$u_i(x,t) = u_i^S(x,t), \quad x \in S, \quad t \geq 0; \tag{35}$$

and the Neumann-type conditions

$$\sigma_i^l(x,t)n_l(x) = g_i(x,t), \quad x \in S, \quad t \geq 0, \quad i = \overline{1,N}. \tag{36}$$

Problem 2. Construct resolving boundary integral equations for the solution of the following boundary value problems.

Initial-boundary value problem I. Find a solution of system (1) that satisfies boundary conditions (33)–(35) and front conditions (5)–(7).

Initial-boundary value problem II. Find a solution of system (1) that satisfies boundary conditions (33), (34), and (36) and front conditions (5)–(7).

These solutions are called *classical*.

Remark. Wavefronts arise if the initial and boundary data do not obey the compatibility conditions

$$w_i(x,0) = u_i^0(x), \quad u_{i\,,\,t}^S(x,0) = u_i^1(x), \quad x \in S.$$

In physical problems, they describe shock waves, which are typical when the external actions (forces) have a shock nature and are described by discontinuous or singular functions.

5. Uniqueness of solutions of BVP

Define the functions

$$W(u) = 0,5C_{ij}^{ml}u_{i\,,\,m}u_{j\,,\,l}, \quad K(u) = 0,5\|u_{,\,t}\|^2,$$
$$E(u) = K(u) + W(u), \quad L(u) = K(u) - W(u),$$

which are called the *densities of internal, kinetic, and total energy* of the system, respectively, and L is *the Lagrangian*.

Theorem 5.1. *If u is a classical solution of the Dirichlet (Neumann) boundary value problem, then*

$$\int_{D_t^-} L(u(x,t))dV(x,t) = \int_{D_t^-} G_i(x,t)u_i(x,t)dV(x,t) +$$

$$+ \int_{D_t} g_i(x,t)u_i^S(x,t)dS(x,t) - \int_{S^-} \left(u_i(x,t)u_{i\,,\,t}(x,t) - u_i^0(x)u_i^1(x)\right)dV(x)$$

Here and below, $dV(x) = dx_1 \ldots dx_N$, $dV(x,t) = dV(x)dt$; $dS(x)$, *and*, $dS(x,t)$ *are the differentials of the area of S and D, respectively*.

Proof. Multiplying (1) by u_i and summing the result over i, after simple algebra, we obtain the expression

$$L = \left(C_{ij}^{ml}u_{j\,,\,m}u_i\right)_{,\,l} - (u_i u_{i\,,\,t})_{,\,t} + G_i u_i.$$

This equality is integrated over D_t taking into account the front discontinuities and using the Gauss-Ostrogradsky theorem and initial conditions (33) and (34) to obtain

$$\int_{D_t^-} L(u(x,t))dV(x,t) = \int_{D_t^-} \left(C_{ij}^{ml} u_{j,m} u_i\right)_{,l} - (u_i u_{i,t})_{,t} dV(x,t) +$$

$$+ \int_{D_t^-} G_i(x,t) u_i(x,t) dV(x,t) = \int_{D_t} \sigma_i^l n_l(x) u_i(x,t) dS(x,t) -$$

$$- \int_S u_i u_{i,t}(x,t) - u_i u_{i,t}(x,0)) dS(x,t) + \int_{D_t^-} G_i(x,t) u_i(x,t) dV(x,t) +$$

$$+ \sum_k \int_{F_k \cap D_t^-} u_i \left[\nu_l^k \sigma_i^l(x,t) - \nu_t^k u_{i,t}(x,t)\right]_{F_k} dF_k(x,t)$$

Here, ν_l^k, *and* ν_t^k are the components of the unit normal vector to the front $F_k(x,t)$ in R^{N+1}, for which we have [17]

$$\nu_t^k = -c_k / \left(\nu_j^k \nu_j^k\right)^{1/2}, \tag{37}$$

where c_k is the velocity of the front. With the notation introduced, the relation (37) and the front condition (7) yield the assertion of the theorem.

It is easy to see that the following result holds true.

