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Chapter

Discrete Time Sliding Mode Control

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Abstract

This chapter discusses the concept of discrete-time sliding mode control (DTSMC) and its design procedure. It also covers how the states are brought to a predefined sliding surface mathematically and kept in a region near to the surface within a small band. This band is termed as an ultimate band in the field of DTSMC, which denotes the degree of robustness. Researchers have been working to find out different approaches to reach to that surface, but the most promising and well-defined way is reaching law approach. The idea of reaching law is discussed briefly in this chapter with examples for better understanding of the design procedure. In this chapter, a small introduction of continuous time sliding mode control (CTSMC) is given. Finally, the current state of the art is presented.

Keywords: sliding mode control, variable structure control, chattering, reaching law, robustness, quasi-sliding band, relative degree, ultimate band

1. Introduction

The elevator statement about sliding mode control (SMC) is that it is one of the robust control design techniques which is mathematically well-structured and assures performance in the presence of certain class of disturbance and uncertainties. Due to this it is used for controlling practical uncertain systems. It is originated from the concept of variable structure control (VSC). The name VSC itself describes that there is more than one structure defining a system which describes the complete behavior of the variable structure systems. In VSC, the control input is logically so chosen that the final closed-loop system behavior becomes stable regardless of the natures of the substructures (stable or unstable). This gives rise to a new system behavior not a part of any of the substructures. This phenomenon of getting a new system behavior is called sliding mode in the domain of variable structure control [1–4].

The design procedure of SMC consists of two steps. The first step is to design a sliding surface appropriately which decides the behavior of the system during sliding. Then a control action is designed so that all the state trajectories are steered to the sliding surface in finite time and then forced to stay on the surface. Once the sliding is established, i.e., the trajectories are on the sliding surface, the system becomes invariant to modelling inaccuracies and exogenous disturbances. The term "invariant" is stronger than robustness as it satisfies certain conditions additionally. The whole design procedure can be observed in three modes or phases, i.e., reaching mode, sliding mode, and steady-state mode. Reaching mode is the phase where the state trajectories are driven to the sliding surface. It is also known as hitting mode or

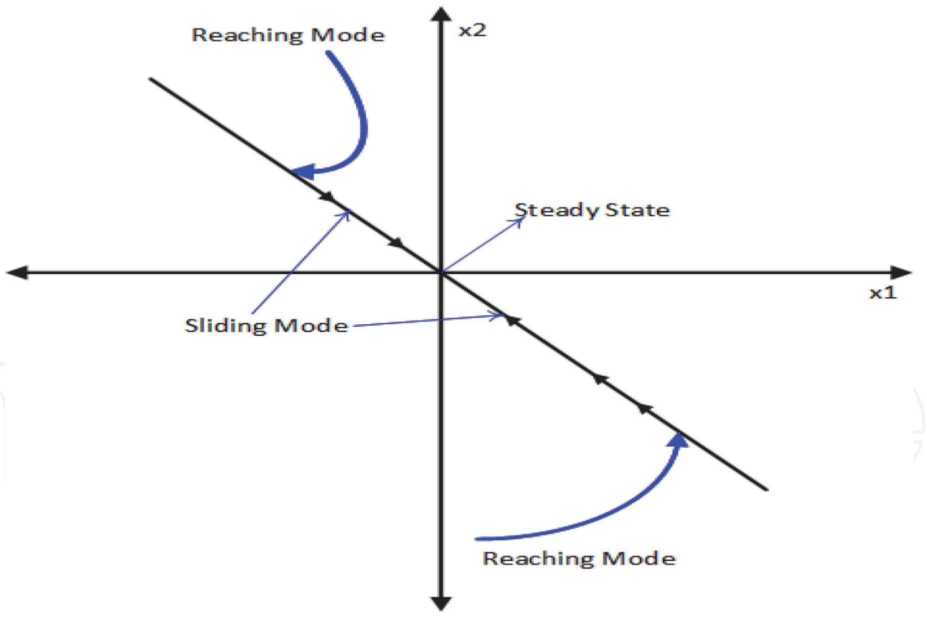

Figure 1.
Trajectories of ideal variable structure systems.

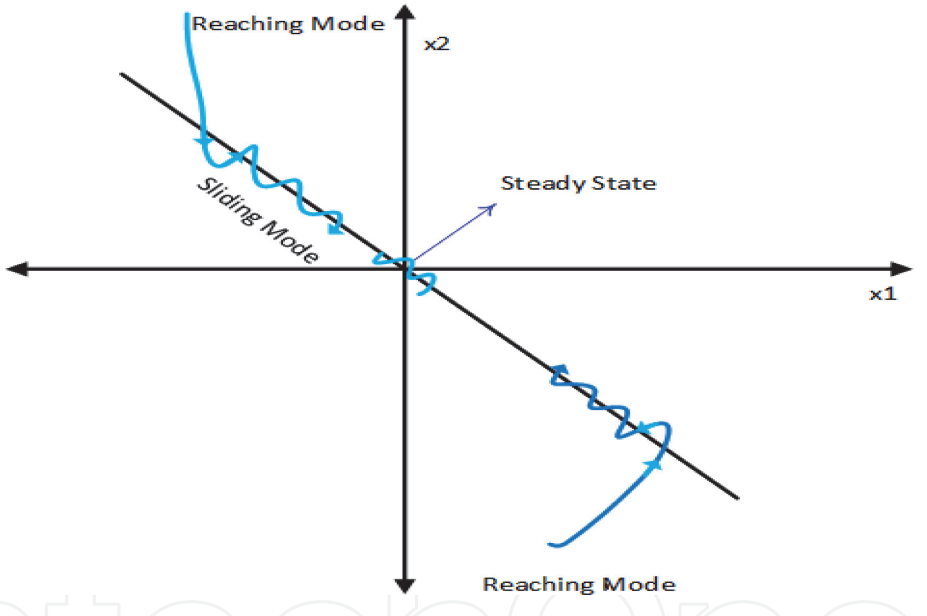

Figure 2.
Trajectories of practical variable structure systems.

non-sliding mode. In sliding mode, the trajectories are restrained and kept moving along the surface towards the equilibrium point or reference point. Finally, in steady-state mode, the system reaches its final state, which would be zero-error state, constant offset state, or limit-cycle state. Different modes of VSC are shown in **Figures 1** and **2**.

SMC is always being judged by its steady-state mode, more specifically for chattering. Chattering is a high-frequency oscillation around the equilibrium point which arises due to the discontinuous nature of the control action. Due to this, the well-designed control action stands unsuitable for many practical applications. This behavior creates a problem of wear and tears in the mechanical parts, vibrations in the machines or flapping of wing vanes in aerospace, hitting effect, etc. Hence, it is unwanted in the light of implementation. The discontinuous nature demanded by the control action cannot be delivered by any real physical actuator due to its finite bandwidth. The numerical computation done by a computer is also limited by certain clock cycles. A lot of works have been done in the field of chattering elimination and reduction. Schemes like continuous approximation around the sliding surface (quasi-sliding mode) [1–3], higher-order sliding mode [5–8], discrete-time sliding modes are a few way outs for the process of chattering. Here in this chapter, the concept of discrete-time sliding mode (DTSM) design is discussed. Readers can explore more in the field of continuous time higher-order sliding mode whose theory is rich and well-structured.

