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Inverse Modeling Problems in Task Enrichment for STEM Courses

Victor Martinez-Luaces, José Antonio Fernández-Plaza and Luis Rico

Abstract

Problem solving is considered as one of the most important topics in STEM (science, technology, engineering, and mathematics) education, and this is especially relevant when problems require modeling skills in order to be solved. Also, it should be noted that in many branches of science and technology, typical problems are posed in an inverse form. Then, combining both characteristics, the so-called inverse modeling problems deserve to be studied deeply, particularly in their potential for task enrichment. For those reasons, since 2016, a research project was carried out, by using inverse modeling problems to develop prospective teacher's task enrichment skills. The results of this experience that took place in 2017 showed nine different groups of proposals where only few participants were very creative, whereas many others posed trivial problems or simply imitated examples analyzed previously. After that, a new research design was proposed during 2018 and implemented during the first months of 2019, with the aim of avoiding—or at least attenuating—those difficulties observed in the previous fieldwork. The new results showed interesting differences and few similarities when compared with the other experience. In this chapter, both experiences are analyzed, and lastly, findings and final conclusions are reported.

Keywords: inverse problems, task enrichment, prospective teachers, analysis of solution strategies, sketches, mathematical modeling

1. Introduction

In the last decade, STEM education has become an important topic, deeply analyzed by several authors, particularly in North America and Europe [1–6].

It must be remarked that the conjunction of the subjects to which the STEM education refers is not arbitrary. Sciences provide a context for reflection, organization, and action. They propose problems and questions that invite exploration and discovery and provide criteria to classify and organize the natural environment, thus allowing us to deepen into its richness and complexity. Technology and engineering offers tools and techniques that make the construction of models and artifacts that resolve conflicts or minimize impacts easier. Mathematics provides a mode of expression and representation and a set of notions and skills that allow interpreting and modeling the environment, providing strategies to invent and solve problems and promote logical and critical thinking. As a consequence, STEM

education permits the students to understand the world and interact with it in a critical, constructive, and efficient manner.

The natural link that exists between mathematics and science—which is at the core of STEM education—establishes important challenges for mathematics and science teachers. In particular, the mathematics teacher should know precisely the meaning of mathematical contents, identify the needs of students, diagnose learning problems, and prepare proposals for intervention and instruction for their approach and resolution. The abovementioned is important in all cases of mathematics teaching but especially important when working with STEM students, due to the strong bond between mathematics, science, and technology [7–9].

Besides, for future teachers to carry out these teaching strategies, it is necessary to look for significant situations in which the mathematical and scientific contents acquire meaning, for which it is essential to deepen their meanings (performing the semantic analysis, according to the method of analysis of content), as well as cognitive aspects (plausible expectations, learning stages, limitations, and opportunities, which constitute cognitive analysis) and instructive aspects. Therefore, the didactic analysis [10–13] becomes an important tool for the teacher to carry out teaching strategies that promote the development of the STEM competence of the students.

For these reasons, educational research must respond to the training needs of university students who are going to be teachers in the coming years in order to promote favorable attitudes toward sciences and mathematics.

One of the challenges consists in developing prospective teacher's task enrichment skills, [14] and for this purpose, inverse problems [15, 16] are especially relevant since in many branches of science and technology, typical problems are posed in an inverse form. In previous works we analyzed the particular cases when modeling skills are combined with inverse problems, and we called them inverse modeling problems [17, 18].

In this chapter we consider inverse modeling problems, focusing on their posing for task enrichment purposes. We describe our research carried out during the last 4 years, when working with prospective teachers at the University of Granada, Spain (UGR), and some of our most recent findings are reported and discussed in the following sections.

2. Theoretical framework

In our research relatively simple problems are proposed to prospective teachers. In all of them, only fundamental concepts of calculus, linear algebra, and geometry are necessary to be considered. The idea is to analyze easy problems, susceptible of being reformulated in the form of an inverse problem by prospective teachers.

It is expected that the reformulations raised by the participants will be richer than the original and will favor a teaching-learning process based more on exploration than repetition of procedures. As Lester and Cai [19] observed: "…teachers can develop worthwhile mathematical tasks by simply modifying problems from the textbooks" (p. 124).