Corollary. *If* $u_i(x,0) = 0, \quad u_{i,t}(x,0) = 0$, *and*

$$\lim_{t\to+\infty} u_{i,l} \to 0, \quad \lim_{t\to+\infty} u_{i,t} \to 0, \quad x \in S^-,$$

then

$$\int_{D^-} L(u(x,t))dV(x,t) = \int_{D^-} G_i(x,t) u_i(x,t) dV(x,t) + \int_D g_i(x,t) u_i^S(x,t) dS(x,t)$$

is proved in the following theorem [17]:

Theorem 5.2. *If u is a classical solution of the Dirichlet (Neumann) boundary value problem, then*

$$\int_{S^-} (E(u,t) - E(u,0)) dV(x) =$$

$$\int_{D_t^-} G_i(x,t) u_{i,t}(x,t) dV(x,t) + \int_{D_t} g_i(x,t) u_{i,t}^S(x,t) dS(x,t).$$

It is easy to see that this theorem implies the uniqueness of the solutions to the initial-boundary value problems in question.

Theorem 5.3. *If a classical solution of the Dirichlet (Neumann) boundary value problem exists and satisfies the conditions*

$$\lim_{t\to+\infty} u_{i,l} \to 0, \quad \lim_{t\to+\infty} u_{i,t} \to 0, \quad \forall x \in S^-,$$

then this solution is unique.

Proof. Since the problem is linear, it suffices to prove the uniqueness of the solution to the homogeneous boundary value problem. If there are two solutions u_1

and u_2, then their difference $u = u_1 - u_2$ satisfies the system of equations with $G = 0$ and the zero initial conditions, i.e.

$$u_i^m(x) = 0 \quad (m = 0, 1).$$

The vector u on the boundary S satisfies the homogeneous boundary conditions

$$u_i(x,t) = 0 \quad \text{or} \quad g_i(x,t) = 0.$$

Theorem 5.2 yields

$$\int_S E(u,t)dS(x) = \int_S (K(u,t) + W(u,t))dS(x) = 0.$$

Since the integrand is positive definite and by the conditions of the theorem, we have $u \equiv 0$. The theorem is proved.

6. Analogues of the Kirchhoff and Green's formulas

Let us assume that S is a smooth boundary with a continuous normal of a set S^-. The characteristic function $H_S^-(x)$ of a set S^- is defined for $x \in S$ as

$$H_S^-(x) = 1/2 \tag{38}$$

The Heaviside function $H(t)$ is extended to zero by setting $H(0) = 1/2$. Define the characteristic function of D^- as

$$H_D^-(x,t) = H_S^-(x)H(t) \tag{39}$$

Accordingly, for u defined on D^-, we introduce the generalized function

$$\hat{u}(x,t) = uH_D^-(x,t), \tag{40}$$

which is defined on the entire space R^{N+1}. Similarly,

$$\hat{G}_k(x,t) = G_k H_D^-(x,t). \tag{41}$$

Let $\hat{U}_i^k(x,t)$ denotes the Green's matrix, i.e. the fundamental solution of Eq. (1) that corresponds to the function $F_i = \delta_i^k \delta(x)\delta(t)$ and satisfies the conditions

$$\hat{U}_i^k(x,0) = 0, \quad \hat{U}_{i,t}^k(x,0) = 0, \quad x \neq 0 \tag{42}$$

For system (1), such a matrix was constructed in [10].

The primitive of Green's matrix with respect to t is defined as

$$\hat{V}_i^k(x,t) = \hat{U}_i^k(x,t) \underset{t}{*} H(t) \quad \Rightarrow \quad \partial_t \hat{V}_i^k = \hat{U}_i^k. \tag{43}$$

Here and below, the star denotes the complete convolution with respect to (x,t), while the variable under the star denotes the incomplete convolution with respect to x or t, respectively. The convolution exists since the supports are semibounded with respect to t. Clearly, the convolution is the solution of Eq. (1) at $F_i = \delta_i^k \delta(x)H(t)$.

Theorem 6.1. *If $u(x,t)$ is a classical solution of the Dirichlet (Neumann) boundary value problem, then the generalized solution $\hat{u}$ can be represented as the the sum of the convolutions*

$$\hat{u}_i = U_i^k * \hat{G}_k + U_i^k \quad *_x \; u_k^1(x)H_S^-(x) +$$
$$+\partial_t U_i^k \underset{x}{*} u_k^0(x)H_S^-(x) + U_i^k * g_k(x,t)\delta_S(x)H(t) - \tag{44}$$
$$-C_{kj}^{ml}\partial_l V_i^k * u_{j,t}(x,t)n_m(x)\delta_S(x)H(t) - C_{kj}^{ml}\partial_l V_i^k \underset{x}{*} u_j^0(x)n_m(x)\delta_S(x).$$

Here, δ_S is a singular generalized function that is a single layer on S (see [2]), and $g_k(x,t)\delta_S(x)H(t)$ is a single layer on D.