2. Discrete-time sliding mode control

Control system designs are streaming from continuous to discrete design with the invention of digital circuitry. High-performance computing devices, portable microprocessors, and plug and play features make the sophisticated design easy to implement. Discrete-time sliding mode control is the obvious transformation from the continuous time sliding mode control for the real-time application. Like continuous time sliding mode control, DTSMC is also easy to design and also well-suited for implementation.

3. Control problem formulation

Consider an uncertain discrete-time system:

$$x(k+1) = Ax(k) + B[u(k) + f(k)] \tag{1}$$

where the states $x(k) \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m, f \in \mathbb{R}^m$, and the output $y \in \mathbb{R}$. $f(k)$ is the disturbance coming from an exogenous system and is upper bounded by f_m. A, B are system matrix and input matrix, respectively, and are having appropriate dimensions. Here the problem is either to stabilize the system, i.e., $\lim_{k\to\infty} x(k) = 0$, or to track a time-varying trajectory, i.e., $\lim_{k\to\infty} x(k) = x_d(k)$, where $x_d(k)$ is the desired trajectory. But tracking can be treated as error stabilization mathematically, i.e., by making $\lim_{k\to\infty} e(k) = 0$ where $e(k) = x(k) - x_d(k)$. The system (1) will be transferred to error space $e(k+1) = Ae(k) + Bu(k) + f(k) + Ax_d(k) - x_d(k+1)$. So here in this chapter, only stabilization is addressed for single-input single-output system.

3.1 Controller design by Gao's reaching law

3.1.1 Design procedure

Here the aim is to design a control law $u(k)$ such that $\lim_{k\to\infty} x(k) = 0$. The first step is to choose a sliding variable as

$$s(k) = c^T x(k) \tag{2}$$

where c is a sliding variable design parameter. Next step is to choose Gao's reaching law [9].

$$s(k+1) = \alpha s(k) - \beta sign(s(k) + d(k) \tag{3}$$

where $\alpha \in (0, 1)$ and $\beta > 0$ and $d(k)$ are assumed to be the same as the uncertain quantity $c^T Bf(k)$ and are bounded by $d_m = |c^T Bf_m|$. A detailed selection procedure of α and β is given in the next section. Using Eqs. (1)–(3), one can write

$$s(k+1) = c^T x(k+1) = c^T Ax(k) + c^T B[u(k) + f(k)] \tag{4}$$

$$c^T Ax(k) + c^T B[u(k) + f(k)] = \alpha s(k) - \beta sign(s(k) + d(k) \tag{5}$$

and control input can be derived as

$$u(k) = -\left(c^T B\right)^{-1}\left[c^T Ax(k) - \alpha s(k) + \beta sign(s(k)\right] \tag{6}$$

By applying this control input (6), states are brought to a band around the sliding surface $s(k) = 0$ by assuming $c^T B$ to be non-singular.

3.1.2 Procedure to choose the sliding variable parameter

The system (1) can be transformed to regular from by using QR factorization method [10]. There exists an invertible linear operator, T, which transforms system (1)–(7):

$$\begin{aligned} x_1(k+1) &= a_{11}x_1(k) + a_{12}x_2(k) \\ x_2(k+1) &= a_{21}x_1(k) + a_{22}x_2(k) + b_2[u(k) + f(k)] \end{aligned} \tag{7}$$

where $a_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $a_{12} \in \mathbb{R}^{(n-m)\times m}$, $a_{21} \in \mathbb{R}^{m\times(n-m)}$, $a_{22} \in \mathbb{R}^{m\times m}$, and $b_2 \in \mathbb{R}^{m\times m}$. b_2 is assumed to be non-singular.

$c^T = [c_1 \ \ I_m]$ should be chosen such that the nominal closed loop system (i.e., without disturbance) should be stable. The sliding variable is chosen as

$$s(k) = c_1 x_1(k) + I_m x_2(k) \tag{8}$$

where $c_1 \in \mathbb{R}^{m\times(n-m)}$ and I_m are a unity matrix of order m. During the period of ideal sliding,

$$\begin{aligned} &c_1 x_1(k) + I_m x_2(k) = 0 \\ &\Rightarrow x_2(k) = -c_1 x_1(k) \end{aligned} \tag{9}$$

Then the system in closed loop is described by

$$x_1(k+1) = (a_{11} - a_{12}c_1)x_1(k) \tag{10}$$

which guarantees the asymptotic stability by choosing negative real value of the spectrum of $(a_{11} - a_{12}c_1)$, i.e., $Re\,[\sigma(a_{11} - a_{12}c_1)] < 0$.

3.1.3 Analysis of reaching law

Reaching law for a continuous plant is given by

$$\dot{s}(t) = -\mu s(t) - k sign(s(t)) \tag{11}$$

The discrete version of Eq. (11) is proposed by Gao [9] as

$$\begin{aligned} s(k+1) - s(k) &= -\mu\tau s(k) - k\tau sign(s(k)) \\ s(k+1) &= (1 - \mu\tau)s(k) - k\tau sign(s(k)) \end{aligned} \tag{12}$$

$$s(k+1) = \alpha s(k) - \beta sign(s(k)) \tag{13}$$

where $\tau > 0$ is the sampling time. $\mu > 0$ and $k > 0$. $\alpha = 1 - \mu\tau$ and $\beta = k\tau > 0$.

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He proposed few attributes of discrete-time variable structure control to get the trajectories of satisfactory nature. The following attributes are the basis of discrete-time reaching law. If the following conditions are satisfied by the control law, then it is said to achieve the discrete-time sliding mode.

1. The discrete-time control drives the state trajectories monotonically towards the sliding surface from anywhere in the state space and crosses the surface in finite time.

2. From the point of crossing the surface, trajectories will cross the surface in each sampling time, which makes a zigzag motion around the surface.

3. The amplitude of the zigzag oscillation about the surface is non-increasing and restrained the trajectories within a priori band.

The motion of the system is said to be quasi-sliding mode if it satisfies the attributes (2) and (3). Ultimate band denotes the steady-state behavior of the system where the trajectories stay within it for all time in future. If the arithmetic value of the ultimate band is zero, then it is called the ideal quasi-sliding mode.

These attributes are fundamental basis on which the concept of DTSMC stands, but many researchers have already designed it in several other ways.

Remark 1: The value of α and β should be chosen such that all the attributes should be satisfied. To satisfy those attributes, $\alpha \in (0, 1)$ must be chosen. For example, for $|\alpha| > 1$, monotonic nature catered by first attribute may be violated. Similarly, for $\alpha = 0$, the sliding variable oscillates in a constant band of β which again violates the first attribute.

Remark 2: The *sign* term in Eq. (13) confirms the satisfaction of the second and third attributes. But β should be chosen appropriately; otherwise the third attribute may not be satisfied. This reaching law is also known as switching reaching law as the sliding variable switches around the sliding surface $s(k) = 0$, i.e., from positive to negative or vice versa. With higher sampling rate, the control input (6) may create a problem during implementation as the actuator cannot be pushed for such oscillation.

Remark 3: For reaching law (3), β must be chosen more than $\frac{1+\alpha}{1-\alpha} d_m$ where $d(k) \le d_m$.

The explanation is given below.

