Chapter

Determinantal Representations of the Core Inverse and Its Generalizations

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Abstract

Generalized inverse matrices are important objects in matrix theory. In particular, they are useful tools in solving matrix equations. The most famous generalized inverses are the Moore-Penrose inverse and the Drazin inverse. Recently, it was introduced new generalized inverse matrix, namely the core inverse, which was late extended to the core-EP inverse, the BT, DMP, and CMP inverses. In contrast to the inverse matrix that has a definitely determinantal representation in terms of cofactors, even for basic generalized inverses, there exist different determinantal representations as a result of the search of their more applicable explicit expressions. In this chapter, we give new and exclusive determinantal representations of the core inverse and its generalizations by using determinantal representations of the Moore-Penrose and Drazin inverses previously obtained by the author.

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1. Introduction

In the whole chapter, the notations $\mathbb{R}$ and $\mathbb{C}$ are reserved for fields of the real and complex numbers, respectively. $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ matrices over $\mathbb{C}$. $\mathbb{C}_r^{m\times n}$ determines its subset of matrices with a rank r. For $\mathbf{A}\in\mathbb{C}^{m\times n}$, the symbols $\mathbf{A}^*$ and $\mathrm{rk}(\mathbf{A})$ specify the conjugate transpose and the rank of $\mathbf{A}$, respectively, $|\mathbf{A}|$ or $det\mathbf{A}$ stands for its determinant. A matrix $\mathbf{A}\in\mathbb{C}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

$\mathbf{A}^\dagger$ means *the Moore-Penrose inverse* of $\mathbf{A}\in\mathbb{C}^{n\times m}$, i.e., the exclusive matrix $\mathbf{X}$ satisfying the following four equations:

$$\mathbf{AXA} = \mathbf{A} \tag{1}$$

$$\mathbf{XAX} = \mathbf{X} \tag{2}$$

$$(\mathbf{AX})^* = \mathbf{AX} \tag{3}$$

$$(\mathbf{XA})^* = \mathbf{XA} \tag{4}$$

For $\mathbf{A}\in\mathbb{C}^{n\times n}$ with index $\mathrm{Ind}\,\mathbf{A} = k$, i.e., the smallest positive number such that $\mathrm{rk}\left(\mathbf{A}^{k+1}\right) = \mathrm{rk}\left(\mathbf{A}^{k}\right)$, *the Drazin inverse* of $\mathbf{A}$, denoted by $\mathbf{A}^d$, is called the unique matrix $\mathbf{X}$ that satisfies Eq. (2) and the following equations,

$$\mathbf{AX} = \mathbf{XA}; \tag{5}$$

$$\mathbf{XA}^{k+1} = \mathbf{A}^k \tag{6}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k. \tag{7}$$

In particular, if $\operatorname{Ind}\mathbf{A} = 1$, then the matrix $\mathbf{X}$ is called *the group inverse*, and it is denoted by $\mathbf{X} = \mathbf{A}^{\#}$. If $\operatorname{Ind}\mathbf{A} = 0$, then $\mathbf{A}$ is nonsingular and $\mathbf{A}^d = \mathbf{A}^\dagger = \mathbf{A}^{-1}$.

It is evident that if the condition (5) is fulfilled, then (6) and (7) are equivalent. We put both these conditions because they will be used below independently of each other and without the obligatory fulfillment of (5).

A matrix $\mathbf{A}$ satisfying the conditions $(i), (j), \ldots$ is called an $\{i, j, \ldots\}$-inverse of $\mathbf{A}$, and is denoted by $\mathbf{A}^{(i,j,\ldots)}$. The set of matrices $\mathbf{A}^{(i,j,\ldots)}$ is denoted $\mathbf{A}\{i, j, \ldots\}$. In particular, $\mathbf{A}^{(1)}$ is called the inner inverse, $\mathbf{A}^{(2)}$ is called the outer inverse, $\mathbf{A}^{(1,2)}$ is called the reflexive inverse, $\mathbf{A}^{(1,2,3,4)}$ is the Moore-Penrose inverse, etc.

For an arbitrary matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, we denote by

- $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{H}^{n\times 1} : \mathbf{Ax} = 0\}$, the kernel (or the null space) of $\mathbf{A}$;
- $\mathcal{C}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^{m\times 1} : \mathbf{y} = \mathbf{Ax}, \mathbf{x} \in \mathbb{H}^{n\times 1}\}$, the column space (or the range space) of $\mathbf{A}$; and
- $\mathcal{R}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^{1\times n} : \mathbf{y} = \mathbf{xA}, \mathbf{x} \in \mathbb{H}^{1\times m}\}$, the row space of $\mathbf{A}$.

$\mathbf{P}_A := \mathbf{AA}^\dagger$ and $\mathbf{Q}_A := \mathbf{A}^\dagger\mathbf{A}$ are the orthogonal projectors onto the range of $\mathbf{A}$ and the range of $\mathbf{A}^*$, respectively.

The core inverse was introduced by Baksalary and Trenkler in [1]. Later, it was investigated by S. Malik in [2] and S.Z. Xu et al. in [3], among others.

Definition 1.1. [1] A matrix $\mathbf{X} \in \mathbb{C}^{n\times n}$ is called the core inverse of $\mathbf{A} \in \mathbb{C}^{n\times n}$ if it satisfies the conditions

$$\mathbf{AX} = \mathbf{P}_A, \textit{and } \mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{A}).$$

When such matrix $\mathbf{X}$ exists, it is denoted as $A^{\oplus}$.

In 2014, the core inverse was extended to the core-EP inverse defined by K. Manjunatha Prasad and K.S. Mohana [4]. Other generalizations of the core inverse were recently introduced for $n \times n$ complex matrices, namely BT inverses [5], DMP inverses [2], CMP inverses [6], etc. The characterizations, computing methods, and some applications of the core inverse and its generalizations were recently investigated in complex matrices and rings (see, e.g., [7–18]).

In contrast to the inverse matrix that has a definitely determinantal representation in terms of cofactors, for generalized inverse matrices, there exist different determinantal representations as a result of the search of their more applicable explicit expressions (see, e.g. [19–25]). In this chapter, we get new determinantal representations of the core inverse and its generalizations using recently obtained by the author determinantal representations of the Moore-Penrose inverse and the Drazin inverse over the quaternion skew field, and over the field of complex numbers as a special case [26–34]. Note that a determinantal representation of the core-EP generalized inverse in complex matrices has been derived in [4], based on the determinantal representation of an reflexive inverse obtained in [19, 20].

The chapter is organized as follows: in Section 2, we start with preliminary introduction of determinantal representations of the Moore-Penrose inverse and the

Drazin inverse. In Section 3, we give determinantal representations of the core inverse and its generalizations, namely the right and left core inverses are established in Section 3.1, the core-EP inverses in Section 3.2, the core DMP inverse and its dual in Section 3.3, and finally the CMP inverse in Section 3.4. A numerical example to illustrate the main results is considered in Section 4. Finally, in Section 5, the conclusions are drawn.

2. Preliminaries

Let $\alpha \coloneqq \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta \coloneqq \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets with $1 \le k \le \min\{m, n\}$. By $\mathbf{A}^{\alpha}_{\beta}$, we denote a submatrix of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with rows and columns indexed by α and β, respectively. Then, $\mathbf{A}^{\alpha}_{\alpha}$ is a principal submatrix of $\mathbf{A}$ with rows and columns indexed by α, and $|\mathbf{A}|^{\alpha}_{\alpha}$ is the corresponding principal minor of the determinant $|\mathbf{A}|$. Suppose that

$$L_{k,n} \coloneqq \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$$

stands for the collection of strictly increasing sequences of $1 \le k \le n$ integers chosen from $\{1, \ldots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, put $I_{r,m}\{i\} \coloneqq \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$ and $J_{r,n}\{j\} \coloneqq \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

The jth columns and the ith rows of $\mathbf{A}$ and $\mathbf{A}^*$ denote $\mathbf{a}_{.j}$ and $\mathbf{a}^*_{.j}$ and $\mathbf{a}_{i.}$ and $\mathbf{a}^*_{i.}$, respectively. By $\mathbf{A}_{i.}(\mathbf{b})$ and $\mathbf{A}_{.j}(\mathbf{c})$, we denote the matrices obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$, and its jth column with the column $\mathbf{c}$.

