Chapter

Higher-Order Kinematics in Dual Lie Algebra

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Abstract

In this chapter, using the ring properties of dual number algebra, vector and tensor calculus, a computing method for the higher-order acceleration vector field properties in general rigid body motion is proposed. The higher-order acceleration field of a rigid body in a general motion is uniquely determined by higher-order time derivative of a dual twist. For the relative kinematics of rigid body motion, equations that allow the determination of the higher-order acceleration vector field are given, using an exponential Brockett-like formula in the dual Lie algebra. In particular cases, the properties for velocity, acceleration, jerk, and jounce fields are given. This approach uses the isomorphism between the Lie algebra of the rigid displacements $se(3)$, of the Special Euclidean group, $S\mathbb{E}_3$, and the Lie algebra of dual vectors. The results are coordinate free and in a closed form.

Keywords: higher-order kinematics, dual algebra, lie group

1. Introduction

The kinematic analysis of multibody systems has been traditionally considered as the determination of the positions, velocities, accelerations, jerks and jounces of their constitutive members. This is an old field with a long history, which has attracted the attention of mathematicians and engineers. Michel Chasles discovered (1834) that any rigid body displacement is equivalent to a screw displacement [1]. Screw theory is an efficient mathematical tool for the study of spatial kinematics. The pioneering work of Ball [2], the treatises of Hunt [3], and Phillips [4] and the multitude of contributions appearing in the literature are evidence of this. The isomorphism between screw theory and the Lie algebra, se(3), of the Special Euclidean group, $S\mathbb{E}_3$, provide with a wealth of results and techniques from modern differential geometry and Lie group theory [5–9].

A kinematic mapping relates the motion of a rigid body to the joint motions of a kinematic chain. Its time derivatives yield the twist, acceleration, jerk and jounce etc. of the body. Time derivatives of the twists of members in a kinematic chain and derivatives of screws are essential operations in kinematics. Recognizing the Lie group nature of rigid body motions, and correspondingly the Lie algebra nature of screws, Karger [5], Rico et al. [6], Lerbet [7] and Müller [8, 9] derived closed form expressions of higher-order time derivatives of twist.

In this chapter, using the tensor calculus and the dual numbers algebra, a new computing method for studying the higher-order accelerations field properties is proposed in the case of the general rigid body motion. For the spatial kinematic chains, equations that allow the determination of the n^{th} order accelerations field are given, using a Brockett-like formula. The crucial observation is that the n^{th} order time derivative of twist of the terminal body in a kinematic chain can be determined by propagating the k^{th} order time derivative of twists of the bodies in the chain, for $k = \overline{0, n}$. The results are coordinate-free and in a closed form.

2. Theoretical consideration on rigid body motion

The general framework of this chapter is a rigid body that moves with respect to a fixed reference frame $\{\mathfrak{R}^0\}$. Consider another reference frame $\{\mathfrak{R}\}$ originated in a point Q that moves together with the rigid body. Let $\boldsymbol{\rho}_Q$ denote the position vector of point Q with respect to frame $\{\mathfrak{R}^0\}$, $\mathbf{v}_Q$ its absolute velocity and $\mathbf{a}_Q$ its absolute acceleration.

Then the vector parametric equation of motion is:

$$\boldsymbol{\rho} = \boldsymbol{\rho}_Q + \mathbf{R}\mathbf{r} \tag{1}$$

where $\boldsymbol{\rho}$ represents the absolute position of a generic point P of the rigid body with respect to $\{\mathfrak{R}^0\}$ and $\mathbf{R} = \mathbf{R}(t)$ is an orthogonal proper tensorial function in $\mathbf{SO}_3^{\mathbb{R}}$. Vector $\mathbf{r}$ is constant and it represents the relative position vector of the arbitrary point P with respect to $\{\mathfrak{R}\}$.

The results of this section succinctly present the velocity and acceleration vector field in rigid body motion. These results lead to the generalization presented in the next section.

With the denotations that were introduced, the vector fields of velocities and accelerations are described by:

$$\begin{cases} \mathbf{v} - \mathbf{v}_Q = \dot{\mathbf{R}}\mathbf{R}^T(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \\ \mathbf{a} - \mathbf{a}_Q = \ddot{\mathbf{R}}\mathbf{R}^T(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \end{cases} \tag{2}$$

Tensors:

$$\begin{cases} \boldsymbol{\Phi}_1 = \dot{\mathbf{R}}\mathbf{R}^T \\ \boldsymbol{\Phi}_2 = \ddot{\mathbf{R}}\mathbf{R}^T \end{cases} \tag{3}$$

represent the **velocity tensor** respectively the **acceleration tensor**. Tensor $\boldsymbol{\Phi}_1 = \tilde{\boldsymbol{\omega}} \in \mathbf{so}_3^{\mathbb{R}}$ is the skew-symmetric tensor associated to the instantaneous angular velocity $\boldsymbol{\omega} \in \mathbf{V}_3^{\mathbb{R}}$. Tensor $\boldsymbol{\Phi}_2 = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}$, where $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ is the instantaneous angular acceleration of the rigid body. One may remark that vectors:

$$\begin{cases} \mathbf{a}_1 = \mathbf{v} - \boldsymbol{\Phi}_1\boldsymbol{\rho} = \mathbf{v}_Q - \boldsymbol{\Phi}_1\boldsymbol{\rho}_Q \\ \mathbf{a}_2 = \mathbf{a} - \boldsymbol{\Phi}_2\boldsymbol{\rho} = \mathbf{a}_Q - \boldsymbol{\Phi}_2\boldsymbol{\rho}_Q \end{cases} \tag{4}$$

do not depend on the choice of point P of the rigid body. They are called the **velocity invariant** respectively the **acceleration invariant** (at a given moment of time).

2.1 The velocity field in rigid body motion

It is described by:

$$\mathbf{v} - \mathbf{v}_Q = \mathbf{\Phi}_1(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \tag{5}$$

The instantaneous angular velocity $\boldsymbol{\omega}$ of the rigid body may be determined as $\boldsymbol{\omega} = \text{vect}\mathbf{\Phi}_1$. The major property that may be highlighted from Eq. (4) is that the velocity of a given point of the rigid may be computed when knowing the velocity tensor $\mathbf{\Phi}_1$ and the velocity invariant $\mathbf{a}_1$:

$$\mathbf{v} = \mathbf{a}_1 + \mathbf{\Phi}_1\boldsymbol{\rho} \tag{6}$$

2.2 The acceleration field in rigid body motion

It is described by

$$\mathbf{a} - \mathbf{a}_Q = \mathbf{\Phi}_2(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \tag{7}$$

The absolute acceleration of a given point of the rigid body may be computed when knowing the acceleration tensor $\mathbf{\Phi}_2$ and the acceleration invariant $\mathbf{a}_2$:

$$\mathbf{a} = \mathbf{a}_2 + \mathbf{\Phi}_2\boldsymbol{\rho} \tag{8}$$

The instantaneous angular acceleration of the rigid body may be determined as:

$$\boldsymbol{\varepsilon} = \text{vect}\mathbf{\Phi}_2 \tag{9}$$

The determinant of tensor $\mathbf{\Phi}_2$ is (see [10]): $\det\mathbf{\Phi}_2 = -(\boldsymbol{\omega} \times \boldsymbol{\varepsilon})^2$. It follows that if $\boldsymbol{\omega} \times \boldsymbol{\varepsilon} \neq \mathbf{0}$, then tensor $\mathbf{\Phi}_2$.
is invertible and its inverse is (see [10]):

$$\mathbf{\Phi}_2^{-1} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\varepsilon})^2}\left[\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} + (\boldsymbol{\omega} \otimes \boldsymbol{\omega})^2 - \widetilde{\widetilde{\boldsymbol{\omega}}^2\boldsymbol{\varepsilon}}\right] \tag{10}$$

It follows that if tensor $\mathbf{\Phi}_2$ is non-singular, then for an arbitrary given acceleration $\mathbf{a}$ we may find a point of the rigid that has this acceleration. Its absolute position is given by (see also Eq. (8)):

$$\boldsymbol{\rho} = \mathbf{\Phi}_2^{-1}(\mathbf{a} - \mathbf{a}_2) \tag{11}$$

Particularly, if $\mathbf{\Phi}_2$ is non-singular, then there exists a point G of zero acceleration, named the **acceleration center**. Its absolute position vector is given by:

$$\boldsymbol{\rho}_G = -\mathbf{\Phi}_2^{-1}\mathbf{a}_2 \tag{12}$$

3. The vector field of the n^{th} order accelerations

This section extends some of the previous considerations to the case of the n^{th} order accelerations. We define the n^{th} order acceleration of a point as:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} \stackrel{\text{def}}{=} \frac{d^n}{dt^n}\boldsymbol{\rho}, n \geq 1 \tag{13}$$

For $n = 1$, it represents the velocity, and for $n = 2$, the acceleration. By derivation with respect to time successively in Eq. (2), it follows that:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \mathbf{R}^{(n)}\mathbf{R}^T\left(\boldsymbol{\rho} - \boldsymbol{\rho}_Q\right), \text{where } \mathbf{R}^{(n)} \stackrel{\text{def}}{=} \frac{d^n}{dt^n}\mathbf{R} \tag{14}$$

We define:

$$\boldsymbol{\Phi}_n \stackrel{\text{def}}{=} \mathbf{R}^{(n)}\mathbf{R}^T \tag{15}$$

the $\boldsymbol{n^{th}}$ **order acceleration tensor** in rigid body motion. A vector invariant is immediately highlighted from Eq. (14) with the denotation (15). Vector:

$$\mathbf{a}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho} = \mathbf{a}_Q^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho}_Q \tag{16}$$

does not depend on the choice of the point of the rigid body for which the acceleration $\mathbf{a}^{[n]}$ is computed. Vector $\mathbf{a}_n$ is named the **invariant vector** of the $\boldsymbol{n^{th}}$ **order accelerations**. Then Eq. (7) may be generalized as it follows:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \boldsymbol{\Phi}_n\left(\boldsymbol{\rho} - \boldsymbol{\rho}_Q\right) \tag{17}$$

The next Theorem gives the fundamental properties of the vector field of the n^{th} order accelerations.

Theorem 1. *In the rigid body motion, at a moment of time t, there exist tensor* $\boldsymbol{\Phi}_n$ *defined by Eq. (15) and vector* $\mathbf{a}_n$ *such as:*

$$\begin{gathered} \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \boldsymbol{\Phi}_n\left(\boldsymbol{\rho} - \boldsymbol{\rho}_Q\right) \\ \mathbf{a}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho} = \mathbf{a}_Q^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho}_Q \end{gathered} \tag{18}$$

for any point P of the rigid body with the absolute position defined by vector $\boldsymbol{\rho}$.