As per the second and third attributes, if $s(k) > 0$, then $s(k+1) < 0$ and $s(k+2) > 0$ must hold. If by applying control input derived in Eq. (6), $s(k)$ becomes approximately zero and considering the system is affected by maximum value of disturbance, i.e., d_m, then one finds from Eq. (3)

$$s(k+2) = \alpha^2 s(k) - \alpha\beta sign(s(k)) - \beta sign(s(k+1)) + \alpha d(k) + d(k+1) \tag{14}$$

For positive and small value of $s(k)$, further from Eq. (14)

$$s(k+2) = -\alpha\beta sign(s(k)) - \beta sign(s(k+1)) + \alpha d(k) + d(k+1) \tag{15}$$

To show $s(k+2) > 0$ considering extreme value of disturbance $-d_m$, the right-hand side of Eq. (15) must be greater than zero:

$$\begin{aligned} &or, \ -\alpha\beta + \beta - \alpha d_m - d_m > 0 \\ &\Rightarrow \beta(1-\alpha) - (1+\alpha) d_m > 0 \\ &\Rightarrow \beta > \frac{1+\alpha}{1-\alpha} d_m \end{aligned} \tag{16}$$

The value of β comes out same for the case $s(k) < 0$, when $s(k+1) > 0$ and $s(k+2) < 0$ must hold.

Remark 4: The ultimate band (δ) for the reaching law (13) is given by $\delta = \frac{\beta}{1+\alpha}$ [10].

By applying the control input, the sliding variable $s(k)$ becomes a very less value, i.e., δ; then for positive value of $s(k)$ and d_m, one finds from Eq. (13)

$$\begin{aligned} &-\delta = \alpha\delta - \beta \\ &\Rightarrow -\delta(1+\alpha) = -\beta \\ &\Rightarrow \delta = \frac{\beta}{1+\alpha} \end{aligned} \tag{17}$$

Similarly, the ultimate band for the reaching law (3) can be derived as $\delta = \beta + d_m$ by taking $s(k) = 0$.

Remark 5: For nominal system (without disturbance) with the reaching law (13), states are converged to zero asymptotically, but the sliding variable is converged to zero in finite time.

Justification: By choosing an appropriate value of α and very small value of β and with the control input in Eq. (6), finite time convergence is achieved. Once it is achieved, then $s(k+1) = s(k) = 0$:

$$s(k+1) = c^T Ax(k) + c^T Bu(k) = 0 \tag{18}$$

Equivalent control is found as

$$u_{eqv}(k) = -\left(c^T B\right)^{-1} c^T Ax(k) \tag{19}$$

Substituting Eq. (19) in system (1), one gets

$$x(k+1) = \left[I - B\left(c^T B\right)^{-1} c^T\right] Ax(k) \tag{20}$$

The value of c should be chosen such that the eigenvalues of should lie within a unit circle. Once this is satisfied, the asymptotic convergence is guaranteed.

Example 1: Let us take a discrete-time state space model:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} [u(k) + f(k)] \tag{21}$$

Here the aim is to stabilize the states by using discrete-time sliding mode control. $f(k)$ is the disturbance which is upper bounded by 0.01. The value of c^T is chosen as $[0.1 \quad -1]$. The value of α is chosen as 0.1, and the value of β is taken as 0.2544 as per Remark 3. The value of ultimate band is found to be 0.4544. Simulation is done in MATLAB/Simulink in discrete setting with sampling time 1 ms. With the control input derived in Eq. (6), stabilization is done within an ultimate band. Initial value of states is taken as $[-1 \quad 1]$. The amount of control effort is calculated by taking $\sum_{k=0}^{T} |u(k)|$, where simulation is run for T seconds.

From **Figures 3** and **4**, it is clear that the sliding variable cross-recrosses the $s(k) = 0$ in each sampling time and reaches the sliding surface in finite time and stays within a band. It can also be seen that it is bounded by the calculated ultimate band. States of the system are within a band and can be seen in **Figure 5**. The control input is shown in **Figure 6** and the control effort is found to be 0.2642 when the simulation is run for 2 s.

3.2 Controller design by Utkin's reaching law

Prof. Drakunov and Prof. Utkin proposed a non-switching reaching law where the sliding variable $s(k)$ reaches to the sliding surface $s(k) = 0$ in one time step rather than in finite time suggested in [9]. It is motivated by the concept of dead-beat control in discrete-time concept where the steady-state output is attained by the minimal use of control law [11]. Reaching law is given as

$$s(k+1) = 0 \tag{22}$$

For uncertain disturbance affected system, reaching law is given as

$$s(k+1) = d(k) \tag{23}$$

For the system (1) and using the reaching law (23), the control law is modified as

$$u(k) = -\left(c^T B\right)^{-1} c^T A x(k) \tag{24}$$

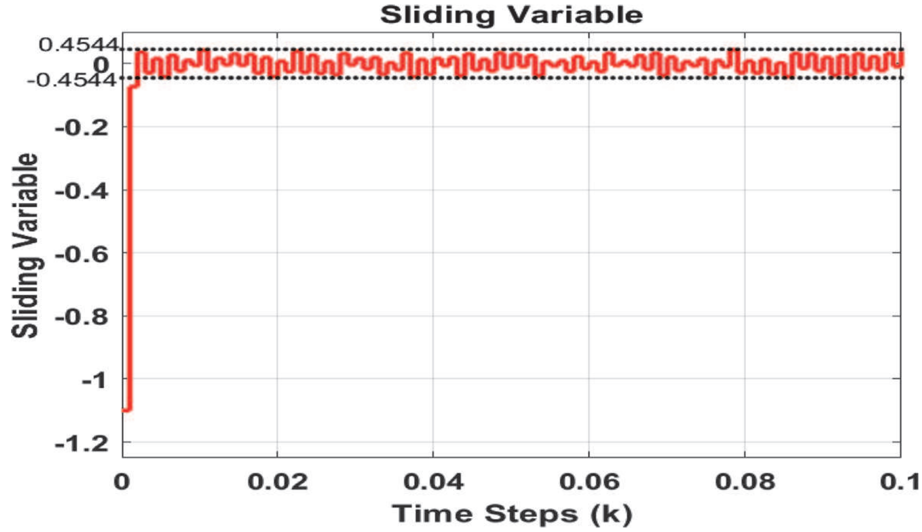

Figure 3.
Sliding variable $s(k)$ evolution for Gao's reaching law.