Theorem 2.1. [28] *If $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, then the Moore-Penrose inverse $\mathbf{A}^{\dagger} = \left(a^{\dagger}_{ij}\right) \in \mathbb{C}^{n\times m}$ possesses the determinantal representations*

$$a^{\dagger}_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}^*_{.j}\right)\right|^{\beta}_{\beta}}{\sum_{\beta\in J_{r,n}} |\mathbf{A}^*\mathbf{A}|^{\beta}_{\beta}} = \tag{8}$$

$$= \frac{\sum_{\alpha\in I_{r,m}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{a}^*_{i.}\right)\right|^{\alpha}_{\alpha}}{\sum_{\alpha\in I_{r,m}} |\mathbf{A}\mathbf{A}^*|^{\alpha}_{\alpha}}. \tag{9}$$

Remark 2.2. For an arbitrary full-rank matrix $\mathbf{A} \in \mathbb{C}^{m\times n}_r$, a row vector $\mathbf{b} \in \mathbb{H}^{1\times m}$, and a column-vector $\mathbf{c} \in \mathbb{H}^{n\times 1}$, we put, respectively,

$$\left|(\mathbf{A}\mathbf{A}^*)_{i.}(\mathbf{b})\right| = \sum_{\alpha\in I_{m,m}\{i\}} \left|(\mathbf{A}\mathbf{A}^*)_{i.}(\mathbf{b})\right|^{\alpha}_{\alpha}, \quad i = 1, \ldots, m,$$

$$|\mathbf{A}\mathbf{A}^*| = \sum_{\alpha\in I_{m,m}} |\mathbf{A}\mathbf{A}^*|^{\alpha}_{\alpha}, \quad \text{when } r = m;$$

$$\left|(\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{c})\right| = \sum_{\beta\in J_{n,n}\{j\}} \left|(\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{c})\right|^{\beta}_{\beta}, \quad j = 1, \ldots, n,$$

$$|\mathbf{A}^*\mathbf{A}| = \sum_{\beta\in J_{n,n}} |\mathbf{A}^*\mathbf{A}|^{\beta}_{\beta}, \quad \text{when } r = n.$$

Corollary 2.3. [21] *Let $\mathbf{A} \in \mathbb{C}^{m\times n}_r$. Then, the following determinantal representations can be obtained*

i. for the projector $\mathbf{Q}_A = \left(q_{ij}\right)_{n\times n}$,

$$q_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.j})\right|_\beta^\beta}{\sum_{\beta\in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} = \frac{\sum_{\alpha\in I_{r,n}\{j\}} \left|(\mathbf{A}^*\mathbf{A})_{j.}(\dot{\mathbf{a}}_{i.})\right|_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_\alpha^\alpha}, \tag{10}$$

where $\dot{\mathbf{a}}_{.j}$ is the jth column and $\dot{\mathbf{a}}_{i.}$ is the ith row of $\mathbf{A}^*\mathbf{A}$; and

ii. for the projector $\mathbf{P}_A = \left(p_{ij}\right)_{m\times m}$,

$$p_{ij} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{i.})\right|_\alpha^\alpha}{\sum_{\alpha\in I_{r,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha} = \frac{\sum_{\beta\in J_{r,m}\{i\}} \left|(\mathbf{A}\mathbf{A}^*)_{.i}(\ddot{\mathbf{a}}_{.j})\right|_\beta^\beta}{\sum_{\beta\in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_\beta^\beta}, \tag{11}$$

where $\ddot{\mathbf{a}}_{i.}$ is the ith row and $\ddot{\mathbf{a}}_{.j}$ is the jth column of $\mathbf{A}\mathbf{A}^*$.

The following lemma gives determinantal representations of the Drazin inverse in complex matrices.

Lemma 2.4. [21] *Let* $\mathbf{A}\in\mathbb{C}^{n\times n}$ *with* $\operatorname{Ind}\mathbf{A} = k$ *and* $\operatorname{rk}\mathbf{A}^{k+1} = \operatorname{rk}\mathbf{A}^k = r$. *Then, the determinantal representations of the Drazin inverse* $\mathbf{A}^d = \left(a_{ij}^d\right)\in\mathbb{C}^{n\times n}$ *are*

$$a_{ij}^d = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)\right|_\beta^\beta}{\sum_{\beta\in J_{r,n}} \left|\mathbf{A}^{k+1}\right|_\beta^\beta} = \tag{12}$$

$$= \frac{\sum_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^{k+1}\right)_{j.}\left(\mathbf{a}_{i.}^{(k)}\right)\right|_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} \left|\mathbf{A}^{k+1}\right|_\alpha^\alpha}, \tag{13}$$

where $\mathbf{a}_{i.}^{(k)}$ is the ith row and $\mathbf{a}_{.j}^{(k)}$ is the jth column of $\mathbf{A}^k$.

Corollary 2.5. [21] *Let* $\mathbf{A}\in\mathbb{C}^{n\times n}$ *with* $\operatorname{Ind}\mathbf{A} = 1$ *and* $\operatorname{rk}\mathbf{A}^2 = \operatorname{rk}\mathbf{A} = r$. *Then, the determinantal representations of the group inverse* $\mathbf{A}^\# = \left(a_{ij}^\#\right)\in\mathbb{C}^{n\times n}$ *are*

$$a_{ij}^\# = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{a}_{.j}\right)\right|_\beta^\beta}{\sum_{\beta\in J_{r,n}} \left|\mathbf{A}^2\right|_\beta^\beta} = \frac{\sum_{\alpha\in I_{r,n}\{j\}} \left|\left(\mathbf{A}^2\right)_{j.}\left(\mathbf{a}_{i.}\right)\right|_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} \left|\mathbf{A}^2\right|_\alpha^\alpha}. \tag{14}$$

3. Determinantal representations of the core inverse and its generalizations

3.1 Determinantal representations of the core inverses

Together with the core inverse in [35], the dual core inverse was to be introduced. Since the both these core inverses are equipollent and they are different only in the position relative to the inducting matrix $\mathbf{A}$, we propose called them as the right and left core inverses regarding to their positions. So, from [1], we have the following definition that is equivalent to Definition 1.1.

Definition 3.1. A matrix $\mathbf{X} \in \mathbb{C}^{n \times n}$ is said to be the right core inverse of $\mathbf{A} \in \mathbb{C}^{n \times n}$ if it satisfies the conditions

$$\mathbf{AX} = \mathbf{P}_A, \textit{and } \mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{A}).$$

When such matrix $\mathbf{X}$ exists, it is denoted as $A^{\oplus}$.

The following definition of the left core inverse can be given that is equivalent to the introduced dual core inverse [35].

Definition 3.2 A matrix $\mathbf{X} \in \mathbb{C}^{n \times n}$ is said to be the left core inverse of $\mathbf{A} \in \mathbb{C}^{n \times n}$ if it satisfies the conditions

$$\mathbf{XA} = \mathbf{Q}_A, \textit{and } \mathcal{R}(\mathbf{X}) = \mathcal{R}(\mathbf{A}). \tag{15}$$

When such matrix $\mathbf{X}$ exists, it is denoted as $A_{\oplus}$.

Remark 3.3. In [35], the conditions of the dual core inverse are given as follows:

$$A_{\oplus}\mathbf{A} = \mathbf{P}_{A^*}, \textit{and } \mathcal{C}(A_{\oplus}) \subseteq \mathcal{C}(\mathbf{A}^*).$$

Since $\mathbf{P}_{A^*} = \mathbf{A}^*(\mathbf{A}^*)^{\dagger} = \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^* = \mathbf{A}^{\dagger}\mathbf{A} = \mathbf{Q}_A$, and $\mathcal{R}(\mathbf{A}) = \mathcal{C}(\mathbf{A}^*)$, then these conditions and (15) are analogous.