Remark 1. *Given the absolute position of a point of the rigid body and knowing* $\boldsymbol{\Phi}_n$ *and* $\mathbf{a}_n$*, its acceleration is computed from:*

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} = \mathbf{a}_n + \boldsymbol{\Phi}_n\boldsymbol{\rho} \tag{19}$$

Remark 2. *Tensor* $\boldsymbol{\Phi}_n$ *and vector* $\mathbf{a}_n$ *generalize the notions of velocity/acceleration tensor respectively velocity/acceleration invariant. They are fundamental in the study of the vector field of the* n^{th} *order accelerations. The recursive formulas for computing* $\boldsymbol{\Phi}_n$ *and* $\mathbf{a}_n$ *are:*

$$\begin{cases} \boldsymbol{\Phi}_{n+1} = \dot{\boldsymbol{\Phi}}_n + \boldsymbol{\Phi}_n\boldsymbol{\Phi}_1 \\ \mathbf{a}_{n+1} = \dot{\mathbf{a}}_n + \boldsymbol{\Phi}_n\mathbf{a}_1 \end{cases}, n \geq 1, \textit{where } \boldsymbol{\Phi}_1 = \tilde{\boldsymbol{\omega}}, \mathbf{a}_1 = \mathbf{v}_Q - \boldsymbol{\Phi}_1\boldsymbol{\rho}_Q \tag{20}$$

Remark 3. *One may remark that from Eq. (20) it follows by direct computation:*

$$\begin{cases} \boldsymbol{\Phi}_n = \boldsymbol{\Phi}_{n-1}\boldsymbol{\Phi}_1 + \left(\dfrac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Phi}_1\right) + \displaystyle\sum_{k=1}^{n-2}\left[\frac{d^k}{dt^{n-1}}\left(\boldsymbol{\Phi}_{n-k-1}\boldsymbol{\Phi}_1\right)\right] \\ \mathbf{a}_n = \boldsymbol{\Phi}_{n-1}\mathbf{a}_1 + \left(\dfrac{d^{n-1}}{dt^{n-1}}\mathbf{a}_1\right) + \displaystyle\sum_{k=1}^{n-2}\left[\frac{d^k}{dt^{n-1}}\left(\boldsymbol{\Phi}_{n-k-1}\mathbf{a}_1\right)\right] \end{cases}, n \geq 3 \tag{21}$$

Remark 4. *By defining the n^{th} order instantaneous n^{th} order angular acceleration of the rigid body $\boldsymbol{\varepsilon}^{[n]} \stackrel{\text{def}}{=} \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\omega}$, it follows from Eq. (21) that its associated skew-symmetric tensor may be expressed as $\tilde{\boldsymbol{\varepsilon}}^{[n]} = \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Phi}_1$. The expression of the instantaneous n^{th} order angular acceleration is:*

$$\boldsymbol{\varepsilon}^{[n]} = vect\left\{\boldsymbol{\Phi}_n - \boldsymbol{\Phi}_{n-1}\boldsymbol{\Phi}_1 - \sum_{k=1}^{n-2}\left[\frac{d^k}{dt^k}(\boldsymbol{\Phi}_{n-k-1}\boldsymbol{\Phi}_1)\right]\right\}, n \geq 3 \tag{22}$$

3.1 Homogenous matrix approach to the field of n^{th} order accelerations

The set of affine maps, $g : \mathbf{V}_3 \to \mathbf{V}_3, g(\mathbf{u}) = \mathbf{R}\mathbf{u} + \mathbf{w}$, where $\mathbf{R}$ is an orthogonal proper tensor and $\mathbf{w}$ a vector in $\mathbf{V}_3$ is a group under composition and it is called *the group of direct affine isometries* or *rigid motions* and it is denoted $S\mathbb{E}_3$. Any rigid finite motion may be described by such a map. Tensor $\mathbf{R}$ models the rotation of the considered rigid body and vector $\mathbf{w}$ its translation. An affine map from $S\mathbb{E}_3$ may be represented with a 4×4 square matrix:

$$g = \begin{bmatrix} \mathbf{R} & \mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix} \tag{23}$$

One may remark that the following relations hold true:

$$\begin{cases} \begin{bmatrix} \mathbf{R}_1 & \mathbf{w}_1 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_2 & \mathbf{w}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{w}_2 + \mathbf{w}_1 \\ \mathbf{0} & 1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{R} & \mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix} \end{cases} \tag{24}$$

We may extend now $S\mathbb{E}_3$ to $S\mathbb{E}_3^{\mathbb{R}}$, the set of the functions with the domain $\mathbb{R}$ and the range $S\mathbb{E}_3$. The parametric vector equation of the rigid body motion (1) may be rewritten with the help of a homogenous matrix function in $S\mathbb{E}_3^{\mathbb{R}}$ like it follows:

$$\begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \boldsymbol{\rho}_Q \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} \tag{25}$$

From Eq. (25), it follows that:

$$\begin{bmatrix} \dot{\boldsymbol{\rho}} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}} & \dot{\boldsymbol{\rho}}_Q \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}} & \dot{\boldsymbol{\rho}}_Q \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\boldsymbol{\rho}_Q \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \tag{26}$$

and by making the computations and taking into account Eqs. (3) and (4) it follows that:

$$\begin{bmatrix} \dot{\boldsymbol{\rho}} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \mathbf{a}_1 \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \tag{27}$$

By using the previous considerations, it follows that Eq. (25) may be extended like:

$$\begin{bmatrix} \mathbf{a}_{\boldsymbol{\rho}}^{[n]} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \tag{28}$$

Eq. (28) represents a unified form of describing the vector field of the n^{th} order accelerations in rigid body motion. The matrix:

$$\boldsymbol{\Psi}_n = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \tag{29}$$

contains both the n^{th} order acceleration tensor $\boldsymbol{\Phi}_n$ and the vector invariant $\mathbf{a}_n$. Eqs. (20) may be put in a compact form:

$$\boldsymbol{\Psi}_{n+1} = \dot{\boldsymbol{\Psi}}_n + \boldsymbol{\Psi}_n \boldsymbol{\Psi}_1, n \geq 1 \tag{30}$$

If follows that $\boldsymbol{\Psi}_n$ may be written as:

$$\boldsymbol{\Psi}_n = \boldsymbol{\Psi}_{n-1}\boldsymbol{\Psi}_1 + \left(\frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Psi}_1\right) + \sum_{k=1}^{n-2}\left[\frac{d^k}{dt^k}(\boldsymbol{\Psi}_n\boldsymbol{\Psi}_1)\right], n \geq 3 \tag{31}$$

4. Symbolic calculus of higher-order kinematics invariants

We will present a method for the symbolic calculation of higher-order kinematics invariants for rigid motion.

Let be $\mathbf{a}_n$ and $\boldsymbol{\Phi}_n$, $n \in \mathbb{N}$ vector invariant, respectively, tensor invariant for the n^{th} order accelerations fields. We denote by

$$\boldsymbol{\Psi}_n = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \tag{32}$$

and we have the following relationship of recurrence:

$$\begin{gathered} \boldsymbol{\Psi}_{n+1} = \dot{\boldsymbol{\Psi}}_n + \boldsymbol{\Psi}_n \boldsymbol{\Psi}_1, n \in \mathbb{N} \\ \boldsymbol{\Psi}_1 = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \end{gathered} \tag{33}$$

The pair of vectors $(\boldsymbol{\omega}, \mathbf{v})$ is also known as *the spatial twist of rigid body.*

Let be $\boldsymbol{\mathcal{A}}$ the matrix ring

$$\boldsymbol{\mathcal{A}} = \left\{ \mathbf{A} \in \mathcal{M}_{3\times 3}(\mathbb{R}) \middle| \mathbf{A} = \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{a} \\ \mathbf{0} & 0 \end{bmatrix}; \boldsymbol{\Phi} \in \mathrm{L}(\mathbf{V}_3, \mathbf{V}_3), \mathbf{a} \in \mathbf{V}_3 \right\} \tag{34}$$

and $\boldsymbol{\mathcal{A}}[\mathrm{X}]$ the set of polynomials with coefficients in the non-commutative ring $\boldsymbol{\mathcal{A}}$. A generic element of $\boldsymbol{\mathcal{A}}[\mathrm{X}]$ has the form

$$\mathrm{P}(\mathrm{X}) = \mathbf{A}_0\mathrm{X}^m + \mathbf{A}_1\mathrm{X}^{m-1} + \ \dots \ + \mathbf{A}_{m-1}\mathrm{X} + \mathbf{A}_m, \mathbf{A}_k \in \boldsymbol{\mathcal{A}}, k = \overline{0, m} \tag{35}$$

Theorem 2. *There is a unique polynomial* $\mathbf{P}_n \in \boldsymbol{\mathcal{A}}[X]$ *such that* $\boldsymbol{\Psi}_\mathrm{n}$ *will be written as*

$$\boldsymbol{\Psi}_n = \mathbf{P}_n(\mathbf{D})\boldsymbol{\Psi}_1, n \in \mathbb{N} \tag{36}$$

where $\mathbf{D} = \frac{d}{dt}$ *is the operator of time derivative.*

Proof: Taking into account Eqs. (36) and (33) we will have the following relationship of recurrence for $\mathbf{P}_n(\mathbf{D})$:

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\mathbf{\Psi}_1) \\ \mathbf{P}_0 = \mathbf{I} \end{cases} \tag{37}$$

Since $\mathbf{\Psi}_1 = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$ it follows the next outcome.

Theorem 3. *There is a unique polynomial with the coefficients in the non-commutative ring* $\mathbf{L}(\mathbf{V}_3, \mathbf{V}_3)$ *such that the vector respectively the tensor invariants of the* n^{th} *order accelerations will be written as*

$$\begin{matrix} \mathbf{a}_n = \mathbf{P}_n \mathbf{v} \\ \mathbf{\Phi}_n = \mathbf{P}_n \tilde{\boldsymbol{\omega}} \end{matrix}, n \in \mathbb{N}^* \tag{38}$$

where $\mathbf{P}_n$ *fulfills the relationship of recurrence*

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\tilde{\boldsymbol{\omega}}), n \in \mathbb{N}^* \\ \mathbf{P}_1 = \mathbf{I} \end{cases} \tag{39}$$

It follows

$$\begin{gathered} \mathbf{P}_2 = \mathbf{D} + \tilde{\boldsymbol{\omega}} \\ \mathbf{P}_3 = \mathbf{D}^2 + \tilde{\boldsymbol{\omega}}\mathbf{D} + 2\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 \\ \mathbf{P}_4 = \mathbf{D}^3 + \tilde{\boldsymbol{\omega}}\mathbf{D}^2 + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\mathbf{D} + 3\ddot{\tilde{\boldsymbol{\omega}}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^3 \end{gathered} \tag{40}$$

Thus, it follows:

- the velocity field invariants

$$\begin{cases} \mathbf{a}_1 = \mathbf{v} \\ \mathbf{\Phi}_1 = \tilde{\boldsymbol{\omega}} \end{cases} \tag{41}$$

- the acceleration field invariants

$$\begin{cases} \mathbf{a}_2 = \dot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\mathbf{v} \\ \mathbf{\Phi}_2 = \dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 \end{cases} \tag{42}$$

- jerk field invariants

$$\begin{cases} \mathbf{a}_3 = \ddot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\mathbf{v} + 2\dot{\tilde{\boldsymbol{\omega}}}\mathbf{v} + \tilde{\boldsymbol{\omega}}^2\mathbf{v} \\ \mathbf{\Phi}_3 = \ddot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 2\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3 \end{cases} \tag{43}$$

- hyper-jerk (jounce) field invariants

$$\begin{cases} \mathbf{a}_4 = \dddot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\ddot{\mathbf{v}} + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\dot{\mathbf{v}} + 3\ddot{\tilde{\boldsymbol{\omega}}}\mathbf{v} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}}\mathbf{v} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}}\mathbf{v} + \tilde{\boldsymbol{\omega}}^3\mathbf{v} \\ \boldsymbol{\Phi}_4 = \dddot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}\ddot{\tilde{\boldsymbol{\omega}}} + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\dot{\tilde{\boldsymbol{\omega}}} + 3\ddot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}}^2 + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^4 \end{cases} \tag{44}$$

Remark 5. *The higher-order time derivative of spatial twist solve completely the problem of determining the field of the n^{th} order acceleration of rigid motion.*

4.1 Higher-order acceleration center and vector invariants of rigid body motion

Equation (16) may be written as

$$\mathbf{a}_{\rho}^{[n]} - \phi_n \boldsymbol{\rho} = \mathbf{a}_Q^{[n]} - \boldsymbol{\Phi}_n \boldsymbol{\rho}_Q, n \in \mathbb{N}^*. \tag{45}$$

This shows us that the vector function

$$\mathbf{I}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[\mathbf{n}]} - \boldsymbol{\Phi}_{\mathbf{n}}\boldsymbol{\rho}, n \in \mathbb{N}^* \tag{46}$$

has the same value in every point of the rigid body under the general spatial motion, at a given moment of time t. It represents a **vector invariant** of the n-th order acceleration field.