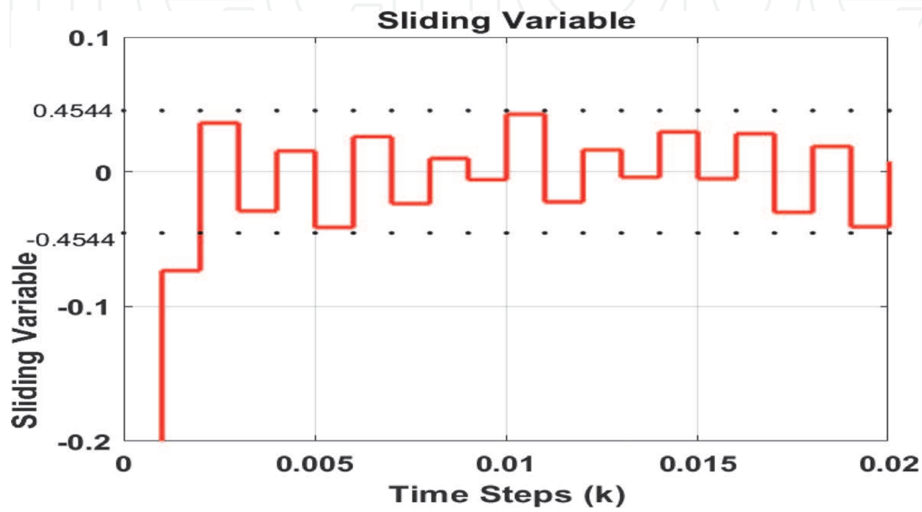

Figure 4.
Magnified part of sliding variable $s(k)$ of ***Figure 3***

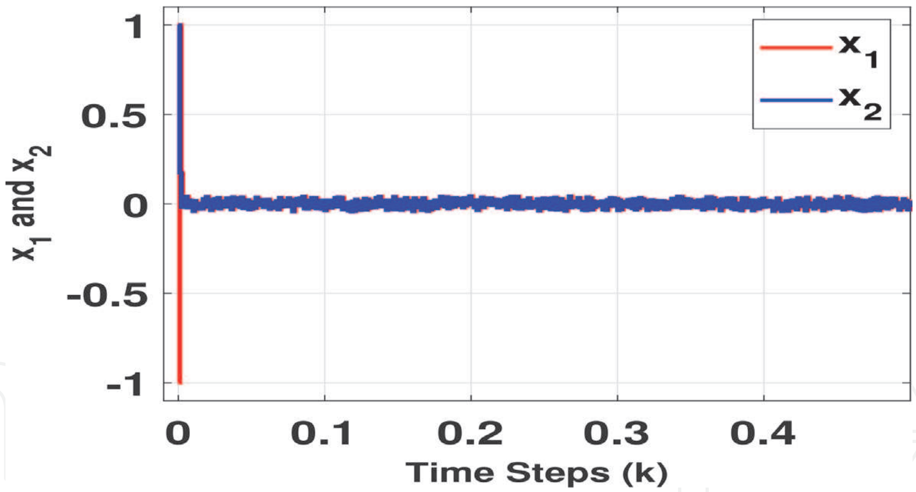

Figure 5.
Evolution of states of the system using Gao's reaching law.

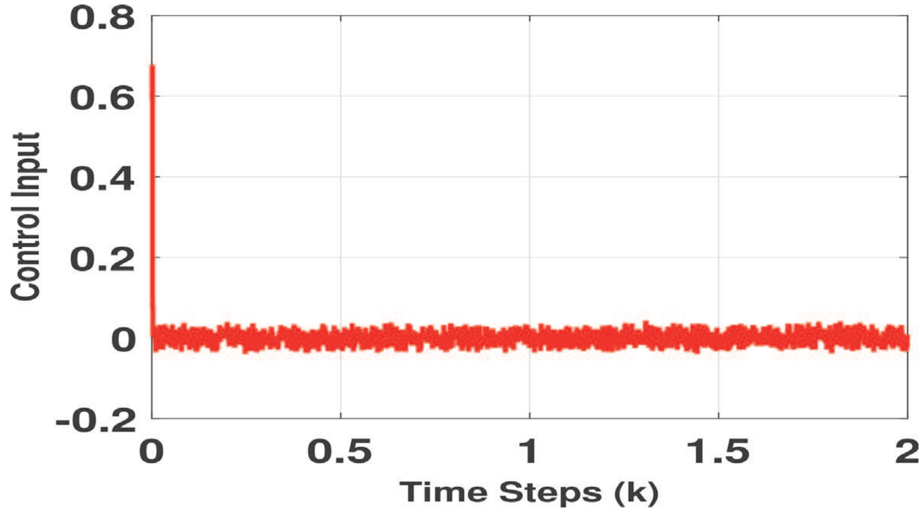

Figure 6.
Control input for Gao's reaching law.

Remark 6: The ultimate band for Eq. (23) is d_m which is lesser than that of ultimate band found from Gao's reaching law.

Remark 7: More control effort may be required as it steers the trajectories to zero in a single step rather than in finite number of steps.

Remark 8: There is no switching demanded across the sliding surface. Hence the control input derived in Eq. (24) becomes more feasible in higher sampling rate.

To reduce the control effort, following control input $u_{mod}(k)$ can be given to the system:

$$u_{mod}(k) = \begin{cases} u(k) & if \quad u(k) \le u_m \\ u_m \dfrac{u(k)}{|u(k)|} & if \quad u(k) > u_m \end{cases} \tag{25}$$

where $u_m > 0$ is the maximum value of control that can be given to the system and $u(k)$ is the control input derived in Eq. (24). In this case the system does not converge to the ultimate band in a single step.

System (21) is considered with the control input derived in Eq. (24) with the same parameters. Ultimate band is calculated as 0.02. From **Figure 7**, it can be

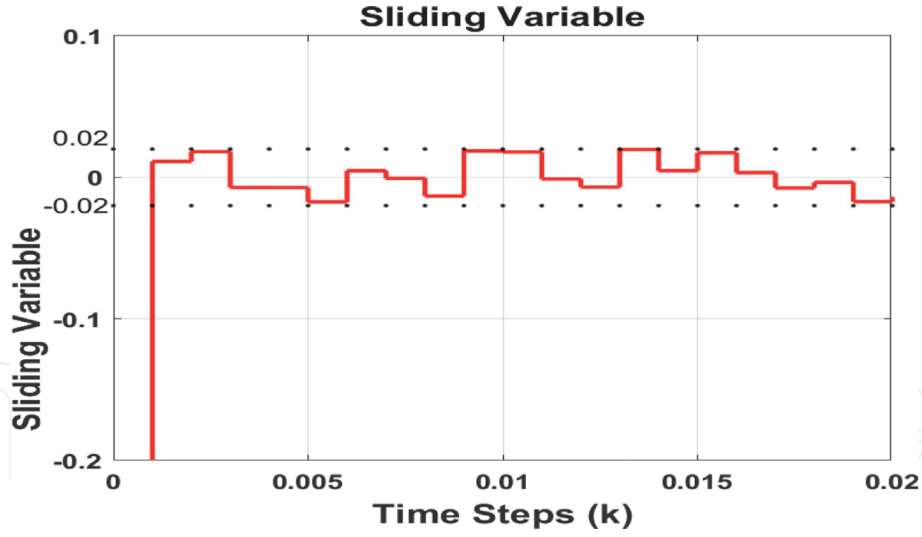

Figure 7.
Sliding variable $s(k)$ evolution for Utkin's reaching law.

noticed that the sliding variable does not have zigzag motion in each sampling time like the sliding variable found in **Figure 4** which shows the non-switching type. Trajectories of states are shown in **Figure 8**. Control input is also non-switching type which makes it more practically implementable and is shown in **Figure 9**. The control effort is numerically found to be 0.0243 which is lesser than that of the Gao's control effort for this case. But it should be noted that the control effort may be higher for other systems. This is explicitly mentioned in the Remark section.

