Chapter

Analytical Applications on Some Hilbert Spaces

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Abstract

In this paper, we establish an uncertainty inequality for a Hilbert space H. The minimizer function associated with a bounded linear operator from H into a Hilbert space K is provided. We come up with some results regarding Hardy and Dirichlet spaces on the unit disk $\mathbb{D}$.

Keywords: Hilbert space, Hardy space, Dirichlet space, uncertainty inequality, minimizer function

1. Introduction

Hilbert spaces are the most important tools in the theories of partial differential equations, quantum mechanics, Fourier analysis, and ergodicity. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions. Saitoh et al. applied the theory of Hilbert spaces to the Tikhonov regularization problems [1, 2]. Matsuura et al. obtained the approximate solutions for bounded linear operator equations with the viewpoint of numerical solutions by computers [3, 4]. During the last years, the theory of Hilbert spaces has gained considerable interest in various fields of mathematical sciences [5–9]. We expect that the results of this paper will be useful when discussing (in Section 2) uncertainty inequality for Hilbert space H and minimizer function associated with a bounded linear operator T from H into a Hilbert space K. As applications, we consider Hardy and Dirichlet spaces as follows.

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk. The Hardy space $H(\mathbb{D})$ is the set of all analytic functions f in the unit disk $\mathbb{D}$ with the finite integral:

$$\int_0^{2\pi} \left| f\left(e^{i\theta}\right)\right|^2 \mathrm{d}\theta. \tag{1}$$

It is a Hilbert space when equipped with the inner product:

$$\langle f, g\rangle_{H(\mathbb{D})} = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\,\mathrm{d}\theta. \tag{2}$$

Over the years, the applications of Hardy space $H(\mathbb{D})$ play an important role in various fields of mathematics [5, 10] and in certain parts of quantum mechanics

[11, 12]. And this space is the background of some applications. For example, in Section 3, we study on $H(\mathbb{D})$ the following two operators:

$$\nabla f(z) = f'(z), \quad Lf(z) = z^2 f'(z) + zf(z), \tag{3}$$

and we deduce uncertainty inequality for this space. Next, we establish the minimizer function associated with the difference operator:

$$T_1 f(z) = \frac{1}{z}(\, f(z) - f(0)). \tag{4}$$

In Section 4, we consider the Dirichlet space $\mathcal{D}(\mathbb{D})$, which is the set of all analytic functions f in the unit disk $\mathbb{D}$ with the finite Dirichlet integral:

$$\int_{\mathbb{D}} |\, f'(z)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}, \quad z = x + iy. \tag{5}$$

It is also a Hilbert space when equipped with the inner product:

$$\langle f, g\rangle_{\mathcal{D}(\mathbb{D})} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{\pi}, \quad z = x + iy. \tag{6}$$

This space is the objective of many applicable works [5, 13–17] and plays a background to our contribution. For example, we study on $\mathcal{D}(\mathbb{D})$ the following two operators:

$$\Lambda f(z) = f'(z) - f'(0), \quad Xf(z) = z^2 f'(z), \tag{7}$$

and we deduce the uncertainty inequality for this space $\mathcal{D}(\mathbb{D})$. And we establish the minimizer function associated with the difference operator:

$$T_2 f(z) = \frac{1}{z}\big(\, f(z) - zf'(0) - f(0)\big). \tag{8}$$

2. Generalized results

Let H be a Hilbert space equipped with the inner product $\langle ., .\rangle_H$. And let A and B be the two operators defined on H. We define the commutator $[A, B]$ by

$$[A, B] := AB - BA. \tag{9}$$

The adjoint of A denoted by A^* is defined by

$$\langle Af, g\rangle_H = \langle f, A^* g\rangle_H, \tag{10}$$

for $f \in \mathrm{Dom}(A)$ and $g \in \mathrm{Dom}(A^*)$.

Theorem 2.1. For $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$, one has

$$\|A^* f\|_H^2 = \|Af\|_H^2 + \langle [A, A^*]\, f, f\rangle_H. \tag{11}$$

Proof. Let $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$. Then AA^*f and A^*Af belong to H. Therefore $[A, A^*]\, f \in H$. Hence one has

$$\|A^*f\|_H^2 = \langle AA^*f, f\rangle_H = \langle A^*Af, f\rangle_H + \langle [A, A^*]f, f\rangle_H \tag{12}$$

$$= \|Af\|_H^2 + \langle [A, A^*]f, f\rangle_H. \quad \square \tag{13}$$

The following result is proved in [18, 19].