The invariant value of vector $\mathbf{I}_n$ is obtained for $\boldsymbol{\rho} = \mathbf{0}$ and it is the n-th order acceleration of the point of the rigid body that passes the origin of the fixed reference frame at a given moment of time: $\mathbf{I}_n = \mathbf{a}_0^{[\mathbf{n}]} \stackrel{\text{def}}{=} \mathbf{a}_n$. Eq. (46) becomes:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} = \mathbf{a}_n + \phi_n \boldsymbol{\rho}. \tag{47}$$

Let be $\boldsymbol{\Phi}_{\mathbf{n}}^*$ be the adjugate tensor of $\boldsymbol{\Phi}_{\mathbf{n}}$ uniquely defined by:$\boldsymbol{\Phi}_{\mathbf{n}}\boldsymbol{\Phi}_{\mathbf{n}}^* = (\det\boldsymbol{\Phi}_{\mathbf{n}})\mathbf{I}$.

From Eq. (46), results another invariant

$$\mathbf{J}_n = \boldsymbol{\Phi}_n^* \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - (\det\boldsymbol{\Phi}_n)\boldsymbol{\rho}, n \in \mathbb{N}^*. \tag{48}$$

The value of this invariant is $\mathbf{J}_n = \boldsymbol{\Phi}_n^* \mathbf{a}_n$.

In the specific case when tensor $\boldsymbol{\Phi}_{\mathbf{n}}$ is non-singular ($\det\boldsymbol{\Phi}_{\mathbf{n}} \neq 0$), from (47) results the position vector having an imposed n-th order acceleration $\mathbf{a}^*$:

$$\boldsymbol{\rho}^* = \boldsymbol{\Phi}_n^{-1}(\mathbf{a}^* - \mathbf{a}_n), n \in \mathbb{N}^*. \tag{49}$$

In a particular case of the **n-th order acceleration center** G_n (i.e. the point that have $\mathbf{a}^* = \mathbf{0}$) on obtain:

$$\boldsymbol{\rho}_{G_n} = -\boldsymbol{\Phi}_n^{-1}\mathbf{a}_n \tag{50}$$

Assuming that the tensor $\boldsymbol{\Phi}_{\mathbf{n}}$ is non-singular, the previous relations lead to a new vector invariant that characterize the accelerations of n-th and m-th order ($n, m \in \mathbb{N}^*$):

$$\mathbf{K}_{m,n} = \mathbf{a}_{\boldsymbol{\rho}}^{[m]} - \boldsymbol{\Phi}_m \boldsymbol{\Phi}_n^{-1} \mathbf{a}_{\boldsymbol{\rho}}^{[n]}, m, n \in \mathbb{N}^*. \tag{51}$$

The value of this invariant is $\mathbf{K}_{m,n} = \mathbf{a}_m - \boldsymbol{\Phi}_m \boldsymbol{\Phi}_n^{-1} \mathbf{a}_n$.

The problem of the determination the adjugate tensor of the n-th acceleration tensor and the conditions in which these tensors are inversable is, as the author knows, still an open problem in theoretical kinematics field. We will propose a method based on the tensors algebra that will give a closed form, coordinate- free solution, dependent to the time derivative of spatial twist.

The vector field of the higher-order acceleration is a non-stationary vector field. Differential operator *div* and *curl* is expressed, taking into account Eq. (47), through the linear invariants of the tensor $\mathbf{\Phi}_n$, as below:

$$\begin{aligned} div\ \boldsymbol{a}_{\varrho}^{[n]} &= trace\ \mathbf{\Phi}_n \\ curl\ \boldsymbol{a}_{\varrho}^{[n]} &= 2vect\ \mathbf{\Phi}_n \end{aligned} \tag{52}$$

Let $\mathbf{\Phi} \in \mathbf{L}(\mathbf{V}_3, \mathbf{V}_3)$ a tensor and we note $\mathbf{t} = \text{vect}\mathbf{\Phi}$ and $\mathbf{S} = \text{sym}\mathbf{\Phi}$. The below theorem takes place.

Theorem 4. *The adjugate tensor and determinant of the tensor* $\mathbf{\Phi}$ *is*:

$$\begin{aligned} \mathbf{\Phi}^* &= \mathbf{S}^* - \widetilde{\mathbf{St}} + \mathbf{t} \otimes \mathbf{t} \\ \det\mathbf{\Phi} &= \det\mathbf{S} + \mathbf{tSt} \end{aligned} \tag{53}$$

Let $\mathbf{\Phi}_\mathrm{n}$ the n-th order acceleration tensor, $\mathbf{\Phi}_\mathrm{n} = \tilde{\mathbf{t}}_n + \mathbf{S}_n$.

The vectors $\mathbf{t}_n$ and the symmetric tensors $\mathbf{S}_n, n \in \mathbb{N}^*$ can be obtained with the below recurrence relation:

$$\begin{cases} \mathbf{t}_{\mathrm{n}+1} = \dot{\mathbf{t}}_\mathrm{n} + \dfrac{1}{2}\left[(\mathrm{trace}\mathbf{\Phi}_\mathrm{n})\mathbf{I} - \mathbf{\Phi}_\mathbf{n}^\mathrm{T}\right]\boldsymbol{\omega} \\ \mathbf{t}_1 = \boldsymbol{\omega} \end{cases} \tag{54}$$

$$\begin{cases} \mathbf{S}_{\mathrm{n}+1} = \dot{\mathbf{S}}_\mathrm{n} + \mathrm{sym}(\mathbf{\Phi}_\mathrm{n}\tilde{\boldsymbol{\omega}}) \\ \mathbf{S}_1 = \mathbf{0} \end{cases} \tag{55}$$

It follows that:

- Velocity field: $\mathbf{\Phi}_1 = \tilde{\boldsymbol{\omega}}, \mathbf{t}_1 = \boldsymbol{\omega}, \mathbf{S}_1 = \mathbf{0}$

$$\begin{aligned} \mathbf{\Phi}_1^* &= \boldsymbol{\omega} \otimes \boldsymbol{\omega} \\ \det\mathbf{\Phi}_1 &= 0 \end{aligned} \tag{56}$$

$\mathbf{\Phi}_1$ is singular for any $\boldsymbol{\omega}$. In this case,

$$\begin{aligned} \mathrm{div}\ \mathbf{a}_\varrho^{[1]} &= 0 \\ \mathrm{curl}\ \mathbf{a}_\varrho^{[1]} &= 2\boldsymbol{\omega} \end{aligned} \tag{57}$$

- Acceleration field: $\mathbf{\Phi}_2 = \tilde{\boldsymbol{\omega}}^2 + \dot{\tilde{\boldsymbol{\omega}}}, \mathbf{t}_2 = \dot{\boldsymbol{\omega}}, \mathbf{S}_2 = \tilde{\boldsymbol{\omega}}^2$

$$\begin{aligned} \mathbf{\Phi}_2^* &= (\boldsymbol{\omega} \otimes \boldsymbol{\omega})^2 - \widetilde{\tilde{\boldsymbol{\omega}}^2\dot{\boldsymbol{\omega}}} + \dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}} \\ \det\mathbf{\Phi}_2 &= -(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2 \end{aligned} \tag{58}$$

$\mathbf{\Phi}_2$ is nonsingular if and only if $\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}} \neq \mathbf{0}$. In this case

$$\mathbf{\Phi}_2^{-1} = \frac{\widetilde{\tilde{\boldsymbol{\omega}}^2\dot{\boldsymbol{\omega}}} - (\boldsymbol{\omega} \otimes \boldsymbol{\omega})^2 - \dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}}}{(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2} \tag{59}$$

$$\operatorname{div} \mathbf{a}_{\varrho}^{[2]} = -2\boldsymbol{\omega}^2$$
$$\operatorname{curl} \mathbf{a}_{\varrho}^{[2]} = 2\dot{\boldsymbol{\omega}}. \tag{60}$$

- Jerk field: $\boldsymbol{\Phi}_3 = \ddot{\tilde{\boldsymbol{\omega}}} + 2\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^3$, $\mathbf{t}_3 = \ddot{\boldsymbol{\omega}} + \frac{1}{2}\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega} - \boldsymbol{\omega}^2\boldsymbol{\omega}$, $\mathbf{S}_3 = \frac{3}{2}\left[\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + \dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}}\right]$,

$$\boldsymbol{\Phi}_3^* = \frac{9}{4}\left[(\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}})^2 + (\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega})^2 + (\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \otimes (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega})\right] - \widetilde{\mathbf{S}_3\mathbf{t}_3} + \mathbf{t}_3 \otimes \mathbf{t}_3$$

$$\det\boldsymbol{\Phi}_3 = \frac{12(\mathbf{t}_3 \times \dot{\boldsymbol{\omega}})(\boldsymbol{\omega} \times \mathbf{t}_3) + 27\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2}{4}. \tag{61}$$

$\boldsymbol{\Phi}_3$ is nonsingular if and only if $4(\dot{\boldsymbol{\omega}} \times \mathbf{t}_3)(\boldsymbol{\omega} \times \mathbf{t}_3) \neq 9\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2$. In this case

$$\boldsymbol{\Phi}_3^{-1} == \frac{9\left[(\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}})^2 + (\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega})^2 + (\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \otimes (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega})\right] - 4\widetilde{\mathbf{S}_3\mathbf{t}_3} + 4\mathbf{t}_3 \otimes \mathbf{t}_3}{12(\mathbf{t}_3 \times \dot{\boldsymbol{\omega}})(\boldsymbol{\omega} \times \mathbf{t}_3) + 27\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2}$$
$$\operatorname{div} \mathbf{a}_{\varrho}^{[3]} = -6\, \boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}}$$
$$\operatorname{curl} \mathbf{a}_{\varrho}^{[3]} = 2\ddot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\omega} - 2\boldsymbol{\omega}^2\boldsymbol{\omega}. \tag{62}$$

- Jounce field:

$$\boldsymbol{\Phi}_4 = \dddot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}\ddot{\tilde{\boldsymbol{\omega}}} + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\dot{\tilde{\boldsymbol{\omega}}} + 3\ddot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}}^2 + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^4$$
$$\mathbf{t}_4 = \dddot{\boldsymbol{\omega}} + \ddot{\boldsymbol{\omega}} \times \boldsymbol{\omega} - 2\boldsymbol{\omega}^2\dot{\boldsymbol{\omega}} - 4(\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}})\boldsymbol{\omega}$$
$$\mathbf{S}_4 = 2\operatorname{sym}\left(2\ddot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}}^2\right) + 3\dot{\tilde{\boldsymbol{\omega}}}^2 + \tilde{\boldsymbol{\omega}}^4 \tag{63}$$

$$\mathbf{S}_4^* = 2\operatorname{sym}\left[3\dot{\tilde{\boldsymbol{\omega}}}(\boldsymbol{\omega} \otimes \mathbf{w})\dot{\tilde{\boldsymbol{\omega}}} - \alpha\boldsymbol{\omega} \otimes \mathbf{w}\right] - \tilde{\boldsymbol{\omega}}(\mathbf{w} \otimes \mathbf{w})\tilde{\boldsymbol{\omega}} - 3\alpha\dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}} + \alpha\boldsymbol{\omega}^4\mathbf{I} \tag{64}$$

$$\boldsymbol{\Phi}_4^* = \mathbf{S}_4^* - \widetilde{\mathbf{S}_4\mathbf{t}_4} + \mathbf{t}_4 \otimes \mathbf{t}_4 \tag{65}$$

$$\det \boldsymbol{\Phi}_4 = \alpha\left[6(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \cdot (\mathbf{w} \times \dot{\boldsymbol{\omega}}) - (\boldsymbol{\omega} \times \mathbf{w})^2 + 2\alpha\boldsymbol{\omega} \cdot \mathbf{w} + 3\alpha\dot{\boldsymbol{\omega}}^2 + \alpha^2\right] - 3(\dot{\boldsymbol{\omega}}, \boldsymbol{\omega}, \mathbf{w})^2 \tag{66}$$

In Eqs. (65) and (66), the following notation has been used:

$$\begin{aligned} \mathbf{w} &= \ddot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times \dot{\boldsymbol{\omega}} - \boldsymbol{\omega}^2\boldsymbol{\omega} \\ \alpha &= \boldsymbol{\omega}^4 - 2\dot{\boldsymbol{\omega}}^2 - 2\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}} \end{aligned} \tag{67}$$