3.3 Controller design by Bartoszewicz's reaching law

Prof. Andrzej Bartoszewicz in [12] suggested a non-switching type reaching law which is linear in nature. Reaching law conditions is given as

$$\begin{aligned} s(k) > \nu &\Rightarrow -\nu \le s(k+1) < s(k) \\ s(k) < -\nu &\Rightarrow s(k) < s(k+1) < \nu \\ or, |s(k)| < \nu &\Rightarrow |s(k+1)| \le \nu \end{aligned} \tag{26}$$

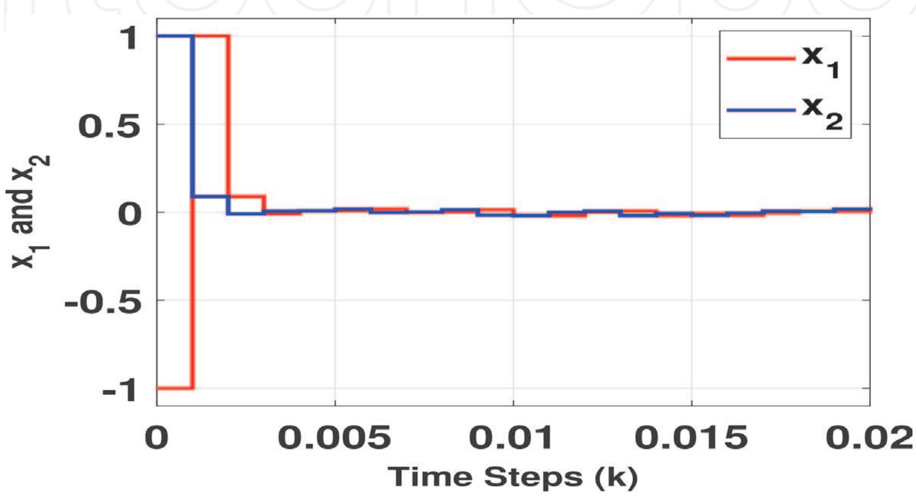

Figure 8.
States of the system using Utkin's reaching law.

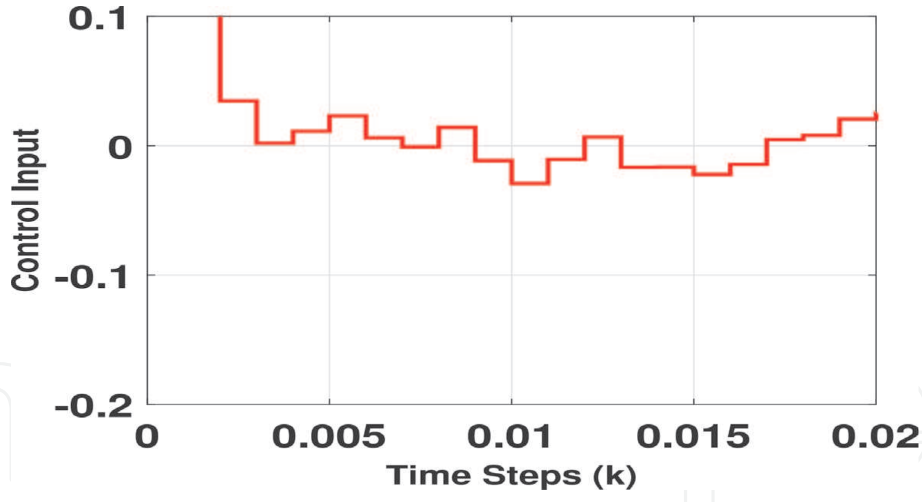

Figure 9.
Control input for Utkin's reaching law.

for $\nu > 0$. Reaching law is proposed by considering a priori function $P_f(k)$ and is given as

$$s(k+1) = P_f(k+1) + d(k)$$
$$P_f(k) = \begin{cases} \dfrac{l* - k}{l*} s(0) & for \quad k < l* \\ 0 & \text{for} \quad k \geq l* \end{cases} \tag{27}$$

where l^* is a positive integer and must satisfy the condition $l* < \frac{s(0)}{2d_m}$. Control input required to stabilize the states in system (1) with this reaching law is derived as

$$u(k) = -\left(c^T B\right)^{-1} \left[c^T Ax(k) - P_f(k+1)\right] \tag{28}$$

Remark 1: The ultimate band for the reaching law (27) is d_m.
Remark 2: Here the states may or may not hit the sliding surface $s(k) = 0$.
Remark 3: Due to the linear control input derived in Eq. (28), the implementation becomes easy for higher sampling rate.

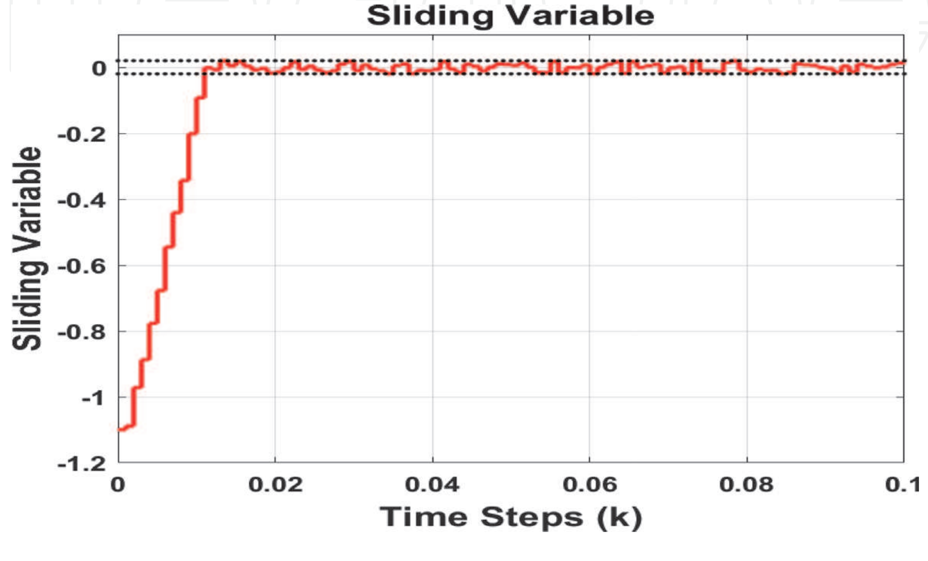

Figure 10.
Sliding variable $s(k)$ evolution for Bartoszewicz's reaching law.