The latter links the work to be done with a traditional area of research in mathematics education, as is the case of problem posing, the first subsection of our theoretical framework. In the end, the other two subsections are devoted to inverse problems and mathematical modeling.

2.1 Problem posing

There is a long tradition in the literature in English regarding problem-solving research, and the work of Brown and Walter [20, 21] and Kilpatrick [22], among

others, represents some of the best known examples. Under the common denomination of "problem posing," these authors include the formulation of new problems and/or the reformulation of problems previously proposed, in a certain format that can be more or less structured [23–26].

A particular case worthy of study occurs when students pose a new problem during the resolution of one of greater complexity [27]. This situation can already be seen in the work of Polya [28] that proposes, as a possible strategy, the approach of the problem in a different way or the establishment of variants, discarding some of its conditions.

In works done by other researchers, the formulation of problems does not have to be linked to the resolution of a specific problem. For example, in some cases the invention of problems is proposed starting from a certain situation or experience [23, 24].

Another option is to combine the two previous approaches and ask students to solve a problem after changing a condition or the final question of the problem, thereby creating a new problem [23].

Other researchers such as Brown and Walter [20, 21] propose a strategy to raise new problems that they call "What if not?" consisting in changing conditions, restrictions, etc. of a certain problem and then generating a new one.

Stoyanova [29] identifies three possible ways in terms of the formulation and invention of problems: free situations and semi-structured and structured situations. In the first of the aforementioned, there are no restrictions on the invention of problems. In the semi-structured, the problem-based approach is proposed, based on any experience or quantitative information. Lastly, in the structured situations, a certain given problem is reformulated or some condition of it is changed.

In our research in Granada, the participants are given a direct problem, which should be reformulated in the form of an inverse problem. Therefore, this can be considered as a structured situation, following the classification given by Stoyanova [29].

2.2 Inverse problems

According to Groetsch's [15, 16] ideas, the process of solving a direct problem can be schematized as in **Figure 1**.

In contrast, inverse problems may have multiple solutions or simply no solutions, thus making them more interesting though consequently more difficult [30]. In essence there are two types of inverse problems; firstly, the causation problem, where the procedure is well-known and the question is concerned with the necessary data in order to obtain a given result. This situation is schematized in **Figure 2**.

The other inverse problem found is the specification problem, where data and result are given and the question is concerned with which procedure can let reach the desired result (output) with the chosen data (input). This process is schematized in **Figure 3**.

Both of these problems are common in the experimental sciences and real-life situations, as noted in previous research [31, 32].
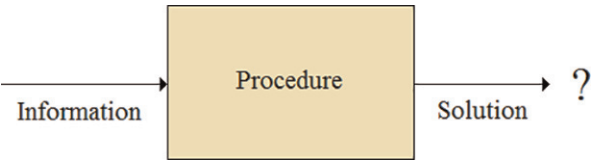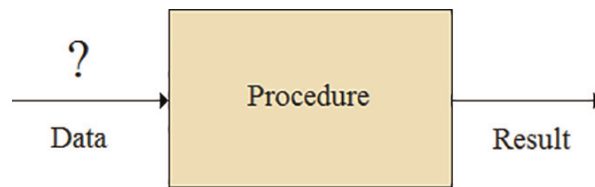

Figure 1.
Scheme for direct problems.

Figure 2.
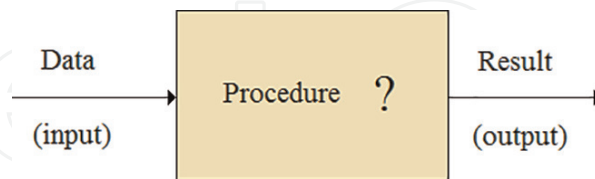*Scheme for inverse causation problems.*

Figure 3.
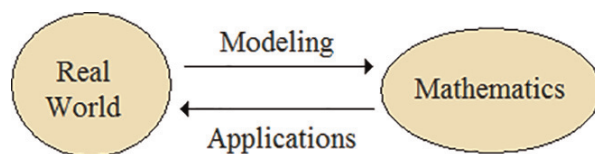*Scheme for inverse specification problems.*

Figure 4.
Comparison scheme between modeling and application problems.