If

$$3(\dot{\boldsymbol{\omega}}, \boldsymbol{\omega}, \mathbf{w})^2 \neq \alpha\left[6(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \cdot (\mathbf{w} \times \dot{\boldsymbol{\omega}}) - (\boldsymbol{\omega} \times \mathbf{w})^2 + 2\alpha\boldsymbol{\omega} \cdot \mathbf{w} + 2\alpha\dot{\boldsymbol{\omega}}^2 + \alpha^2\right] \tag{68}$$

then $\boldsymbol{\Phi}_4$ is inversible and

$$\boldsymbol{\Phi}_4{}^{-1} = \frac{\boldsymbol{\Phi}_4^*}{\det\boldsymbol{\Phi}_4} \tag{69}$$

In the hypothesis (68), there is jounce center, determined by

$$\begin{aligned} \boldsymbol{\rho}_{G_4} &= -\boldsymbol{\Phi}_4^{-1}\mathbf{a}_4 \\ \operatorname{div} \mathbf{a}_\varrho^{[4]} &= -2\left(4\boldsymbol{\omega}\cdot\ddot{\boldsymbol{\omega}} + 3\dot{\boldsymbol{\omega}}^2 + \boldsymbol{\omega}^4\right) \\ \operatorname{curl} \mathbf{a}_\varrho^{[4]} &= 2\dddot{\boldsymbol{\omega}} + 2\ddot{\boldsymbol{\omega}}\times\boldsymbol{\omega} - 4\boldsymbol{\omega}^2\dot{\boldsymbol{\omega}} - 8(\boldsymbol{\omega}\cdot\dot{\boldsymbol{\omega}})\boldsymbol{\omega} \end{aligned} \tag{70}$$

5. Dual algebra in rigid body kinematics

In this section, we will present some algebraic properties for dual numbers, dual vectors and dual tensors. More details can be found in [10–25].

5.1 Dual numbers

Let the set of real dual numbers to be denoted by

$$\underline{\mathbb{R}} = \left\{ \mathrm{a} + \varepsilon \mathrm{a}_0 \middle| \mathrm{a}, \mathrm{a}_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\} \tag{71}$$

where $\mathrm{a} = \operatorname{Re}(\underline{\mathrm{a}})$ is the real part of $\underline{\mathrm{a}}$ and $\mathrm{a}_0 = \operatorname{Du}(\underline{\mathrm{a}})$ the dual part. The sum and product between dual numbers generate a ring with zero divisors structure for $\underline{\mathbb{R}}$.

Any differentiable function $\mathrm{f} : \mathrm{S} \subset \mathbb{R} \to \mathbb{R}, \mathrm{f} = \mathrm{f}(\mathrm{a})$ can be completely defined on $\underline{\mathrm{S}} \subset \underline{\mathbb{R}}$ such that:

$$\mathrm{f} : \underline{\mathrm{S}} \subset \underline{\mathbb{R}} \to \underline{\mathbb{R}}; \mathrm{f}(\underline{\mathrm{a}}) = \mathrm{f}(\mathrm{a}) + \varepsilon \mathrm{a}_0 \mathrm{f}'(\mathrm{a}) \tag{72}$$

Based on the previous property, two of the most important functions have the following expressions: $\cos\underline{\mathrm{a}} = \cos\mathrm{a} - \varepsilon\mathrm{a}_0\sin\mathrm{a}$; $\sin\underline{\mathrm{a}} = \sin\mathrm{a} + \varepsilon\mathrm{a}_0\cos\mathrm{a}$;

5.2 Dual vectors

In the Euclidean space, the linear space of free vectors with dimension 3 will be denoted by $\mathbf{V}_3$. The ensemble of dual vectors is defined as:

$$\underline{\mathbf{V}_3} = \left\{ \mathbf{a} + \varepsilon \mathbf{a}_0; \mathbf{a}, \mathbf{a}_0 \in \mathbf{V}_3, \varepsilon^2 = 0, \varepsilon \neq 0 \right\} \tag{73}$$

where $\mathbf{a} = \operatorname{Re}(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = \operatorname{Du}(\underline{\mathbf{a}})$ the dual part. For any three dual vectors $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}$, the following notations will be used for the basic products: $\underline{\mathbf{a}}\cdot\underline{\mathbf{b}}$—scalar product, $\underline{\mathbf{a}}\times\underline{\mathbf{b}}$—cross product and $\langle\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}\rangle = \underline{\mathbf{a}}\cdot(\underline{\mathbf{b}}\times\underline{\mathbf{c}})$—triple scalar product. Regarding algebraic structure, $(\underline{\mathbf{V}}_3, +, \cdot_{\underline{\mathbb{R}}})$ is a free $\underline{\mathbb{R}}$-module [13, 14].

The magnitude of $|\underline{\mathbf{a}}|$, denoted by $|\underline{\mathbf{a}}|$, is the dual number computed from

$$\underline{\mathbf{a}} = \begin{cases} \|\mathbf{a}\| + \varepsilon \dfrac{\mathbf{a}_0\cdot\mathbf{a}}{\|\mathbf{a}\|}, \ \operatorname{Re}(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \varepsilon\|\mathbf{a}_0\|, \ \operatorname{Re}(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{74}$$

where $\|.\|$ is the Euclidean norm. For any dual vector $\underline{\mathbf{a}} \in \underline{\mathbf{V}}_3$, if $|\underline{\mathbf{a}}| = 1$ then $\underline{\mathbf{a}}$ is called unit dual vector.

5.3 Dual tensors

An $\underline{\mathbb{R}}$-linear application of $\underline{\mathbf{V}}_3$ into $\underline{\mathbf{V}}_3$ is called an Euclidean dual tensor:

$$\underline{\mathbf{T}}(\underline{\lambda}_1\underline{\mathbf{v}}_1 + \underline{\lambda}_2\underline{\mathbf{v}}_2) = \underline{\lambda}_1\underline{\mathbf{T}}(\underline{\mathbf{v}}_1) + \underline{\lambda}_2\underline{\mathbf{T}}(\underline{\mathbf{v}}_2), \forall\underline{\lambda}_1, \underline{\lambda}_2 \in \underline{\mathbb{R}}, \forall\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\mathbf{V}}_3 \tag{75}$$

Let $\mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3)$ be the set of dual tensors, then any dual tensor $\underline{\mathbf{T}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3)$ can be decomposed as $\underline{\mathbf{T}} = \mathbf{T} + \varepsilon\mathbf{T}_0$, where $\mathbf{T}, \mathbf{T_0} \in \mathbf{L}(\mathbf{V}_3, \mathbf{V}_3)$ are real tensors. Also, the dual transposed tensor, denoted by $\underline{\mathbf{T}}^{\mathrm{T}}$, is defined by

$$\underline{\mathbf{v}}_1 \cdot (\underline{\mathbf{T}}\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_2 \cdot \left(\underline{\mathbf{T}}^{\mathrm{T}}\underline{\mathbf{v}}_1\right), \forall\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\mathbf{V}}_3 \tag{76}$$

while $\forall\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \in \underline{\mathbf{V}}_3,\ \operatorname{Re}\left\langle\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3\right\rangle \neq 0$ the determinant is

$$\left\langle\underline{\mathbf{T}}\underline{\mathbf{v}}_1, \underline{\mathbf{T}}\underline{\mathbf{v}}_2, \underline{\mathbf{T}}\underline{\mathbf{v}}_3\right\rangle = \det\underline{\mathbf{T}}\left\langle\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3\right\rangle. \tag{77}$$

Orthogonal dual tensor maps are a powerful instrument in the study of the rigid motion with respect to an inertial and non-inertial reference frames.

Let the orthogonal dual tensor set be denoted by:

$$\underline{\mathbb{SO}}_3 = \left\{\underline{\mathbf{R}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \middle| \underline{\mathbf{R}}\underline{\mathbf{R}}^{\mathrm{T}} = \underline{\mathbf{I}}, \det\underline{\mathbf{R}} = 1\right\} \tag{78}$$

where $\underline{\mathbb{SO}_3}$ is the set of special orthogonal dual tensors and $\underline{\mathbf{I}}$ is the unit orthogonal dual tensor.

Theorem 5 (Structure Theorem). *For any* $\underline{\mathbf{R}} \in \underline{\mathbb{SO}}_3$ *a unique decomposition is viable*

$$\underline{\mathbf{R}} = (\mathbf{I} + \varepsilon\tilde{\boldsymbol{\rho}})\mathbf{R} \tag{79}$$

where $\boldsymbol{R} \in S\mathbb{O}_3$ *and* $\boldsymbol{\rho} \in \boldsymbol{V}_3$ *are called* ***structural invariants.***

Taking into account the Lie group structure of $\underline{\mathbb{SO}}_3$ and the result presented in previous theorem, it can be concluded that any orthogonal dual tensor $\underline{\mathbf{R}} \in \underline{\mathbb{SO}}_3$ can be used globally parameterize displacements of rigid bodies.

Theorem 6 (Representation Theorem). *For any orthogonal dual tensor* $\underline{\mathbf{R}}$ *defined as in Eq. (79), a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *can be computed to have the following equation*, [13–15]:

$$\underline{\mathbf{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \mathbf{I} + \sin\underline{\alpha}\tilde{\underline{\mathbf{u}}} + (1 - \cos\underline{\alpha})\tilde{\underline{\mathbf{u}}}^2 = \exp\left(\underline{\alpha}\tilde{\underline{\mathbf{u}}}\right) \tag{80}$$

The parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the **natural invariants** of $\underline{\mathbf{R}}$**.** The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi-Chasles axis [13, 14], while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translated distance d.

The Lie algebra of the Lie group $\underline{\mathbb{SO}}_3$ is the skew-symmetric dual tensor set denoted by $\underline{\mathbb{so}}_3 = \left\{\tilde{\underline{\boldsymbol{\alpha}}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \middle| \tilde{\underline{\boldsymbol{\alpha}}} = -\tilde{\underline{\boldsymbol{\alpha}}}^{\mathrm{T}}\right\}$, where the internal mapping is $\langle\tilde{\underline{\boldsymbol{\alpha}}}_1, \tilde{\underline{\boldsymbol{\alpha}}}_2\rangle = \widetilde{\tilde{\underline{\boldsymbol{\alpha}}}_1\underline{\boldsymbol{\alpha}}_2}$.

The link between the Lie algebra $\underline{\mathbb{so}}_3$, the Lie group $\underline{S\mathbb{O}}_3$, and the exponential map is given by the following.

Theorem 7. *The mapping*

$$exp\colon \underline{\mathbb{so}}_3 \to \underline{S\mathbb{O}}_3,\ exp\left(\tilde{\underline{\boldsymbol{\alpha}}}\right) = e^{\tilde{\underline{\boldsymbol{\alpha}}}} = \sum_{k=0}^{\infty}\frac{\tilde{\underline{\boldsymbol{\alpha}}}^k}{k!} \tag{81}$$

is well defined and onto.

$$\log\colon \underline{\mathbb{SO}}_3 \to \underline{\mathbb{so}}_3,\ \log\underline{\mathbf{R}} = \left\{\tilde{\underline{\boldsymbol{\psi}}} \in \underline{\mathbb{so}}_3 \middle| \exp\left(\tilde{\underline{\boldsymbol{\psi}}}\right) = \underline{\mathbf{R}}\right\} \tag{82}$$

and is the inverse of Eq. (81).

Based on Theorems 6 and 7, for any orthogonal dual tensor $\underline{\mathbf{R}}$, a dual vector $\underline{\boldsymbol{\psi}} = \underline{\alpha}\,\underline{\mathbf{u}} = \boldsymbol{\psi} + \varepsilon\boldsymbol{\psi}_0$ can be computed and represents the **screw dual vector**, which embeds the screw axis and screw parameters.