Theorem 2.2. Let A and B be the self-adjoint operators on a Hilbert space H. Then

$$\|(A-a)f\|_H \|(B-b)f\|_H \geq \frac{1}{2}|\langle [A, B]f, f\rangle_H|, \tag{14}$$

for all $f \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$, and all $a, b \in \mathbb{R}$.

Theorem 2.3. Let $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$. For all $a, b \in \mathbb{R}$, one has

$$\|(A + A^* - a)f\|_H \|(A - A^* + ib)f\|_H \geq |\|Af\|_H^2 - \|A^*f\|_H^2|, \tag{15}$$

where i is the imaginary unit.

Proof. Let us consider the following two operators on $\mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$ by

$$P = A + A^*, \quad Q = i(A - A^*). \tag{16}$$

It follows that, for $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$, we have $Pf, Qf \in H$. The operators P and Q are self-adjoint and $[P, Q] = -2i[A, A^*]$. Thus the inequality (15) follows from Theorems 2.1 and 2.2. $\square$

Theorem 2.4. Let $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$. Then

$$\Delta_H^+(f)\Delta_H^-(f) \geq \|f\|_H^4 \left(\|Af\|_H^2 - \|A^*f\|_H^2\right)^2, \tag{17}$$

where

$$\Delta_H^{\pm}(f) = \|f\|_H^2 \|(A \pm A^*)f\|_H^2 - |\langle (A \pm A^*)f, f\rangle_H|^2. \tag{18}$$

Proof. Let $f \in \mathrm{Dom}(AA^*) \cap \mathrm{Dom}(A^*A)$. The operator P given by (16) is self-adjoint; then for any real a, we have

$$\|(P-a)f\|_H^2 = \|Pf\|_H^2 + a^2\|f\|_H^2 - 2a\langle Pf, f\rangle_H. \tag{19}$$

This shows that

$$\min_{a \in \mathbb{R}} \|(P-a)f\|_H^2 = \|Pf\|_H^2 - \frac{|\langle Pf, f\rangle_H|^2}{\|f\|_H^2}, \tag{20}$$

and the minimum is attained when $a = \frac{\langle Pf, f\rangle_H}{\|f\|_H^2}$. In other words, we have

$$\min_{a \in \mathbb{R}} \|(A + A^* - a)f\|_H^2 = \|(A + A^*)f\|_H^2 - \frac{|\langle (A + A^*)f, f\rangle_H|^2}{\|f\|_H^2}. \tag{21}$$

Similarly

$$\min_{b \in \mathbb{R}} \|(A - A^* + ib)f\|_H^2 = \|(A - A^*)f\|_H^2 - \frac{|\langle (A - A^*)f, f\rangle_H|^2}{\|f\|_H^2}. \tag{22}$$

Then by (15), (21), and (22), we deduce the inequality (17). □

Let $\lambda > 0$ and let $T : H \to K$ be a bounded linear operator from H into a Hilbert space K. Building on the ideas of Saitoh [2], we examine the minimizer function associated with the operator T.

Theorem 2.5. For any $k \in K$ and for any $\lambda > 0$, the problem

$$\inf_{f \in H} \left\{ \lambda \|f\|_H^2 + \|Tf - k\|_K^2 \right\} \tag{23}$$

has a unique minimizer given by

$$f_{\lambda,k}^* = (\lambda I + T^* T)^{-1} T^* k. \tag{24}$$

Proof. The problem (23) is solved elementarily by finding the roots of the first derivative $D\Phi$ of the quadratic and strictly convex function $\Phi(f) = \lambda \|f\|_H^2 + \|Tf - k\|_K^2$. Note that for convex functions, the equation $D\Phi(f) = 0$ is a necessary and sufficient condition for the minimum at f. The calculation provides

$$D\Phi(f) = 2\lambda f + 2T^*(Tf - k), \tag{25}$$

and the assertion of the theorem follows at once. □

Theorem 2.6. If $T : H \to K$ is an isometric isomorphism; then for any $k \in K$ and for any $\lambda > 0$, the problem

$$\inf_{f \in H} \left\{ \lambda \|f\|_H^2 + \|Tf - k\|_K^2 \right\} \tag{26}$$

has a unique minimizer given by

$$f_{\lambda,h}^* = \frac{1}{\lambda + 1} T^{-1} k. \tag{27}$$

Proof. We have $T^* = T^{-1}$ and $T^* T = I$. Thus, by (24), we deduce the result. □