Due to [1], we introduce the following sets of quaternion matrices

$$\mathbb{C}_n^{\mathrm{CM}} = \left\{\mathbf{A} \in \mathbb{C}^{n \times n} : \operatorname{rk} \mathbf{A}^2 = \operatorname{rk} \mathbf{A}\right\},$$
$$\mathbb{C}_n^{\mathrm{EP}} = \left\{\mathbf{A} \in \mathbb{C}^{n \times n} : \mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}\mathbf{A}^{\dagger}\right\} = \{\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{A}^*)\}.$$

The matrices from $\mathbb{C}_n^{\mathrm{CM}}$ are called group matrices or core matrices. If $\mathbf{A} \in \mathbb{C}_n^{\mathrm{EP}}$, then clearly $\mathbf{A}^{\dagger} = \mathbf{A}^{\#}$. It is known that the core inverses of $\mathbf{A} \in \mathbb{C}^{n \times n}$ exist if and only if $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$ or $\operatorname{Ind} \mathbf{A} = 1$. Moreover, if $\mathbf{A}$ is nonsingular, $\operatorname{Ind} \mathbf{A} = 0$, then its core inverses are the usual inverse. Due to [1], we have the following representations of the right and left core inverses.

Lemma 3.4. [1] *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$. *Then,*

$$\mathbf{A}^{\oplus} = \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\dagger}, \tag{16}$$

$$\mathbf{A}_{\oplus} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\#} \tag{17}$$

Remark 3.5. In Theorems 3.6 and 3.7, we will suppose that $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$ but $\mathbf{A} \notin \mathbb{C}_n^{\mathrm{EP}}$. Because, if $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$ and $\mathbf{A} \in \mathbb{C}_n^{\mathrm{EP}}$ (in particular, $\mathbf{A}$ is Hermitian), then from Lemma 3.4 and the definitions of the Moore-Penrose and group inverses, it follows that $\mathbf{A}^{\oplus} = \mathbf{A}_{\oplus} = \mathbf{A}^{\#} = \mathbf{A}^{\dagger}$.

Theorem 3.6. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$ *and* $\operatorname{rk} \mathbf{A}^2 = \operatorname{rk} \mathbf{A} = s$. *Then, its right core inverse has the following determinantal representations*

$$a_{ij}^{\oplus,r} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{u}_{i.}^{(1)}\right)\right|_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s,n}} \left|\mathbf{A}^2\right|_{\beta}^{\beta} \sum_{\alpha \in I_{s,n}} \left|\mathbf{A}\mathbf{A}^*\right|_{\alpha}^{\alpha}} = \tag{18}$$

$$= \frac{\sum_{\beta \in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{u}_{.j}^{(2)}\right)\right|_{\beta}^{\beta}}{\sum_{\beta \in J_{s,n}} \left|\mathbf{A}^2\right|_{\beta}^{\beta} \sum_{\alpha \in I_{s,n}} \left|\mathbf{A}\mathbf{A}^*\right|_{\alpha}^{\alpha}}, \tag{19}$$

where

$$\mathbf{u}_{i.}^{(1)} = \left[\sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\tilde{\mathbf{a}}_{.f}\right)\right|_\beta^\beta\right] \in \mathbb{C}^{1\times n},\ f = 1, \dots, n$$

$$\mathbf{u}_{.j}^{(2)} = \left[\sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\tilde{\mathbf{a}}_{l.}\right)\right|_\alpha^\alpha\right] \in \mathbb{C}^{n\times 1},\ l = 1, \dots, n.$$

are the row and column vectors, respectively. Here $\tilde{\mathbf{a}}_{.f}$ and $\tilde{\mathbf{a}}_{l.}$ are the fth column and lth row of $\tilde{\mathbf{A}} \coloneqq \mathbf{A}^2\mathbf{A}^*$.

Proof. Taking into account (16), we have for $^{\#}A$,

$$a_{ij}^{\#,r} = \sum_{l=1}^n \sum_{f=1}^n a_{il}^{\#} a_{lf} a_{fj}^{\dagger}. \tag{20}$$

By substituting (14) and (15) in (20), we obtain

$$a_{ij}^{\#,r} = \sum_{l=1}^n \frac{\sum_{f=1}^n \sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{a}_{.f}\right)\right|_\beta^\beta a_{fl}}{\sum_{\beta\in J_{s,n}} \left|\mathbf{A}^2\right|_\beta^\beta} \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{a}_{l.}^*\right)\right|_\alpha^\alpha}{\sum_{\alpha\in I_{s,n}} \left|\mathbf{A}\mathbf{A}^*\right|_\alpha^\alpha} =$$

$$\frac{\sum_{f=1}^n \sum_{l=1}^n \sum_{\beta\in J_{s,n}\{j\}} \left|\left(\mathbf{A}^2\right)_{.j}\left(\mathbf{e}_{.f}\right)\right|_\beta^\beta \tilde{a}_{fl} \sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{e}_{l.}\right)\right|_\alpha^\alpha}{\sum_{\beta\in J_{s,n}} \left|\mathbf{A}^2\right|_\beta^\beta \sum_{\alpha\in I_{s,n}} \left|\mathbf{A}\mathbf{A}^*\right|_\alpha^\alpha},$$

where $\mathbf{e}_{.l}$ and $\mathbf{e}_{l.}$ are the unit column and row vectors, respectively, such that all their components are 0, except the lth components which are 1; $\tilde{a}_{lf}$ is the (lf)th element of the matrix $\tilde{\mathbf{A}} \coloneqq \mathbf{A}^2\mathbf{A}^*$.

Let

$$u_{il}^{(1)} \coloneqq \sum_{f=1}^n \sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{e}_{.f}\right)\right|_\beta^\beta \tilde{a}_{fl} = \sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\tilde{\mathbf{a}}_{.l}\right)\right|_\beta^\beta, \quad i, l = 1, \dots, n.$$

Construct the matrix $\mathbf{U}_1 = \left(u_{il}^{(1)}\right) \in \mathbb{H}^{n\times n}$. It follows that

$$\sum_l u_{il}^{(1)} \sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{e}_{l.}\right)\right|_\alpha^\alpha = \sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{u}_{i.}^{(1)}\right)\right|_\alpha^\alpha,$$

where $\mathbf{u}_{i.}^{(1)}$ is the ith row of $\mathbf{U}_1$. So, we get (18). If we first consider

$$u_{if}^{(2)} \coloneqq \sum_l \tilde{a}_{fl} \sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{e}_{l.}\right)\right|_\alpha^\alpha = \sum_{\alpha\in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\tilde{\mathbf{a}}_{f.}\right)\right|_\alpha^\alpha, \quad f, j = 1, \dots, n.$$

and construct the matrix $\mathbf{U}_2 = \left(u_{if}^{(2)}\right) \in \mathbb{H}^{n\times n}$, then from

$$\sum_{f=1}^n \sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{e}_{.f}\right)\right|_\beta^\beta u_{if}^{(2)} = \sum_{\beta\in J_{s,n}\{i\}} \left|\left(\mathbf{A}^2\right)_{.i}\left(\mathbf{u}_{f}^{(2)}\right)\right|_\beta^\beta,$$

it follows (19). $\square$

Taking into account (17), the following theorem on the determinantal representation of the left core inverse can be proved similarly.