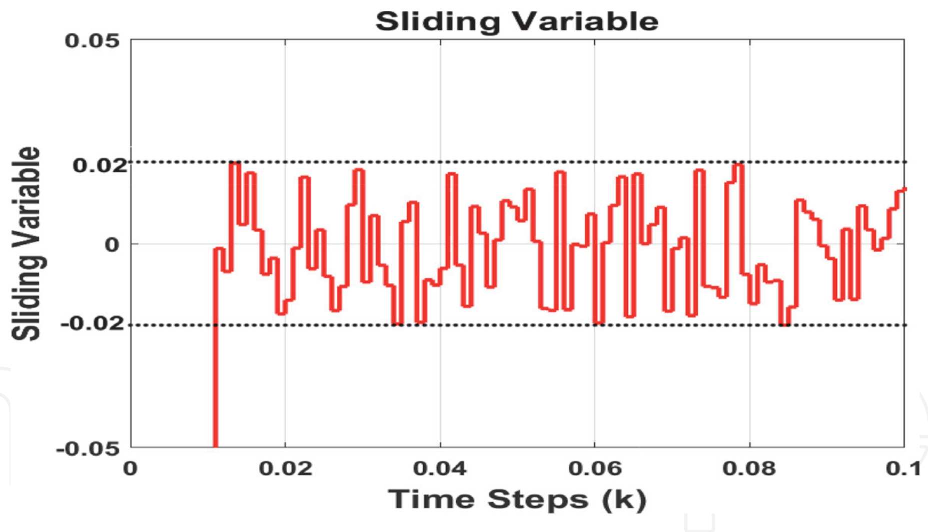

Figure 11.
Magnified part of sliding variable $s(k)$ of ***Figure 10****.*

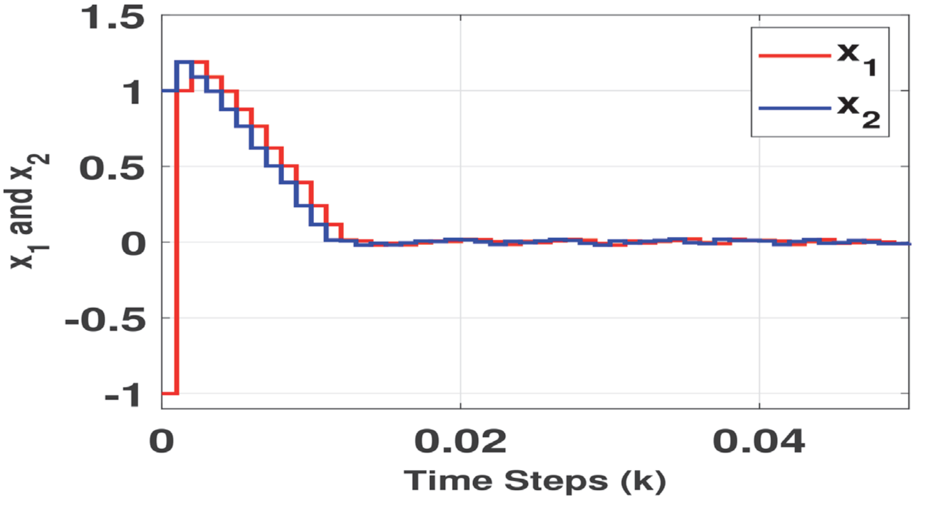

Figure 12.
Evolution of states of the system using Bartoszewicz's reaching law.

Remark 4: The term l^* shows the rate of decay and is a tuning parameter which does a control bargain in terms of amount of control effort and faster convergence. Lesser the value of l^*, more the control input and vice versa.

By taking the same example as in Eq. (21), control input derived in Eq. (28) is used for stabilization. l^* is chosen as 0.1. Sliding variable is shown in **Figure 10**. Ultimate band is found to be 0.02 which is clearly visible in the magnified part of sliding variable shown in **Figure 11**. States stay within a band near to zero and the trajectories are shown in **Figure 12**. Control input is shown in **Figure 13** and the control effort is found to be 0.037. The remark 2 explanation can be seen in **Figure 11**. If we take $l^* = 0.1$, then control effort will be 0.14. Hence the designer should take a good care before choosing the value of l^*.

4. Relative degree two discrete-time sliding variable

Higher relative degree-based reaching laws are explored in the search for better robustness in terms of ultimate band and finding the benefits of using the delayed

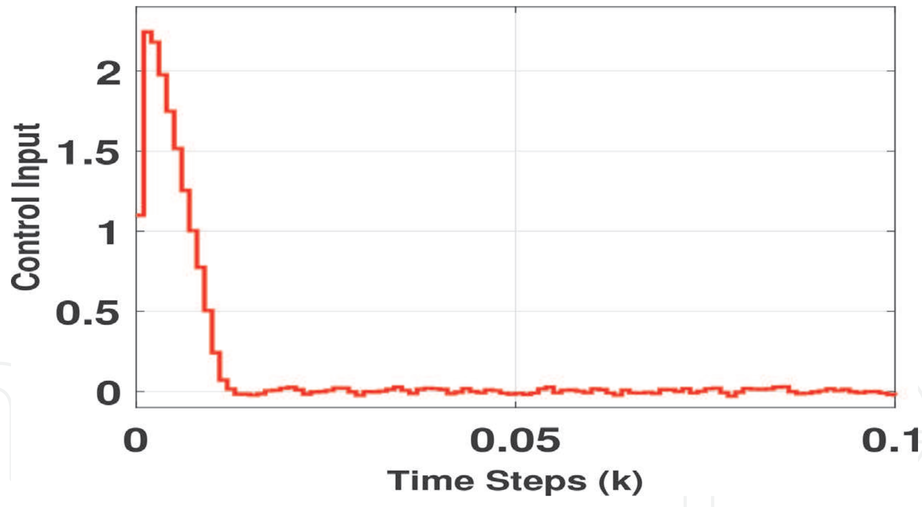

Figure 13.
Control input for Bartoszewicz's reaching law.

output instead of using the current output of interest. Many advancements are done in this domain [13–17]. But here only the relative degree two (RD2) is explained briefly. Readers are encouraged to study the advancement in this domain (from the reference citations above).

The concept of reaching law discussed in the Section 3 is of relative degree one (RD1) as the control input appears at the unit delay of the output. Similarly, in RD2 the control input and output are just two steps far. In general, relative degree r of an output means that the control input $u(k)$ appears first time at the rth delay of the output. Here the sliding variable is denoted as $s_2(k)$ to signify the relative degree two. The sliding variable is considered as $s_2(k) = c_2^T x(k)$, where c_2 is chosen such that $c_2^T B = 0$ but $c_2^T AB \neq 0$. With this sliding variable, control input does not appear on the $(k+1)$th instant but appears first time in the $(k+2)$th instant of $s_2(k)$. Reaching law for the sliding variable is suggested in [13]. Using the system (1) and with sliding variable $s_2(k) = c_2^T x(k)$, one can get

$$s_2(k+1) = c_2^T x(k+1) = c_2^T Ax(k) + c_2^T B[u(k) + f(k)] = c_2^T Ax(k) \tag{29}$$

Here the control input does not appear in $s_2(k+1)$ but appears in $s_2(k+2)$. Hence we should check for $s_2(k+2)$:

$$s_2(k+2) = c_2^T Ax(k+1) = c_2^T A^2 x(k) + c_2^T AB[u(k) + f(k)] \tag{30}$$

Here the control input appears in the dynamics of $s_2(k+2)$, so it is RD2:

$$\begin{aligned} s_2(k+2) = {} & \alpha^2 s_2(k) - \alpha\beta_2 sign(s_2(k)) \\ & -\beta_2 sign(s_2(k+1)) + d_2(k) \end{aligned} \tag{31}$$

where $|d_2(k)| \leq d_{2m} = |c_2^T AB| f_m$. The reaching law (31) is analyzed, and the dynamics of states during reaching and at steady-state are explained via the following lemmas [13], and estimate of robustness is given by the calculation of ultimate band:

Lemma 1 [13]: If $\beta_2 > \frac{d_{2m}}{1+\alpha}$ and $sign(s_2(k+1)) = sign(s_2(k))$, then $|s_2(k+2)|$ is strictly smaller than $|s_2(k)|$ or $s_2(k+2)$ crosses the hyperplane $s_2(k) = 0$.