2.3 Mathematical modeling

In the preliminary discussion document to the International Commission on Mathematical Instruction (ICMI) Study 14 [33], the term "modeling" focuses on the direction that goes from the real world toward mathematics, whereas the term "applications" implies the opposite direction. In addition, the term "modeling" emphasizes the process that is taking place, while the word "applications" stresses the object involved, particularly real-life cases that are susceptible to mathematical manipulation. Taking into account these ideas, we arrive at the following schema (**Figure 4**).

An extended discussion about modeling and application problems in our previous research can be found in papers [31, 34].

3. Our previous experiences at UGR master courses

In the University of Granada, the research was designed to work with one or more groups of prospective mathematics teachers for secondary education. Taking into account the available options, we chose to work with the students of groups A and B from the course named "Learning and Teaching Mathematics in Secondary School," included in the Master's Degree in Teaching Secondary Education [14].

In the 2016–2017 academic year at the University of Granada, group A consisted of 33 students, and 41 students formed group B, with regular class attendance. Two of the master courses' university professors collaborated on our research.

In a first class, the prospective teachers of both groups worked on a problem about the filling of a swimming pool. In the first session of the fieldwork, the aforementioned problem was proposed—in the form of a direct problem—and future professors were asked to reformulate it as part of a task enrichment proposal to be used in secondary school courses.

The productions of the prospective teachers of both groups underwent a first analysis, and among all the reformulations presented, three of them were highlighted and selected since they had been posed spontaneously in an inverse form. They were particularly interesting, one of them was proposed by a participant from group A, and the other two were proposed by members of group B.

Then, in a second work session, showing these reformulations, they were given a brief explanation about direct problems and inverse problems. At the end, prospective teachers proposed a new direct problem: the sheep problem.

Unlike what happened with the problem of the pool, in this case the participants were asked to reformulate the problem in an inverse manner for task enrichment purposes.

When those prospective teachers worked with the sheep problem, nine different groups of inverse problems were identified—some of them including up to four variants—and in almost all cases the participants added to their proposal the corresponding task analysis. The productions and the most creative reformulations were analyzed in a previous book chapter [14]. Brief descriptions of the nine groups of inverse problems are the following:

1. Reformulations based on the inverse function, asking the length of the rope given the ratio of area

2. Trivial reformulations

3. Inverse problem asking the location of the peg at which the sheep is tied

4. Inverse problem asking the side of the square

5. Optimization problem, including two sheep and asking for the length of rope in which the accessible area without intersection is maximum

6. Sequential inverse problem, in which, from a given R_0 given and an accessible area A_0, the student has to define a sequence of R_n, in which, the area between R_{n-1} and R_n, that is to say, $A_n - A_{n-1}$ is A_0, and find out the value of n such that it is not possible to find the corresponding R_n

7. Incremental problem, in which, given a length of rope, the student asks the increment in the length in order to increase the accessible area by a certain percentage

8. Dynamic problem, in which the student includes new magnitudes, such as speed

9. Equivalent area problem, given different locations of the peg, in which the student asks the length of rope such that the accessible area remains invariant

After that experience, it was observed that several prospective teachers were particularly creative in both the reformulation itself and in the tasks enrichment; however, the vast majority opted for a standard approach and, in some cases, for a trivialization of the problem.

For these reasons, a new research design was proposed during 2018 and implemented during the first months of 2019, with the aim of avoiding—or at least attenuating—those difficulties observed in the previous fieldwork. As an example,

in the year 2017 fieldwork, the participants were not asked to solve their proposed problems, so this was an aspect that needs to be improved in further research.

The new results showed interesting differences and few similarities which are analyzed in the next sections.

4. Fieldwork and results

In this section we start considering the sheep problem in its original version, posed in a direct form. After that we show some of the most creative reformulations proposed by the prospective teachers who participate in the fieldwork at UGR. Finally, we conclude with some general remarks about the productions of the prospective teachers in this experience.

It is important to mention that the following results represent part of the general research about task enrichment by prospective teachers (see, for instance, [14]). In this opportunity our work is focused on the mathematical content of the proposals. Other aspects of the didactic analysis, like the cognitive and the instructional dimensions of the enriched tasks, will be part of further research.

4.1 The sheep problem

In this problem, a sheep is grazing in a square field with side length L. The sheep is tied at the point $(L/2, 0)$, and the rope attached to the sheep has a length R as shown in **Figure 5**.

