Chapter

Exact Traveling Wave Solutions of One-Dimensional Parabolic-Parabolic Models of Chemotaxis

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Abstract

In this chapter we consider several different parabolic-parabolic systems of chemotaxis which depend on time and one space coordinate. For these systems we obtain the exact analytical solutions in terms of traveling wave variables. Not all of these solutions are acceptable for biological interpretation, but there are solutions that require detailed analysis. We find this interesting, since chemotaxis is present in the continuous mathematical models of cancer growth and invasion (Anderson, Chaplain, Lolas, et al.) which are described by the systems of reaction–diffusion-taxis partial differential equations, and the obtaining of exact solutions to these systems seems to be a very interesting task, and a more detailed analysis is possible in a future study.

Keywords: parabolic-parabolic system, exact solution, soliton solution, Patlak-Keller-Segel model, chemotaxis

1. Introduction

This chapter uses the publications of Shubina M.V.:

1. Exact Traveling Wave Solutions of One-Dimensional Parabolic-Parabolic Models of Chemotaxis, Russian J Math Phys., Maik Nauka/Interperiodica Publishing (Russian Federation), 25(3), 383–395, 2018.

2. The 1D parabolic-parabolic Patlak-Keller-Segel model of chemotaxis: The particular integrable case and soliton solution, J Math Phys., 57(9), 091501, 2016.

Chemotaxis, or the directed cell (bacteria or other organisms) movement up or down a chemical concentration gradient, plays an important role in many biological and medical fields such as embryogenesis, immunology, cancer growth, and invasion. The macroscopic classical model of chemotaxis was proposed by Patlak in 1953 [1] and by Keller and Segel in the 1970s [2–4]. Since then, the mathematical modeling of chemotaxis has been widely developed. This model is described by the system of coupled nonlinear partial differential equations. Proceeding from the study of the properties of these equations, it is concluded that the model demonstrates a deep mathematical structure. The survey of Horstmann [5] provides a detailed

introduction into the mathematics of the Patlak-Keller-Segel model and summarizes different mathematical results; the detailed reviews also can be found in the textbooks of Suzuki [6] and Perthame [7]. In the review of Hillen and Painter [8], a number of variations of the original Patlak-Keller-Segel model are explored in detail. The authors study their formulation from a biological perspective, summarize key results on their analytical properties, and classify their solution forms [8]. It should be noted that interest in the Patlak-Keller-Segel model does not weaken and new works appear devoted to the study of various properties of equations and their solutions [9–12] and the links below.

In this chapter we investigate a number of different models describing chemotaxis. The aim of this paper is to obtain exact analytical solutions of these models. For one-dimensional parabolic-parabolic systems under consideration, we present these solutions in explicit form in terms of traveling wave variables. Of course, not all of the solutions obtained can have appropriate biological interpretation since the biological functions must be nonnegative in all domains of definition. However some of these solutions are positive and bounded, and their analysis requires further investigation. Despite the large number of works devoted to the systems under consideration and their properties, as well as the properties of their solutions, it seems to us that the solutions obtained in this paper are new.

The Patlak-Keller-Segel model describes the space–time evolution of a cell density $u(t, \vec{r})$ and a concentration of a chemical substance $v(t, \vec{r})$. The general form of this model is:

$$\begin{cases} u_t - \nabla(\delta_1 \nabla u - \eta_1 u \nabla \phi(v)) = 0 \\ v_t - \delta_2 \nabla^2 v - f(u, v) = 0, \end{cases}$$

where $\delta_1 > 0$ and $\delta_2 \geq 0$ are cell and chemical substance diffusion coefficients, respectively, and η_1 is a chemotaxis coefficient; when $\eta_1 > 0$, this is an attractive chemotaxis ("positive taxis"), and when $\eta_1 < 0$, this is a repulsive ("negative") one [13, 14]. $\phi(v)$ is the chemosensitivity function, and $f(u, v)$ characterizes the chemical growth and degradation. These functions are taken in different forms that correspond to some variations of the original Patlak-Keller-Segel model. We follow the reviews of Hillen and Painter [8] and of Wang [15] and consider the models presented therein.

This paper is concerned with one-dimensional simplified models when the coefficients δ_1, δ_2, and η_1 are positive constants, $x \in \Re, t \geq 0$, $u = u(x, t)$, and $v = v(x, t)$.

2. Signal-dependent sensitivity model

Let us start with a model that allows nonnegative bounded solutions that may be of interest from a biological point of view. Now consider the "logistic" model, one of versions of signal-dependent sensitivity model [8] with the chemosensitivity functions $\phi(v) = (1 + b) \ln(v + b)$, where $b = const$, and $f(u, v) = \tilde{\sigma} u - \tilde{\beta} v$. In the review [5] one can see a mathematical analysis of this model. When $b = 0$ and $\tilde{\beta} = 0$, the existence of traveling waves was established in [16, 17]. The replacements of $t \to \delta_1 t$ and $u \to \sigma \frac{\tilde{\sigma}}{\delta_1} u$ give $\delta_1 = 1$, $\alpha = \frac{\delta_2}{\delta_1}$, $\beta = \frac{\tilde{\beta}}{\delta_1}$, and $\sigma = \pm 1$. We also set $\eta = \frac{\eta_1(1+b)}{\delta_1}$, $1 + b > 0$, as well as $\phi(v) = \ln|v + b|$. It should be noted that a sign of σ may effect on the mathematical properties of the system. So, $\sigma = 1$ corresponds to an increase of a chemical substance, proportional to cell density, whereas $\sigma = -1$ corresponds to its decrease. And as we shall see later, various solutions correspond to these two cases.

After the above replacements, the model reads:

$$\begin{cases} u_t - u_{xx} + \eta\left(u\frac{v_x}{v+b}\right)_x = 0 \\ v_t - \alpha v_{xx} - \sigma u + \beta v = 0. \end{cases} \tag{1}$$

If we introduce the function $v = v + b$, in terms of traveling wave variable $y = x - ct$, where $c = const$, this system has the form:

$$\begin{cases} u_y + cu - \eta u(\ln(v))_y + \lambda = 0 \\ \alpha v_{yy} + cv_y - \beta v + \beta b + \sigma u = 0, \end{cases} \tag{2}$$

where $u = u(y)$, $v = v(y)$, and λ is an integration constant.
In this chapter we will consider the case of $\lambda = 0$. Then Eq. (2) gives:

$$u = C_u e^{-cy} v^{\eta}, \tag{3}$$

C_u is a constant and we will examine the following equation for v:

$$\alpha v_{yy} + cv_y - \beta v + \beta b + \sigma C_u e^{-cy} v^{\eta} = 0. \tag{4}$$

Since η is a positive constant, we consider two cases: $\eta = 1$ [Eq. (4) is a linear nonhomogeneous equation] and $\eta \neq 1$.

A. $\eta = 1$

Let us begin with $\eta = 1$. We introduce the new variable z and the new function w:

$$\begin{aligned} z &= \left(\frac{4\sigma C_u}{\alpha c^2}\right)^{\frac{1}{2}} e^{-\frac{cy}{2}} \\ w &= \left(\frac{4\sigma C_u}{\alpha c^2}\right)^{\frac{\alpha-2}{4\alpha}} v e^{\frac{cy}{2\alpha}} \end{aligned} \tag{5}$$

and Eq. (4) becomes:

$$z^2 w_{zz} + z w_z + w\left(z^2 - \nu^2\right) = \Lambda z^{-\frac{1}{\alpha}}, \tag{6}$$

where $\nu^2 = \frac{1}{\alpha^2}\left(1 + \frac{4\alpha\beta}{c^2}\right)$ and $\Lambda = -\frac{4\beta b}{\alpha c^2}\left(\frac{4\sigma C_u}{\alpha c^2}\right)^{\frac{1}{4}}$. Eq. (6) is the Lommel differential equation [18, 19] with $\mu = -1 - \frac{1}{\alpha}$, and we consider $\sigma C_u > 0$. Since this is a linear inhomogeneous second-order differential equation, one can integrate it by the method of variation of parameters. We assume a solution in the form:

$$w(z) = C_J(z) J_\nu(z) + C_Y(z) Y_\nu(z),$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions and $C_J(z)$ and $C_Y(z)$ are the functions of z that satisfy the equations:

$$\begin{aligned} J_\nu(z)\left(C_J(z)\right)_z + Y_\nu(z)\left(C_Y(z)\right)_z &= 0 \\ \left(J_\nu(z)\right)_z\left(C_J(z)\right)_z + \left(Y_\nu(z)\right)_z\left(C_Y(z)\right)_z &= \Lambda z^{-\frac{1}{\alpha}}. \end{aligned}$$