3. The Hardy space $H(\mathbb{D})$

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk. The Hardy space $H(\mathbb{D})$ is the set of all analytic functions f in the unit disk $\mathbb{D}$ with the finite integral:

$$\int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 \mathrm{d}\theta. \tag{28}$$

It is a Hilbert space when equipped with the inner product:

$$\langle f, g \rangle_{H(\mathbb{D})} = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g(e^{i\theta})}\, \mathrm{d}\theta. \tag{29}$$

If $f, g \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g \rangle_{H(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}. \tag{30}$$

The set $\{z^n\}_{n=0}^{\infty}$ forms an Hilbert's basis for the space $H(\mathbb{D})$.
The Szegő kernel S_z given for $z \in \mathbb{D}$, by

$$S_z(w) = \sum_{n=0}^{\infty} \overline{z}^n w^n = \frac{1}{1-\overline{z}w}, \quad w \in \mathbb{D}, \tag{31}$$

is a reproducing kernel for the Hardy space $H(\mathbb{D})$, meaning that $S_z \in H(\mathbb{D})$, and for all $f \in H(\mathbb{D})$, we have $\langle f, S_z \rangle_{H(\mathbb{D})} = f(z)$.

For $z \in \mathbb{D}$, the function $u(z) = S_{\overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem:

$$u'(z) = w(zu'(z) + u(z)), \quad w \in \mathbb{D}, \quad u(0) = 1. \tag{32}$$

In the next of this section, we define the operators ∇, $\mathfrak{R}$, and L on $H(\mathbb{D})$ by

$$\nabla f(z) = f'(z), \quad \mathfrak{R} f(z) = zf'(z), \quad Lf(z) = z^2 f'(z) + zf(z). \tag{33}$$

These operators satisfy the commutation rule:

$$[\nabla, L] = \nabla L - L\nabla = 2\mathfrak{R} + I, \tag{34}$$

where I is the identity operator.

We define the Hilbert space $U(\mathbb{D})$ as the space of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{U(\mathbb{D})}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f'\left(e^{i\theta}\right) \right|^2 \mathrm{d}\theta < \infty. \tag{35}$$

If $f \in U(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|f\|_{U(\mathbb{D})}^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2. \tag{36}$$

Thus, the space $U(\mathbb{D})$ is a subspace of the Hardy space $H(\mathbb{D})$.

Theorem 3.1.

i. For $f \in U(\mathbb{D})$, then ∇f, $\mathfrak{R} f$ and Lf belong to $H(\mathbb{D})$.

ii. $\nabla^* = L$.

iii. For $f \in U(\mathbb{D})$, one has

$$\|Lf\|_{H(\mathbb{D})}^2 = \|\nabla f\|_{H(\mathbb{D})}^2 + \|f\|_{H(\mathbb{D})}^2 + 2\langle \mathfrak{R} f, f \rangle_{H(\mathbb{D})}. \tag{37}$$

Proof.

i. Let $f \in U(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\nabla f(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, \quad \mathfrak{R} f(z) = \sum_{n=1}^{\infty} n a_n z^n, \tag{38}$$

and

$$Lf(z) = \sum_{n=1}^{\infty} n a_{n-1} z^n. \tag{39}$$

Therefore

$$\|\nabla f\|^2_{H(\mathbb{D})} = \sum_{n=0}^{\infty} (n+1)^2 |a_{n+1}|^2 = \|f\|^2_{U(\mathbb{D})}, \tag{40}$$

$$\|\Re f\|^2_{H(\mathbb{D})} = \sum_{n=1}^{\infty} n^2 |a_n|^2 = \|f\|^2_{U(\mathbb{D})}, \tag{41}$$

and

$$\|Lf\|^2_{H(\mathbb{D})} = \sum_{n=0}^{\infty} (n+1)^2 |a_n|^2 \leq |f(0)|^2 + 4\|f\|^2_{U(\mathbb{D})}. \tag{42}$$