Proof. Applying the operator L_{ij} to $\hat{u}(x,t)$, using the differentiation rules for generalized functions, and taking into account the equalities

$$\partial_j H_D^- = -n_j\delta_S(x)H(t), \partial_t H_D^- = \delta(t)H_S^-(x),$$

and the front conditions (5) and (6), we obtain

$$L_{kj}(\partial_x,\partial_t)\hat{u}_j(x,t) = \hat{G}_k(x,t) + u_k^1(x)H_S^-(x)\delta(t) +$$
$$+u_k^0(x)H_S^-(x)\dot{\delta}(t) + g_k(x,t)\delta_S(x)H(t) - C_{kj}^{ml}\{u_j(x,t)n_m(x)\delta_S(x)H(t)\}_{,l}$$

Next, we use the properties of Green's matrix to construct a weak solution of Eq. (1) in the form of the convolution

$$\hat{w}_i(x,t) = U_i^k * \hat{G}_k + \hat{U}_i^k \underset{x}{*} u_k^1(x)H_S^-(x) + \partial_t\hat{U}_i^k \underset{x}{*} u_k^0(x)H_S^-(x) +$$
$$+\hat{U}_i^k * g_k(x,t)\delta_S(x)H(t) - C_{kj}^{ml}\hat{U}_i^k * \left(u_j(x,t)n_m(x)\delta_S(x)H(t)\right)_{,l}. \tag{45}$$

The last convolution can be transformed using the relation (43) and applying the differentiation rules for convolutions and generalized functions:

$$C_{kj}^{ml}\partial_t\hat{V}_i^k * \left(u_j n_m(x)\delta_S(x)H(t)\right)_{,l} = C_{kj}^{ml}\partial_l\hat{V}_i^k * \left(u_j n_m(x)\delta_S(x)H(t)\right)_{,t} =$$
$$= C_{kj}^{ml}\partial_l\hat{V}_i^k * \Big(u_{j,t}n_m(x)\delta_S(x)H(t) + u_j^0(x)n_m(x)\delta_S(x)\delta(t)\Big) =$$
$$= C_{kj}^{ml}\partial_t\hat{V}_i^k * u_{j,t}n_m(x)\delta_S(x)H(t) + C_{kj}^{ml}\partial_l\hat{V}_i^k \underset{x}{*} u_j^0(x)n_m(x)\delta_S(x)$$

Let us show that $\hat{w}_i(x,t) = \hat{u}_i(x,t)$. Indeed, $\forall\varphi\in D_N\left(R^{N+1}\right)$

$$(\hat{w}_i,\varphi_i) = \Big(\hat{U}_i^k * \hat{F}_k,\varphi_i\Big) = \Big(\hat{U}_i^k * L_{kj}(\partial_x,\partial_t)\hat{u}_j,\varphi_i\Big) =$$
$$= \Big(L_{kj}(\partial_x,\partial_t)\hat{U}_i^k * \hat{u}_j,\varphi_i\Big) = \Big(\delta_i^j\delta(x,t) * \hat{u}_j,\varphi_i\Big) = (\hat{u}_i,\varphi_i).$$

Here, $\hat{F}_k$ denotes the right-hand side of (44). Since $(\hat{u}_i,\varphi_i) = 0$, if $supp\,\varphi\in D^+$, it follows that $\hat{w}_i(x,t) = 0,\ x\notin D^-$. This implies the assertion of the theorem, since the solution of the problem is unique.

Given initial and boundary values (33)–(36), the above formula recovers the solution in the domain. For this reason, it can be called an analogue of the Kirchhoff and Green formulas for solutions of hyperbolic systems (1). It gives a weak solution of the problems.

To represent this formula in integral form and use it for the construction of boundary integral equations for solutions of the initial-boundary value problems, we examine the properties of the functional matrices involved.

7. Singular boundary integral equations

Lemma 7.1 (analogue of the Gauss formula). *If S is an arbitrary closed Lyapunov surface in R^N, then*

$$\int_S T_k^{i(s)}(y-x,n(y))dS(y) = \delta_k^i H_S^-(x)$$

For $x \in S$, the integral is singular and is understood in the sense of its principal value.

Proof. Convolution Eq. (27) with $H_S^-(x)$ and using the differentiation rules for convolutions yields

$$L_{ij}(\partial_x, 0)U_j^{k(s)} * H_S^-(x) + \delta_k^i H_S^-(x) =$$

$$= -C_{ij}^{ml} U_{jk,l}^s * n_m \delta_i^k H_S^-(x) = \int_S T_k^{i(s)}(x-y,n(y))dS(y) + \delta_k^i H_S^-(x) = 0$$

Using (29), we obtain the formula in the lemma. Since $T_k^{i(s)}$ is regular for $x \notin S$, the formula holds for such x. Let us prove the validity of this formula for boundary points.