In **Figure 5**, A represents the area of the sector where the sheep may graze, $r = \dfrac{R}{L}$ is the ratio of the rope length to field side length, and represents the fraction of the total area accessible for the sheep. It can be observed that f is a function of the ratio r that can be obtained by integration techniques.

Typical exercises consist in supplying students with this figure and asking them to obtain f corresponding to one or more values for r. However, a more interesting approach is to ask the students to draw a diagram hoping they realize that the problem can be solved as an intersection of circles and squares. Four different situations may happen:

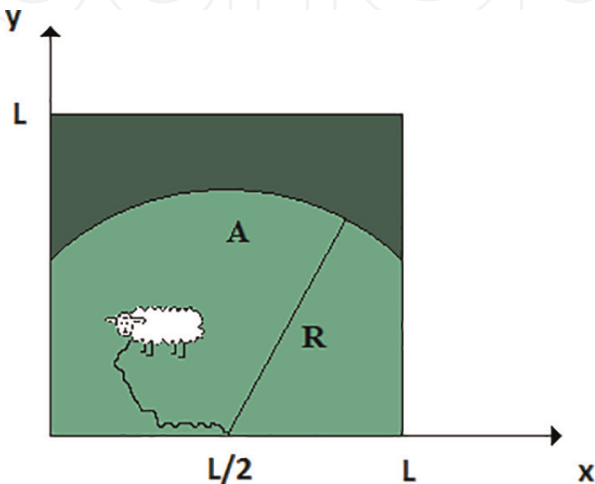- When $r \in [0, 1/2)$, the sheep does not reach the lateral edges of the field.

Figure 5.
Part of the field accessible for the sheep.

- When $r \in [1/2, 1)$, the sheep can reach the lateral edges, but not the upper one.

- When $r \in \left[1, \dfrac{\sqrt{5}}{2}\right)$, the sheep can reach the top edge of the field, but not the whole field.

- When $r \geq \dfrac{\sqrt{5}}{2}$, the sheep can graze all around the field.

This problem requires modeling and integral calculation, and it can be easily converted into an inverse problem. It is obvious that for every value of $r \geq 0$ there exists a unique value of A, but more challenging is to ask the question from another angle. For instance, for any value of A, does a corresponding unique value of r exist or not? To solve this problem, the function $f(r)$ must be studied in terms of continuity and growth with $r \geq 0$, in order to ensure its invertibility.

4.2 The new fieldwork design

As it was mentioned, the new fieldwork was designed in order to avoid, or at least attenuate, the difficulties observed in the previous experience, carried out during year 2017. In particular, both experimental designs had three main differences:

- Prospective teachers were asked to solve the original direct problem, before proposing their inverse reformulations.

- Before proposing to them this new task, several examples about inverse problems were discussed. However, none of them were about the sheep problem. The main reason for this decision was to avoid simple imitation or adaptation of a given model.

- Prospective teachers were asked to solve their own reformulated problem—or at least write a sketch of the solution—with the aim of reducing the number of non-well-posed problems.

This new design produced different responses that cannot be included in the previous nine groups observed during the year 2017 experience. Some of the most creative and new proposals are analyzed in the following subsection.

4.3 Some of the most creative reformulations for the sheep problem

As already stated, some of the prospective teachers' productions cannot be classified into the nine groups observed in the previous fieldwork. The following examples illustrate this situation.

Example 1: An unusual specification problem.

One of the prospective teachers solved the direct problem by integration, observing that the circumference equation can be written as $\left(\frac{L}{2} - x\right)^2 + y^2 = R^2$ and then the area accessible for the sheep is $A = \int_0^L \sqrt{R^2 - \left(\frac{L}{2} - x\right)^2} \, dx$. After that, he solves the integral by using the change of variables $x = \frac{L}{2} - R \sin t$ and several well-known trigonometric formulas to obtain the following long formula:

$A = -\frac{R^2}{2}\left[\arcsin\left(\frac{-L}{2R}\right) - \arcsin\left(\frac{L}{2R}\right) + \frac{1}{2}\left(\sin\left[2\arcsin\left(\frac{-L}{2R}\right)\right] - \sin\left[2\arcsin\left(\frac{L}{2R}\right)\right]\right)\right]$ It is easy to observe that this formula can be simplified, but the prospective teacher leaves it in the long version, as shown above.