Considering that Wronskian $W(J_\nu, Y_\nu)(z) = \frac{2}{\pi z}$, we obtain:

$$C_J(z) = c_J - \frac{\Lambda\pi}{2}\int z^{-1-\frac{1}{\alpha}} Y_\nu(z)dz$$

$$C_Y(z) = c_Y + \frac{\Lambda\pi}{2}\int z^{-1-\frac{1}{\alpha}} J_\nu(z)dz,$$

where c_J and c_Y are constants. If both of the numbers $-\frac{1}{\alpha} \pm \nu$ are positive, the lower limits in the integrals may be taken to be zero. Then a particular integral of Lommel equation "proceeding in ascending powers of z" is $s_{\mu,\nu}(z)$ [19]; if one considers a solution of Lommel equation "in the form of descending series," one obtains the function $S_{\mu,\nu}(z)$ [19] [see Eq. (8)]. Thus, quoting Watson [19] "...and so, of Lommel's two functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$, it is frequently more convenient to use the latter." Then the general solution of Eq. (6) has the form:

$$w(z) = C_J J_\nu(z) + C_Y Y_\nu(z) + \Lambda S_{\mu,\nu}(z), \tag{7}$$

where C_J and C_Y are constants,

$$\begin{aligned} S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right) \\ &\quad \left[\sin\left(\frac{\pi}{2}(\mu-\nu)\right)J_\nu(z) - \cos\left(\frac{\pi}{2}(\mu-\nu)\right)Y_\nu(z)\right], \\ s_{\mu,\nu}(z) &= \frac{z^{\mu+1}}{\left[(\mu+1)^2-\nu^2\right]}{}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) \end{aligned} \tag{8}$$

are Lommel functions, and ${}_1F_2$ is the generalized hypergeometric function [18, 19]. Further, substituting the initial variable y and the function v [see Eq. (5)] into Eq. (7), we obtain a formal solution.

1. $b = 0$

We first consider the case $b = 0$. Then $\upsilon = v \geq 0$ and $C_u > 0$. Eq. (6) becomes homogeneous, and for $\sigma = 1$, its general solution is:

$$w(z) = C_J J_\nu(z) + C_Y Y_\nu(z). \tag{9}$$

However one can check that the function $u = u(y)$ diverges as $cy \to -\infty$ for all ν.

Consider now $\sigma = -1$. For $v(y)$ to be real, let $\alpha = 2$. Then Eq. (6) becomes the modified Bessel equation; the analysis of solution behavior at $\pm\infty$ leads to suitable solutions for $v(y)$ and $u(y)$:

$$\begin{aligned} v(y) &= e^{-\frac{cy}{4}} K_\nu\left(\sqrt{\frac{2C_u}{c^2}} e^{-\frac{cy}{2}}\right) \\ u(y) &= C_u e^{-\frac{5cy}{4}} K_\nu\left(\sqrt{\frac{2C_u}{c^2}} e^{-\frac{cy}{2}}\right) \end{aligned} \tag{10}$$

with restrictions $\nu \leq \frac{1}{2}$ and $\beta \leq 0$. So one can see that $v(y) \to 0$ as $cy \to -\infty$ for all $\nu \leq \frac{1}{2}$; $v(y) \to 0$ for $\nu < \frac{1}{2}$ and $v(y) \to \sqrt[4]{\frac{\pi^2 c^2}{8C_u}}$ for $\nu = \frac{1}{2}$ as $cy \to \infty$ and $u(y) \to 0$ as $y \to \pm\infty$ for all $\nu \leq \frac{1}{2}$. The curves of these functions are presented in **Figures 1** and **2**, and

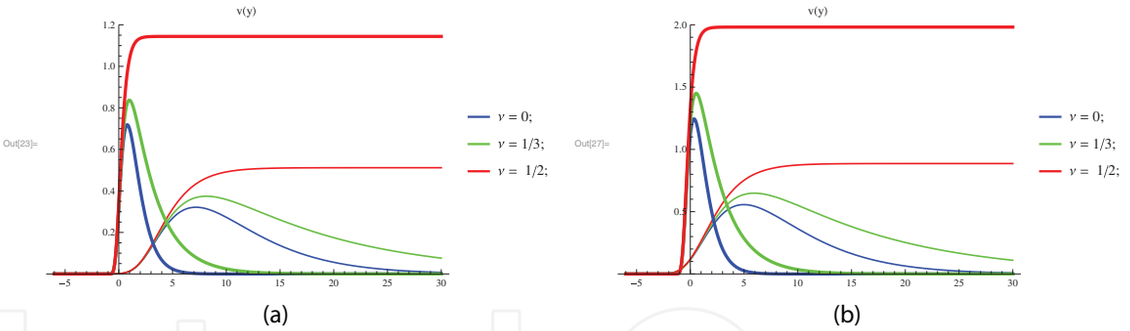

Figure 1.
(a) $v(y)$; $c = 1$; $c = 5$; $C_u = 18$. (b) $v(y)$; $c = 1$; $c = 5$; $C_u = 2$.

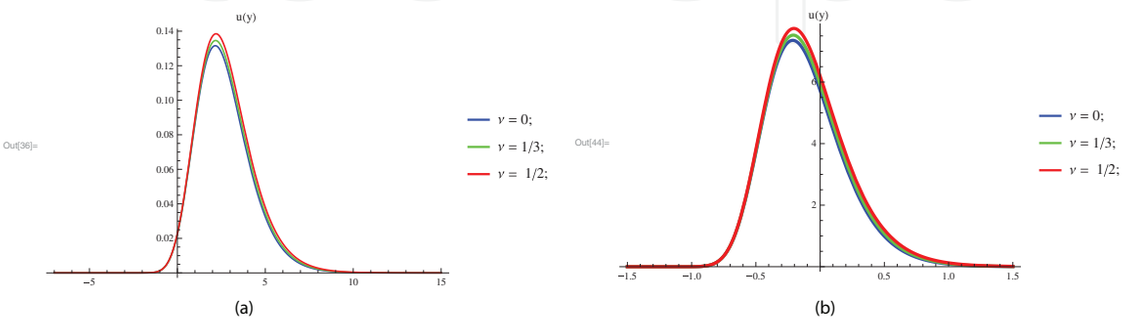

Figure 2.
(a) $u(y)$; $c = 1$; $C_u = 18$. (b) $u(y)$; $c = 5$; $C_u = 18$.

the plots for $c = 5$ are thicker than for $c = 1$. Thus, the solution obtained may be considered as a biologically appropriated one, and this requires further investigation.

2. $b > 0$

Let us return to Eq. (6) with $\Lambda \neq 0$. The analysis of solution asymptotic forms at $\pm\infty$ [18, 19] gives the following expressions for $v(y)$ and $u(y)$:

$$\begin{aligned} v(y) + b &= -\frac{4\beta b}{\alpha c^2}\left(\frac{4\sigma C_u}{\alpha c^2}\right)^{\frac{1}{2\alpha}} e^{-\frac{cy}{2\alpha}} S_{\mu,\nu}\left(\sqrt{\frac{4\sigma C_u}{\alpha c^2}} e^{-\frac{cy}{2}}\right) \\ u(y) &= -C_u \frac{4\beta b}{\alpha c^2}\left(\frac{4\sigma C_u}{\alpha c^2}\right)^{\frac{1}{2\alpha}} e^{-cy\left(1+\frac{1}{2\alpha}\right)} S_{\mu,\nu}\left(\sqrt{\frac{4\sigma C_u}{\alpha c^2}} e^{-\frac{cy}{2}}\right) \end{aligned} \tag{11}$$

with $\sigma C_u > 0$ and $\nu < \frac{1}{\alpha}$. The latter condition leads to the requirement $-\frac{c^2}{4\alpha} \leq \beta < 0$. The $v(y) \to -b$, $u(y) \to -\frac{\beta b}{\sigma}$ as $cy \to -\infty$ and $v(y) \to 0$ and $u(y) \to 0$ as $cy \to \infty$. Thus, one can see that for $b > 0$, $\sigma = 1$, and $C_u > 0$, $u(y) \geq 0$ is satisfied but $v(y) < 0$. These functions are presented in **Figures 3** and **4**. It should be noted that $\nu \neq \frac{1}{\alpha}$ or $\beta \neq 0$ because of the pole in Γ function.

3. $b < 0$

Using the analysis of Eq. (11), one can see that the condition $b < 0$ along with $\sigma = -1$ and $C_u < 0$ ($\sigma C_u > 0$) leads to the fact that the function $u(y)$ has not changed, but $v(y)$ becomes positive on all domains of definition. This function is presented in **Figure 5**.

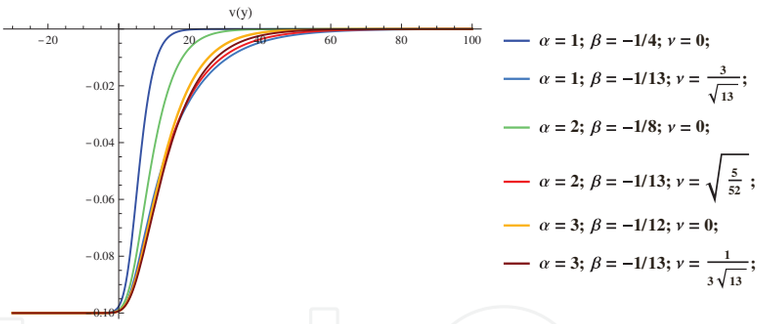

Figure 3.
$v(y)$; $c = 1$; $C_u = 9$; $\sigma = 1$; $b = 0.1$.