Proof: For $sign(s_2(k+1)) = sign(s_2(k)) = 1$, from Eq. (31) we find

$$s_2(k+2) \le \alpha^2 s_2(k) - (1+\alpha)\beta_2 + d_{2m} < s_2(k) \tag{32}$$

since $\beta_2 > \frac{d_{2m}}{(1+\alpha)}$.

For $sign(s_2(k+1)) = sign(s_2(k)) = -1$, from Eq. (31) we find

$$s_2(k+2) \ge \alpha^2 s_2(k) + (1+\alpha)\beta_2 - d_{2m} > s_2(k) \tag{33}$$

From the above two inequalities, it is clear that $|s_2(k+2)| < |s_2(k)|$ or $sign(s_2(k+2)) = -sign(s_2(k+1)) = -sign(s_2(k))$, meaning that $s_2(k+2)$ crosses the hyperplane.

The above lemma signifies that if both $x(k)$ and $x(k+1)$ lie on the same side of the sliding hyperplane, then the state at the next sample instant, i.e., $x(k+2)$, is either on the same side and nearer to the surface or lies on the opposite side of the sliding hyperplane. With increasing k, there exists an instant where the states will cross the sliding hyperplane, $s_2(k) = 0$ for a finite value of k.

Lemma 2 [13]: If $\beta_2 > \frac{d_{2m}}{1-\alpha}$ and $sign(s_2(k+1)) = -sign(s_2(k))$, then $sign(s_2(k+2)) = sign(s_2(k))$.

Proof: With $sign(s_2(k+1)) = -sign(s_2(k))$, from Eq. (31), we get

$$\begin{aligned} s_2(k+2) &= \alpha^2 s_2(k) - \alpha\beta_2 \text{sign}(s_2(k)) \\ &\quad - \beta_2 \text{sign}(s_2(k+1)) + d_2(k) \\ &= \alpha^2 s_2(k) - \alpha\beta_2 \text{sign}(s_2(k)) + \beta_2 \text{sign}(s_2(k)) + d_2(k) \\ &= \alpha^2 s_2(k) + (1-\alpha)\beta_2 \text{sign}(s_2(k)) + d_2(k) \end{aligned} \tag{34}$$

Since $\beta_2 > \frac{d_{2m}}{1-\alpha}$, then for any $|d_2(k)| < d_{2m}$, we get $sign(s_2(k+2)) = sign(s_2(k))$.

This lemma shows that $\beta_2 > \frac{d_{2m}}{1-\alpha}$ is the necessary and sufficient condition for crossing and recrossing the sliding hyperplane at each successive instant, i.e., achieving the quasi-sliding mode as defined in [9]. This is because the condition on β_2 in Lemma 1 is already covered by β_2 in Lemma 2.

The ultimate band δ_2 for the sliding surface $s_2(k)$ indicates the robustness of the system. It is the maximum value that $s_2(k)$ can attain on either side of $s_2(k) = 0$ and can be calculated by putting $s_2(k) = \delta_2$ and maximizing the disturbance in a bid to maximize the value of $s_2(k+2)$. Hence

$$\delta_2 = \alpha^2 \delta_2 - \alpha\beta_2 + \beta_2 + d_{2m} \tag{35}$$

This leads to

$$\delta_2 = \frac{(1-\alpha)\beta_2 + d_{2m}}{(1-\alpha^2)} \tag{36}$$

4.1 Design procedure

Here the aim is to design a control law $u(k)$ such that $\lim_{k\to\infty} x(k) = 0$. Initially a sliding variable is chosen as

$$s_2(k) = c_2^T x(k) \tag{37}$$

where c_2 is a design parameter. The next step is to choose the RD2 reaching law [13]:

$$s_2(k+2) = \alpha^2 s_2(k) - \alpha\beta_2 sign(s_2(k)) - \beta_2 sign(s_2(k+1)) + d_2(k) \quad (38)$$

where $\alpha \in (0,1)$ and $\beta_2 > \frac{d_{2m}}{1-\alpha}$ and $d_2(k)$ are assumed to be the same as $c^T ABf(k)$ and are bounded by $d_m = |c^T ABf_m|$. Using Eqs. (1), (37), and (38), one can write

$$s_2(k+1) = c_2^T x(k+1) = c_2^T Ax(k) + c_2^T B[u(k) + f(k)] = c_2^T Ax(k) \quad (39)$$

$$s_2(k+2) = c_2^T Ax(k+1) = c_2^T A^2 x(k) + c_2^T AB[u(k) + f(k)] \quad (40)$$

$$\begin{aligned} c_2^T A^2 x(k) + c_2^T AB[u(k) + f(k)] = \alpha^2 s_2(k) - \alpha\beta_2 sign(s_2(k)) \\ -\beta_2 sign(s_2(k+1)) + d_2(k) \end{aligned} \quad (41)$$

Control input is derived as

$$u(k) = -\left(c^T AB\right)^{-1}\left[c_2^T A^2 x(k) - \alpha^2 s_2(k) + \alpha\beta_2 sign(s_2(k)) + \beta_2 sign(s_2(k+1))\right] \quad (42)$$

By applying this control input (42), states are brought to zero by assuming $c^T B$ is non-singular.

Remark 1: Once the sliding happens, $s_2(k)$ becomes zero. This guarantees $x_1(k) = 0$ and $x_2(k) = 0$ in the same time instant. This is shown in [13]. In the presence of disturbance, finite time bounded stability is achieved instead of finite time stability [13].

Remark 2: The ultimate band δ_2 found in case of RD2 for the reaching law (12) is always smaller than the ultimate band δ_1 found in case of RD1 for the reaching law (13).

This can be shown mathematically with the help of Eqs. (16), (17) and (36):

$$\delta_1 = \beta + d_{m1} > \frac{2d_{m1}}{1-\alpha} \quad (43)$$

where $d_{m1} = \|c^T B\| f_m$.

$$\delta_2 = \frac{(1-\alpha)\beta_2 + d_{m1}}{(1-\alpha^2)} > \frac{2d_{m2}}{1-\alpha^2} \quad (44)$$

where $d_{m2} = \|c^T AB\| f_m$. By multiplying $\rho > 1$ in the right-hand side of inequalities (43) and (44), relationships can be transformed to equalities:

$$\delta_1 = \rho\frac{2d_{m1}}{1-\alpha} \quad (45)$$

$$\delta_2 = \rho\frac{2d_{m2}}{1-\alpha^2} \quad (46)$$

$$\frac{\delta_2}{\delta_1} = \frac{2d_{m2}}{2d_{m1}(1+\alpha)} \le \frac{p}{(1+\alpha)} \quad (47)$$

where $p = \|cA_{12}\| > 0$, it is proved that $\delta_2 < \delta_1$. Detailed proof is explained in [8].