After this classical solution, his inverse reformulation proposes to get the solution in a geometrical way and compare the final result with the one obtained by integration.

This is a very interesting specification problem, since the data and final result are known and he asks for another procedure in order to get the desired result.

When the prospective teacher solves his own reformulation, he divides the area accessible for the sheep into three parts, a circular sector and two triangles, as it can be observed in **Figure 6**.

In **Figure 6** both triangles have a height h that can be easily obtained by Pythagoras' theorem, giving $h = \sqrt{R^2 - \frac{L^2}{4}}$ and then the area of each triangle is $\frac{1}{2}\frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}$. After that, the angle α in both triangles is determined by using trigonometric concepts, for instance, $\alpha = arctan\left(\frac{2h}{L}\right) = arctan\left(\frac{2}{L}\sqrt{R^2 - \frac{L^2}{4}}\right)$, and so, the angle of the circular sector, $\pi - 2\alpha$, is easily obtained. Then, the accessible area can be written as

$$A = \frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}} + \pi R^2 \frac{\pi - 2arctan\left(\frac{2}{L}\sqrt{R^2 - \frac{L^2}{4}}\right)}{2\pi}.$$

The prospective teacher shows that both formulas give the same results for particular values like $R = \frac{L}{2}$ and $R = \frac{L}{\sqrt{2}}$. The participant ends his work observing that "as it was expected, both methods gave the same results."

It is important to remark that in the previous experience, carried out in 2017, all the reformulations (i.e., the nine groups and their variants) corresponded to causation inverse problems. None of them proposed an inverse specification problem as in this creative production.

Example 2: An arc length inverse problem.

Another prospective teacher solved the direct problem by using integrals and the same change of variables showed above. Nevertheless, he used other trigonometric formulas, and he takes advantage of symmetry arguments to get a different formula:
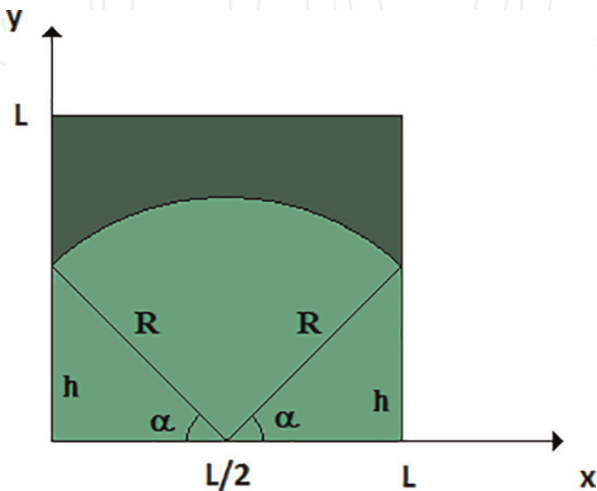

Figure 6.
The accessible area divided into three parts.

$$
A = \begin{cases}
\dfrac{1}{2}\pi R^2 & if \quad 0 \le R \le \dfrac{L}{2} \\[4mm]
R^2\left(\dfrac{L}{2R}\sqrt{1 - \left(\dfrac{L}{2R}\right)^2} + \arcsin\left(\dfrac{L}{2R}\right)\right) & if \quad \dfrac{L}{2} \le R \le L
\end{cases}.
$$

So, as it can be observed, he considered two different situations, depending on the comparison between R and $\frac{L}{2}$. It is important to mention that other radii greater than L are not considered.

In the inverse reformulation, he proposes to give this piecewise function as part of the data. He informs that the shepherd decides to eliminate the rope and, instead of it, he wants to build a circular fence like in **Figure 5**, i.e., the same as in the original problem. This fence costs 15 €/m, and the prospective teacher asks for the final cost as a function of variable R.

As it can be easily observed, the problem could be solved in a direct way, by using the arc length formula and then calculating the corresponding integral and lastly multiplying by the cost per meter. However, this solution does not use the given area function—the input of this problem—so it cannot be considered as the solution required, at least for this proposed reformulation.