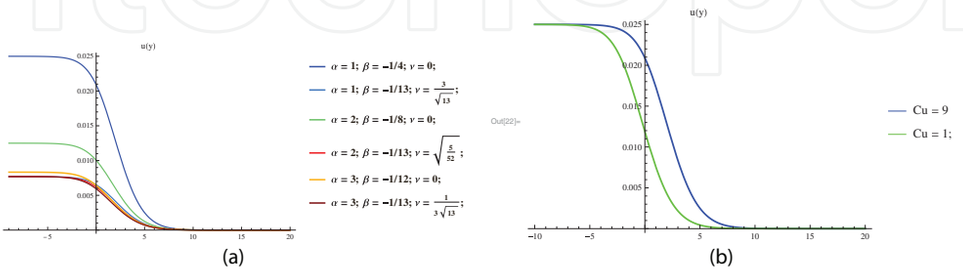

Figure 4.
(a) $u(y)$; $c = 1$; $C_u = 9$; $\sigma = 1$; $b = 0.1$. (b) $u(y)$; $c = 1$; $\sigma = 1$; $b = 0.1$; $\alpha = 1$; $\beta = -1/4$; $\nu = 0$.

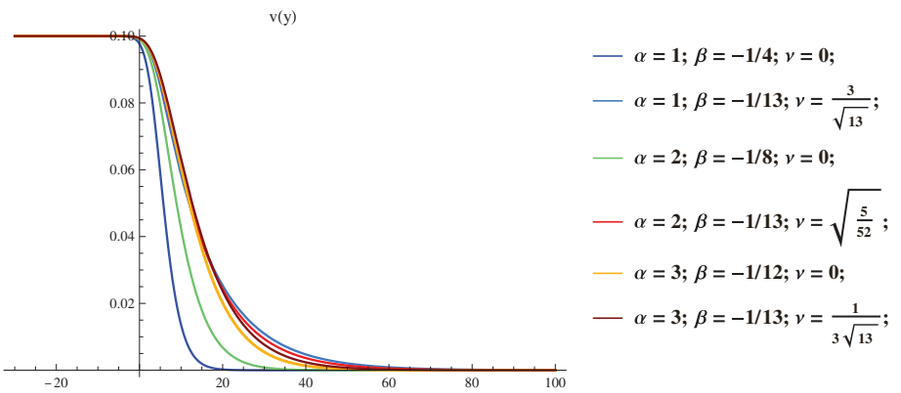

Figure 5.
$v(y)$; $c = 1$; $C_u = -9$; $\sigma = -1$; $b = -0.1$.

B. $\eta \neq 1$

Let us return to Eq. (4) and rewrite it in terms of the variable $\xi = e^{-\frac{cy}{\alpha}}$:

$$\xi^2 v_{\xi\xi} - \frac{\alpha\beta}{c^2} v + \frac{\sigma\alpha C_u}{c^2} \xi^\alpha v^\eta = -\frac{\alpha\beta b}{c^2}. \tag{12}$$

To integrate this equation, we use the Lie group method of infinitesimal transformations [20]. We find a group invariant of a second prolongation of one-parameter symmetry group vector of (12), and with its help, we transform Eq. (12) into an equation of the first order. It turns out that nontrivial symmetry group requires some conditions:

$$\begin{aligned} \frac{\alpha\beta b}{c^2} &= 0, \\ \beta &= \frac{(\alpha-2)(\alpha+\eta+1)c^2}{\alpha(\eta+3)^2} \end{aligned} \tag{13}$$

and we consider the case $b = 0$. Thus, $v = v$, and for:

$$z = \frac{v^{\frac{1-\eta}{\alpha}}}{y}$$
$$w = v_y\, v^{-\frac{\alpha+\eta-1}{\alpha}} \tag{14}$$

we obtain the Abel equation of the second kind:

$$w_z\left[(1-\eta)w - \alpha z\right] + (\alpha+\eta-1)z^{-1}w^2 + \alpha z\left(-\frac{\alpha\beta}{c^2} + \frac{\sigma\alpha C_u}{c^2}z^{-\alpha}\right) = 0. \tag{15}$$

Then we find the solutions of Eq. (15) in parametric form [21] with the parameter t. Now we consider the case $2\alpha+\eta \neq 1$. A combination of substitutions leads to:

$$z = \left(-\frac{(\eta+3)\left[(\eta+1)t^2 + \frac{2\sigma\alpha C_u}{c^2}\right]}{2(2\alpha+\eta-1)}\frac{\vartheta_t(t)}{\vartheta(t)}\right)^{\frac{2}{\alpha}}$$
$$w = z^{\frac{2-\alpha}{2}}\left(t + \frac{2(2\alpha+\eta+1)}{(\eta-1)(\eta+3)}z^{\frac{\alpha}{2}}\right) + \frac{\alpha}{1-\eta}z, \tag{16}$$

where we take

$$\vartheta(t) > 0 \text{ and } (2\alpha+\eta-1)\,\vartheta_t(t) < 0, \tag{17}$$

and Eq. (15) becomes an equation for the function $\vartheta(t)$. Solving it, for $\sigma C_u > 0$, we obtain:

$$\vartheta(t) = \tilde{C}_\vartheta\left(\frac{2\sigma\alpha C_u}{c^2}\right)^{-\frac{\eta+3}{2(\eta+1)}} t\,{}_2F_1\left(\frac{1}{2}, \frac{\eta+3}{2(\eta+1)}; \frac{3}{2}; -\frac{(\eta+1)c^2}{2\sigma\alpha C_u}t^2\right) + C_\vartheta, \tag{18}$$

where $\tilde{C}_\vartheta$ and C_ϑ are constants and ${}_2F_1$ is the hypergeometric Gauss function. Further we obtain the solutions of initial Eqs. (3)–(4) in parametric form:

$$y(t) = -\frac{\alpha(\eta+3)}{c(2\alpha+\eta-1)}\ \ln\left(\vartheta(t)\right)$$
$$v(t) = \left(-\frac{\tilde{C}_\vartheta(\eta+3)}{2(2\alpha+\eta-1)}\right)^{\frac{2}{1-\eta}}\left((\eta+1)t^2 + \frac{2\sigma\alpha C_u}{c^2}\right)^{-\frac{1}{\eta+1}}\left(\vartheta(t)\right)^{\frac{2-\alpha}{2\alpha+\eta-1}} \tag{19}$$
$$u(t) = C_u\left(-\frac{\tilde{C}_\vartheta(\eta+3)}{2(2\alpha+\eta-1)}\right)^{\frac{2}{1-\eta}}\left((\eta+1)t^2 + \frac{2\sigma\alpha C_u}{c^2}\right)^{-\frac{1}{\eta+1}}\left(\vartheta(t)\right)^{\frac{\alpha\eta+2\alpha+2}{2\alpha+\eta-1}}$$

where the constant $\tilde{C}_\vartheta$ is chosen so that $(2\alpha+\eta-1)\tilde{C}_\vartheta < 0$, which is consistent with Eq. (17). Using the asymptotic representation of hypergeometric Gauss function as $t \to \pm\infty$ [18], we can take:

$$C_\vartheta > |\tilde{C}_\vartheta|\ \frac{\pi}{2\sqrt{\eta+1}}\left(\frac{2\sigma\alpha C_u}{c^2}\right)^{-\frac{1}{\eta+1}}\frac{\Gamma\left(\frac{1}{\eta+1}\right)}{\Gamma\left(\frac{\eta+3}{2(\eta+1)}\right)} \tag{20}$$

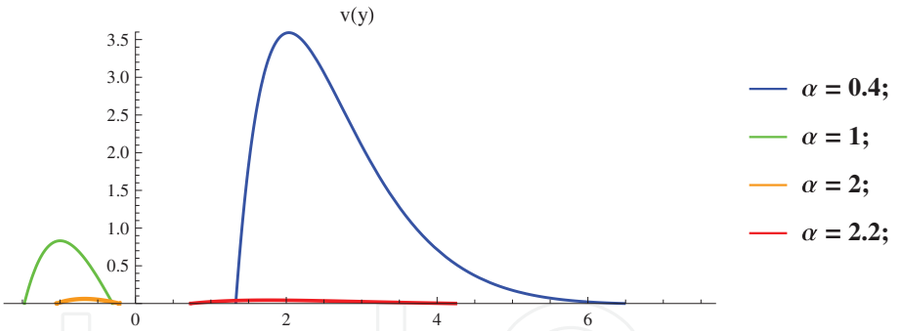

Figure 6.
$v(y)$; $\eta = 0.1$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.4$; $|\tilde{C}_\vartheta| = 1$.