Consequently ∇f, $\Re f$, and Lf belong to $H(\mathbb{D})$.

ii. For $f, g \in U(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle \nabla f, g \rangle_{H(\mathbb{D})} = \sum_{n=0}^{\infty} (n+1) a_{n+1} \overline{b_n} = \sum_{n=1}^{\infty} n a_n \overline{b_{n-1}} = \langle f, Lg \rangle_{H(\mathbb{D})}. \tag{43}$$

Thus $\nabla^* = L$.

iii. Let $f \in U(\mathbb{D})$. By (ii) and (34), we deduce that

$$\|Lf\|^2_{H(\mathbb{D})} = \langle \nabla Lf, f \rangle_{H(\mathbb{D})} \tag{44}$$

$$= \langle L\nabla f, f \rangle_{H(\mathbb{D})} + \langle [\nabla, L] f, f \rangle_{H(\mathbb{D})} \tag{45}$$

$$= \|\nabla f\|^2_{H(\mathbb{D})} + \|f\|^2_{H(\mathbb{D})} + 2\langle \Re f, f \rangle_{H(\mathbb{D})}. \quad \square \tag{46}$$

Theorem 3.2. Let $f \in U(\mathbb{D})$. For all $a, b \in \mathbb{R}$, one has

$$\|(\nabla + L - a) f\|_{H(\mathbb{D})} \|(\nabla - L + ib) f\|_{H(\mathbb{D})} \geq \|f\|^2_{H(\mathbb{D})} + 2\langle \Re f, f \rangle_{H(\mathbb{D})}. \tag{47}$$

Theorem 3.3. Let T_1 be the difference operator defined on $H(\mathbb{D})$ by

$$T_1 f(z) = \frac{1}{z}(f(z) - f(0)). \tag{48}$$

i. The operator T_1 maps continuously from $H(\mathbb{D})$ to $H(\mathbb{D})$, and

$$\|T_1 f\|_{H(\mathbb{D})} \leq \|f\|_{H(\mathbb{D})}. \tag{49}$$

ii. For $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, we have

$$T_1^* f(z) = z f(z), \quad T_1^* T_1 f(z) = f(z) - f(0). \tag{50}$$

iii. For any $h \in H(\mathbb{D})$ and for any $\lambda > 0$, the problem

$$\inf_{f \in H(\mathbb{D})} \left\{ \lambda \|f\|^2_{H(\mathbb{D})} + \|T_1 f - h\|^2_{H(\mathbb{D})} \right\} \tag{51}$$

has a unique minimizer given by

$$f^*_{\lambda,h}(z) = \frac{1}{\lambda + 1} z h(z), \quad z \in \mathbb{D}. \tag{52}$$

Proof.

i. If $f \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $T_1 f(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ and

$$\|T_1 f\|_{H(\mathbb{D})}^2 = \sum_{n=1}^{\infty} |a_n|^2 \leq \|f\|_{H(\mathbb{D})}^2. \tag{53}$$

ii. If $f, g \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle T_1 f, g \rangle_{H(\mathbb{D})} = \sum_{n=0}^{\infty} a_{n+1} \overline{b_n} = \sum_{n=1}^{\infty} a_n \overline{b_{n-1}} = \langle f, T_1^* g \rangle_{H(\mathbb{D})}, \tag{54}$$

where $T_1^* g(z) = zg(z)$, for $z \in \mathbb{D}$. And therefore

$$T_1^* T_1 f(z) = z T_1 f(z) = f(z) - f(0). \tag{55}$$

iii. From Theorem 2.5 we have

$$\left(\lambda I + T_1^* T_1\right) f_{\lambda,h}^*(z) = T_1^* h(z). \tag{56}$$

By (ii) we deduce that

$$(\lambda + 1) f_{\lambda,h}^*(z) - f_{\lambda,h}^*(0) = zh(z). \tag{57}$$

And from this equation, $f_{\lambda,h}^*(0) = 0$. Hence

$$f_{\lambda,h}^*(z) = \frac{1}{\lambda + 1} zh(z). \quad \square \tag{58}$$

4. The Dirichlet space $\mathcal{D}(\mathbb{D})$

The Dirichlet space $\mathcal{D}(\mathbb{D})$ is the set of all analytic functions f in the unit disk $\mathbb{D}$ with the finite Dirichlet integral:

$$\int_{\mathbb{D}} \left| f'(z) \right|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}, \quad z = x + iy. \tag{59}$$