The form of $\underline{\boldsymbol{\psi}}$ implies that $\underline{\tilde{\boldsymbol{\psi}}} \in \log\underline{\mathbf{R}}$. The types of rigid displacements that can be parameterized by $\underline{\boldsymbol{\psi}}$ are:

- roto-translation if $\boldsymbol{\psi} \neq 0, \boldsymbol{\psi}_0 \neq 0$ and $\boldsymbol{\psi}\cdot\boldsymbol{\psi}_0 \neq 0 \Leftrightarrow$ if $\left|\underline{\boldsymbol{\psi}}\right| \in \underline{\mathbb{R}}$ and $\left|\underline{\boldsymbol{\psi}}\right| \notin \{\varepsilon\mathbb{R}\}$;
- pure translation if if $\boldsymbol{\psi} = 0$ and $\boldsymbol{\psi}_0 \neq 0 \Leftrightarrow$ if $\left|\underline{\boldsymbol{\psi}}\right| \in \underline{\varepsilon\mathbb{R}}$;
- pure rotation if $\boldsymbol{\psi} \neq 0$ and $\boldsymbol{\psi}\cdot\boldsymbol{\psi}_0 = 0 \Leftrightarrow$ if $\left|\underline{\boldsymbol{\psi}}\right| \in \mathbb{R}$.

Theorem 8. *The natural invariants* $\underline{\alpha} = \alpha + \varepsilon d, \underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *can be used to directly recover the structural invariants* $\mathbf{R}$ *and* $\boldsymbol{\rho}$ *from Eq. (79):*

$$\begin{aligned} \mathbf{R} &= \mathrm{I} + \sin\alpha\tilde{\mathbf{u}} + (1-\cos\alpha)\tilde{\mathbf{u}}^2 \\ \boldsymbol{\rho} &= d\mathbf{u} + \sin\alpha\mathbf{u}_0 + (1-\cos\alpha)\mathbf{u}\times\mathbf{u}_0 \end{aligned} \tag{83}$$

Theorem 9 (isomorphism theorem). *The special Euclidean group* $(S\mathbb{E}_3,\cdot)$ *and* $(\underline{S\mathbb{O}}_3,\cdot)$ *are connected via the isomorphism of the Lie groups*

$$\begin{aligned} &\Phi : S\mathbb{E}_3 \to \underline{S\mathbb{O}}_3 \\ &\Phi(g) = (\mathbf{I} + \varepsilon\tilde{\boldsymbol{\rho}})\mathbf{R} \end{aligned} \tag{84}$$

where $g = \begin{bmatrix} \mathbf{R} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathbf{R} \in \underline{S\mathbb{O}}_3$, $\boldsymbol{\rho} \in \mathbf{V}_3$.

Remark 6. *The inverse of* $\boldsymbol{\Phi}$ is

$$\boldsymbol{\Phi}^{-1} : \underline{S\mathbb{O}}_3 \leftrightarrow S\mathbb{E}_3; \boldsymbol{\Phi}^{-1}(\underline{\mathbf{R}}) = \begin{bmatrix} \mathbf{R} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{85}$$

where $\mathbf{R} = \operatorname{Re}(\underline{\mathbf{R}}), \boldsymbol{\rho} = \operatorname{vect}\left(\operatorname{Du}(\underline{\mathbf{R}})\cdot\mathbf{R}^T\right)$.

6. Higher-order kinematics in dual Lie algebra

Being the rigid body motion given by the following parametric equation in a given reference frame:

$$\begin{cases} \boldsymbol{\rho} = \boldsymbol{\rho}(t) \in \mathbf{V}_3 \\ \mathbf{R} = \mathbf{R}(t) \in \mathbf{S}\mathbb{O}_3 \end{cases} \tag{86}$$

with $t \in \mathbf{I} \subseteq \mathbb{R}$ is time variable.

The dual orthogonal tensor that describes the rigid body motion is [13, 24]:

$$\underline{\mathbf{R}} = (\mathbf{I} + \varepsilon\tilde{\boldsymbol{\rho}})\mathbf{R} \tag{87}$$

In relation (87), the skew symmetric tensor associated to the vector $\boldsymbol{\rho}$ is denoted by $\tilde{\boldsymbol{\rho}}$.

It can be easily demonstrated [14, 15], that:

$$
\begin{aligned}
&\underline{\mathbf{R}}\underline{\mathbf{R}}^{\mathrm{T}} = \underline{\mathbf{I}} \\
&\det\underline{\mathbf{R}} = 1 \\
&\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = (\underline{\mathbf{R}}\underline{\mathbf{a}}) \cdot (\underline{\mathbf{R}}\underline{\mathbf{b}}), \forall \underline{\mathbf{a}}, \underline{\mathbf{b}} \in \underline{\mathbf{V}}_3 \\
&\underline{\mathbf{R}}(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \underline{\mathbf{R}}\ (\underline{\mathbf{a}}) \times \underline{\mathbf{R}}\ (\underline{\mathbf{b}}), \forall \underline{\mathbf{a}}, \underline{\mathbf{b}} \in \underline{\mathbf{V}}_3
\end{aligned} \tag{88}
$$

The tensor **R** transports the dual vectors from the body frame in the space frame with the conservation of the dual angles and the relative orientation of lines that corresponds to the dual vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

The dual angular velocity for the rigid body motion (86) is given by (87):

$$\underline{\boldsymbol{\omega}} = \mathrm{vect}\dot{\underline{\mathbf{R}}}\underline{\mathbf{R}}^{\mathrm{T}} \tag{89}$$

It can be demonstrated that:

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v} \tag{90}$$

where

$$\boldsymbol{\omega} = \mathrm{vect}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{91}$$

is the instantaneous angular velocity of the rigid body and

$$\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{92}$$

is the linear velocity of a point of the rigid body that coincides with the origin of the reference frame at that given moment.

The dual angular velocity $\underline{\boldsymbol{\omega}}$ completely characterizes the distribution of the velocity field of the rigid body. The pair $(\boldsymbol{\omega}, \mathbf{v})$ is called "the twist of the rigid body motion" [13, 14].

Being:

$$\underline{\mathbf{V}}_{\boldsymbol{\rho}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}_{\boldsymbol{\rho}} \tag{93}$$

the dual velocity for a point localized in the reference frame by the position vector $\boldsymbol{\rho}$.

In (93), $\boldsymbol{\omega}$ is the instantaneous angular velocity of the rigid body and $\mathbf{v}_{\boldsymbol{\rho}}$ is the linear velocity of the point. Using the next equation,

$$\mathrm{e}^{\varepsilon\tilde{\boldsymbol{\rho}}} = \mathbf{I} + \varepsilon\tilde{\boldsymbol{\rho}} \tag{94}$$

from (90), (92)–(94), results:

$$\mathrm{e}^{\varepsilon\tilde{\boldsymbol{\rho}}}\underline{\mathbf{V}}_{\boldsymbol{\rho}} = \underline{\boldsymbol{\omega}} \tag{95}$$

Consequently, $\mathrm{e}^{\varepsilon\tilde{\mathbf{r}}}\underline{\mathbf{V}}_{\mathrm{r}}$ is an invariant having the same value for any $\mathbf{r}$.

Writing this invariant in two different points of the rigid body (noted with P and Q), results that:

$$\mathrm{e}^{\varepsilon\tilde{\mathbf{r}}_P}\underline{\mathbf{V}}_{\mathrm{P}} = \mathrm{e}^{\varepsilon\tilde{\mathbf{r}}_Q}\underline{\mathbf{V}}_{\mathrm{Q}} \tag{96}$$

From (97), results:

$$\underline{\mathbf{V}}_{\mathrm{P}} = \mathrm{e}^{\varepsilon \widetilde{\mathrm{PQ}}} \underline{\mathbf{V}}_{\mathrm{Q}} \tag{97}$$

with $\mathrm{PQ} = \boldsymbol{\rho}_{\mathrm{Q}} - \boldsymbol{\rho}_{\mathrm{P}}$.
Relation (97) is true for any P and Q.
Analogue with Eq. (95), the following invariants take place:

$$\begin{aligned} e^{\varepsilon\tilde{\boldsymbol{\rho}}}\underline{\mathbf{A}}_{\boldsymbol{\rho}} &= \underline{\dot{\boldsymbol{\omega}}}, \forall \boldsymbol{\rho} \in \mathbf{V}_3 \\ e^{\varepsilon\tilde{\boldsymbol{\rho}}}\underline{\mathbf{J}}_{\boldsymbol{\rho}} &= \underline{\ddot{\boldsymbol{\omega}}}, \forall \boldsymbol{\rho} \in \mathbf{V}_3 \\ e^{\varepsilon\tilde{\boldsymbol{\rho}}}\underline{\mathbf{H}}_{\boldsymbol{\rho}} &= \underline{\dddot{\boldsymbol{\omega}}}, \forall \boldsymbol{\rho} \in \mathbf{V}_3 \end{aligned} \tag{98}$$

where we denoted

$$\begin{aligned} \underline{\mathbf{A}}_{\boldsymbol{\rho}} &= \dot{\boldsymbol{\omega}} + \varepsilon \mathbf{A}_{\boldsymbol{\rho}} \\ \underline{\mathbf{J}}_{\boldsymbol{\rho}} &= \ddot{\boldsymbol{\omega}} + \varepsilon \mathbf{J}_{\boldsymbol{\rho}} \\ \underline{\mathbf{H}}_{\boldsymbol{\rho}} &= \dddot{\boldsymbol{\omega}} + \varepsilon \mathbf{H}_{\boldsymbol{\rho}} \end{aligned} \tag{99}$$

with $\mathbf{A}_{\boldsymbol{\rho}}, \mathbf{J}_{\boldsymbol{\rho}}, \mathbf{H}_{\boldsymbol{\rho}}$ the reduced acceleration, reduced jerk, respectively the reduced hyper-jerk (jounce), in a point given by the position vector $\boldsymbol{\rho}$:

$$\begin{aligned} \mathbf{A}_{\boldsymbol{\rho}} &= \mathbf{a}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{v}_{\boldsymbol{\rho}} \\ \mathbf{J}_{\boldsymbol{\rho}} &= \mathbf{j}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{a}_{\boldsymbol{\rho}} - 2\dot{\boldsymbol{\omega}} \times \mathbf{v}_{\boldsymbol{\rho}} \\ \mathbf{H}_{\boldsymbol{\rho}} &= \mathbf{h}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{j}_{\boldsymbol{\rho}} - 3\dot{\boldsymbol{\omega}} \times \mathbf{a}_{\boldsymbol{\rho}} - 3\ddot{\boldsymbol{\omega}} \times \mathbf{v}_{\boldsymbol{\rho}} \end{aligned} \tag{100}$$

In (100), $\mathbf{a}_{\boldsymbol{\rho}}$, $\mathbf{j}_{\boldsymbol{\rho}}$ and $\mathbf{h}_{\boldsymbol{\rho}}$ are, respectively, the acceleration, the jerk, and the hyper-jerk (jounce), in a point given by the position vector $\boldsymbol{\rho}$.
Analogue with Eq. (97) the following equations take place:

$$\begin{aligned} \underline{\mathbf{A}}_{\mathrm{P}} &= \mathrm{e}^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{A}}_{\mathrm{Q}} \\ \underline{\mathbf{J}}_{\mathrm{P}} &= \mathrm{e}^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{J}}_{\mathrm{Q}} \quad , \\ \underline{\mathbf{H}}_{\mathrm{P}} &= \mathrm{e}^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{H}}_{\mathrm{Q}} \end{aligned} \tag{101}$$