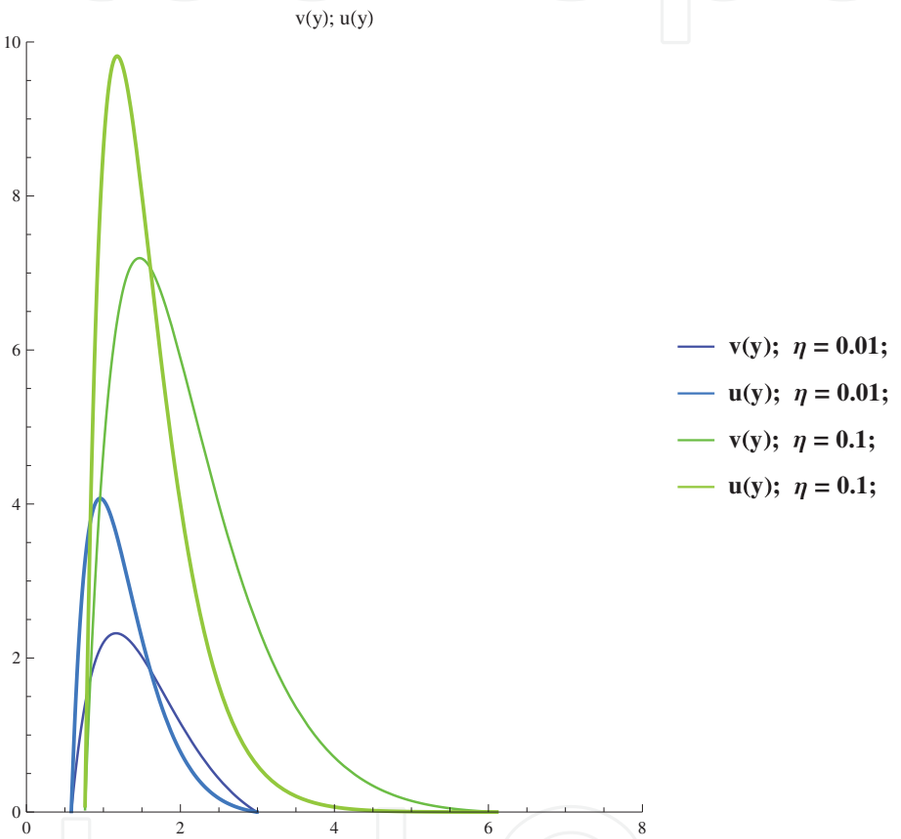

Figure 7.
$v(y)$; $u(y)$; $\alpha = 0.4$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.35$; $|\tilde{C}_\vartheta| = 1$.

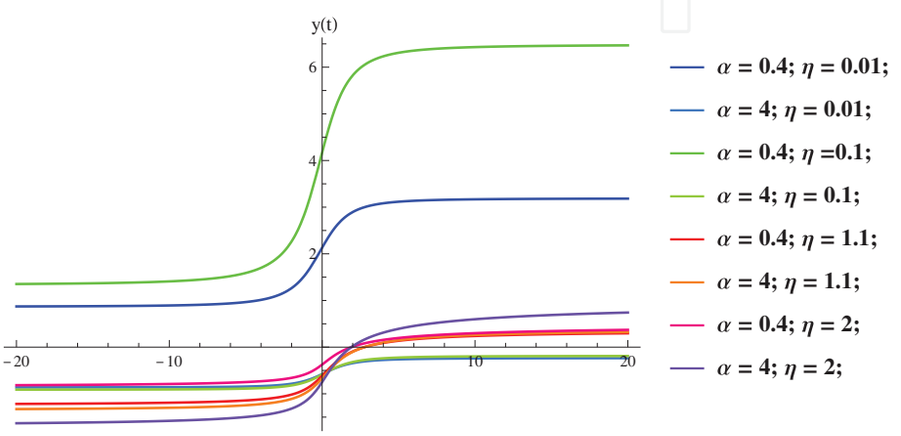

Figure 8.
$y(t)$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.4$; $|\tilde{C}_\vartheta| = 1$.

in order for y, v, and u to be real. Then one can see that all functions in Eq. (19) are continuous bounded ones for $t \in \Re$ and v, u are positive. Hence, one may try to biologically interpret the functions $v(y)$ and $u(y)$, and this requires further investigation. In **Figure 6** one may see the different curves $v(y)$ for $\eta = 0.1$ and different α. **Figure 7** demonstrates $v(y)$ and $u(y)$ for two values $\eta : \eta = 0.1$ and $\eta = 0.01$, see **Figure 7**. Further, for larger values of α and η, it seems more convenient to present the curves $y(t)$, $v(t)$, and $u(t)$ to analyze them (see **Figures 8–10**). One can see from Eq. (13) that $\beta \gtrless 0$ when $\alpha \gtrless 2$, and the case of $\beta = 0$ and $\alpha = 2$ is presented in **Figure 11**.

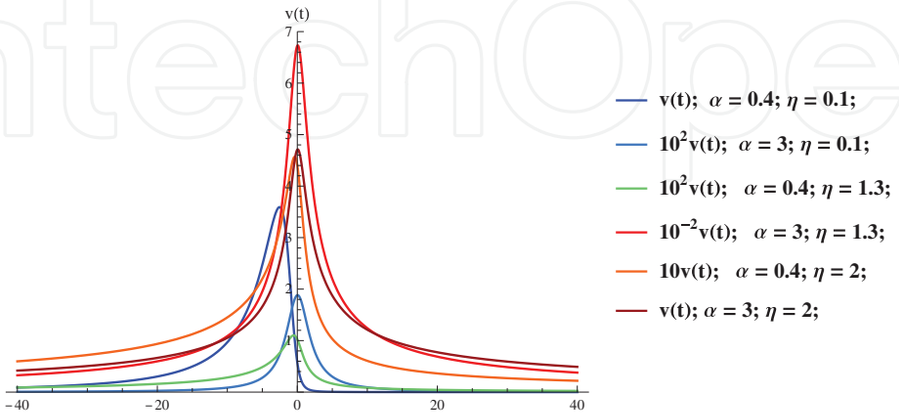

Figure 9.
$v(t)$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.4$; $|\tilde{C}_\vartheta| = 1$.

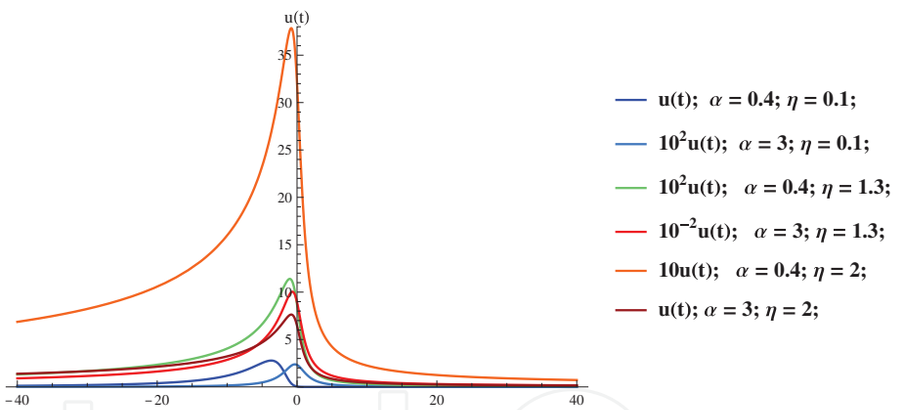

Figure 10.
$u(t)$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.4$; $|\tilde{C}_\vartheta| = 1$.

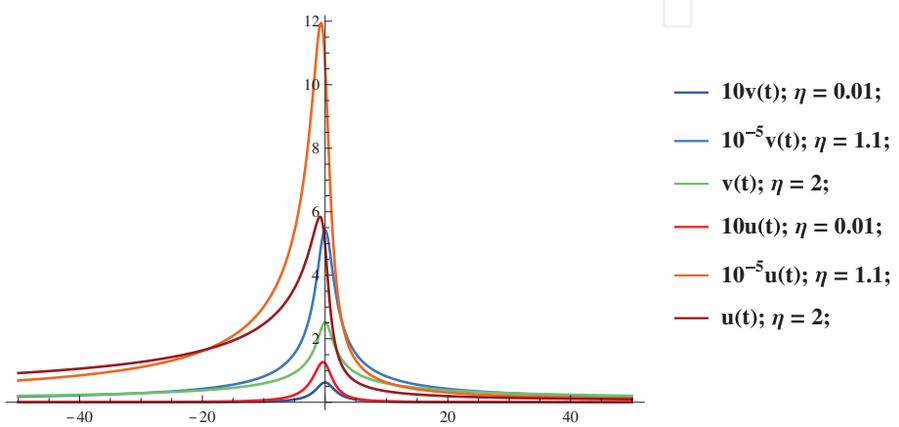

Figure 11.
$v(t)$; $u(t)$; $\alpha = 2$; $\frac{\sigma\alpha C_u}{c^2} = 2$; $c = 1$; $C_\vartheta = 1.4$; $|\tilde{C}_\vartheta| = 1$.