It is a Hilbert space when equipped with the inner product:

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{D})} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}, \quad z = x + iy. \tag{60}$$

If $f, g \in \mathcal{D}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{D})} = a_0 \overline{b_0} + \sum_{n=1}^{\infty} n a_n \overline{b_n}. \tag{61}$$

The set $\left\{1, \frac{z^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ forms an Hilbert's basis for the space $\mathcal{D}(\mathbb{D})$.
The function K_z given for $z \in \mathbb{D}$, by

$$K_z(w) = 1 + \log\left(\frac{1}{1 - \bar{z}w}\right), \quad w \in \mathbb{D}, \tag{62}$$

is a reproducing kernel for the Dirichlet space $\mathcal{D}(\mathbb{D})$, meaning that $K_z \in \mathcal{D}(\mathbb{D})$, and for all $f \in \mathcal{D}(\mathbb{D})$, we have $\langle f, K_z \rangle_{\mathcal{D}(\mathbb{D})} = f(z)$.

For $z \in \mathbb{D}$, the function $u(z) = K_{\bar{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem:

$$\frac{u'(z) - u'(0)}{z} = wu'(z), \quad w \in \mathbb{D}, \quad u(0) = 1. \tag{63}$$

In the next of this section, we define the operators Λ, $\mathfrak{R}$, and X on $\mathcal{D}(\mathbb{D})$ by

$$\Lambda f(z) = f'(z) - f'(0), \quad \mathfrak{R} f(z) = z f'(z), \quad Xf(z) = z^2 f'(z). \tag{64}$$

These operators satisfy the following commutation relation:

$$[\Lambda, X] = \Lambda X - X\Lambda = 2\mathfrak{R}. \tag{65}$$

We define the Hilbert space $V(\mathbb{D})$ as the space of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{V(\mathbb{D})} = \int_{\mathbb{D}} \left| f'(z) \right|^2 |z|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{\pi} < \infty, \quad z = x + iy. \tag{66}$$

If $f \in V(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|f\|^2_{V(\mathbb{D})} = \sum_{n=1}^{\infty} n^3 |a_n|^2. \tag{67}$$

Thus, the space $V(\mathbb{D})$ is a subspace of the Dirichlet space $\mathcal{D}(\mathbb{D})$.

Theorem 4.1.

i. For $f \in V(\mathbb{D})$, then Λf, $\mathfrak{R} f$, and Xf belong to $\mathcal{D}(\mathbb{D})$.

ii. $\Lambda^* = X$.

iii. For $f \in V(\mathbb{D})$, one has

$$\|Xf\|^2_{\mathcal{D}(\mathbb{D})} = \|\Lambda f\|^2_{\mathcal{D}(\mathbb{D})} + 2\langle \mathfrak{R} f, f \rangle_{\mathcal{D}(\mathbb{D})}. \tag{68}$$

Proof.

i. Let $f \in V(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\Lambda f(z) = \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n, \quad \mathfrak{R} f(z) = \sum_{n=1}^{\infty} n a_n z^n, \tag{69}$$

and

$$Xf(z) = \sum_{n=2}^{\infty} (n-1) a_{n-1} z^n. \tag{70}$$

Therefore

$$\|\Lambda f\|^2_{\mathcal{D}(\mathbb{D})} = \sum_{n=1}^{\infty} n(n+1)^2 |a_{n+1}|^2 \leq \sum_{n=2}^{\infty} n^3 |a_n|^2 \leq \|f\|^2_{V(\mathbb{D})}, \tag{71}$$

$$\|\mathfrak{R}f\|^2_{\mathcal{D}(\mathbb{D})} = \sum_{n=1}^{\infty} n^3|a_n|^2 = \|f\|^2_{V(\mathbb{D})}, \tag{72}$$

and

$$\|Xf\|^2_{\mathcal{D}(\mathbb{D})} = \sum_{n=1}^{\infty}(n+1)n^2|a_n|^2 \le 2\|f\|^2_{V(\mathbb{D})}. \tag{73}$$