The lines corresponding to the dual vectors $\underline{\dot{\boldsymbol{\omega}}}, \underline{\ddot{\boldsymbol{\omega}}}, \underline{\dddot{\boldsymbol{\omega}}}$ represent the loci, where the vectors $\mathbf{A}_{\boldsymbol{\rho}}, \mathbf{J}_{\boldsymbol{\rho}}, \mathbf{H}_{\boldsymbol{\rho}}$ have the minimum module value. Supplementary,

$$\begin{aligned} \min_{\boldsymbol{\rho} \in \mathrm{V}_3} \left\| \mathbf{A}_{\boldsymbol{\rho}} \right\| &= \left| \mathrm{Du} \left| \underline{\dot{\boldsymbol{\omega}}} \right| \right| \\ \min_{\boldsymbol{\rho} \in \mathrm{V}_3} \left\| \mathbf{J}_{\boldsymbol{\rho}} \right\| &= \left| \mathrm{Du} \left| \underline{\ddot{\boldsymbol{\omega}}} \right| \right| \\ \min_{\boldsymbol{\rho} \in \mathrm{V}_3} \left\| \mathbf{H}_{\boldsymbol{\rho}} \right\| &= \left| \mathrm{Du} \left| \underline{\dddot{\boldsymbol{\omega}}} \right| \right| \end{aligned} \tag{102}$$

Interesting is the fact that for the plane motion $\min \left\| \mathbf{A}_{\boldsymbol{\rho}} \right\| = \min \left\| \mathbf{J}_{\boldsymbol{\rho}} \right\| = \min \left\| \mathbf{H}_{\boldsymbol{\rho}} \right\| = 0$ because $\mathrm{Du}\left|\underline{\dot{\boldsymbol{\omega}}}\right| = \mathrm{Du}\left|\underline{\ddot{\boldsymbol{\omega}}}\right| = \mathrm{Du}\left|\underline{\dddot{\boldsymbol{\omega}}}\right| = 0$

All properties are extended for higher-order accelerations. The vector $\underline{\boldsymbol{\omega}}^{(n)} = \frac{d^n \underline{\boldsymbol{\omega}}}{dt^n}, n \in \mathbb{N}$ describes completely the helicoidally field of the n order reduced accelerations, for $n \in \mathbb{N}$:

$$e^{\varepsilon\tilde{\boldsymbol{\rho}}}\underline{\mathbf{A}}_{\boldsymbol{\rho}}^{[n]} = \underline{\boldsymbol{\omega}}^{(n)} \tag{103}$$

In Eq. (103) $\underline{\mathbf{A}}_{\boldsymbol{\rho}}^{[n]}$ denote the n^{th} order of the dual reduced acceleration in a given point by the position vector $\boldsymbol{\rho}$.

It follows that the dual part of the n^{th} order differentiation of $\underline{\boldsymbol{\omega}}^{(n)}$

$$\underline{\boldsymbol{\omega}}^{(n)} = \boldsymbol{\omega}^{(n)} + \boldsymbol{\varepsilon}\mathbf{v}^{(n)} \tag{104}$$

is the n^{th} order reduced acceleration of that point of the rigid body that at the given time pass by the origin of the reference frame.

From equation

$$\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{105}$$

it follows that

$$\mathbf{v}^{(n)} = \boldsymbol{\rho}^{(n+1)} - \sum_{k=0}^{n} \mathrm{C}_n^k \boldsymbol{\omega}^{(k)} \times \boldsymbol{\rho}^{(n-k)}, n \in \mathbb{N} \tag{106}$$

with the following notations

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} \triangleq \boldsymbol{\rho}^{(n+1)}, n \in \mathbb{N} \tag{107}$$

for the $n \in \mathbb{N}$ order acceleration of the point given by the position vector $\boldsymbol{\rho}$ and

$$\mathbf{A}_{\boldsymbol{\rho}}^{[n]} \triangleq \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \sum_{k=0}^{n-1} \mathrm{C}_n^k \boldsymbol{\omega}^{(n-k)} \mathbf{a}_{\boldsymbol{\rho}}^{[k]} \tag{108}$$

for the n^{th} order reduced acceleration of the same point the equation:

$$\mathbf{A}_{\boldsymbol{\rho}}^{[n]} = \mathbf{v}^{(n)} + \boldsymbol{\omega}^{(n)} \times \boldsymbol{\rho} \tag{109}$$

which proves the character of the helicoidally field of the n^{th} order reduced accelerations field.

For $\boldsymbol{\rho} = 0$, the relations between the n^{th} order reduced acceleration and the n order acceleration from point O, the origin of the reference frame, are written

$$\mathbf{A}_{\mathbf{0}}^{[n]} = \mathbf{v}^{(n)} = \mathbf{a}_n - \sum_{k=1}^{n-1} \mathrm{C}_n^k \boldsymbol{\omega}^{(n-k)} \mathbf{a}_k, n \in \mathbb{N} \tag{110}$$

The invert of previous equation is written:

$$\mathbf{a}_n = \mathbf{P}_n(\mathbf{v}), n \in \mathbb{N} \tag{111}$$

where $\mathbf{P}_n$ is the polynomial with the coefficients in the ring of the second order Euclidean tensors and the polynomials $\mathbf{P}_n[\mathbf{D}]$ follow the recurrence equation:

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\tilde{\boldsymbol{\omega}}) \\ \mathbf{P}_0 = \mathbf{I} \end{cases} \tag{112}$$

it follows successively

$$
\begin{aligned}
\mathbf{P}_1 &= \tilde{\boldsymbol{\omega}} \\
\mathbf{P}_2 &= \mathbf{D} + \tilde{\boldsymbol{\omega}} \\
\mathbf{P}_3 &= \mathbf{D}^2 + \tilde{\boldsymbol{\omega}}\mathbf{D} + 2\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 \\
\mathbf{P}_4 &= \mathbf{D}^3 + \tilde{\boldsymbol{\omega}}\mathbf{D}^2 + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\mathbf{D} + 3\ddot{\tilde{\boldsymbol{\omega}}} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3
\end{aligned}
\tag{113}
$$

If we denote $\mathbf{T} = (\mathbf{v}, \tilde{\boldsymbol{\omega}})$ and by $\boldsymbol{\Psi}_n = (\mathbf{a}_n, \boldsymbol{\Phi}_n)$, $n \in \mathbb{N}^*$, for the case of the velocities, accelerations, jerks and jounces, on obtain (**Figure 1**):

$$
\begin{bmatrix} \mathbf{T} \\ \dot{\mathbf{T}} \\ \ddot{\mathbf{T}} \\ \dddot{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\tilde{\boldsymbol{\omega}} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -2\dot{\tilde{\boldsymbol{\omega}}} & -\tilde{\boldsymbol{\omega}} & \mathbf{I} & \mathbf{0} \\ -3\ddot{\tilde{\boldsymbol{\omega}}} & -3\dot{\tilde{\boldsymbol{\omega}}} & -\tilde{\boldsymbol{\omega}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \\ \boldsymbol{\Psi}_3 \\ \boldsymbol{\Psi}_4 \end{bmatrix} \tag{114}
$$

$$
\begin{bmatrix} \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \\ \boldsymbol{\Psi}_3 \\ \boldsymbol{\Psi}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \tilde{\boldsymbol{\omega}} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ 2\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 & \tilde{\boldsymbol{\omega}} & \mathbf{I} & \mathbf{0} \\ 3\ddot{\tilde{\boldsymbol{\omega}}} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3 & 3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 & \tilde{\boldsymbol{\omega}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \dot{\mathbf{T}} \\ \ddot{\mathbf{T}} \\ \dddot{\mathbf{T}} \end{bmatrix} \tag{115}
$$

$$
\begin{aligned}
\mathbf{T}^{(\mathrm{n}-\mathbf{1})} &= \boldsymbol{\Psi}_{\mathrm{n}} - \sum_{\mathrm{k}=1}^{\mathrm{n}-1} \mathbf{C}_{\mathrm{n}-\mathbf{1}}^{\mathrm{k}} \tilde{\boldsymbol{\omega}}^{(\mathrm{n}-1-\mathrm{k})} \boldsymbol{\Psi}_{\mathrm{k}} \\
\boldsymbol{\Psi}_{\mathrm{n}} &= \mathbf{P}_{\mathrm{n}} \mathbf{T}, \mathrm{n} \in \mathbb{N}^*
\end{aligned}
\tag{116}
$$

Theorem 10. *The n^{th} order accelerations field of a rigid body in a general motion is uniquely determined by the k^{th} order time derivative of a dual twist $\underline{\boldsymbol{\omega}}$, $k = \overline{0, n-1}$.*

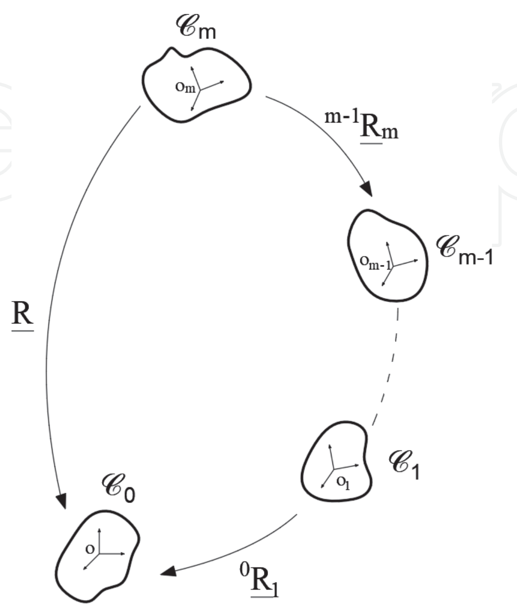

Figure 1.
Higher-order time derivative of dual twist.

7. Higher-order kinematics of spatial chain using dual Lie algebra

Consider a spatial kinematic chain of the bodies $C_k, k = \overline{0, m}$ where the relative motion of the rigid body C_k with respect to C_{k-1} is given by the proper orthogonal tensor ${}^{k-1}\underline{\mathbf{R}}_k \in \underline{\mathbf{SO}}_3^{\mathbb{R}}$. The relative motion properties of the body C_m with respect to C_0 are described by the orthogonal dual tensor (**Figure 2**):

$$\underline{\mathbf{R}} = {}^0\underline{\mathbf{R}}_1 \cdot {}^1\underline{\mathbf{R}}_2 \dots {}^{m-1}\underline{\mathbf{R}}_m \tag{117}$$

Instantaneous dual angular velocity (dual twist) of the rigid body in relation to the reference frame it will be given by the equation

$${}_0\underline{\boldsymbol{\omega}}_m = \text{vect}\dot{\underline{\mathbf{R}}}\underline{\mathbf{R}}^T \tag{118}$$

It follows from (110) and (111) that:

$${}_0\underline{\boldsymbol{\omega}}_m = \underline{\boldsymbol{\Omega}}_1 + {}^0\mathbf{R}_1\underline{\boldsymbol{\Omega}}_2 + \dots + {}^0\mathbf{R}_1 \cdot {}^1\mathbf{R}_2 \dots {}^{m-2}\mathbf{R}_{m-1}\underline{\boldsymbol{\Omega}}_m \tag{119}$$

where

$$\underline{\boldsymbol{\Omega}}_k = \text{vect}\,{}^{k-1}\dot{\underline{\mathbf{R}}}_k\,{}^{k-1}\underline{\mathbf{R}}_k^T \tag{120}$$

Using the denotation

$$\underline{\boldsymbol{\omega}}_k = {}^0\underline{\mathbf{R}}_1 \cdot {}^1\underline{\mathbf{R}}_2 \dots {}^{k-2}\underline{\mathbf{R}}_{k-1}\underline{\boldsymbol{\Omega}}_k \tag{121}$$

Eq. (118) will be written

$${}_0\underline{\boldsymbol{\omega}}_m = \underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \dots + \underline{\boldsymbol{\omega}}_m \tag{122}$$