3. Logarithmic sensitivity

The model with logarithmic chemosensitivity function $\phi(v) \sim \ln v$ is also studied. For the case of $f(u,v) = -v^m u + \tilde{\beta} v$, where $\tilde{\beta} = const$, an extensive analysis is performed in [15]. This survey is focused on different aspects of traveling wave solutions. When $m = 0$, this model coincides with Eq. (1) for $b = 0$. When $\tilde{\beta} = 0$ and $m = 1$, the system was studied in [22, 23]. The complete analysis for $\tilde{\beta} = 0$ is performed in [15]. An existence of global solution is established in [24].

Now we consider the system with $\phi(v) = \ln v$ and $f(u,v) = \tilde{\sigma} v u - \tilde{\beta} v$. Similarly, a replacement of $t \to \delta_1 t$ and $u \to \sigma \frac{\tilde{\sigma}}{\delta_1} u$ gives $\delta_1 = 1$, $\eta = \frac{\eta_1}{\delta_1}$, $\alpha = \frac{\delta_2}{\delta_1}$, $\beta = \frac{\tilde{\beta}}{\delta_1}$, and $\sigma = \pm 1$. Then the model has the form:

$$\begin{cases} u_t - u_{xx} + \eta\left(u \frac{v_x}{v}\right)_x = 0 \\ v_t - \alpha v_{xx} - \sigma v u + \beta v = 0. \end{cases} \tag{21}$$

Let us rewrite the system (21) in terms of the function $v(x,t) = \ln v(x,t)$:

$$\begin{cases} u_t - u_{xx} + \eta(u v_x)_x = 0 \\ v_t - \alpha v_{xx} - \alpha (v_x)^2 + \beta - \sigma u = 0, \end{cases} \tag{22}$$

Then in terms of the traveling wave variable $y = x - ct$, where $c = const$, Eq. (22) has the form:

$$\begin{cases} u_y + cu - \eta u v_y + \lambda = 0 \\ \alpha v_{yy} + \alpha \left(v_y\right)^2 + c v_y - \beta + \sigma u = 0, \end{cases} \tag{23}$$

where $u = u(y)$, $v = v(y)$, and λ is an integration constant. To integrate Eq. (23), we tested this system on the Painlevé ODE test. One can show that for $\eta > 0$, it passes this test only if $\alpha = 2$ with the additional condition $\lambda = -\sigma c \beta \left(1 + \frac{\eta}{2}\right)$ [25]. If we express $u(y)$ as $v(y)$ from Eq. (23), we obtain an equation only for $v(y)$; for $\alpha = 2$, it has the form:

$$2v_{yyy} + 3cv_{yy} + \left(c^2 + \eta\beta\right)v_y + 2(2-\eta)v_y v_{yy} + 2(2-\eta)\left(v_y\right)^2 - 2\eta\left(v_y\right)^3 - c\beta - \sigma\lambda = 0. \tag{24}$$

For $\lambda = -\sigma c \beta \left(1 + \frac{\eta}{2}\right)$, this equation can be linearized. It becomes equivalent to the following linear equation for F:

$$F_y + cF = 0, \text{ where } F(y) = e^{2v}\left(2v_{yy} + cv_y - \eta\left(v_y\right)^2 + \frac{\eta\beta}{2}\right) \tag{25}$$

that gives the equation for $v(y)$:

$$2v_{yy} + cv_y - \eta\left(v_y\right)^2 + \frac{\eta\beta}{2} = C_F e^{-2v - cy}, \tag{26}$$

where $C_F = const$. If we rewrite Eq. (26) in terms of the variable $\xi = e^{-\frac{cy}{2}}$ for the function $\Psi(\xi) = e^{-\frac{\eta}{2}v}$, we obtain an equation similar to Eq. (12) with zero right-hand side:

$$\xi^2 \Psi_{\xi\xi} - \frac{\eta^2 \beta}{2c^2} \Psi + \frac{\eta C_F}{c^2} \xi^2 \Psi^{\frac{4}{\eta}+1} = 0. \tag{27}$$

Using the result of the symmetry group analysis of Eq. (12), we can write the solution for $\beta = 0$ [see Eq. (19)]:

$$\begin{aligned} y(t) &= -\frac{2}{c} \ln(\vartheta(t)) \\ v(t) &= \frac{|\tilde{C}_\vartheta|}{2} \left(\frac{2(\eta+2)}{\eta} t^2 + \frac{2\eta C_F}{c^2} \right)^{\frac{1}{\eta+2}} \end{aligned} \tag{28}$$

where $\vartheta(t)$ is given in Eq. (18) and $u(y)$ may be expressed from Eq. (23). However one may see that $v \to \infty$ as $t \to \pm\infty$, and this solution is unacceptable as a biological function.

Another possibility to solve this equation exactly is to put C_F equal to zero. When $C_F = 0$, that means $F(y) = 0$, for $\beta \neq 0$; Eq. (27) can be linearized by $\xi = e^\tau$ [21]. Its solution has three forms according to a sign of the expression $D = \frac{2\eta^2\beta}{c^2} + 1$. Since v should be a nonnegative and bounded function as $cy \to \pm\infty$, the only suitable solution is:

$$v(y) = e^{\frac{c}{2\eta}y} \left(C_- e^{-\frac{c\sqrt{D}}{4}y} + C_+ e^{\frac{c\sqrt{D}}{4}y} \right)^{-\frac{2}{\eta}} \tag{29}$$

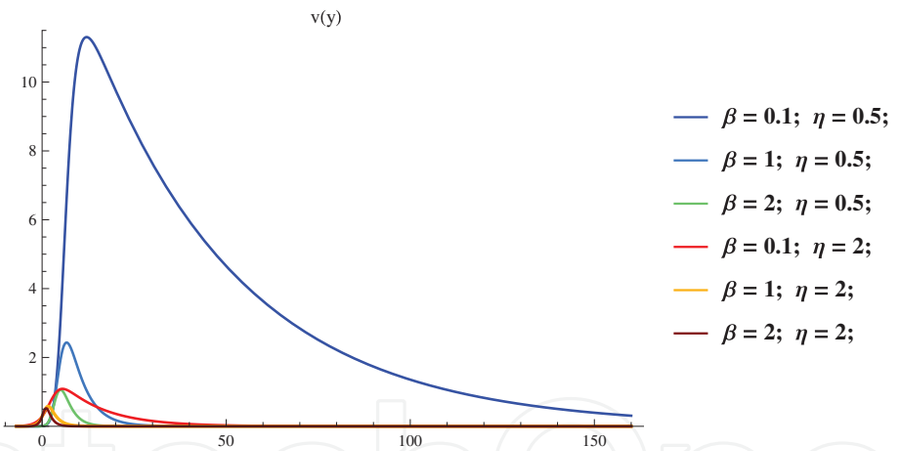

Figure 12.
$v(y)$; $c = 1$.

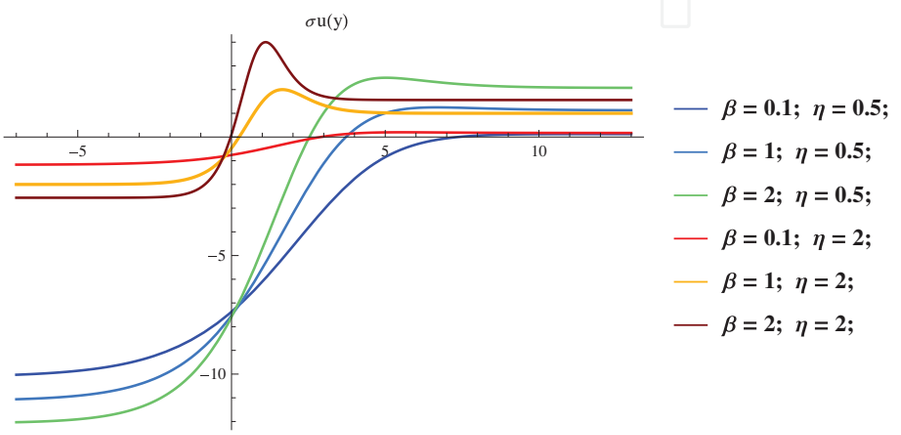

Figure 13.
$\sigma u(y)$; $c = 1$.

where $C_{\pm}$ are positive constants and $\beta > 0$. Unfortunately, the corresponding solution for $u(y)$ is alternating and has the form:

$$u(y) = -\frac{\sigma c^2(\eta+2)}{2\eta^2}\left(C_-^2\left(1+\sqrt{D}\right)e^{-\frac{c\sqrt{D}}{4}y} + C_+^2\left(1-\sqrt{D}\right)e^{\frac{c\sqrt{D}}{4}y} - \frac{4\eta^2\beta}{c^2}C_-C_+\right)\left(C_- e^{-\frac{c\sqrt{D}}{4}y} + C_+ e^{\frac{c\sqrt{D}}{4}y}\right)^{-\frac{2}{\eta}}. \tag{30}$$

It is easy to see that $\sigma u(y) \to \frac{c^2(\eta+2)}{2\eta^2}\left(-1 \pm \sqrt{D}\right)$ as $cy \to \pm\infty$. These functions are presented in **Figures 12** and **13**.