Let $x \in S$. Define $O_\varepsilon(x) = \{y \in S : \|y-x\| < \varepsilon\}$, $S_\varepsilon(x) = S - O_\varepsilon(x)$, $\Gamma_\varepsilon(x) = \{y : \|y-x\| = \varepsilon\}$, $\Gamma_\varepsilon^-(x) = \Gamma_\varepsilon(x) \cap S^-$, *and* $\Gamma_\varepsilon^+(x) = \Gamma_\varepsilon(x) \cap S^+$.

Similarly, we obtain

$$\int_{S_\varepsilon} T_k^{i(s)}(y-x,n(y))dS(y) + \int_{\Gamma_\varepsilon^-} T_k^{i(s)}(y-x,n(y))dS(y) = 0$$

$$\int_{S_\varepsilon} T_k^{i(s)}(y-x,n(y))dS(y) + \int_{\Gamma_\varepsilon^+} T_k^{i(s)}(y-x,n(y))dS(y) = \delta_k^i$$

Since the outward normals to $\Gamma_\varepsilon^-(x)$ and $\Gamma_\varepsilon^+(x)$ at opposite points y^- and y^+ of the sphere $\Gamma_\varepsilon(x)$ coincide, i.e. $n(y^-) = (x-y^-)/\varepsilon = (y^+-x)/\varepsilon = n(y^+)$, while $(y^+-x) = -(y^- - x)$, we take into account the asymptotics of $T_k^{i(s)}$ and, according to Theorem 3.5, sum these two equalities and pass to the limit as $\varepsilon \to 0$, to obtain equality (30) for boundary points. The lemma is proved.

For $M = 1$ and $L_{1j}(\partial_x, 0) = \partial_j \partial_j = \Delta$, this formula coincides with the Gauss formula for the double-layer potential of Laplace equation (see [2]).

Consider formula (44). Formally, it can be represented in the integral form

$$\hat{u}_k(x,t) = \int_D \left(T_k^i(x-y,n(y),t-\tau)u_i(y,t) + U_k^i(x-y,t-\tau)g_i(y,\tau)\right)dD(y,\tau) +$$

$$+ \int_{S^-} \left(U_{k,t}^i(x-y,t)u_i^0(y) + U_k^i(x-y,t)u_i^1(y)\right)dV(y) + U_k^i * \hat{G}_i$$

Under zero initial conditions, this formula coincides in form with the generalized Green formula for elliptic systems. However, the singularities of Green's matrix of the wave equations prevent us from using it for the construction of solutions to boundary value problems, since the integrals on the right-hand side do not exist because T_k^i has strong singularities on the fronts. However, the primitives of the matrix introduced in Section 3 can be used to construct integral representations of formula (44).

Theorem 7.1. *If u is a classical solution of the boundary value problem, then*

$$\hat{u}_k = U_k^i(x,t) * G_i(x,t) + U_k^i(x,t) * g_i(x,t)\delta_s(x)H(t) -$$
$$- \int_S T_k^{i(s)}(x-y)u_i(y,t)dS(y) - \int_S dS(y) \int_0^t W_k^{i(d)}(x-y,n(y),t-\tau)u_{i,t}(y,\tau)d\tau -$$
$$- \int_S W_k^{i(d)}(x-y,n(y),t)u_i^0(y)dS(y) + \left(U_k^i(x,t) \quad *_x \; u_i^0(y)H_S^-(x)\right)_{,t}$$

For $x \in S$, the integral is singular and is understood in the sense of its principal value.

Proof. For even N, the integral representation (42) has the form

$$\hat{u}_k = \int_S dS(y) \int_0^t \left(U_k^i(x-y,t-\tau)g_i(y,\tau) - W_k^i(x-y,n(y),t-\tau)u_{i,\tau}(y,\tau)\right)d\tau -$$
$$- \int_S W_k^i(x-y,n(y),t)u_i^0(y)dS(y) + \partial_t \int_{S^-} U_k^i(x-y,t)u_i^0(y)dS^-(y) +$$
$$+ \int_{S^-} U_k^i(x-y,t)u_i^1(y)dV(y) + \int_{D^-} U_k^i(x-y,t-\tau)G_i(y,\tau)dV(y)d\tau$$

Here, all the integrals are regular for interior points and singular for boundary points.