4.2 Results and discussions

System (21) is again taken for showing the results of RD2 reaching law-based design. Here $c_2^T = [1 \quad 0]$ is chosen. α and β are taken as 0.1 and 0.02544, respectively. The ultimate band is calculated as 0.04131 shown in **Figure 15** which is very

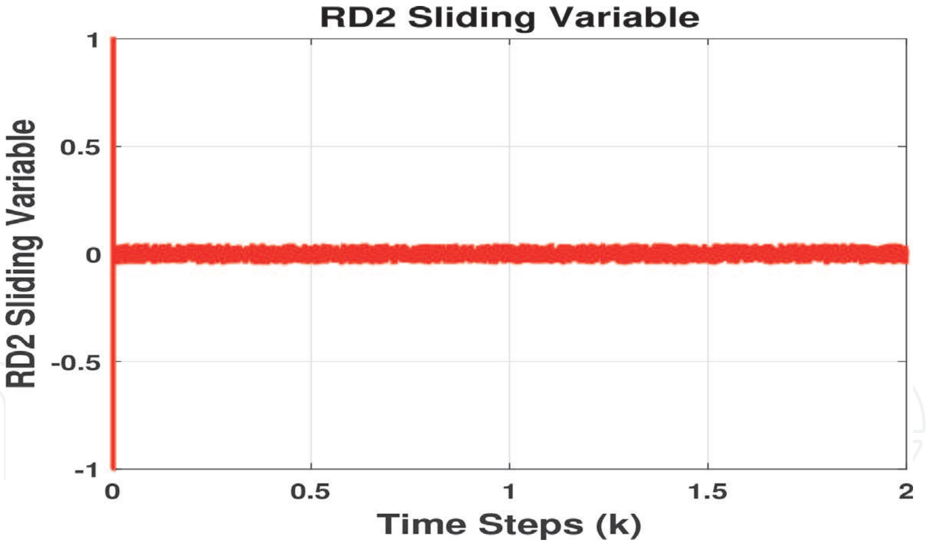

Figure 14.
Sliding variable s(k) evolution for RD2.

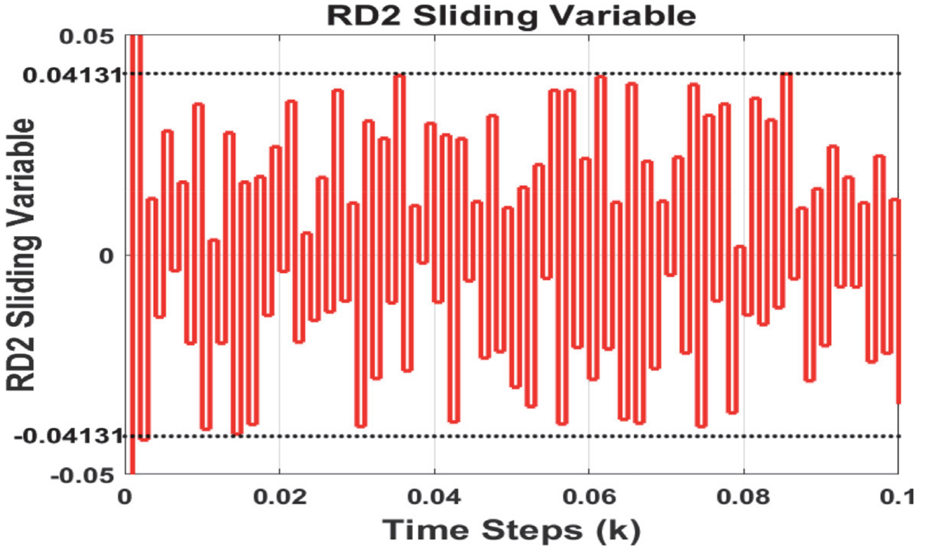

Figure 15.
Magnified part of sliding variable s(k) of **Figure 14**.

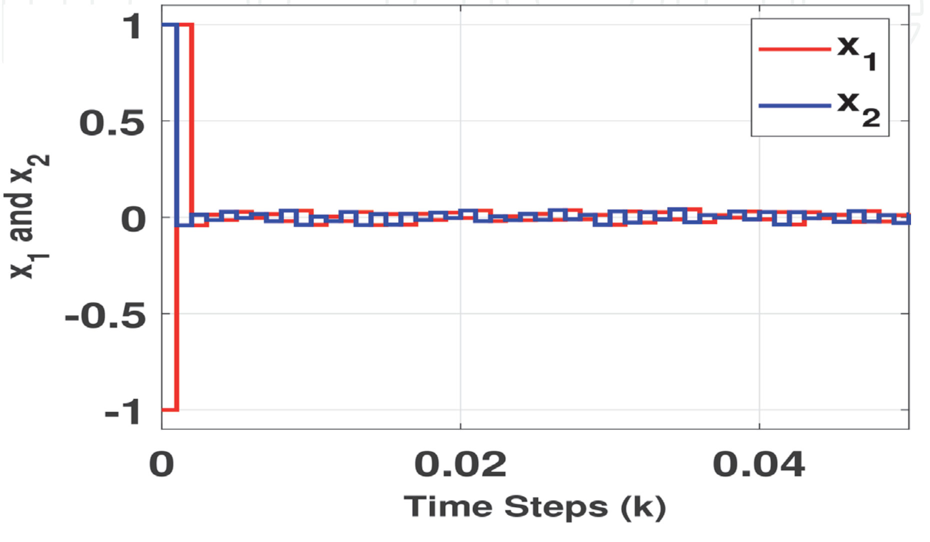

Figure 16.
States of the system using RD2 sliding variable.

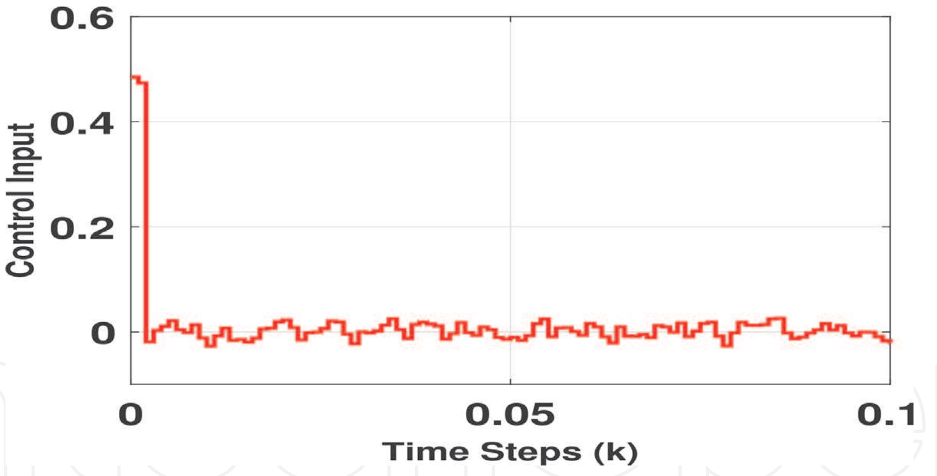

Figure 17.
Control input using RD2 sliding variable.

less than ultimate band found in the case of Gao's reaching law as 0.4544 shown in **Figure 4**. Time series data of RD2 sliding variable is shown in **Figure 14**, and the magnified part is shown in **Figure 15**. The states are finite-time bounded within a band too which is shown in **Figure 16**. The control input required to stabilize is given in **Figure 17**, and the amount of control effort is found to be 0.0225.

5. Conclusions

In this chapter, three most popular reaching laws, i.e., Gao's, Utkin's, and Bartoszewicz's reaching law in relative degree one, are discussed. In addition to that state-of-the-art research in relative degree two sliding variable for Gao's is discussed. Comparison shows better performance in terms of finite time ultimate boundedness of states and reduced ultimate band of state variable in case of RD2. The concept of ultimate band, finite-time bounded stability and requirement of control effort for all the reaching laws are briefly explained. Examples are given with simulation results for all the cases which show the behavior of the closed-loop system.

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