4. Linear sensitivity

Let us consider the system with linear function $\phi(v) \sim v$. When $f(u,v) = u - v$, the system is called the minimal chemotaxis model following the nomenclature of [26]. This model is often considered with $f(u,v) = \tilde{\sigma}u - \tilde{\beta}v$ ($\tilde{\sigma}$ and $\tilde{\beta}$ are constants), and it is studied in many papers. As was proved in [27, 28], the solutions of this system are global and bounded in time for one space dimension. The case of positive $\tilde{\sigma}$ and nonnegative $\tilde{\beta}$ is studied in [29–33]. As we noted earlier, a sign of $\tilde{\sigma}$ may effect on the mathematical properties of the system, which changes its solvability conditions [34].

Now we consider the linear chemosensitivity function $\phi(v) = v$ and $f(u,v) = \tilde{\sigma}u - \tilde{\beta}v$. The replacement of $t \to \delta_1 t$, $v \to \frac{\eta_1}{\delta_1}v$, and $u \to \sigma\frac{\tilde{\sigma}\eta_1}{\delta_1^2}u$ leads to $\delta_1 = \eta_1 = 1$, $\alpha = \frac{\delta_2}{\delta_1}$, $\beta = \frac{\tilde{\beta}}{\delta_1}$, and $\sigma = \pm 1$. Then the system has the form:

$$\begin{cases} u_t - u_{xx} + (uv_x)_x = 0 \\ v_t - \alpha v_{xx} + \beta v - \sigma u = 0. \end{cases} \tag{31}$$

This system reduces to the system of ODEs in terms of traveling wave variable $y = x - ct$, where $c = const$:

$$\begin{cases} u_y + cu - uv_y + \lambda = 0 \\ \alpha v_{yy} + cv_y - \beta v + \sigma u = 0, \end{cases} \tag{32}$$

where $u = u(y)$, $v = v(y)$, and λ is an integration constant. As shown in [35], this system passes the Painlevé ODE test only if $\alpha = 2$ and $\beta = 0$. Let us focus on this case.

It is convenient to solve Eq. (32) in terms of variable:

$$z = \frac{\kappa}{|c|}\, e^{-\frac{cy}{2}}, \tag{33}$$

where $\kappa > 0$ is an arbitrary constant. Then for v and u, we obtain the solutions in the form:

$$v = -\ln\left[\frac{|c|}{\kappa} z Z_\nu^2(z)\right]$$
$$u = c^2 z^2\left(1 - \frac{1}{4}(v_z)^2\right) - \frac{\lambda}{c}, \text{where } \nu^2 = \frac{1}{4} - \frac{\lambda}{c^3}. \tag{34}$$

The function $Z_\nu(z)$ satisfies the modified Bessel's equation and can be present as a linear combination of Infeld's and Macdonald's functions.

Using the series expansion of the Infeld's function, as well as theirs asymptotic behavior [36], one may obtain the following asymptotic forms for $e^{v_\nu(z)}$ and $u_\nu(z)$:

$$z \to \infty : \quad e^{v_\nu(z)} \to 0; \; u_\nu(z) \to 0. \tag{35}$$

$$z \to 0 : \quad e^{v_\nu(z)} \to \begin{cases} \infty, & 0 \le \nu < \frac{1}{2}; \\ \frac{\kappa}{|c| C^2} \frac{8\pi}{(\pi+2)^2}, & \nu = \frac{1}{2}; \\ 0, & \nu > \frac{1}{2}; \end{cases} \tag{36}$$

$$u_\nu(z) \to c^2 \left(\nu - \frac{1}{2} \right), \tag{37}$$

where the expression for $\nu = \frac{1}{2}$ agrees with Eq. (39).

So, the exact solution obtained has the form:

$$\begin{aligned} v &= -\ln \left[e^{-\frac{cy}{2}} A^2 \left(I_\nu \left(\frac{\kappa}{|c|} e^{-\frac{cy}{2}} \right) + B K_\nu \left(\frac{\kappa}{|c|} e^{-\frac{cy}{2}} \right) \right)^2 \right] \\ u &= -\sigma \left((v_y)^2 - \kappa^2 e^{-cy} + \frac{\lambda}{c} \right), \text{ where } \nu^2 = \frac{1}{4} - \frac{\lambda}{c^3}, \end{aligned} \tag{38}$$

where $\kappa > 0$, A, and B are arbitrary constants and the functions I_ν and K_ν are Infeld's and Macdonald's functions, respectively. This solution is not satisfactory from the biological point of view, since $v(y)$ is an alternating function for any ν. However it seems interesting because of the following: in the case of $\nu = \frac{1}{2}$ and $B = \frac{2+\pi}{2\pi}$ in terms of $e^{-\frac{cy}{2}}$, its form coincides with the well-known Korteweg-de Vries soliton.

Consider now the class of solutions with half-integer index $\nu = n + \frac{1}{2}$, when $Z_\nu(z)$ can be expressed in hyperbolic functions. The requirement of absence of divergence $u \to -\infty$ for finite z leads to the following form for $Z_{n+\frac{1}{2}}(z)$:

$$Z_{n+\frac{1}{2}}(z) = \begin{cases} C z^{n+\frac{1}{2}} \left(\frac{d}{z dz} \right)^n \frac{\cosh(z+\zeta)}{z}, & n = 2k, \\ C z^{n+\frac{1}{2}} \left(\frac{d}{z dz} \right)^n \frac{\sinh(z+\zeta)}{z}, & n = 2k+1; \; k = 0, 1 \ldots; \\ \zeta = \frac{1}{2} \ln \frac{2}{\pi}, \; C = const. \end{cases} \tag{39}$$

At first let us consider the solutions obtained for $e^{v_{n+\frac{1}{2}}}$ and $u_{n+\frac{1}{2}}$ as functions of z. We begin with $n = 0$ or $\nu = \frac{1}{2}$. It is interesting to present the expressions for $e^{u_{\frac{1}{2}}(z)}$ and $u_{\frac{1}{2}}(z)$:

$$e^{v_{\frac{1}{2}}(z)} = \frac{\kappa}{C^2 |c|} sech^2(z+\zeta) \tag{40}$$

$$u_{\frac{1}{2}}(z) = z^2 c^2 sech^2(z+\zeta), \tag{41}$$

where Eq. (40) appears the one-soliton solution exactly the same as the well-known one of the Korteweg-de Vries equation. Returning to the variable y:

$$e^{v\left(e^{-\frac{cy}{2}}\right)} = \frac{\kappa}{C^2|c|}\, sech^2\left(\frac{\kappa}{|c|}\, e^{-\frac{cy}{2}} + \frac{1}{2}\ln\frac{2}{\pi}\right)$$

$$u(y) = \frac{\sigma(\pi B - 1)\kappa^2 e^{-cy}}{\left(\sinh\left(\frac{\kappa}{|c|}\, e^{-\frac{cy}{2}}\right) + \frac{\pi}{2} B e^{-\frac{\kappa}{|c|} e^{-\frac{cy}{2}}}\right)^2}. \tag{42}$$

One can see that for $\sigma = 1$ (an increase of a chemical substance), the cell density $u(y) \geq 0$ for $B \geq \frac{1}{\pi}$ and that for $B > 0$ $u(y)$ is the solitary continuous solution vanishing as $y \to \pm\infty$, whereas for $B < 0$ $u(y)$ has a point of discontinuity. One can say that when $B < 0$, we obtain "blow-up" solution in the sense that it goes to infinity for finite y, and this is true for different ν.

The expressions for $n \geq 1$ become more complicated, and one can see the solitonic behavior of $e^{v_{n+\frac{1}{2}}(z)}$ and the curves for $u_{n+\frac{1}{2}}(z)$ in **Figures 14** and **15**.

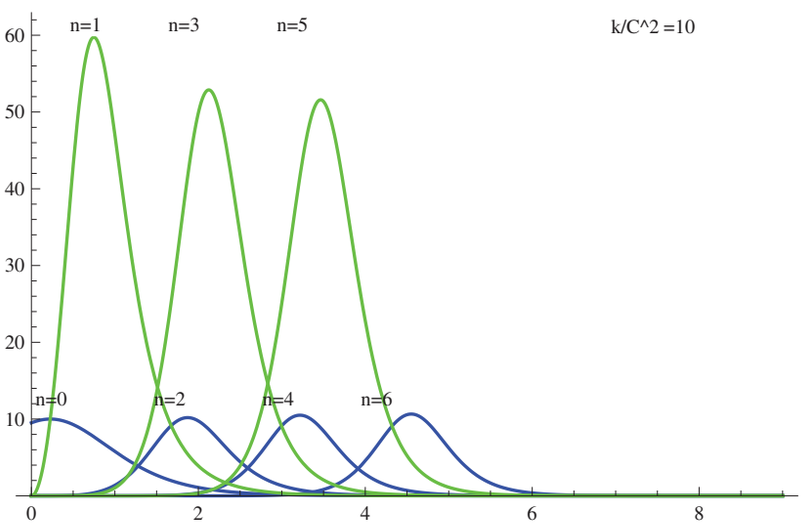

Figure 14.
$e^{v_{n+\frac{1}{2}}(z)}$; $n = 0, \ldots 6$; $c = 1$.