The prospective teacher solves his own problem by differentiating the given function, since he claims that $L = \frac{dA}{dR}$. This statement—given without any demonstration or justification—is not true for every region in $\mathbf{R}^2$, although it is correct in this case, since it is composed of a circular sector and triangles. A more detailed discussion about when the derivative of the area gives the perimeter can be read at [35]. Finally, after obtaining $L(R)$ by differentiation, the price is easily obtained multiplying by the cost per meter.

In our previous experience, in 2017, only one prospective teacher proposed a reformulation involving arc length in a problem where two sheep were running, one along a straight line and the other along the circumference. In that problem, the accessible area did not appear as an input, so the presence of the arc length can be considered as the unique weak connection between both proposals.

As it was commented, the participant solution is not general, though it is simple, short, elegant, and accurate for this particular situation. As in the previous example, this proposal cannot be included in one of the nine groups observed in 2017.

Example 3: An inversion by intervals.

One of the prospective teachers proposed another interesting inverse reformulation that cannot be classified in the nine groups previously obtained in 2017.

Firstly, he solves the direct problem, changing the axis position such that the new origin is located in the point where the sheep is tied. Due to this change, the accessible area can be obtained as $A = \int_{-L/2}^{L/2}\sqrt{R^2 - x^2}dx = \left[\frac{x}{2}\sqrt{R^2 - x^2} + \frac{R^2}{2}\arcsin\left(\frac{x}{R}\right)\right]_{-L/2}^{L/2}$.

After some algebraic manipulations, he arrives at $f(r) = \frac{1}{2}\sqrt{r^2 - \frac{1}{4}} + r^2\arcsin\left(\frac{1}{2r}\right)$.

Later, he will use this formula for solving the corresponding inverse problem.

In his reformulation, the prospective teacher proposes that between 60 and 70% of the field is needed for the sheep to graze and between 30 and 40% is needed for a pumpkin plantation, and he asks for at least one value of r that makes both percentages possible.

In the solution of his own inverse problem, he makes mistakes in the derivative of $f(r)$; nevertheless, his conclusion about the growth of this function is obviously right. For this reason he tries with particular values, like $f(1) \cong 0.956$ and

$f(0.5) \cong 0.3926$, and after some iterations, he concludes that $f(0.7) \cong 0.625$, so $r = 0.7$ is a possible answer.

The proposal itself seems to be not as creative as the previous ones (Examples 1 and 2), but it is the only one that asked for the pre-image of an interval, an interesting topic related to continuity, monotony, and derivatives, among other important calculus concepts.

Example 4: General inversion of a vector function.

Another prospective teacher tries to solve the direct problem by integration. For this purpose he puts the area accessible for the sheep as $A = \int_0^L \sqrt{R^2 - \left(\frac{L}{2} - x\right)^2} dx$, and he proposes the change of variables $x = \frac{L}{2} - R \sin t$. Unfortunately some mistakes when using trigonometric formulas led him to a wrong result $f(r) = \frac{5}{4} r^2 \arcsin\left(\frac{L^2}{2} r\right)$. He does not use this function in the inverse problem proposal, but it appears again in the corresponding solution.

The reformulation considers the same situation schematized in **Figure 5**, like in the original problem, and he asks to obtain L and R for given values of f and r. In other words, the input is the vector (L, R), and the expected output is another vector (f, r); then, it corresponds to a vector function inversion.

The solution is wrong and easier than it should be, since he considers the function previously obtained $f(r) = \frac{5}{4} r^2 \arcsin\left(\frac{L^2}{2} r\right)$, which is simpler than the correct one: $f(r) = \frac{1}{2}\sqrt{r^2 - \frac{1}{4}} + r^2 \arcsin\left(\frac{1}{2r}\right)$. If the wrong $f(r)$ is utilized, it follows that $\frac{4f}{5r^2} = \arcsin\left(\frac{L^2}{2} r\right)$, and then L can be obtained as $L = \sqrt{\frac{2}{r} \sin\left(\frac{4f}{5r^2}\right)}$. Finally, this L can be multiplied by r to obtain R.

It is obvious that the inverse problem was unwittingly simplified; however, it is an interesting proposal and the only one of this type in both years 2017 and 2019. Another important characteristic is that it requires a general inversion, since the input vector (L, R) is a generic vector of the vector space $\mathbf{R}^2$.

Example 5: Problems that ask for a sketch of the region.