Theorem 3.7. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathrm{CM}}$ *and* $\operatorname{rk}\mathbf{A}^2 = \operatorname{rk}\mathbf{A} = s$. *Then for its left core inverse* $({}_{\#}A) = \left(a_{ij}^{\#,l}\right)$, *we have*

$$a_{ij}^{\#,l} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left| \left(\mathbf{A}^2\right)_{j.}\left(\mathbf{v}_{i.}^{(1)}\right) \right|_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{s,n}} \left|\mathbf{A}^2\right|_{\alpha}^{\alpha}} = \frac{\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{v}_{.j}^{(2)}\right) \right|_{\beta}^{\beta}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{s,n}} \left|\mathbf{A}^2\right|_{\alpha}^{\alpha}},$$

where

$$\mathbf{v}_{i.}^{(1)} = \left[\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i}(\overline{\mathbf{a}}_{.f}) \right|_{\beta}^{\beta} \right] \in \mathbb{C}^{1\times n},\ f = 1, \dots, n$$

$$\mathbf{v}_{.j}^{(2)} = \left[\sum_{\alpha \in I_{s,n}\{j\}} \left| \left(\mathbf{A}^2\right)_{j.}(\overline{\mathbf{a}}_{l.}) \right|_{\alpha}^{\alpha} \right] \in \mathbb{C}^{n\times 1},\ l = 1, \dots, n.$$

Here $\overline{\mathbf{a}}_{.f}$ and $\overline{\mathbf{a}}_{l.}$ are the fth column and lth row of $\overline{\mathbf{A}} \coloneqq \mathbf{A}^*\mathbf{A}^2$.

3.2 Determinantal representations of the core-EP inverses

Similar as in [4], we introduce two core-EP inverses.

Definition 3.8. A matrix $\mathbf{X} \in \mathbb{C}^{n\times n}$ is said to be the right core-EP inverse of $\mathbf{A} \in \mathbb{C}^{n\times n}$ if it satisfies the conditions

$$\mathbf{XAX} = \mathbf{A}, \textit{and}\ \mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{X}^*) = \mathcal{C}\left(\mathbf{A}^d\right).$$

It is denoted as $\mathbf{A}^{\textcircled{\dagger}}$.

Definition 3.9. A matrix $\mathbf{X} \in \mathbb{C}^{n\times n}$ is said to be the left core-EP inverse of $\mathbf{A} \in \mathbb{C}^{n\times n}$ if it satisfies the conditions

$$\mathbf{XAX} = \mathbf{A}, \textit{and}\ \mathcal{R}(\mathbf{X}) = \mathcal{R}(\mathbf{X}^*) = \mathcal{R}\left(\mathbf{A}^d\right).$$

It is denoted as $\mathbf{A}_{\textcircled{\dagger}}$.

Remark 3.10. Since $\mathcal{C}\left((\mathbf{A}^*)^d\right) = \mathcal{R}\left(\mathbf{A}^d\right)$, then the left core inverse $\mathbf{A}_{\textcircled{\dagger}}$ of $\mathbf{A} \in \mathbb{C}^{n\times n}$ is similar to the $*$core inverse introduced in [4], and the dual core-EP inverse introduced in [35].

Due to [4], we have the following representations the core-EP inverses of $\mathbf{A} \in \mathbb{C}^{n\times n}$,

$$\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\{2,3,6a\}}\ \text{and}\ \mathcal{C}(\mathbf{A}^{\textcircled{\dagger}}) \subseteq \mathcal{C}\left(\mathbf{A}^k\right),$$
$$\mathbf{A}_{\textcircled{\dagger}} = \mathbf{A}^{\{2,4,6b\}}\ \text{and}\ \mathcal{R}(\mathbf{A}_{\textcircled{\dagger}}) \subseteq \mathcal{R}\left(\mathbf{A}^k\right).$$

Thanks to [35], the following representations of the core-EP inverses will be used for their determinantal representations.

Lemma 3.11. *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *and* $\operatorname{Ind}\mathbf{A} = k$. *Then*

$$\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^k \left(\mathbf{A}^{k+1}\right)^{\dagger}, \tag{21}$$

$$\mathbf{A}_{\textcircled{\dagger}} = \left(\mathbf{A}^{k+1}\right)^{\dagger} \mathbf{A}^k. \tag{22}$$

Moreover, if $\operatorname{Ind}\mathbf{A} = 1$, then we have the following representations of the right and left core inverses

$$A^{\oplus} = \mathbf{A}\left(\mathbf{A}^2\right)^{\dagger}, \tag{23}$$

$$A_{\oplus} = \left(\mathbf{A}^2\right)^{\dagger}\mathbf{A}. \tag{24}$$

Theorem 3.12. *Suppose* $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\operatorname{Ind}\mathbf{A} = k$, $\operatorname{rk}\mathbf{A}^k = s$, *and there exist* $\mathbf{A}^{\textcircled{\dagger}}$ *and* $\mathbf{A}_{\textcircled{\dagger}}$. *Then* $\mathbf{A}^{\textcircled{\dagger}} = \left(a_{ij}^{\textcircled{\dagger},r}\right)$ *and* $\mathbf{A}_{\textcircled{\dagger}} = \left(a_{ij}^{\textcircled{\dagger},l}\right)$ *possess the determinantal representations, respectively,*

$$a_{ij}^{\textcircled{\dagger},r} = \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left| \left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\hat{\mathbf{a}}_{i.}) \right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{s,n}} \left| \mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^* \right|_{\alpha}^{\alpha}}, \tag{25}$$

$$a_{ij}^{\textcircled{\dagger},l} = \frac{\sum_{\beta\in J_{s,n}\{i\}} \left| \left(\left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1}\right)_{.i}(\check{\mathbf{a}}_{.j}) \right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s,n}} \left| \left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1} \right|_{\beta}^{\beta}}, \tag{26}$$

where $\hat{\mathbf{a}}_{i.}$ is the ith row of $\hat{\mathbf{A}} = \mathbf{A}^k\left(\mathbf{A}^{k+1}\right)^*$ and $\check{\mathbf{a}}_{.j}$ is the jth column of $\check{\mathbf{A}} = \left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^k$.

Proof. Consider $\left(\mathbf{A}^{k+1}\right)^{\dagger} = \left(a_{ij}^{(k+1,\dagger)}\right)$ and $\mathbf{A}^k = \left(a_{ij}^{(k)}\right)$. By (21),

$$a_{ij}^{\textcircled{\dagger},r} = \sum_{t=1}^{n} a_{it}^{(k)} a_{tj}^{(k+1,\dagger)}.$$

Taking into account (9) for the determinantal representation of $\left(\mathbf{A}^{k+1}\right)^{\dagger}$, we get

$$a_{ij}^{\textcircled{\dagger},r} = \sum_{t=1}^{n} a_{it}^{(k)} \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left| \left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\mathbf{a}_{t.}^{(k+1,*)}\right) \right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{r,m}} \left| \mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^* \right|_{\alpha}^{\alpha}},$$

where $\mathbf{a}_{t.}^{(k+1,*)}$ is the tth row of $\left(\mathbf{A}^{k+1}\right)^*$. Since $\sum_{t=1}^{n} a_{it}^{(k)}\mathbf{a}_{t.}^{(k+1,*)} = \hat{\mathbf{a}}_{i.}$, then it follows (25).

The determinantal representation (26) can be obtained similarly by integrating (8) for the determinantal representation of $\left(\mathbf{A}^{k+1}\right)^{\dagger}$ in (22). □

Taking into account the representations (23)-(24), we obtain the determinantal representations of the right and left core inverses that have more simpler expressions than they are obtained in Theorems 3.6 and 3.7.