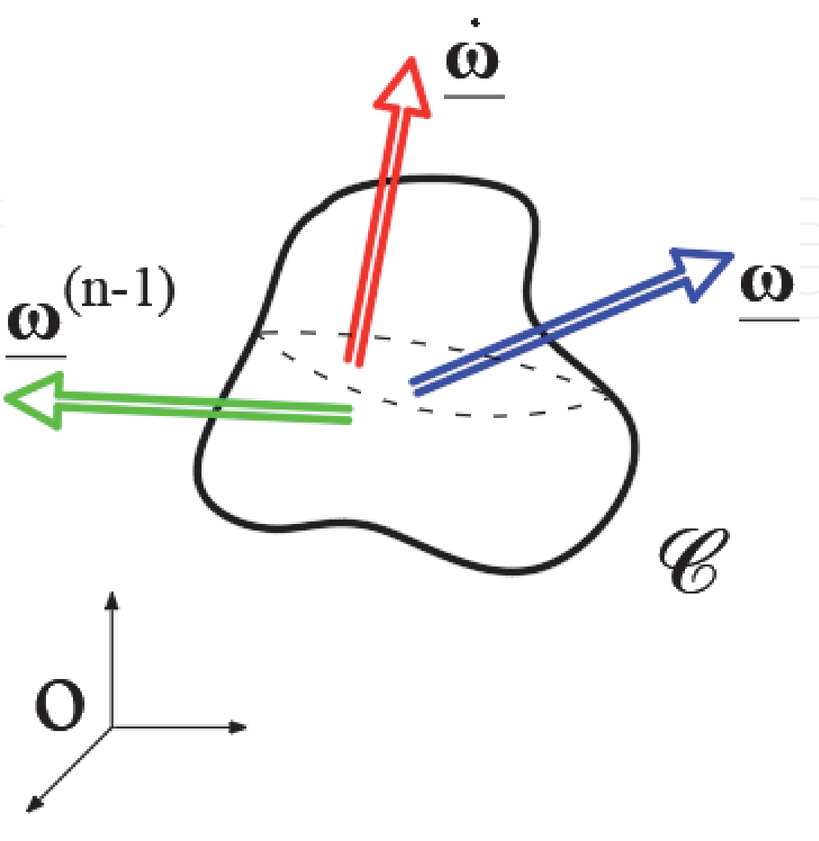

Figure 2.
Orthogonal dual tensors of relative rigid body motion.

where $\underline{\boldsymbol{\omega}}_k$ is the dual twist of the relative motion of the body C_k in relation to the body C_{k-1} observed from the body C_0.

Remark 7. *For* $m = 2$, ${}_0\underline{\boldsymbol{\omega}}_2 = \underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2$, *we will obtain the space replica of* ***Aronhold-Kennedy Theorem:*** *the instantaneous screw axis for the three relative rigid body motions has in every moment a common perpendicular, at any given time. The common perpendicular is line that corresponds to the dual vector* $\underline{\boldsymbol{\omega}}_1 \times \underline{\boldsymbol{\omega}}_2$.

To determine the field of the n^{th} order accelerations of a rigid body C_m we have to determine the ${}_0\underline{\boldsymbol{\omega}}_m^{(n)}$, $n \in \mathbb{N}$.

We denote $\underline{\boldsymbol{\omega}}_p^{[n]} = {}^0\underline{\mathbf{R}}_1 {}^1\underline{\mathbf{R}}_2 \dots {}^{p-2}\underline{\mathbf{R}}_{p-1} \underline{\boldsymbol{\Omega}}_p^{(n)}$ the $\mathrm{n^{th}}$ order derivative of the relative dual twist $\underline{\boldsymbol{\Omega}}_p$, resolved in the body frame of C_0.

In order to determine the $\mathrm{n^{th}}$ order accelerations field of a rigid body C_m, we have to determine the ${}_0\underline{\boldsymbol{\omega}}_m^{(n)}$, $n \in \mathbb{N}^*$.

To compute ${}_0\underline{\boldsymbol{\omega}}_m^{(n)}, n \in \mathbb{N}^*$ we will use the following

Lemma: *If* $\underline{\boldsymbol{\omega}}_\mathrm{p} = \underline{\mathbf{R}}\, \underline{\boldsymbol{\Omega}}_\mathrm{p}$ *with* $\underline{\mathbf{R}} \in \underline{\mathbf{SO}}_3^{\mathbb{R}}$ *and* $\underline{\boldsymbol{\omega}}_\mathrm{p}, \underline{\boldsymbol{\Omega}}_\mathrm{p} \in \underline{\mathbf{V}}_3^{\mathbb{R}}$, *then*

$$\underline{\boldsymbol{\omega}}_\mathrm{p}^{(\mathrm{n})} = \mathbf{p}_\mathrm{n}(\underline{\boldsymbol{\omega}})\underline{\boldsymbol{\omega}}_\mathrm{p}, \mathrm{p} = \overline{1, \mathrm{n}} \tag{123}$$

where $\mathbf{p}_\mathrm{n}(\underline{\boldsymbol{\omega}})$ *are polynomials of the differential operator* $\mathbf{D} = \frac{\mathrm{d}}{\mathrm{dt}}$, *with coefficients in the non-commutative ring of Euclidian dual tensors.*

$$\mathbf{p}_\mathrm{n}(\underline{\boldsymbol{\omega}}) = \sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{C}_\mathrm{n}^\mathrm{k} \underline{\boldsymbol{\Phi}}_{\mathrm{n-k}} \mathbf{D}^{[\mathrm{k}]}, \tag{124}$$

where C_n^k *is the binomial coefficient*, $\mathbf{D}^{[\mathrm{k}]}\boldsymbol{\omega}_\mathrm{p} = \boldsymbol{\omega}_\mathrm{p}^{[\mathrm{k}]}$ *and* $\underline{\boldsymbol{\Phi}}_\mathrm{p}$ *are dual tensors*

$$\underline{\boldsymbol{\Phi}}_\mathrm{p} = \underline{\mathbf{R}}^{(\mathrm{p})} \underline{\mathbf{R}}^\mathrm{T}, \underline{\mathbf{R}} \in \underline{\mathbf{SO}}_3^{\mathbb{R}}, \mathrm{p} = \overline{0, \mathrm{n}}, \tag{125}$$

which follow the recurrence equation:

$$\begin{cases} \underline{\boldsymbol{\Phi}}_{\mathrm{p}+1} = \dot{\underline{\boldsymbol{\Phi}}}_\mathrm{p} + \underline{\boldsymbol{\Phi}}_\mathrm{p} \tilde{\underline{\boldsymbol{\omega}}} \\ \underline{\boldsymbol{\Phi}}_0 = \underline{\mathbf{I}} \end{cases}, \mathrm{p} \in \mathbb{N}. \tag{126}$$

Theorem 11. *The following equation takes place*

$$\begin{aligned} {}_0\underline{\boldsymbol{\omega}}_m^{(n)} = \underline{\boldsymbol{\omega}}_1{}^{(n)} + \mathbf{p}_n(\underline{\boldsymbol{\omega}}_1)\underline{\boldsymbol{\omega}}_2 + \mathbf{p}_n(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2)\underline{\boldsymbol{\omega}}_3 + \dots \\ + \mathbf{p}_n(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \dots + \underline{\boldsymbol{\omega}}_{m-1})\underline{\boldsymbol{\omega}}_m; \forall n \in \mathbb{N} \end{aligned} \tag{127}$$

where $\mathbf{p}_n(\boldsymbol{\omega})$ *are polynomials of the derivative operator* $\mathbf{D} = \frac{d}{dt}$, *with coefficients in the non-commutative ring of Euclidian dual tensors*

$$\mathbf{p}_n(\underline{\boldsymbol{\omega}}) = \sum_{k=0}^{n} \mathrm{C}_n^k \underline{\boldsymbol{\Phi}}_{n-k} \mathbf{D}^{[k]} \tag{128}$$

where C_n^k *is the binomial coefficient*, $\mathbf{D}^{[\mathrm{k}]}\underline{\boldsymbol{\omega}}_\mathrm{p} = \underline{\boldsymbol{\omega}}_\mathrm{p}^{[\mathrm{k}]}$ and $\underline{\boldsymbol{\Phi}}_p$ *are dual tensors*

$$\underline{\boldsymbol{\Phi}}_p = \underline{\mathbf{R}}^{(p)} \cdot \underline{\mathbf{R}}^T, \underline{\mathbf{R}} \in \underline{\mathbf{SO}}_3^{\mathbb{R}}, p = \overline{0, n} \tag{129}$$

which follow the recurrence equation:

$$\begin{cases} \underline{\boldsymbol{\Phi}}_{p+1} = \dot{\underline{\boldsymbol{\Phi}}}_p + \underline{\boldsymbol{\Phi}}_p \underline{\tilde{\boldsymbol{\omega}}} \\ \underline{\boldsymbol{\Phi}}_0 = \underline{\mathbf{I}} \end{cases}, p \in \mathbb{N}. \tag{130}$$

Other equivalent forms of Eq. (127) are the following recursive formulas (**Figures 3** and **4**):

$$\underline{\boldsymbol{\omega}}_m^{(n)} = \underline{\boldsymbol{\omega}}_1^{(n)} + \mathbf{p}_n\left({}_0\underline{\boldsymbol{\omega}}_1\right)\underline{\boldsymbol{\omega}}_2 + \mathbf{p}_n\left({}_0\underline{\boldsymbol{\omega}}_2\right)\underline{\boldsymbol{\omega}}_3 + \dots + \mathbf{p}_n\left({}_0\underline{\boldsymbol{\omega}}_{m-1}\right)\underline{\boldsymbol{\omega}}_m, \forall n \in \mathbb{N} \tag{131}$$

The previous equations are valid in the most general situation where there are no kinematic links between the rigid bodies $C_1, C_2, \dots, C_m$.

The following identity can be proved:

$$\underline{\boldsymbol{\Phi}}_k\left(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \dots + \underline{\boldsymbol{\omega}}_{p-1}\right) = \sum_{k_1+k_2+\dots+k_{p-1}=k} \mathrm{C}_{\mathrm{n}}^{\mathrm{k}_1,\dots,\mathrm{k}_{\mathrm{p}-1}} \underline{\boldsymbol{\Phi}}_{k_1}(\underline{\boldsymbol{\omega}}_1)\underline{\boldsymbol{\Phi}}_{k_2}(\underline{\boldsymbol{\omega}}_2)\dots\underline{\boldsymbol{\Phi}}_{k_{p-1}}(\underline{\boldsymbol{\omega}}_{p-1}) \tag{132}$$

where $\mathrm{C}_{\mathrm{n}}^{\mathrm{k}_1,\dots,\mathrm{k}_{\mathrm{p}-1}} = \frac{n!}{k_1!\dots k_{p-1}!}$ is the multinomial coefficient.

From Eq. (131), on obtain the closed form non-recursive coordinate-free formula:

$$\begin{aligned} {}_0\underline{\boldsymbol{\omega}}_{\mathrm{m}}^{(\mathrm{n})} &= \underline{\boldsymbol{\omega}}_1^{[\mathrm{n}]} + \underline{\boldsymbol{\omega}}_2^{[\mathrm{n}]} + \dots + \underline{\boldsymbol{\omega}}_{\mathrm{m}}^{[\mathrm{n}]} + \\ &+ \sum_{p=2}^{m}\sum_{k=1}^{n} \mathrm{C}_n^k \sum_{\mathrm{k}_1+\dots+\mathrm{k}_{\mathrm{p}-1}=\mathrm{k}} \mathrm{C}_{\mathrm{n}}^{\mathrm{k}_1,\dots,\mathrm{k}_{\mathrm{p}-1}} \underline{\boldsymbol{\Phi}}_{\mathrm{k}_1}(\underline{\boldsymbol{\omega}}_1)\dots\underline{\boldsymbol{\Phi}}_{\mathrm{k}_{\mathrm{p}-1}}(\underline{\boldsymbol{\omega}}_{\mathrm{p}-1})\underline{\boldsymbol{\omega}}_{\mathrm{p}}^{[\mathrm{n}-\mathrm{k}]}, \end{aligned} \tag{133}$$

where

$$\begin{aligned} \underline{\boldsymbol{\Phi}}_0(\underline{\boldsymbol{\omega}}) &= \underline{\mathbf{I}} \\ \underline{\boldsymbol{\Phi}}_1(\underline{\boldsymbol{\omega}}) &= \underline{\tilde{\boldsymbol{\omega}}} \end{aligned} \tag{134}$$

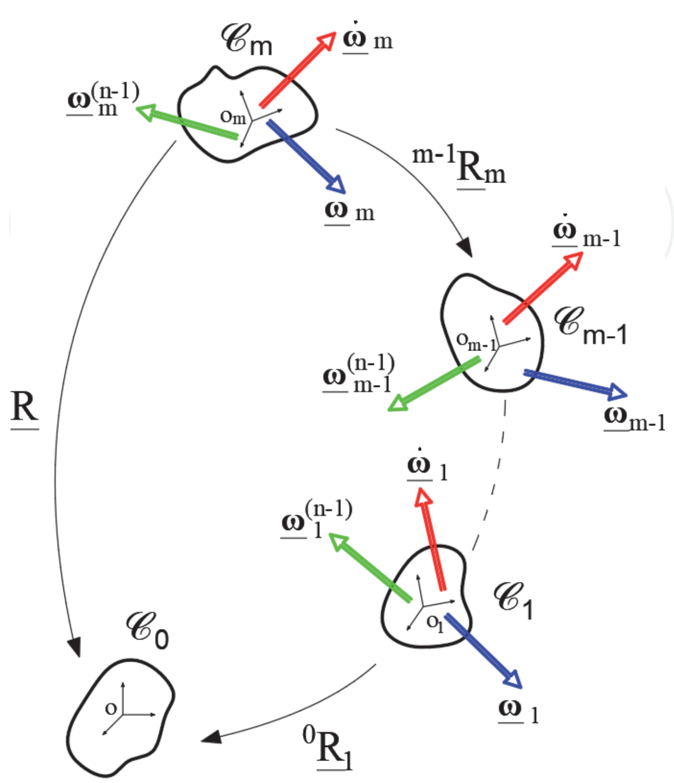

Figure 3.
Higher-order time derivative of dual twist of relative motion.