Four prospective teachers (two in each group) proposed inverse problems that ask for a sketch of the region, with different variants.

In the first one, the axis are chosen such that the sheep is tied at the origin of the coordinate system, and so, the accessible area can be obtained as $A = \int_{-L/2}^{L/2} \sqrt{R^2 - x^2} dx$. In her reformulation, the prospective teacher gives the integral, and the problem consists in doing a sketch of the region, including an identification of its elements.

The second one is similar, but there are no axis changes, and the accessible area is given by the formula $A = \int_{-\arcsin(L/2R)}^{\arcsin(L/2R)} R^2 \cos\theta \sqrt{1 - \sin^2\theta}\, d\theta$. The prospective teacher informs that this formula was obtained after performing a change of variable $x = R\sin\theta + \frac{L}{2}$, and, like in the previous case, she asks for the region involved.

The last two cases are different since in both of them the student is asked to propose a criteria that allow to choose between regions **(a)** and **(b)**, for any given value of the accessible area A (see **Figure 7**). The solution can be obtained by considering the limit case, i.e., $R = \frac{L}{2}$ and then $A = \frac{1}{2}\pi R^2 = \frac{1}{2}\pi\left(\frac{L}{2}\right)^2 = \frac{1}{8}\pi L^2$. So the requested criteria is very simple: if $A > \frac{1}{8}\pi L^2$, the region is like **(b)**, whereas if $A < \frac{1}{8}\pi L^2$, the region is like **(a)**.

It can be noted that in those cases, the problem has a weak connection with the sheep problem, since the context about the sheep, the rope, etc. can be eliminated from the proposals, and the solution remains unchanged.
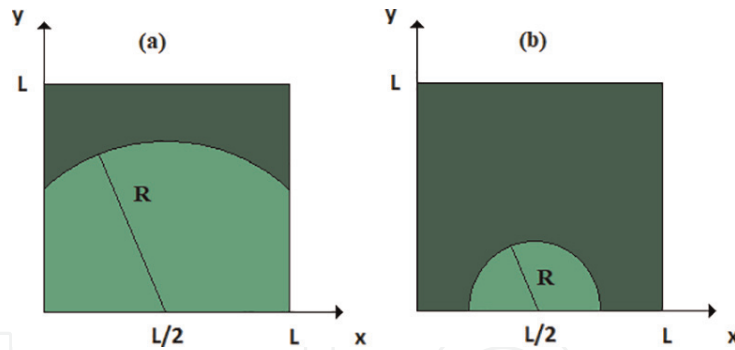
Figure 7.
Two possible cases for the accessible area.

Example 6: Interpretation problems.

A couple of prospective teachers proposed a different kind of inverse problem, where the elements of **Figure 5** are explained. After that, one of them gives the following function as a data for the inverse problem:

$f(r) = \frac{1}{2}\left[\sqrt{r^2 - \frac{1}{4}} + 2r^2 \arcsin\left(\frac{1}{2r}\right)\right]$. She only mentions that f and r express relations between the variables and the problem consists in determining the meaning of both.

In the last one, the situation proposed is almost the same, but there is a mistake in the given function f; so, the correct answer will be that r has no meaning.

As it happened in the other examples commented above, this kind of inverse problem did not appear in the year 2017.

5. Discussion

Problem posing has an important role to play in the STEM classroom. For instance, Beal and Cohen [36] suggested "... significant changes to the existing model of education, in which students would move from passive consumers of educational resources that have been developed by others to creators of rich, innovative and authentic STEM content...".

Moreover, problem posing has a positive influence on students' ability to solve word problems and provided a chance to gain insight into students' understanding of mathematical concepts and processes.

Some researchers have found evidences that students' experience with problem posing enhances their perception of the subject; causes excitement and motivation; improves students' thinking, problem solving skills, attitudes, and confidence in mathematics; and contributes to a broader understanding of mathematical concepts [37].