Remark. If N is odd, then, since U is singular, the integrals involving U are still written in the form of a convolution, which is taken according to the convolution rules depending on the form of U. For the wave equation of odd dimension, such representations were constructed in [4].

It is easy to see that, for zero initial data, the last three integrals (in the convolution) vanish.

Applying Theorem 3.5, by virtue of (31), the second term can be represented as

$$\int_S dS(y) \int_0^t W_k^i(x-y,n(y),t-\tau)d_\tau u_i(y,\tau) =$$
$$= \int_S T_k^{i(s)}(x-y)\left(u_i(y,t) - u_i^0(y)\right)dS(y)$$
$$+ \int_S dS(y) \int_0^t W_k^{i(d)}(x-y,n(y),t-\tau)u_{i,\tau}(y,\tau)d\tau$$

Here, the first integral is singular for $x \in S$ and exists in the sense of its principal value by Lemma 7.1, while the second integral is regular. Then for interior points, we obtain the formula of the theorem.

Let us show that the equality holds in the sense of definition (37) for boundary points as well.

Let $x^* \in S$, $x \in S^-$ and $x \to x^*$. Then, since the convolutions containing U_k^i and $W_k^{i(d)}$ are continuous, we obtain

$$\lim_{x \to x^*} u_k(x,t) = u_k(x^*,t) =$$

$$= \lim_{x \to x^*} \int_S T_k^{i(s)}(y-x)u_i(y,t)dS(y) + \int_S W_k^{i(d)}(x^* - y, n(y), t)u_i^0(y)dS(y) -$$

$$- \int_S dS(y) \int_0^t \left(U_k^i(x^* - y, t-\tau)g_i(y,\tau) + W_k^{i(d)}(x^* - y, n(y), t-\tau)u_{i,\tau}(y,\tau) \right) d\tau +$$

$$+ \int_{S^-} U_k^i(x^* - y, t)u_i^1(y)dV(y) + \int_{S^-} \left(U_k^i(x^* - y, t)u_i^0(y) \right)_{,t} dV(y) +$$

$$+ \int_{D^-} U_k^i(x^* - y, t-\tau)G_i(y,\tau)dV(y)d\tau$$

By Lemma 7.1, the limit on the right-hand side can be transformed into

$$\int_S T_k^{i(s)}(y-x^*)(u_i(y,t) - u_i(x^*,t))dS(y) + u_i(x^*,t)\delta_k^i =$$

$$= V.P.\int_S T_k^{i(s)}(y-x^*)u_i(y,t)dS(y) - u_i(x^*,t)V.P.\int_S T_k^{i(s)}(y-x^*)dS(y) +$$

$$+ u_i(x^*,t)\delta_k^i = V.P.\int_S T_k^{i(s)}(y-x^*)u_i(y,t)dS(y) + 0,5u_i(x^*,t)\delta_k^i$$

Adding up and combining like terms, we derive the formula of the theorem for boundary points. The theorem is proved.

The formula on the boundary yields boundary integral equations for solving initial-boundary value problems.

Theorem 7.2. *The classical solution of the Dirichlet (Neumann) initial-boundary value problem for $x \in S$ and $t > 0$ satisfies the singular boundary integral equations* ($k = \overline{1,M}$)

$$0,5u_k(x,t) = U_k^i(x,t) * G_i(x,t) + U_k^i(x,t) * g_i(x,t)\delta_s(x)H(t) -$$

$$- V.P.\int_S T_k^{i(s)}(x-y)u_i(y,t)dS(y) - \int_S dS(y) \int_0^t W_k^{i(d)}(x-y, n(y), t-\tau)u_{i,t}(y,\tau)d\tau -$$

$$- \int_S W_k^{i(d)}(x-y, n(y), t)u_i^0(y)dS(y) + \left(U_k^i(x,t) \underset{x}{*} u_i^0(y)H_S^-(x) \right)_{,t} + U_i^k \underset{x}{*} u_k^1(x)H_S^-(x).$$

From these equations, we can determine the unknown boundary functions of the corresponding initial-boundary value problem. Next, the formulas of Theorem 7.1 are used to determine the solution inside the domain.

8. Conclusions

The solvability of the obtained systems of BIEs in a particular class of functions is an independent problem in functional analysis. These equations can be numerically solved using the boundary element method. In special cases of nonstationary boundary value problems in elasticity theory ($M = N = 2, 3$), these equations were solved in [4, 6–8].

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