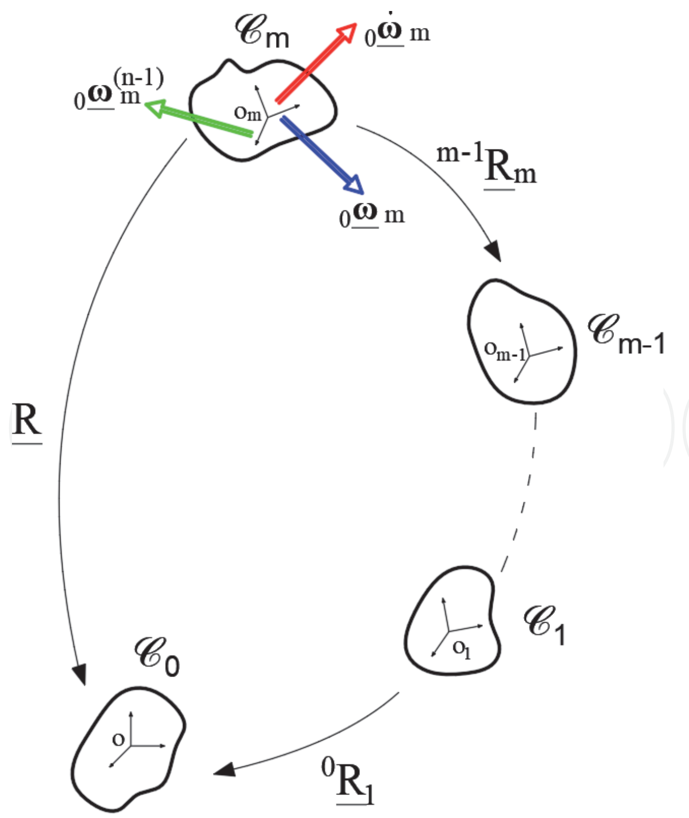

Figure 4.
Higher-order time derivative of dual twist of relative motion on terminal body.

$$\underline{\boldsymbol{\Phi}}_2(\underline{\boldsymbol{\omega}}) = \tilde{\underline{\boldsymbol{\omega}}}^{[1]} + \tilde{\underline{\boldsymbol{\omega}}}^2$$

$$\underline{\boldsymbol{\Phi}}_3(\underline{\boldsymbol{\omega}}) = \tilde{\underline{\boldsymbol{\omega}}}^{[2]} + \tilde{\underline{\boldsymbol{\omega}}}\tilde{\underline{\boldsymbol{\omega}}}^{[1]} + 2\tilde{\underline{\boldsymbol{\omega}}}^{[1]}\tilde{\underline{\boldsymbol{\omega}}} + \tilde{\underline{\boldsymbol{\omega}}}^3$$

$$\underline{\boldsymbol{\Phi}}_4(\underline{\boldsymbol{\omega}}) = \tilde{\underline{\boldsymbol{\omega}}}^{[3]} + \tilde{\underline{\boldsymbol{\omega}}}\,\tilde{\underline{\boldsymbol{\omega}}}^{[2]} + \left(3\tilde{\underline{\boldsymbol{\omega}}}^{[1]} + \tilde{\underline{\boldsymbol{\omega}}}^2\right)\tilde{\underline{\boldsymbol{\omega}}}^{[1]} + 3\tilde{\underline{\boldsymbol{\omega}}}^{[2]}\,\tilde{\underline{\boldsymbol{\omega}}} + 3\tilde{\underline{\boldsymbol{\omega}}}^{[1]}\,\tilde{\underline{\boldsymbol{\omega}}}^2 + 2\tilde{\underline{\boldsymbol{\omega}}}\,\tilde{\underline{\boldsymbol{\omega}}}^{[1]}\tilde{\underline{\boldsymbol{\omega}}} + \tilde{\underline{\boldsymbol{\omega}}}^4$$

$$\dots$$

$$\underline{\boldsymbol{\Phi}}_n(\underline{\boldsymbol{\omega}}) = \mathbf{P}_n\tilde{\underline{\boldsymbol{\omega}}}, \mathrm{n} \in \mathbb{N}^* \tag{135}$$

8. Higher-order kinematics for general 2C manipulator

We'll apply the general results obtained in the previous chapter for the particular case of four degrees of freedom 2C general manipulator. In this case the relative motions of three bodies C_0, C_1, C_2 are given, the spatial motion of the terminal body C_2 been described by dual orthogonal tensor as it follows:

$$^0\underline{\mathbf{R}}_2 = {}^0\underline{\mathbf{R}}_1{}^1\underline{\mathbf{R}}_2 \tag{136}$$

where

$$^0\underline{\mathbf{R}}_1 = e^{\underline{\alpha}_1(t)_0\tilde{\underline{\mathbf{u}}}_1} \tag{137}$$

$$^1\underline{\mathbf{R}}_2 = e^{\underline{\alpha}_2(t)_1\tilde{\underline{\mathbf{u}}}_2} \tag{138}$$

In Eqs. (138) and (139), the dual angles $\underline{\alpha}_1(t)$ and $\underline{\alpha}_2(t)$ are four times differentiable functions, and unit dual vectors $_0\underline{\mathbf{u}}_1$ and $_1\underline{\mathbf{u}}_2$ being constant. To simplify the writing, we will denote:

$$ {}_0\underline{\mathbf{u}}_1 = \cdot\underline{\mathbf{u}}_1 \tag{139} $$

$$ {}_0\underline{\mathbf{u}}_2 = \left(I + \sin\underline{\alpha}_1{}_0\tilde{\mathbf{u}}_1 + (1 - \cos\underline{\alpha}_1)_0\tilde{\mathbf{u}}_1^2\right)_1\underline{\mathbf{u}}_2 = \cdot\underline{\mathbf{u}}_2 \tag{140} $$

$$ \underline{\omega}_1 = \dot{\alpha}_1(t) + \varepsilon\dot{d}_1(t) \tag{141} $$

$$ \underline{\omega}_2 = \dot{\alpha}_2(t) + \varepsilon\dot{d}_2(t) \tag{142} $$

According to the observations from Section 6, the vector field of the velocity, the acceleration, the jerk, the jounce is uniquely determined by the dual vectors $\underline{\boldsymbol{\omega}}, \dot{\underline{\boldsymbol{\omega}}}, \ddot{\underline{\boldsymbol{\omega}}}, \dddot{\underline{\boldsymbol{\omega}}}$. Taking into account Eq. (133), we will have:

$$ \underline{\boldsymbol{\omega}} = \underline{\omega}_1\underline{\mathbf{u}}_1 + \underline{\omega}_2\underline{\mathbf{u}}_2 \tag{143} $$

$$ \dot{\underline{\boldsymbol{\omega}}} = \dot{\underline{\omega}}_1\underline{\mathbf{u}}_1 + \dot{\underline{\omega}}_2\underline{\mathbf{u}}_2 + \underline{\omega}_1\underline{\omega}_2\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2 \tag{144} $$

$$ \ddot{\underline{\boldsymbol{\omega}}} = \ddot{\underline{\omega}}_1\underline{\mathbf{u}}_1 + \ddot{\underline{\omega}}_2\underline{\mathbf{u}}_2 + (2\underline{\omega}_1\dot{\underline{\omega}}_2 + \dot{\underline{\omega}}_1\underline{\omega}_2)\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2 + \underline{\omega}_1^2\underline{\omega}_2\underline{\mathbf{u}}_1 \times (\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2) \tag{145} $$

$$ \dddot{\underline{\boldsymbol{\omega}}} = \dddot{\underline{\omega}}_1\underline{\mathbf{u}}_1 + \dddot{\underline{\omega}}_2\underline{\mathbf{u}}_2 + \left(\ddot{\underline{\omega}}_1\underline{\omega}_2 + 3\dot{\underline{\omega}}_1\dot{\underline{\omega}}_2 + 3\underline{\omega}_1\ddot{\underline{\omega}}_2 - \underline{\omega}_1^3\underline{\omega}_2\right)\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2 + 3\left(\underline{\omega}_1^2\dot{\underline{\omega}}_2 + \dot{\underline{\omega}}_1\underline{\omega}_1\underline{\omega}_2\right)\underline{\mathbf{u}}_1 \times (\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2) \tag{146} $$

$$ \begin{aligned} \dddot{\underline{\boldsymbol{\omega}}} = {} & \left[\dddot{\underline{\omega}}_1 + 3\left(\underline{\omega}_1^2\dot{\underline{\omega}}_2 + \dot{\underline{\omega}}_1\underline{\omega}_1\underline{\omega}_2\right)\underline{\mathbf{u}}_1 \cdot \underline{\mathbf{u}}_2\right]\underline{\mathbf{u}}_1 + \left[\dddot{\underline{\omega}}_2 - 3\left(\underline{\omega}_1^2\dot{\underline{\omega}}_2 + \dot{\underline{\omega}}_1\underline{\omega}_1\underline{\omega}_2\right)\right]\underline{\mathbf{u}}_2 \\ & + \left(\ddot{\underline{\omega}}_1\underline{\omega}_2 + 3\dot{\underline{\omega}}_1\dot{\underline{\omega}}_2 + 3\underline{\omega}_1\ddot{\underline{\omega}}_2 - \underline{\omega}_1^3\underline{\omega}_2\right)\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}_2 \end{aligned} \tag{147} $$

Similarly, the results for six degrees of freedom general 3 C manipulator can be obtained, the calculus being a little longer.

9. Conclusions

The higher-order kinematics properties of rigid body in general motion had been deeply studied. Using the isomorphism between the Lie group of the rigid displacements $S\mathbb{E}_3$ and the Lie group of the orthogonal dual tensors $\underline{S\mathbb{O}}_3$, a general method for the study of the field of arbitrary higher-order accelerations is described. It is proved that all information regarding the properties of the distribution of high-order accelerations are contained in the n-th order derivatives of the dual twist of the rigid body. These derivatives belong to the Lie algebra associated to the Lie group $\underline{S\mathbb{O}}_3$.

For the case of the spatial relative kinematics, equations that allow the determination of the n-th order field accelerations are given, using a Brockett-like formulas specific to the dual algebra. In particular cases the properties for velocity, acceleration, jerk, hyper-jerk (jounce) fields are given.

The obtained results interest the theoretical kinematics, jerk and jounce analysis in the case of parallel manipulations, control theory and multibody kinematics.

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