Consequently Λf, $\mathfrak{R}f$, and Xf belong to $\mathcal{D}(\mathbb{D})$.

ii. For $f, g \in V(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle \Lambda f, g\rangle_{\mathcal{D}(\mathbb{D})} = \sum_{n=1}^{\infty} n(n+1)a_{n+1}\overline{b_n} = \sum_{n=2}^{\infty} n(n-1)a_n\overline{b_{n-1}} = \langle f, Xg\rangle_{\mathcal{D}(\mathbb{D})}. \tag{74}$$

iii. Let $f \in V(\mathbb{D})$. By (ii) and (65), we deduce that

$$\|Xf\|^2_{\mathcal{D}(\mathbb{D})} = \langle \Lambda Xf, f\rangle_{\mathcal{D}(\mathbb{D})} \tag{75}$$

$$= \langle X\Lambda f, f\rangle_{\mathcal{D}(\mathbb{D})} + \langle [\Lambda, X]f, f\rangle_{\mathcal{D}(\mathbb{D})} \tag{76}$$

$$= \|\Lambda f\|^2_{\mathcal{D}(\mathbb{D})} + 2\langle \mathfrak{R}f, f\rangle_{\mathcal{D}(\mathbb{D})}. \quad \square \tag{77}$$

Theorem 4.2. Let $f \in V(\mathbb{D})$. For all $a, b \in \mathbb{R}$, one has

$$\|(\Lambda + X - a)f\|_{\mathcal{D}(\mathbb{D})}\|(\Lambda - X + ib)f\|_{\mathcal{D}(\mathbb{D})} \ge 2\langle \mathfrak{R}f, f\rangle_{\mathcal{D}(\mathbb{D})}. \tag{78}$$

Theorem 4.3. Let T_2 be the difference operator defined on $\mathcal{D}(\mathbb{D})$ by

$$T_2 f(z) = \frac{1}{z}\big(f(z) - zf'(0) - f(0)\big). \tag{79}$$

i. The operator T_2 maps continuously from $\mathcal{D}(\mathbb{D})$ to $\mathcal{D}(\mathbb{D})$, and

$$\|T_2 f\|_{\mathcal{D}(\mathbb{D})} \le \|f\|_{\mathcal{D}(\mathbb{D})}. \tag{80}$$

ii. For $f \in \mathcal{D}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$T_2^* f(z) = \sum_{n=2}^{\infty} \frac{n-1}{n} a_{n-1} z^n, \quad T_2^* T_2 f(z) = \sum_{n=2}^{\infty} \frac{n-1}{n} a_n z^n. \tag{81}$$

iii. For any $d \in \mathcal{D}(\mathbb{D})$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathcal{D}(\mathbb{D})} \left\{ \lambda \|f\|^2_{\mathcal{D}(\mathbb{D})} + \|T_2 f - d\|^2_{\mathcal{D}(\mathbb{D})} \right\} \tag{82}$$

has a unique minimizer given by

$$f^*_{\lambda,d}(z) = \langle d, \Psi_z\rangle_{\mathcal{D}(\mathbb{D})}, \quad z \in \mathbb{D}, \tag{83}$$

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{\overline{z}^{n+1}}{\lambda(n+1)+n} w^n, \quad w \in \mathbb{D}. \tag{84}$$

Proof.

i. If $f \in \mathcal{D}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $T_2 f(z) = \sum_{n=1}^{\infty} a_{n+1} z^n$ and

$$\|T_2 f\|^2_{\mathcal{D}(\mathbb{D})} = \sum_{n=2}^{\infty} (n-1)|a_n|^2 \leq \sum_{n=2}^{\infty} n|a_n|^2 \leq \|f\|^2_{\mathcal{D}(\mathbb{D})}. \tag{85}$$

ii. If $f, g \in \mathcal{D}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle T_2 f, g \rangle_{\mathcal{D}(\mathbb{D})} = \sum_{n=1}^{\infty} n a_{n+1} \overline{b_n} = \sum_{n=2}^{\infty} (n-1) a_n \overline{b_{n-1}} = \left\langle f, T_2^* g \right\rangle_{\mathcal{D}(\mathbb{D})}, \tag{86}$$

where

$$T_2^* g(z) = \sum_{n=2}^{\infty} \frac{n-1}{n} b_{n-1} z^n, \quad z \in \mathbb{D}. \tag{87}$$