Corollary 3.13. *Let* $\mathbf{A} \in \mathbb{C}_s^{n\times n}$, $\operatorname{Ind}\mathbf{A} = 1$, *and there exist* $A^{\oplus}$ *and* $A_{\oplus}$. *Then* $A^{\oplus} = \left(a_{ij}^{\oplus,r}\right)$ *and* $A_{\oplus} = \left(a_{ij}^{\oplus,l}\right)$ *can be expressed as follows*

$$a_{ij}^{\oplus,r} = \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left| \left(\mathbf{A}^2\left(\mathbf{A}^2\right)^*\right)_{j.}(\hat{\mathbf{a}}_{i.}) \right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{s,n}} \left|\mathbf{A}^2\left(\mathbf{A}^2\right)^*\right|_{\alpha}^{\alpha}},$$

$$a_{ij}^{\oplus,l} = \frac{\sum_{\beta\in J_{s,n}\{i\}} \left| \left(\left(\mathbf{A}^2\right)^*\mathbf{A}^2\right)_{.i}(\check{\mathbf{a}}_{.j}) \right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s,n}} \left|\left(\mathbf{A}^2\right)^*\mathbf{A}^2\right|_{\beta}^{\beta}},$$

where $\hat{\mathbf{a}}_{i.}$ is the ith row of $\hat{\mathbf{A}} = \mathbf{A}\left(\mathbf{A}^2\right)^*$ and $\check{\mathbf{a}}_{.j}$ is the jth column of $\check{\mathbf{A}} = \left(\mathbf{A}^2\right)^*\mathbf{A}$.

3.3 Determinantal representations of the DMP and MPD inverses

The concept of the DMP inverse in complex matrices was introduced in [2] by S. Malik and N. Thome.

Definition 3.14. [2] Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\operatorname{Ind}\mathbf{A} = k$. A matrix $\mathbf{X} \in \mathbb{C}^{n\times n}$ is said to be the DMP inverse of $\mathbf{A}$ if it satisfies the conditions

$$\mathbf{XAX} = \mathbf{X}, \mathbf{XA} = \mathbf{A}^d\mathbf{A}, \textit{and } \mathbf{A}^k\mathbf{X} = \mathbf{A}^k\mathbf{A}^{\dagger}. \tag{27}$$

It is denoted as $\mathbf{A}^{d,\dagger}$.

Due to [2], if an arbitrary matrix satisfies the system of Eq. (27), then it is unique and has the following representation

$$\mathbf{A}^{d,\dagger} = \mathbf{A}^d\mathbf{A}\mathbf{A}^{\dagger}. \tag{28}$$

Theorem 3.15. *Let* $\mathbf{A} \in \mathbb{C}_s^{n\times n}$, $\operatorname{Ind}\mathbf{A} = k$, *and* $\operatorname{rk}\left(\mathbf{A}^k\right) = s_1$. *Then, its DMP inverse* $\mathbf{A}^{d,\dagger} = \left(a_{ij}^{d,\dagger}\right)$ *has the following determinantal representations.*

$$a_{ij}^{d,\dagger} = \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left| (\mathbf{AA}^*)_{j.}\left(\mathbf{u}_{i.}^{(1)}\right) \right|_{\alpha}^{\alpha}}{\sum_{\beta\in J_{s_1,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta} \sum_{\alpha\in I_{s,n}} |\mathbf{AA}^*|_{\alpha}^{\alpha}} = \tag{29}$$

$$= \frac{\sum_{\beta\in J_{s_1,n}\{i\}} \left| \left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{u}_{.j}^{(2)}\right) \right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s_1,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta} \sum_{\beta\in J_{s,n}} |\mathbf{AA}^*|_{\beta}^{\beta}}, \tag{30}$$

where

$$\mathbf{u}_{i.}^{(1)} = \left[\sum_{\beta\in J_{s_1,n}\{i\}} \left| \left(\mathbf{A}^{k+1}\right)_{.i}(\tilde{\mathbf{a}}_{.f}) \right|_{\beta}^{\beta}\right] \in \mathbb{C}^{1\times n},\ f = 1, \ldots, n,$$

$$\mathbf{u}_{.j}^{(2)} = \left[\sum_{\alpha\in I_{s,n}\{j\}} \left| (\mathbf{AA}^*)_{j.}(\tilde{\mathbf{a}}_{l.}) \right|_{\alpha}^{\alpha}\right] \in \mathbb{C}^{n\times 1},\ l = 1, \ldots, n.$$

Here, $\tilde{\mathbf{a}}_{.f}$ and $\hat{\mathbf{a}}_{l.}$ are the fth column and the lth row of $\tilde{\mathbf{A}} := \mathbf{A}^{k+1}\mathbf{A}^*$.

Proof. Taking into account (28) for $\mathbf{A}^{d,\dagger}$, we get

$$a_{ij}^{d,\dagger} = \sum_{l=1}^{n}\sum_{f=1}^{n} a_{il}^{d} a_{lf} a_{fj}^{\dagger}. \tag{31}$$

By substituting (12) and (9) for the determinantal representations of $\mathbf{A}^d$ and $\mathbf{A}^\dagger$ in (31), we get

$$\begin{aligned} a_{ij}^{d,\dagger} &= \\ &\sum_{l=1}^{n}\sum_{f=1}^{n} \frac{\sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.l}^{(k)}\right)\right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s_1,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}} a_{lf} \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{a}_{f.}^*\right)\right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}} = \\ &\sum_{l=1}^{n}\sum_{f=1}^{n} \frac{\sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}(\mathbf{e}_{.l})\right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s_1,n}} \left|\mathbf{A}^{k+1}\right|_{\beta}^{\beta}} \tilde{a}_{lf} \frac{\sum_{\alpha\in I_{s,n}\{j\}} \left((\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{f.})\right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}}, \end{aligned} \tag{32}$$

where $\mathbf{e}_{.l}$ and $\mathbf{e}_{l.}$ are the lth unit column and row vectors, and $\tilde{a}_{lf}$ is the (lf)th element of the matrix $\tilde{\mathbf{A}} = \mathbf{A}^{k+1}\mathbf{A}^*$. If we put

$$u_{if}^{(1)} := \sum_{l=1}^{n}\sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}(\mathbf{e}_{.l})\right|_{\beta}^{\beta} \tilde{a}_{lf} = \sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}(\tilde{\mathbf{a}}_{.f})\right|_{\beta}^{\beta},$$

as the fth component of the row vector $\mathbf{u}_{i.}^{(1)} = \left[u_{i1}^{(1)}, \ldots, u_{in}^{(1)}\right]$, then from

$$\sum_{f=1}^{n} u_{if}^{(1)} \sum_{\alpha\in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{f.})\right|_{\alpha}^{\alpha} = \sum_{\alpha\in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{u}_{i.}^{(1)}\right)\right|_{\alpha}^{\alpha},$$

it follows (29). If we initially obtain

$$u_{lj}^{(2)} := \sum_{f=1}^{n} \tilde{a}_{lf} \sum_{\alpha\in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{f.})\right|_{\alpha}^{\alpha} = \sum_{\alpha\in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\tilde{\mathbf{a}}_{l.})\right|_{\alpha}^{\alpha},$$

as the lth component of the column vector $\mathbf{u}_{.j}^{(2)} = \left[u_{1j}^{(2)}, \ldots, u_{nj}^{(2)}\right]$, then from

$$\sum_{l=1}^{n}\sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}(\mathbf{e}_{.l})\right|_{\beta}^{\beta} u_{lj}^{(2)} = \sum_{\beta\in J_{s_1,n}\{i\}} \left|\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{u}_{.j}^{(2)}\right)\right|_{\beta}^{\beta},$$

it follows (30). □

The name of the DMP inverse is in accordance with the order of using the Drazin inverse (D) and the Moore-Penrose (MP) inverse. In that connection, it would be logical to consider the following definition.

Definition 3.16. Suppose $\mathbf{A}\in\mathbb{C}^{n\times n}$ and $\operatorname{Ind}\mathbf{A} = k$. A matrix $\mathbf{X}\in\mathbb{C}^{n\times n}$ is said to be the MPD inverse of $\mathbf{A}$ if it satisfies the conditions

$$\mathbf{XAX} = \mathbf{X}, \mathbf{AX} = \mathbf{AA}^d, \textit{and } \mathbf{XA}^k = \mathbf{A}^\dagger\mathbf{A}^k.$$

It is denoted as $\mathbf{A}^{\dagger,d}$.