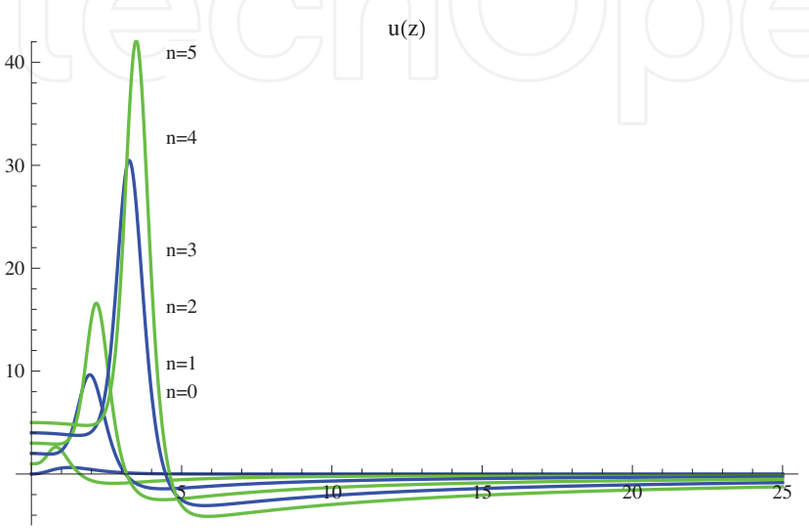

Figure 15.
$u_{n+\frac{1}{2}}(z)$; $n = 0, \ldots 5$; $c = 1$.

The explicit form of our solution in terms of the variable y can be obtained by direct substitution of Eq. (33) into Eq. (39), where $\frac{\lambda}{c} = -c^2 n(n+1)$. The resulting formulae are complicated and slightly difficult for analytic analysis; it seems to be more convenient to present the plots.

For $n = 0$ in the function $e^{v_{\frac{1}{2}}(y)}$, we have the "step" whose altitude depends on the values of velocity c and arbitrary constant κ. One may see that these curves become higher and shift to the right with different rates for the rising κ. The $u_{\frac{1}{2}}(y)$ is the positive function whose altitude and sharpness of peak depend on c (see **Figures 16** and **17**).

For $n \geq 1$ we can see that the solitonic behavior of $e^{v_{n+\frac{1}{2}}(y)}$ is retained for different values of c and κ; the curves become higher and more tight, and they shift to the right also with an increase of c and κ. For the cell density $u_{n+\frac{1}{2}}(y)$, the obtained solution has the negative section converging to zero for $cy \to -\infty$ (**Figures 18–21**).

The curves for the concentration of the chemical substance $v_{n+\frac{1}{2}}(y)$ are presented in **Figure 22**. Since $v_{n+\frac{1}{2}}(y)$ has to be positive (nonnegative), we see that these functions do not satisfy this requirement in all domains of definition.

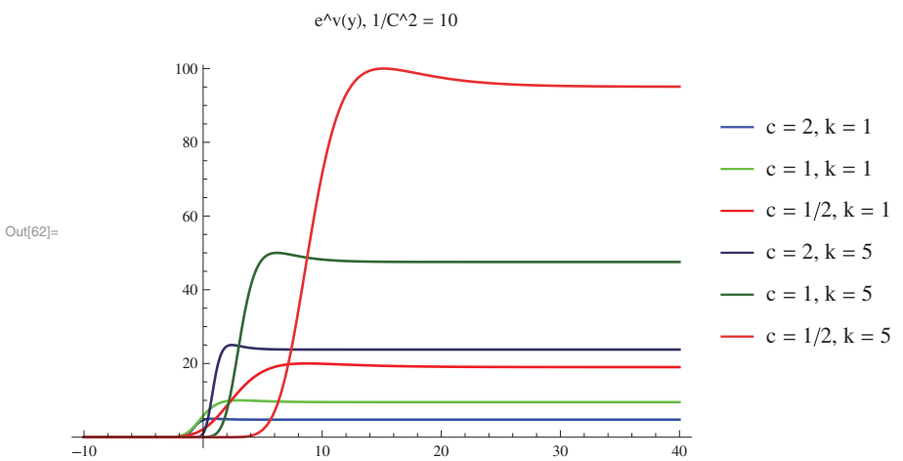

Figure 16.
$e^{v_{n+\frac{1}{2}}(y)}$; $n = 0$.

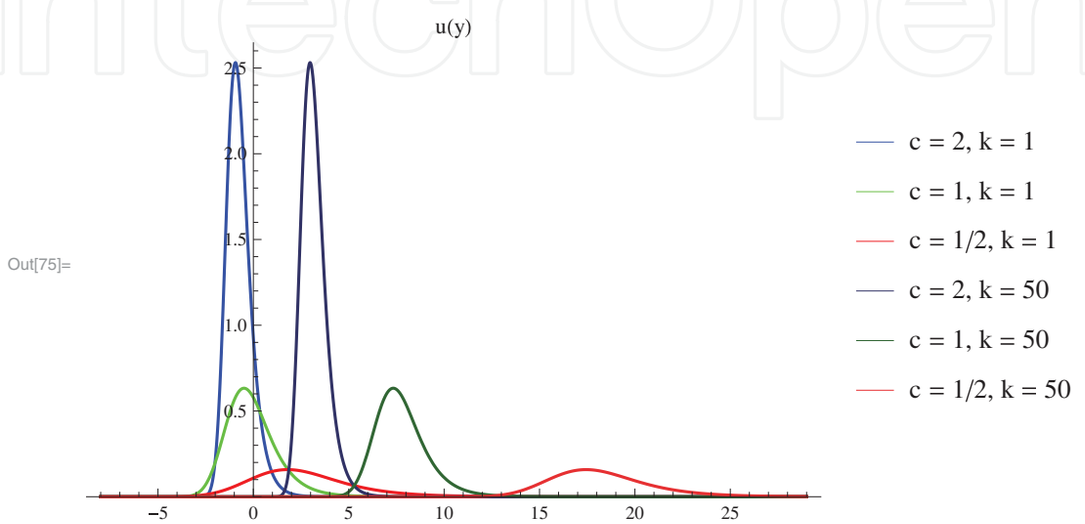

Figure 17.
$u_{n+\frac{1}{2}}(y)$; $n = 0$.