Some of these capabilities were observed in our fieldwork with prospective teachers, particularly in the selected examples, where some responses were very creative. This is an important aspect not only for teaching but also as a prominent feature of mathematical activity. As Poincaré said "Mathematicians may solve some problems that have been posed for them by others or may work on problems that have been identified as important problems in the literature, but it is more common for them to formulate their own problems, based on their personal experience and interests" [38]. In the same way, Hadamard [39] identified the ability to find key research questions as an indicator of exceptional mathematical talent. For instance, a challenging question appears in the solution of Example 2 of the previous section, where the arc length is obtained as the derivative of the area accessible to the sheep. The question about the accuracy of this procedure led us to an interesting research question whose answer is not trivial (see [35] for a general discussion).

In our study there were challenging proposals which not always were possible to be solved by the prospective teachers. This fact is in accordance with Silver and collaborators, who remarked that "... mathematicians certainly pose mathematical problems or conjectures that they are not certain they can solve (e.g., Goldbach's Conjecture), and research with adult subjects has found that they often pose mathematical problems that they could not solve on their own" [27]. This situation takes place in Example 4, where the inverse reformulation was solved only because the problem was unwittingly simplified.

Taking into account that the participants are prospective teachers, it is important for them to be able to create new problems to work on their classes. Moreover, as Silver noted "Problem posing has figured prominently in some inquiry-oriented instruction that has freed students and teachers from the textbook as the main source of wisdom and problems in a school mathematics course" [23]. For this purpose, Kilpatrick [22] argued that one of the basic cognitive processes involved in problem posing is association: "[Because] knowledge is represented as a network of associated ideas, that network can be used to generate problems by taking a concept node in the network and raising questions about its associates" (p. 136). In our research, unexpected association of ideas was found in several proposals, like in Example 1 where different ideas from calculus and trigonometry were combined in a very creative proposition.

Finally, motivation is a very important issue which was also relevant in this experience. It deserves to be mentioned that some of the prospective teachers pose up to three different reformulations of the given direct problem, which can be considered as a result of a motivating activity. This fact was also observed by Winograd who reported that students in his study appeared to be highly motivated to pose problems that their classmates would find interesting or difficult [40].

6. Conclusions

The first immediate conclusion is that the results of both experiences—carried out in 2017 and 2019—are absolutely different.

In 2017 the prospective teachers imitated previous examples provided for the first problem, i.e., the filling of a swimming pool. When they were asked to reformulate the second one (the sheep problem), there were no examples that can be imitated, but they followed the same ideas that they used in their proposal for the swimming pool, like inverting the function, changing the geometry, or including obstacles, among others.

On one hand, in 2019 the previous examples were very simple and concerned other mathematics topics like proportions, arithmetic and geometric sequences, and solving for unknown sides and angles in right triangles. On the other hand, the prospective teachers were asked to solve the direct problem before proposing their own reformulation. So, in this new fieldwork, their experience was more involved in the mathematical solution of the direct problem than other inverse reformulations. It can be observed that this fact led the proposals in many different ways. For instance, they gave a formula in the reformulated problem and asked for an interpretation, a sketch of the corresponding region, or another way to get the same result without using integrals.

Other important difference is the use (or not) of external variables, which can be physical (time and velocity), chemical (amount of fertilizer and herbicide), economical (cost of a fence), or biological variables (like kilograms of grass per day). Those variables were widely used in 2017; however, in 2019 they only appear in a few cases, like in the Example 2, in Section 4.

Another remarkable observation is that the proposals corresponding to 2019 are usually more challenging from a pure mathematical viewpoint. For instance, they ask for different types of solutions (analytical, geometrical, etc.), and they need the analysis of monotony, existence of a pre-image, solving nonlinear equations, etc. They also ask for more conceptual issues, like identify a region or give a meaning to one or more given variables in a certain formula, among other options that were absolutely unusual in 2017.

It can also be observed that the proposals in year 2017 were more practical, i.e., hands-on problems more involved with other disciplines and more connected with the reality and its mathematical modeling, whereas in 2019 they are more conceptual, mathematically challenging, and self-contained.

As a general conclusion, it seems that the prospective teachers tend to propose the reformulations based on their own recent experiences. If these experiences consist in working with previous examples, they try to imitate them, and if their experience consists mainly in solving the direct problem, they tend to use this solution—or the process that led to it—as the main input for problem posing.

Finally, it is difficult to say that one of these experiences yielded better results than the other. In fact, in one of these experiences, certain characteristics predominated, while in the other one, different characteristics were observed. As a consequence, the resulting proposals more than antagonistic can be regarded as truly complementary.

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