The matrix $\mathbf{A}^{\dagger,d}$ is unique, and it can be represented as

$$\mathbf{A}^{\dagger,d} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{d}. \tag{33}$$

Theorem 3.17. *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}_{s}$, $\operatorname{Ind}\mathbf{A} = k$, *and* $\operatorname{rk}\mathbf{A}^{k} = s_1$. *Then, its MPD inverse* $\mathbf{A}^{\dagger,d} = \left(a_{ij}^{\dagger,d}\right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,d} = \frac{\sum_{\beta\in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{v}_{.j}^{(1)}\right)\right|_{\beta}^{\beta}}{\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\beta\in I_{s_1,n}} \left|\mathbf{A}^{k+1}\right|_{\alpha}^{\alpha}} = \frac{\sum_{\alpha\in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{k+1}\right)_{j.}\left(\mathbf{v}_{i.}^{(2)}\right)\right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{s_1,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\alpha\in I_{s,n}} \left|\mathbf{A}^{k+1}\right|_{\alpha}^{\alpha}},$$

where

$$\mathbf{v}_{.j}^{(1)} = \left[\sum_{\alpha\in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{k+1}\right)_{j.}(\hat{\mathbf{a}}_{l.})\right|_{\alpha}^{\alpha}\right] \in \mathbb{C}^{n\times 1},\ l = 1, \dots, n$$

$$\mathbf{v}_{i.}^{(2)} = \left[\sum_{\beta\in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\hat{\mathbf{a}}_{.f}\right)\right|_{\beta}^{\beta}\right] \in \mathbb{C}^{1\times n},\ l = 1, \dots, n.$$

Here, $\hat{\mathbf{a}}_{l.}$ and $\hat{\mathbf{a}}_{.f}$ are the lth row and the fth column of $\hat{\mathbf{A}} := \mathbf{A}^*\mathbf{A}^{k+1}$.
Proof. The proof is similar to the proof of Theorem 3.15. □

3.4 Determinantal representations of the CMP inverse

Definition 3.18. [6] Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ has the core-nilpotent decomposition $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, where $\operatorname{Ind}\mathbf{A}_1 = \operatorname{Ind}\mathbf{A}$, $\mathbf{A}_2$ is nilpotent, and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1 = 0$. The CMP inverse of $\mathbf{A}$ is called the matrix $\mathbf{A}^{c,\dagger} := \mathbf{A}^{\dagger}\mathbf{A}_1\mathbf{A}^{\dagger}$.

Lemma 3.19. [6] *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$. *The matrix* $\mathbf{X} = \mathbf{A}^{c,\dagger}$ *is the unique matrix that satisfies the following system of equations:*

$$\mathbf{XAX} = \mathbf{X},\ \mathbf{AXA} = \mathbf{A}_1,\ \mathbf{AX} = \mathbf{A}_1\mathbf{A}^{\dagger},\ \textit{and}\ \mathbf{XA} = \mathbf{A}^{\dagger}\mathbf{A}_1.$$

Moreover,

$$\mathbf{A}^{c,\dagger} = \mathbf{Q}_A\mathbf{A}^{d}\mathbf{P}_A. \tag{34}$$

Taking into account (34), it follows the next theorem about determinantal representations of the quaternion CMP inverse.

Theorem 3.20. *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}_{s}$, $\operatorname{Ind}\mathbf{A} = m$, *and* $\operatorname{rk}(\mathbf{A}^{m}) = s_1$. *Then, the determinantal representations of its CMP inverse* $\mathbf{A}^{c,\dagger} = \left(a_{ij}^{c,\dagger}\right)$ *can be expressed as*

$$a_{ij}^{c,\dagger} = \frac{\sum_{\beta\in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{v}_{.j}^{(l)}\right)\right|_{\beta}^{\beta}}{\left(\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}\right)^2 \sum_{\beta\in J_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_{\beta}^{\beta}} \tag{35}$$

$$a_{ij}^{c,\dagger} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} \left(\mathbf{w}_{i.}^{(l)} \right) \right|_{\alpha}^{\alpha}}{\left(\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha} \right)^2 \sum_{\beta \in J_{s_1,n}} |\mathbf{A}^{m+1}|_{\beta}^{\beta}} \tag{36}$$

for all $l = 1, 2$, where

$$\mathbf{v}_{.j}^{(1)} = \left[\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} (\hat{\mathbf{u}}_{t.}) \right|_{\alpha}^{\alpha} \right] \in \mathbb{C}^{n \times 1}, t = 1, \dots, n, \tag{37}$$

$$\mathbf{w}_{i.}^{(1)} = \left[\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} (\hat{\mathbf{u}}_{.k}) \right|_{\beta}^{\beta} \right] \in \mathbb{C}^{1 \times n}, k = 1, \dots, n, \tag{38}$$

$$\mathbf{v}_{.j}^{(2)} = \left[\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}^*\mathbf{A})_{j.} (\tilde{\mathbf{g}}_{t.}) \right|_{\alpha}^{\alpha} \right] \in \mathbb{C}^{n \times 1}, t = 1, \dots, n, \tag{39}$$

$$\mathbf{w}_{i.}^{(2)} = \left[\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} (\tilde{\mathbf{g}}_{.k}) \right|_{\beta}^{\beta} \right] \in \mathbb{C}^{1 \times n}, k = 1, \dots, n. \tag{40}$$

Here, $\hat{\mathbf{u}}_{t.}$ is the tth row and $\hat{\mathbf{u}}_{.k}$ is the kth column of $\hat{\mathbf{U}} := \mathbf{U}\mathbf{A}\mathbf{A}^*$, $\tilde{\mathbf{g}}_{t.}$ is the tth row and $\tilde{\mathbf{g}}_{.k}$ is the kth column of $\tilde{\mathbf{G}} := \mathbf{A}^*\mathbf{A}\mathbf{G}$, and the matrices $\mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$ and $\mathbf{G} = (g_{ij}) \in \mathbb{H}^{n \times n}$ are such that

$$u_{ij} = \sum_{\alpha \in I_{s_1,n}\{j\}} \left| \left(\mathbf{A}^{m+1} \right)_{j.} (\hat{\mathbf{a}}_{i.}) \right|_{\alpha}^{\alpha}, \quad g_{ij} = \sum_{\beta \in J_{s_1,n}\{i\}} \left| \left(\mathbf{A}^{m+1} \right)_{.i} (\tilde{\mathbf{a}}_{.j}) \right|_{\beta}^{\beta},$$

where $\hat{\mathbf{a}}_{i.}$ is the ith row of $\hat{\mathbf{A}} := \mathbf{A}^*\mathbf{A}^{m+1}$ and $\tilde{\mathbf{a}}_{.j}$ is the jth column of $\tilde{\mathbf{A}} := \mathbf{A}^{m+1}\mathbf{A}^*$.
Proof. Taking into account (34), we get

$$a_{ij}^{c,\dagger} = \sum_{l=1}^{n} \sum_{k=1}^{n} q_{il}^{A} a_{lk}^{d} p_{kj}^{A}, \tag{41}$$

where $\mathbf{Q}_A = (q_{il}^A)$, $\mathbf{A}^d = (a_{il}^d)$, and $\mathbf{P}_A = (p_{il}^A)$.