And therefore

$$T_2^* T_2 f(z) = \sum_{n=2}^{\infty} \frac{n-1}{n} a_n z^n. \tag{88}$$

iii. We put $d(z) = \sum_{n=0}^{\infty} d_n z^n$ and

$$f^*_{\lambda,d}(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{89}$$

From (ii) and the equation

$$\left(\lambda I + T_2^* T_2\right) f^*_{\lambda,d}(z) = T_2^* d(z), \tag{90}$$

we deduce that

$$c_1 = c_0 = 0, \quad c_n = \frac{n-1}{\lambda n + n - 1} d_{n-1}, \quad n \geq 2. \tag{91}$$

Thus

$$f^*_{\lambda,d}(z) = \sum_{n=1}^{\infty} \frac{n d_n}{\lambda(n+1) + n} z^{n+1} = \langle d, \Psi_z \rangle_{\mathcal{D}(\mathbb{D})}, \quad z \in \mathbb{D}. \quad \square \tag{92}$$

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References

[1] Saitoh S. The Weierstrass transform and an isometry in the heat equation. Applicable Analysis. 1983;**16**:1-6

[2] Saitoh S. Best approximation, Tikhonov regularization and reproducing kernels. Kodai Mathematical Journal. 2005;**28**:359-367

[3] Matsuura T, Saitoh S, Trong DD. Inversion formulas in heat conduction multidimensional spaces. Journal of Inverse and Ill-Posed Problems. 2005;**13**: 479-493

[4] Matsuura T, Saitoh S. Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley-Wiener spaces. Applicable Analysis. 2006;**85**:901-915

[5] Paulsen VI. An Introduction to the Theory of Reproducing Kernel Hilbert Spaces. Cambridge: Cambridge University Press; 2016

[6] Saitoh S. Approximate real inversion formulas of the Gaussian convolution. Applicable Analysis. 2004; **83**:727-733

[7] Soltani F. Operators and Tikhonov regularization on the Fock space. Integral Transforms and Special Functions. 2014;**25**(4):283-294

[8] Soltani F. Inversion formulas for the Dunkl-type Segal-Bargmann transform. Integral Transforms and Special Functions. 2015;**26**(5):325-339

[9] Soltani F. Dunkl multiplier operators on a class of reproducing kernel Hilbert spaces. Journal of Mathematical Research with Applications. 2016;**36**(6): 689-702

[10] Tuan VK, Hong NT. Interpolation in the Hardy space. Integral Transforms and Special Functions. 2013;**24**(8): 664-671

[11] Bohm A, Bui HV. The marvelous consequences of Hardy spaces in quantum physics. Geometric Methods in Physics. 2013;**30**(1):211-228

[12] Mouayn Z. Resolution of the identity of the classical Hardy space by means of Barut-Girardello coherent states. Mathematical Physics. 2012;**2012**: Article ID 530473, 12 p

[13] Arcozzi N, Rochberg R, Sawyer ET, Wick BD. The Dirichlet space: A survey. New York Journal of Mathematics. 2011; **17a**:45-86

[14] Chartrand R. Toeplitz operators on Dirichlet-type spaces, J. Operator Theory. 2002;**48**(1):3-13

[15] Geng L, Tong C, Zeng H. Some linear isometric operators on the Dirichlet space. Applied Mathematics & Information Sciences. 2012;**6**(1):265-270

[16] Geng LG, Zhou ZH, Dong XT. Isometric composition operators on weighted Dirichlet-type spaces. Journal of Inequalities and Applications. 2012; **2012**:23

[17] Martin MJ, Vukotic D. Isometries of the Dirichlet space among the composition operators. Proceedings of the American Mathematical Society. 2005;**134**:1701-1705

[18] Folland G. Harmonic analysis on phase space. In: Annals of Mathematics Studies. Vol. 122. Princeton, New Jersey: Princeton University Press; 1989

[19] Gröchenig K. Foundations of Time-frequency Analysis. Boston: Birkhäuser; 2001