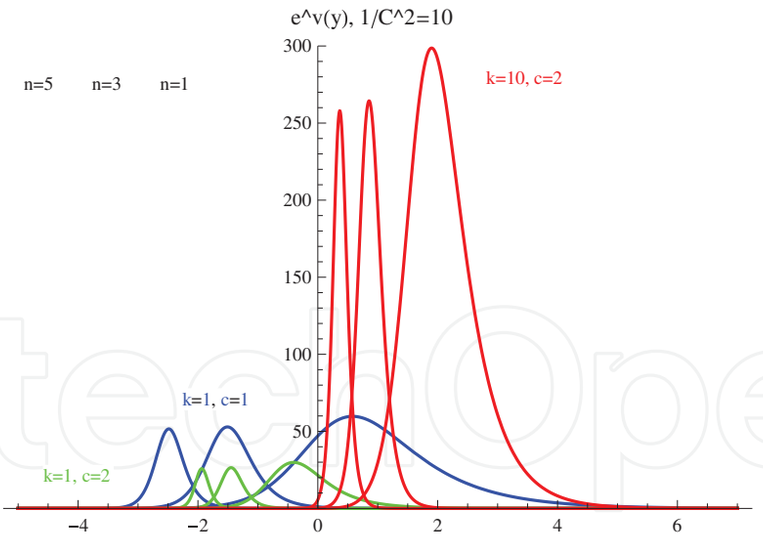

Figure 18.
$e^{v_{n+\frac{1}{2}}(y)}; n = 1; 3; 5.$

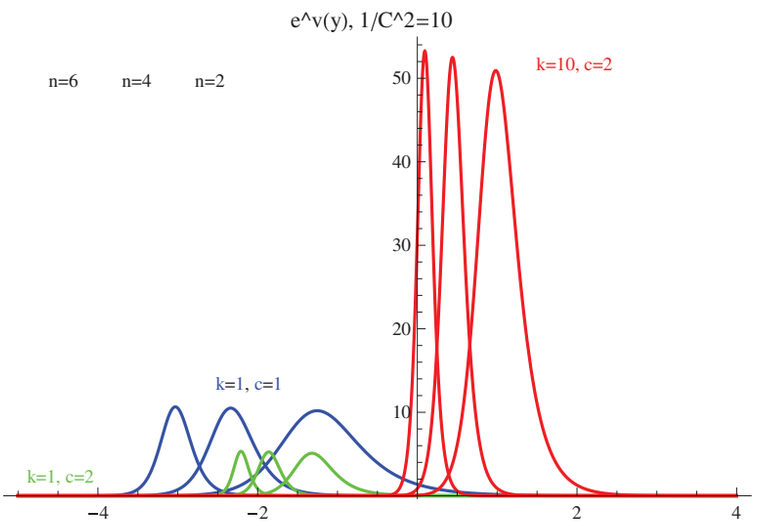

Figure 19.
$e^{v_{n+\frac{1}{2}}(y)}; n = 2; 4; 6.$

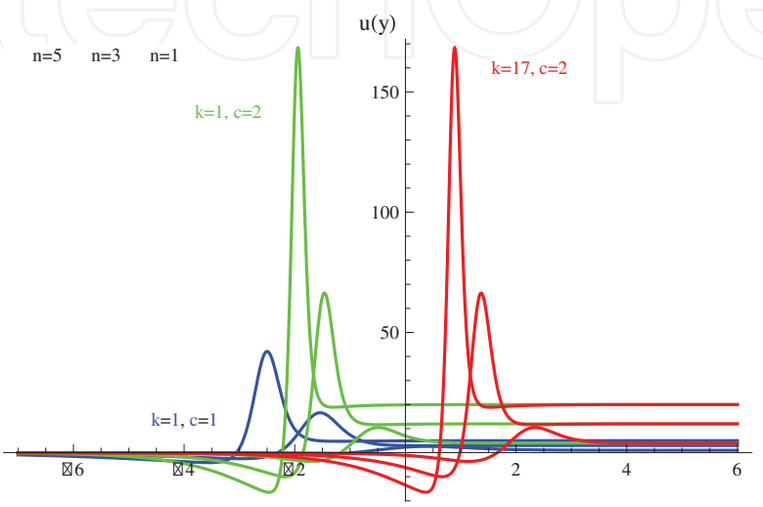

Figure 20.
$u_{n+\frac{1}{2}}(y); n = 1; 3; 5.$

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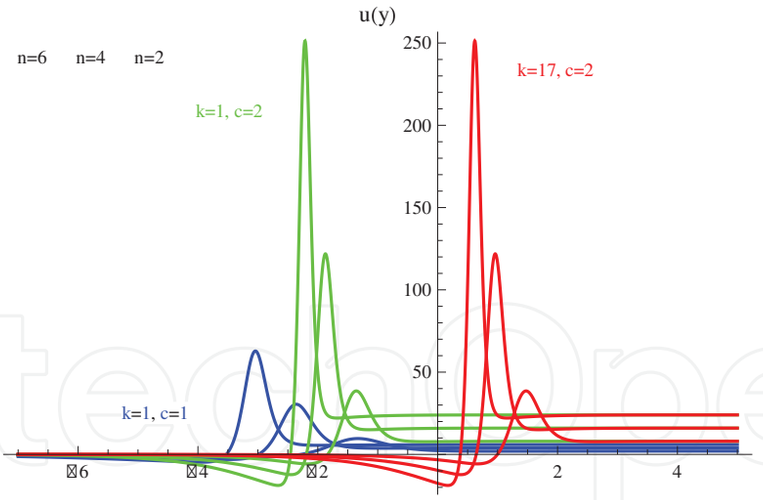

Figure 21.
$u_{n+\frac{1}{2}}(y); n = 2;4;6.$

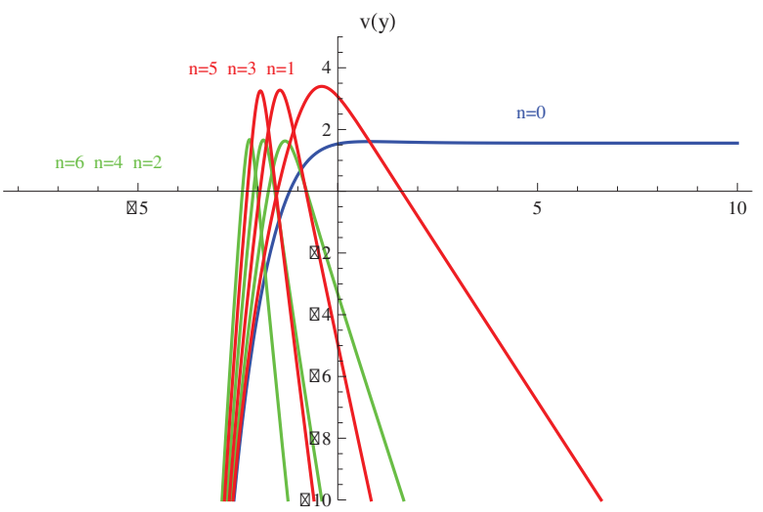

Figure 22.
$v_{n+\frac{1}{2}}(y); n = 0,1,...6.$

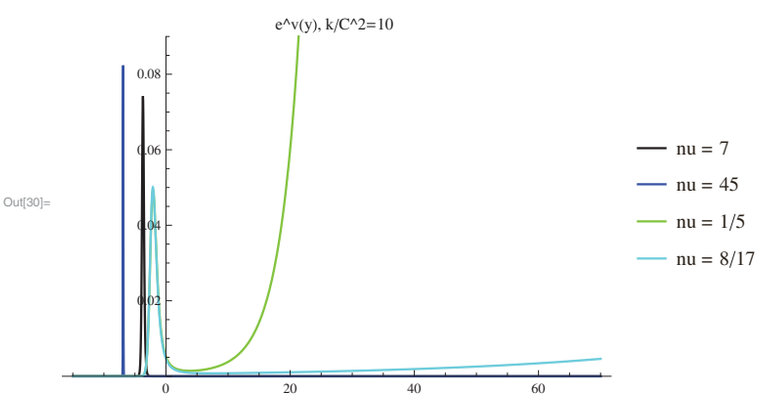

Figure 23.
$e^{v_\nu(y)}; \nu = 1/5;8/17;7;45.$

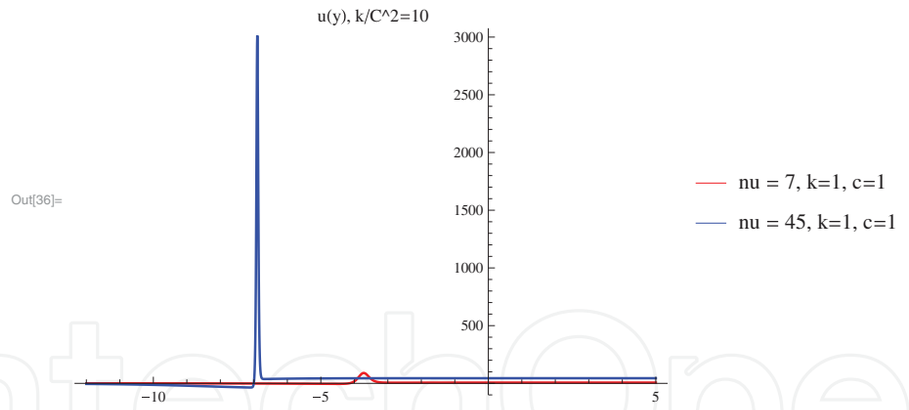

Figure 24.
$u_\nu(y)$; $\nu = 7; 45$.

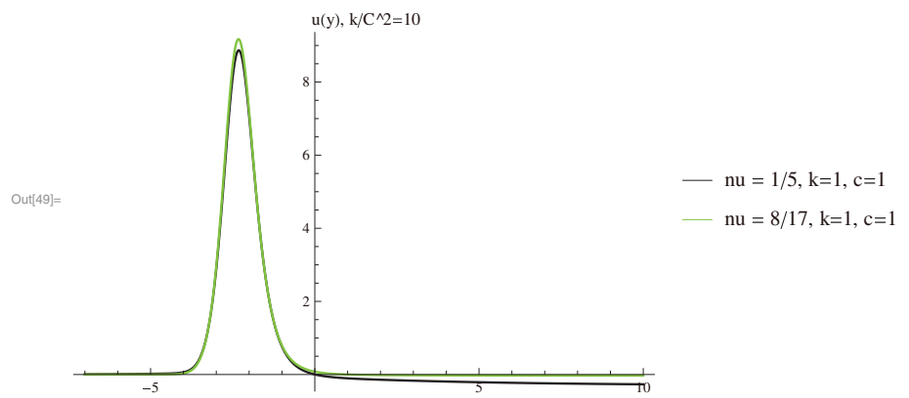

Figure 25.
$u_\nu(y)$; $\nu = 1/5; 8/17$.

In conclusion it seems interesting to present the plots for $e^{v_\nu(y)}$ and $u_\nu(y)$ for different values of ν (**Figures 23–25**). It is interesting to see that there are irregular solutions for $e^{v_\nu(y)}$; however, the corresponding solutions for $u_\nu(y)$ are regular [see Eqs. (35)–(37)].

5. Conclusion

We investigate three different one-dimensional parabolic-parabolic Patlak-Keller-Segel models. For each of them, we obtain the exact solutions in terms of traveling wave variables. Not all of these solutions are acceptable for biological interpretation, but there are solutions that require detailed analysis. It seems interesting to consider the latter for the experimental values of the parameters and see their correspondence with experiment. This question requires further investigations.

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