a. Taking into account the expressions (13), (10), and (11) for the determinantal representations of $\mathbf{A}^d$, $\mathbf{Q}_A$, and $\mathbf{P}_A$, respectively, we have

$$a_{ij}^{c,\dagger} = \sum_{l} \sum_{t} \frac{\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} (\dot{\mathbf{a}}_{.t}) \right|_{\beta}^{\beta}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}} \frac{\sum_{\alpha \in I_{s_1,n}\{l\}} \left| \left(\mathbf{A}^{m+1} \right)_{l.} \left(\mathbf{a}_{t.}^{(m)} \right) \right|_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{s_1,n}} |\mathbf{A}^{m+1}|_{\alpha}^{\alpha}} \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{l.}) \right|_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}},$$

where $\dot{\mathbf{a}}_{.t}$ is the tth column of $\mathbf{A}^*\mathbf{A}$, $\ddot{\mathbf{a}}_{l.}$ is the lth row of $\mathbf{A}\mathbf{A}^*$, and $\mathbf{a}_{t.}^{(m)}$ is the tth row of $\mathbf{A}^m$. So, it is clear that

$$a_{ij}^{c,\dagger} = \sum_{l} \sum_{t} \sum_{k} \frac{\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^*\mathbf{A})_{.i} (\mathbf{e}_{.t}) \right|_{\beta}^{\beta} \hat{a}_{tk} \sum_{\alpha \in I_{s_1,n}\{l\}} \left| \left(\mathbf{A}^{m+1} \right)_{l.} (\mathbf{e}_{k.}) \right|_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{s_1,n}} |\mathbf{A}^{m+1}|_{\alpha}^{\alpha}} \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{l.}) \right|_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}},$$

where $\mathbf{e}_{.t}$ is the tth unit column vector, $\mathbf{e}_{k.}$ is the kth row vector, and $\hat{a}_{tk}$ is the (tk)th element of $\hat{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{m+1}$.
Denote

$$u_{tl} := \sum_k \hat{a}_{tk} \sum_{\alpha \in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{m+1}\right)_{l.}(\mathbf{e}_{k.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{m+1}\right)_{l.}(\hat{\mathbf{a}}_{t.})\right|_\alpha^\alpha \tag{42}$$

as the tth component of a column vector $\mathbf{u}_{.l} = [u_{1l}, \ldots, u_{nl}]$. Then from

$$\sum_t \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta u_{tl} = \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{u}_{.l})\right|_\beta^\beta,$$

we have

$$a_{ij}^{c,\dagger} = \sum_l \frac{\sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{u}_{.l})\right|_\beta^\beta \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{l.})\right|_\alpha^\alpha}{\sum_{\beta \in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_\alpha^\alpha \sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}.$$

Construct the matrix $\mathbf{U} = (u_{tl}) \in \mathbb{H}^{n\times n}$, where u_{tl} is given by (42), and denote $\hat{\mathbf{U}} := \mathbf{U}\mathbf{A}\mathbf{A}^*$. Then, taking into account that $|\mathbf{A}^*\mathbf{A}|_\beta^\beta = |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha$, we have

$$a_{ij}^{c,\dagger} = \frac{\sum_t \sum_k \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta \hat{u}_{tk} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha}{\left(\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2 \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_\alpha^\alpha}.$$

If we put that

$$v_{tj}^{(1)} := \sum_k \hat{u}_{tk} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\hat{\mathbf{u}}_{t.})\right|_\alpha^\alpha$$

is the tth component of a column vector $\mathbf{v}_{.j}^{(1)} = \left[v_{1j}^{(1)}, \ldots, v_{nj}^{(1)}\right]$, then from

$$\sum_t \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta v_{tj}^{(1)} = \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{v}_{.j}^{(1)}\right)\right|_\beta^\beta,$$

it follows (35) with $\mathbf{v}_{.j}^{(1)}$ given by (37). If we initially put

$$w_{ik}^{(1)} := \sum_t \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta \hat{u}_{tk} = \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\hat{\mathbf{u}}_{.k})\right|_\beta^\beta$$

as the kth component of the row vector $\mathbf{w}_{i.}^{(1)} = \left[w_{i1}^{(1)}, \ldots, w_{in}^{(1)}\right]$, then from

$$\sum_k w_{ik}^{(1)} \sum_{\alpha \in I_{s,n}\{j\}} \left|\left(\mathbf{A}^2\right)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s,n}\{j\}} \left|\left(\mathbf{A}^2\right)_{j.}\left(\mathbf{w}_{i.}^{(1)}\right)\right|_\alpha^\alpha,$$

it follows (36) with $\mathbf{w}_{i.}^{(1)}$ given by (38).

b. By using the determinantal representation (12) for $\mathbf{A}^d$ in (41), we have

$$a_{ij}^{c,\dagger} = \sum_k \sum_t \frac{\sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.t})\right|_\beta^\beta}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} \frac{\sum_{\beta \in J_{s_1,n}\{t\}} \left|\left(\mathbf{A}^{m+1}\right)_{.t}\left(\mathbf{a}_{.k}^{(m)}\right)\right|_\beta^\beta}{\sum_{\beta \in J_{s,n}} \left|\mathbf{A}^{m+1}\right|_\beta^\beta} \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{k.})\right|_\alpha^\alpha}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}.$$

Therefore,

$$a_{ij}^{c,\dagger} = \sum_l \sum_k \sum_t \frac{\sum_{\beta \in J_{s,n}\{i\}} \left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.t})\right)_\beta^\beta}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} \times$$

$$\frac{\sum_{\beta \in J_{s_1,n}\{t\}} \left|\left(\mathbf{A}^{m+1}\right)_{.t}(\mathbf{e}_{.k})\right|_\beta^\beta}{\sum_{\beta \in J_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_\beta^\beta} \tilde{a}_{kl} \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{l.})\right|_\alpha^\alpha}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}.$$

where $\mathbf{e}_{.k}$ is the kth unit column vector, $\mathbf{e}_{l.}$ is the lth unit row vector, and $\tilde{a}_{kl}$ is the (kl)th element of $\tilde{\mathbf{A}} = \mathbf{A}^{m+1}\mathbf{A}^*$.

If we denote

$$g_{tl} := \sum_l \sum_{\beta \in J_{s_1,n}\{t\}} \left|\left(\mathbf{A}^{m+1}\right)_{.t}(\mathbf{e}_{.k})\right|_\beta^\beta \tilde{a}_{kl} = \sum_{\beta \in J_{s_1,n}\{t\}} \left|\left(\mathbf{A}^{m+1}\right)_{.t}(\tilde{\mathbf{a}}_{.l})\right|_\beta^\beta \tag{43}$$

as the lth component of a row vector $\mathbf{g}_{t.} = [g_{t1}, \ldots, g_{tn}]$, then

$$\sum_l g_{tl} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{l.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{g}_{t.})\right|_\alpha^\alpha.$$

From this, it follows that

$$a_{ij}^{c,\dagger} = \sum_t \frac{\sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.t})\right|_\beta^\beta \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*|_{j.}(\mathbf{g}_{t.})\right)_\alpha^\alpha}{\sum_{\beta \in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_\alpha^\alpha \sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}.$$

Construct the matrix $\mathbf{G} = (g_{tl}) \in \mathbb{H}^{n \times n}$, where g_{tl} is given by (43). Denote $\tilde{\mathbf{G}} := \mathbf{A}^*\mathbf{A}\mathbf{G}$. Then,

$$a_{ij}^{c,\dagger} = \frac{\sum_t \sum_k \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta \tilde{g}_{tk} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha}{\sum_{\beta \in J_{s,n}} \left(|\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2 \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{m+1}\right|_\alpha^\alpha}.$$

If we denote

$$v_{tj}^{(2)} := \sum_k \tilde{g}_{tk} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\tilde{\mathbf{g}}_{t.})\right|_\alpha^\alpha$$

as the tth component of a column vector $\mathbf{v}_{.j}^{(2)} = \left[v_{1j}^{(2)}, \dots, v_{nj}^{(2)}\right]$, then

$$\sum_t \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta v_{tj}^{(2)} = \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{v}_{.j}^{(2)}\right)\right|_\beta^\beta.$$

Thus, we have (35) with $\mathbf{v}_{.j}^{(2)}$ given by (39).

If, now, we denote

$$w_{ik}^{(2)} := \sum_t \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.t})\right|_\beta^\beta \tilde{g}_{tk} = \sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{g}}_{.k})\right|_\beta^\beta$$

as the kth component of a row vector $\mathbf{w}_{i.}^{(2)} = \left[w_{i1}^{(2)}, \dots, w_{in}^{(2)}\right]$, then

$$\sum_k w_{ik}^{(2)} \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{e}_{k.})\right|_\alpha^\alpha = \sum_{\alpha \in I_{s,n}\{j\}} \left|(\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{w}_{i.}^{(2)}\right)\right|_\alpha^\alpha.$$

So, finally, we have (36) with $\mathbf{w}_{i.}^{(2)}$ given by (40).

4. An example

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -i & i & i \\ -i & -i & -i \end{bmatrix}.$$

Since

$$\mathbf{A}\mathbf{A}^* = \begin{bmatrix} 4 & 2\mathbf{i} & 2\mathbf{i} \\ -2\mathbf{i} & 3 & -1 \\ -2\mathbf{i} & -1 & 3 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 2-2i & 0 & 0 \\ -2-2i & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 4-4i & 0 & 0 \\ -4-4i & 0 & 0 \end{bmatrix},$$

then $\operatorname{rk}\mathbf{A} = 2$ and $\operatorname{rk}\mathbf{A}^2 = \operatorname{rk}\mathbf{A}^3 = 1$, and $k = \operatorname{Ind}\mathbf{A} = 2$ and $r_1 = 1$. So, we shall find $\mathbf{A}^{\oplus}$ and $\mathbf{A}_{\oplus}$ by (25) and (26), respectively.

Since

$$\hat{\mathbf{A}} = \mathbf{A}^2\left(\mathbf{A}^3\right)^* = 16\begin{bmatrix} 2 & 1+\mathbf{i} & -1+\mathbf{i} \\ 1-\mathbf{i} & 1 & \mathbf{i} \\ -1-\mathbf{i} & \mathbf{i} & 1 \end{bmatrix},$$

then by (25),

$$a_{11}^{\oplus,r} = \frac{\sum_{\alpha \in I_{1,3}\{1\}} \left|\left(\mathbf{A}^3\left(\mathbf{A}^3\right)^*\right)_{1.}(\hat{\mathbf{a}}_{1.})\right|_\alpha^\alpha}{\sum_{\alpha \in I_{1,3}} \left|\mathbf{A}^3\left(\mathbf{A}^3\right)^*\right|_\alpha^\alpha} = \frac{1}{4}.$$

By similarly continuing, we get

$$\mathbf{A}^{\oplus}=\frac{1}{8}\begin{bmatrix} 2 & 1+\mathbf{i} & -1+\mathbf{i} \\ 1-\mathbf{i} & 1 & \mathbf{i} \\ -1-\mathbf{i} & \mathbf{i} & 1 \end{bmatrix}.$$

By analogy, due to (26), we have

$$\mathbf{A}_{\oplus}=\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The DMP inverse $\mathbf{A}^{d,\dagger}$ can be found by Theorem 3.15. Since

$$\tilde{\mathbf{A}}=\mathbf{A}^3\mathbf{A}^*=4\begin{bmatrix} 4 & 2\mathbf{i} & 2\mathbf{i} \\ 2-2\mathbf{i} & 1+\mathbf{i} & 1+\mathbf{i} \\ -2-2\mathbf{i} & 1-\mathbf{i} & 1-\mathbf{i} \end{bmatrix}.$$

and $\operatorname{rk}\left(\mathbf{A}^3\right)=1$, then

$$\mathbf{u}_1^{(1)}=\tilde{\mathbf{a}}_{1.},\ \mathbf{u}_2^{(1)}=\tilde{\mathbf{a}}_{2.},\ \mathbf{u}_3^{(1)}=\tilde{\mathbf{a}}_{3.}.$$

Furthermore, by (29),

$$a_{11}^{d,\dagger}=\frac{\sum_{\alpha\in I_{2,3}\{1\}}\left|(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{u}_{1.}^{(1)}\right)\right|_{\alpha}^{\alpha}}{\sum_{\beta\in J_{1,3}}\left|\mathbf{A}^3\right|_{\beta}^{\beta}\sum_{\alpha\in I_{2,3}}\left|\mathbf{A}\mathbf{A}^*\right|_{\alpha}^{\alpha}}=\frac{1}{192}\left(det\begin{bmatrix} 16 & 8\mathbf{i} \\ -2\mathbf{i} & 3 \end{bmatrix}+det\begin{bmatrix} 16 & 8\mathbf{i} \\ -2\mathbf{i} & 3 \end{bmatrix}\right)=\frac{1}{3}.$$

By similarly continuing, we get

$$\mathbf{A}^{d,\dagger}=\frac{1}{12}\begin{bmatrix} 4 & 2\mathbf{i} & 2\mathbf{i} \\ 2-2\mathbf{i} & 1+\mathbf{i} & 1+\mathbf{i} \\ -2-2\mathbf{i} & 1-\mathbf{i} & 1-\mathbf{i} \end{bmatrix}.$$

Similarly by Theorem 3.17, we get

$$\mathbf{A}^{\dagger,d}=\frac{1}{4}\begin{bmatrix} 2 & 0 & 0 \\ -\mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 \end{bmatrix}.$$

Finally, by theorem, we find the CMP inverse $\mathbf{A}^{c,\dagger}=\left(a_{ij}^{c,\dagger}\right)$. Since $\operatorname{rk}\mathbf{A}^3=1$, then $\mathbf{G}=\tilde{\mathbf{A}}$ and

$$\tilde{\mathbf{G}} = \mathbf{A}^*\mathbf{A}\tilde{\mathbf{A}} = 16\begin{bmatrix} 6 & 3\mathbf{i} & 3\mathbf{i} \\ -2\mathbf{i} & 1 & 1 \\ -2\mathbf{i} & 1 & 1 \end{bmatrix}.$$

Furthermore, by (40),

$$w_{11}^{(2)} = \sum_{\beta \in J_{2,3}\{1\}} \left|(\mathbf{A}^*\mathbf{A})_{.1}\left(\tilde{\mathbf{g}}_{.1}\right)\right|_{\beta}^{\beta} = \left(det\begin{bmatrix} 6 & 0 \\ -2\mathbf{i} & 2 \end{bmatrix} + det\begin{bmatrix} 6 & 0 \\ -2\mathbf{i} & 2 \end{bmatrix}\right) = 24.$$

By similar calculations, we get

$$\mathbf{w}_{1.}^{(2)} = [384, 96\mathbf{i}, 96\mathbf{i}],\ \ \mathbf{w}_{2.}^{(2)} = [-192\mathbf{i}, 96, 96],\ \ \mathbf{w}_{3.}^{(2)} = [-192\mathbf{i}, 96, 06].$$

So, by (36), we get

$$\begin{aligned} a_{11}^{c,\dagger} &= \frac{\sum_{\alpha \in I_{2,3}\{1\}} \left|(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{w}_{1.}^{(2)}\right)\right|_{\alpha}^{\alpha}}{\left(\sum_{\alpha \in I_{2,3}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}\right)^2 \sum_{\beta \in J_{1,3}} |\mathbf{A}^3|_{\beta}^{\beta}} \\ &= \frac{1}{4608}\left(det\begin{bmatrix} 384 & 192\mathbf{i} \\ -2\mathbf{i} & 3 \end{bmatrix} + det\begin{bmatrix} 384 & 192\mathbf{i} \\ -2\mathbf{i} & 3 \end{bmatrix}\right) = \frac{1}{3}. \end{aligned}$$

By similarly continuing, we derive

$$\mathbf{A}^{c,\dagger} = \frac{1}{12}\begin{bmatrix} 4 & 2\mathbf{i} & 2\mathbf{i} \\ -2\mathbf{i} & 1 & 1 \\ -2\mathbf{i} & 1 & 1 \end{bmatrix}.$$

5. Conclusions

In this chapter, we get the direct method to find the core inverse and its generalizations that are based on their determinantal representations. New determinantal representations of the right and left core inverses, the right and left core-EP inverses, the DMP, MPD, and CMP inverses are derived.

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